

Space-time multi-domain spectral collocation method for the Fisher's equation

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Abstract. A space-time multi-domain Legendre-Gauss-Lobatto collocation method is developed to solve Fisher's equation with nonhomogeneous boundary conditions, addressing aliasing errors caused by nonlinear terms. For large-scale problems in the spatial domain, a multi-domain collocation method is proposed, where appropriate boundary conditions are imposed at the common interfaces of adjacent subdomains to ensure the global smoothness of the numerical solution. A quasi-orthogonal approximation is constructed to fit both the function values at the interval endpoints and those at interior points. The convergence and stability of the scheme are analyzed using the Petrov-Galerkin spectral method with numerical integration. Numerical experiments demonstrate the effectiveness and accuracy of the proposed method.

Keywords: space-time spectral collocation methods for Fisher's equations; Legendre-Gauss-Lobatto nodes; composite multi-domain quasi-orthogonal approximation and the Petrov-Galerkin spectral method.

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1 Introduction

Fisher's equation [4], a reaction-diffusion model proposed in 1937, applies to physics, biology, chemistry, and economics [8, 16, 31]. Recent extensions integrate neural and complex networks to analyze propagation in complex systems [15]. Existing solution methods include the exp-function [9], split Newton iteration [12], finite difference [10, 21], meshless [22], first integral [17], and Jacobi wavelet collocation [18] methods. Spectral methods are widely employed [11], often combining spectral spatial discretization with finite difference time integration [28, 30].

In numerical computation, spectral method selection depends on problem characteristics. The spectral-Galerkin method addresses linear problems with constant or polynomial coefficients on rectangular domains; the Petrov-

Galerkin method handles odd-order derivatives or non-homogeneous boundary conditions; and the spectral collocation method applies to variable-coefficient or nonlinear problems.

Researchers have extensively employed spectral collocation with Legendre and Chebyshev-Gauss-Lobatto nodes for ODEs, nonlinear PDEs, and integral equations [19, 23, 24, 26, 27, 32, 33], as well as fractional-order equations [1, 2, 3]. Spectral methods achieve high accuracy with fewer nodes than finite difference schemes. To minimize errors in both space and time, spatio-temporal spectral and spectral element methods have been developed [13, 25, 34], including specific applications to the KdV equation [14]. For large-scale problems where high approximation orders increase computational cost, domain decomposition methods provide an efficient alternative.

Building on the work mentioned above, this paper explores the binary multi-domain collocation method for the Fisher's equation in bounded domains. We now turn our attention to the initial boundary value problem of the Fisher's equation [30].

$$\begin{cases} \partial_t u - \partial_x^2 u = u(1-u), x \in \mathbb{D} = (a, b), t \in (0, T], \\ u(a, t) = g_1(t), u(b, t) = g_2(t), t \in [0, T], \\ u(x, 0) = g_0(x), x \in \mathbb{D}. \end{cases} \quad (1.1)$$

Regarding Equation (1.1), we consider the case $h = b - a \gg 1$. Using Lagrange interpolation polynomials, we partition Equation (1.1) into multiple domains to construct a spectral collocation scheme. In each sub-domain, Legendre-Gauss-Lobatto nodes are used for both time and space, forming corresponding nonlinear systems. To ensure continuity at the boundaries of adjacent sub-domains, constraint conditions are applied, ensuring equal function and first derivative values at common boundaries, thus transforming Equation (1.1) into a nonlinear matrix equation. The numerical solution is obtained iteratively.

Numerical experiments show that the multi-domain collocation method achieves spectral accuracy and significantly outperforms the single-domain method in error precision, proving the algorithm's effectiveness. By combining multi-domain pseudo-orthogonal approximation with the Petrov-Galerkin method and numerical integration, we establish the algorithm's convergence and stability. The main advantages of this approach are as follows:

- We apply spectral collocation in both space and time for evolution equations, balancing node distributions. Unlike finite difference methods, this approach mitigates aliasing errors from nonlinear terms and reduces system dimensionality, thereby enhancing computational efficiency.
- For large-scale problems, we develop a multi-domain spectral collocation method that decomposes the spatial domain into subdomains. This strategy significantly improves accuracy over single-domain approaches while maintaining high efficiency despite increased implementation complexity.
- We establish a rigorous convergence analysis for both single- and multi-domain schemes using quasi-orthogonal approximation and the Petrov-

Galerkin method with numerical integration. Theoretical findings on stability and convergence are validated by numerical experiments.

- Distinct from sequential discretization methods [28], our approach employs bivariate interpolation to directly transform nonlinear differential equations into matrix systems, yielding superior efficiency. Furthermore, unlike the time-axis decomposition in [25], our multi-domain strategy focuses on spatial decomposition.
- We present a novel error analysis for the multi-domain method, leveraging quasi-orthogonal operators tailored for interior point interpolation combined with Petrov-Galerkin numerical integration. To our knowledge, such an analysis is absent in current literature.

The next section reviews the differentiation matrix for Lagrange interpolation using Legendre-Gauss-Lobatto nodes, discrete inner products, and interpolation results, and introduces the multi-domain pseudo-orthogonal approximation as the basis for convergence analysis. Sections 3 and 4 present the spatiotemporal spectral collocation scheme for Fisher's equation in single and multiple domains, with numerical results and convergence and stability analysis. The paper concludes with a brief summary.

2 Preliminaries

In this section, we review the differentiation matrix related to Legendre interpolation and its interpolation approximation, as well as the pseudo-orthogonal approximation results.

2.1 Legendre interpolation and its differentiation matrix

We note $\frac{\partial u}{\partial \xi} = \partial_\xi u$. Let $L_M(\xi)$ ($\xi \in I = (-1, 1)$) represent the M order Legendre polynomial, where $\xi_0 = -1$, $\xi_M = 1$ and ξ_m ($m = 1, 2, \dots, M-1$) is the root of $\partial_\xi L_M(\xi) = 0$, then the interpolation basis function with ξ_m as the node is as follows (cf. [14])

$$\varphi_m(\xi) = \frac{(\xi^2 - 1)\partial_\xi L_M(\xi)}{M(M+1)L_M(\xi_m)(\xi - \xi_m)}.$$

Denoted $\mathcal{P}_M(I)$ is the set of polynomials whose order does not exceed M , and for $u(\xi) \in C(I)$, its interpolated polynomial is:

$$P_M(\xi) = \sum_{m=0}^M u_m \varphi_m(\xi), u_m = u(\xi_m), \xi \in I.$$

Write $P_M(\xi)$ as a function of ξ , and find its n -th derivative to obtain:

$$\partial_\xi^n P_M(\xi) = \sum_{m=0}^M u_m \partial_\xi^n \varphi_m(\xi).$$

The entries of first-order differentiation matrix $D = (d_{km})_{0 \leq m, k \leq M}$ and the n - order ($n \geq 2$) differentiation matrix $D^{(n)} = (d_{km}^{(n)})_{0 \leq m, k \leq M}$ of $\varphi_m(\xi)$ are as follows:

$$d_{km} = \partial_\xi \varphi_m(\xi_k), \quad d_{km}^{(n)} = \partial_\xi^n \varphi_m(\xi_k). \quad (2.1)$$

The entries of the first-order differentiation matrix D can be determined by the explicit formulas below, see (3.68) and (3.203) of literature [20],

$$d_{km} = \begin{cases} \frac{L_M(\xi_k)}{(\xi_k - \xi_m)L_M(\xi_m)}, & k \neq m, \\ -M(M+1)/4, & k = m = 0, \\ M(M+1)/4, & k = m = M, \\ 0, & 0 < k = m < M. \end{cases}$$

The n -th differentiation matrix is the n power of the first-order differential matrix, namely $D^{(n)} = D^n$.

2.2 Legendre interpolation approximation on domain $\mathbb{D}$

For any integer $r \geq 0$, we define the space $H^r(\mathbb{D})$ and its norm $\|u\|_{r, \mathbb{D}}$ as usual. Let $\mathcal{P}_M(\mathbb{D})$ be similar to $\mathcal{P}_M(I)$. Let's make $\rho_{M, I, j}$ ($0 \leq j \leq M$) denote the Christoffel number of the corresponding node ξ_j , $j = 1, 2, \dots, M$, we consider the discrete inner product and the discrete norm on the $L^2(\mathbb{D})$. Let $\rho_{N, \mathbb{D}, j} = \frac{b-a}{2} \rho_{M, I, j}$ ($0 \leq j \leq M$) be the Christoffel number corresponding to node $x_j = \frac{b-a}{2} \xi_j + \frac{b+a}{2}$, $j = 1, 2, \dots, M$. Let's introduce the discrete inner product and the discrete norm on the $L^2(\mathbb{D})$:

$$(f, g)_{M, \mathbb{D}} = \sum_{j=0}^M f(y_j)g(x_j)\rho_{M, \mathbb{D}, j}, \quad \|f\|_{M, \mathbb{D}} = (f, f)_{M, \mathbb{D}}^{\frac{1}{2}}.$$

From literature [29], we can get:

$$(\varphi, \phi)_{M, \mathbb{D}} = (\varphi, \phi)_{\mathbb{D}}, \quad \forall \varphi, \phi \in \mathcal{P}_{2M-1}(\mathbb{D}), \quad (2.2)$$

$$\|\varphi\|_{\mathbb{D}} \leq \|\varphi\|_{M, \mathbb{D}} \leq \sqrt{2 + 1/M} \|\varphi\|_{\mathbb{D}}, \quad \forall \varphi \in \mathcal{P}_M(\mathbb{D}). \quad (2.3)$$

Let $\chi^{(\gamma, \delta)}(\xi) = (1-\xi)^\gamma(1+\xi)^\delta$ be the Jacobi weight function, $\xi = \frac{2x-a-b}{h} \in I$, and $\chi_{\mathbb{D}}^{\gamma, \delta}(x) = \chi^{(\gamma, \delta)}(\xi)|_{\xi = \frac{2x-a-b}{h}}$. For any $v \in C(\bar{\mathbb{D}})$, the interpolation approximation $\pi_{L, M, \mathbb{D}} v \in \mathcal{P}_M(\mathbb{D})$, is defined as:

$$\pi_{L, M, \mathbb{D}} v(x_j) = v(x_j), \quad j = 0, 1, \dots, M.$$

From literature [29] and the transform of the above, we have that if $v \in C(\bar{\mathbb{D}})$, $(1 - (\xi(x))^2)^{\frac{r-1}{2}} \partial_x^r v \in L^2(\mathbb{D})$ and the integer $r \geq 1$, then:

$$\|\pi_{L, M, \mathbb{D}} v - v\|_{\mathbb{D}} \leq c \left(\frac{h}{M}\right)^r \|\partial_x^r v\|_{\chi_{\mathbb{D}}^{(r-1, r-1)}, \mathbb{D}}, \quad h = b - a. \quad (2.4)$$

$$|(w, \psi)_{\mathbb{D}} - (w, \psi)_{M, \mathbb{D}}| \leq c \left(\frac{h}{M}\right)^r \|\partial_x^r w\|_{\chi_{\mathbb{D}}^{(r-1, r-1)}, \mathbb{D}} \|p\|_{\mathbb{D}}, \quad \psi \in \mathcal{P}_M(\mathbb{D}). \quad (2.5)$$

We introduce the quasi-orthogonal mapping on $\mathbb{D}$. Let

$$\mathcal{F}(\mathbb{D}) = H^1(\mathbb{D}) \cap \{w \mid \text{there is finite traces of } w \text{ at } x = a, b\},$$

$$\mathcal{F}_0(\mathbb{D}) = \{w \mid w \in \mathcal{F}(\mathbb{D}), w(a) = w(b) = 0\}, \quad \mathcal{P}_N^0(\mathbb{D}) = \mathcal{P}_N(\mathbb{D}) \cap \mathcal{F}_0(\mathbb{D}).$$

The orthogonal mapping $\pi_{M,\mathbb{D}}^{1,0} : \mathcal{F}_0(\mathbb{D}) \rightarrow \mathcal{P}_M^0(\mathbb{D})$ is defined by

$$(\partial_x(\pi_{M,\mathbb{D}}^{1,0}w - w), \partial_x\phi)_{\mathbb{D}} = 0, \quad \forall \phi \in \mathcal{P}_M^0(\mathbb{D}).$$

For any $w \in \mathcal{F}(\mathbb{D})$, we set $\tilde{w}(x) = w(x) - \hat{w}(x) \in \mathcal{F}_0(\mathbb{D})$, $\hat{w}(x) = \frac{w(b)-w(a)}{b-a}(x-a) + w(a)$. The quasi-orthogonal projection $\pi_{M,\mathbb{D}}^{1,*}$ is defined by

$$\pi_{M,\mathbb{D}}^{1,*}w = \pi_{M,\mathbb{D}}^{1,0}\tilde{w} + \hat{w}(x).$$

Define the quasi-orthogonal projection $\pi_{M,I}^{1,*}w(\xi)$ in the same way as Lemma 4 in [7]. Obviously, $\pi_{M,\mathbb{D}}^{1,*}w(x) = \pi_{M,I}^{1,*}w(\xi)|_{\xi=\frac{2x-a-b}{h}}$. From the Lemma 2.4 in literature [7] and $\xi(x) = 2x - a - b/h$, we give the following result.

Lemma 1. *If $w \in \mathcal{F}(\mathbb{D})$, $\partial_x^r w \in L^2_{\chi_{\mathbb{D}}^{(r-1,r-1)}}(\mathbb{D})$ and integer $r \geq 1$, then*

$$\|\partial_x^\mu(\pi_{M,\mathbb{D}}^{1,*}w - w)\|_{\mathbb{D}} \leq c\left(\frac{h}{M}\right)^{r-\mu} \|\partial_x^r w\|_{\chi_{\mathbb{D}}^{(r-1,r-1)},\mathbb{D}}, \quad \mu = 0, 1. \quad (2.6)$$

Remark 1. The quasi-orthogonal approximation $\pi_{M,\mathbb{D}}^{1,*}$ of the function $w \in H^1(\mathbb{D})$ is the sum of the orthogonal projection $\pi_{M,\mathbb{D}}^{1,0}$ of the function $\tilde{w}(x) \in H_0^1(\mathbb{D})$ that is homogeneous at the endpoint and the interpolation

$$\hat{w}(x)|_{x=a,b} = \pi_{L,M,\mathbb{D}}w(x)|_{x=a,b} = w(x)|_{x=a,b},$$

at the endpoint. In other words, the quasi-orthogonal operator $\pi_{M,\mathbb{D}}^{1,*}$ is a homogeneous orthogonal operator $\pi_{M,\mathbb{D}}^{1,0}$ plus the boundary's interpolation operator $\pi_{L,M,\mathbb{D}}$. Obviously, the quasi-orthogonal approximation satisfies:

$$\pi_{M,\mathbb{D}}^{1,*}w(x)|_{x=a,b} = w(x)|_{x=a,b}.$$

We need the following embedding result, see Theorem B.5 of [20].

Lemma 2. *Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, and let $r > \frac{n}{2}$. Then there exists a positive constant C_0 such that for $\forall u \in H^r(\Omega)$ ([20]),*

$$\|u\|_{L^\infty(\Omega)} \leq C_0 \|u\|_{r,\Omega}. \quad (2.7)$$

The combination of the two lemmas above gives the following result.

Lemma 3. *If $w \in H^1(\mathbb{D})$, $\partial_x^r w \in L^2_{\chi_{\mathbb{D}}^{(r-1,r-1)}}(\mathbb{D})$ and integer $r \geq 1$, then*

$$\|\pi_{M,\mathbb{D}}^{1,*}w\|_{L^\infty(\mathbb{D})} \leq \|w\|_{L^\infty(\mathbb{D})} + c\left(\frac{h}{M}\right)^{r-\mu} \|\partial_x^r w\|_{\chi_{\mathbb{D}}^{(r-1,r-1)},\mathbb{D}}, \quad \mu = 0, 1. \quad (2.8)$$

Similarly,

$$\|\pi_{L,M,\mathbb{D}}w\|_{L^\infty(\mathbb{D})} \leq \|w\|_{L^\infty(\mathbb{D})} + c\left(\frac{h}{M}\right)^r \|\partial_x^r w\|_{\chi_{\mathbb{D}}^{(r-1,r-1)},\mathbb{D}}.$$

Next, using Theorem 2.7, and Lemma 4.10 of [5], we present an inverse inequalities used in later parts.

Lemma 4. $\forall \varphi \in \mathcal{P}_M(\mathbb{D})$ and integer $q \geq 1, p \geq 2$, then

$$\|\varphi\|_{L^{2q}(\mathbb{D})} \leq c(\sqrt{\frac{2}{h}}M)^{1-\frac{1}{q}}\|\varphi\|_{\mathbb{D}}, \quad \|\varphi\|_{L^p(\mathbb{D}),M} \leq (\sqrt{\frac{6}{h}}M)^{1-\frac{2}{p}}\|\varphi\|_{\mathbb{D}}.$$

3 The binary collocation method of the Fisher's equation

This section presents the binary collocation scheme to Fisher's equation in a single domain, along with the corresponding numerical results. The convergence and stability of the algorithmic scheme are also proven.

3.1 Single-domain collocation scheme and numerical results

We perform transformation

$$x = \frac{b-a}{2}y + \frac{b+a}{2}, \quad y \in I_y = (-1, 1), \quad t = \frac{T}{2}s + \frac{T}{2}, \quad s \in I_s = (-1, 1],$$

which converts $(y, s) \in \Lambda = I_y \times I_s$ into $(x, t) \in \Omega = \mathbb{D} \times (0, T)$. The Legendre interpolation approximation method of Equation (1.1) is to find the polynomial $u_{M,N}(y, s) \in \mathcal{P}_{M,N}(\Lambda) = \mathcal{P}_N(I_y) \times \mathcal{P}_M(I_s)$ such that

$$\begin{cases} \frac{2}{T}\partial_s u_{M,N}(y, s) - \left(\frac{2}{b-a}\right)^2 \partial_y^2 u_{M,N}(y, s) = -u_{M,N}^2(y, s) + u_{M,N}(y, s), \\ y = y_k, \quad k = 1, 2, \dots, N-1, \quad s = s_l, \quad l = 1, 2, \dots, M, \\ u_{M,N}(-1, s) = g_1(s), \quad u_{M,N}(1, s) = g_2(s), \quad s = s_l, \quad l = 1, 2, \dots, M, \\ u_{M,N}(y, -1) = g_0(y), \quad y = y_k, \quad k = 0, 1, 2, \dots, N. \end{cases} \quad (3.1)$$

The numerical solution is written as:

$$u_{M,N}(y, s) = \sum_{n=0}^N \sum_{m=0}^M \hat{u}_{m,n} \varphi_n(y) \varphi_m(s), \quad \hat{u}_{m,n} = u(y_n, s_m). \quad (3.2)$$

Substituting the numerical solution (3.2) into (3.1), using the properties of Lagrange interpolation basis functions and the differentiation matrix (2.1) gives:

$$\begin{cases} \frac{2}{T} \sum_{m=1}^M \hat{u}_{m,k} d_{l,m} - \frac{4}{(b-a)^2} \sum_{n=1}^{N-1} \hat{u}_{l,n} d_{k,n}^{(2)} + (\hat{u}_{l,k})^2 - \hat{u}_{l,k} \\ = \left(\frac{2}{b-a}\right)^2 (\hat{u}_{l,0} d_{k,0}^{(2)} + \hat{u}_{l,N} d_{k,N}^{(2)}) - \frac{2}{T} \hat{u}_{0,k} \hat{d}_{l,0}, \\ k = 1, 2, \dots, N-1, l = 1, 2, \dots, M. \end{cases} \quad (3.3)$$

We expand Equation (3.3) according to $l = 1, 2, \dots, M$, and $k = 1, 2, \dots, N-1$. The system of Equations (3.3) can be transformed into a nonlinear matrix equation:

$$AX - XB^T + X \cdot^* X - X = F - D, \quad (3.4)$$

where “ $\cdot$ ” represents the multiplication of the corresponding element of the matrix, and the matrices A, B, D, F and X are placed in Appendix A. For the convenience of programming, by stretching and transposing the matrices X, D and F , we define them as Y, R and H , respectively. Therefore, Equation (3.4) can be equivalently rewritten as:

$$(A \otimes E_{N-1} - E_M \otimes B)Y = H - R - Y \cdot *Y + Y,$$

where “ $\otimes$ ” denotes the Kronecker product and E_N represents the N -order identity matrix.

The numerical errors are measured using the L^∞ -norm and the normalized L^2 -norm, respectively. Let $u(x_k, t_l)$ be the exact solution of (3.1), and $u_{M,N}(x_k, t_l)$ be the numerical solution, we have that

$$E_{M,N} = \max_{1 \leq l \leq M, 1 \leq k \leq N-1} |u(x_k, t_l) - u_{M,N}(x_k, t_l)|,$$

$$L_{M,N}^2 = \left(\frac{1}{|T||\Omega|} \int_T \int_\Omega |u(x, t) - u_{M,N}(x, t)|^2 dx dt \right)^{1/2}.$$

As observed in [30], the exact solution of the Fisher’s equation is:

$$u(x, t) = \frac{1}{4} \left(1 - \tanh \frac{\sqrt{6}}{12} \left(x - \frac{5\sqrt{6}}{6} t \right) \right)^2, \quad g_0(x) = \frac{1}{4} \left(1 - \tanh \frac{\sqrt{6}}{12} x \right)^2,$$

$$g_i(t) = \frac{1}{4} \left(1 - \tanh \frac{\sqrt{6}}{12} \left(a_i - \frac{5\sqrt{6}}{6} t \right) \right)^2, \quad i=1, 2 \text{ with } a_1 = a, a_2 = b.$$

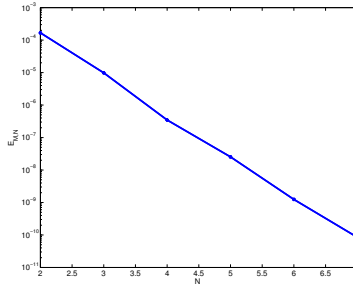


Figure 1. L^∞ -error in the spatial direction of a single domain.

Figure 1 depicts the maximum error $E_{M,N}$ versus the spatial interpolation order N for single-domain simulations with parameters ($M=13, T=2, a=-1, b=1$). In this case ($b-a=2$), the error decays as N increases, confirming the algorithm’s high spatial accuracy, particularly for smaller domain widths.

Figure 2 depicts the L^2 -error and the maximum error $E_{M,N}$ versus the spatial interpolation order N for single-domain simulations with parameters ($M=12, T=2, a=0, b=55$). The results indicate that the error in the L^2 -norm is slightly lower than in the L^∞ -norm, consistent with empirical

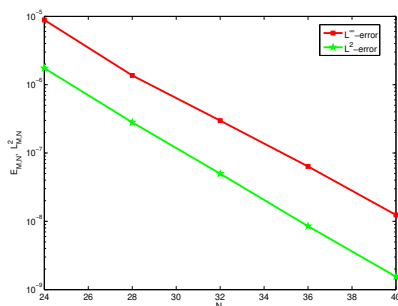


Figure 2. L^2 vs. L^∞ – error in the spatial direction of a single domain.

observations. However, for the larger domain in Figure 2 ($b-a = 55$), numerical errors persist despite increasing N , highlighting the challenges posed by wide domains.

Figure 3 fixes the spatial degree at $N = 40$ (using the parameters of Figure 2) and illustrates the variation of $E_{M,N}$ with the temporal interpolation degree M . The results demonstrate that the algorithm effectively approximates Fisher's equation over time, achieving high temporal precision.

In Table 1, we present the L^2 -errors of scheme (3.1) with parameters $a = 0$, $b = 3$, $M = 35$, and $N = 5$ at different time instances t . The results confirm that scheme (3.1) maintains numerical stability over extended time intervals, and these numerical findings are consistent with the theoretical analysis presented in Subsection 3.3.

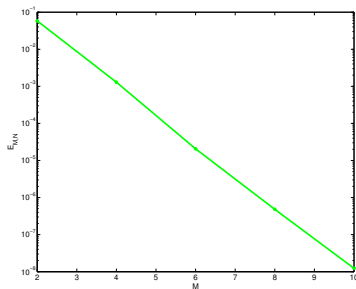


Figure 3. L^∞ – error in the time direction of a single domain computation.

Table 1. The numerical error of scheme (3.1) with $a = 0, b = 3$.

t	0	1	4	7	10	13	15
Error	3.04e-07	3.23e-07	4.18e-07	4.18e-07	4.22e-07	4.22e-07	3.87e-07

3.2 Single domain error analysis

Typically, only the semi-discrete form of the error analysis of (3.1) is considered. For this purpose, set

$$u_N(y, s) = \sum_{n=0}^N u_N(y_n, s) \varphi_n(y), \quad u_N(y_n, s) = \sum_{m=0}^M \hat{u}_{m,n} \varphi_m(s).$$

Let $u_N(x, t) = u_N(y(x), s(t))$. Then (3.1) is transformed into

$$\begin{cases} \partial_t u_N(x, t) - \partial_x^2 u_N(x, t) = u_N(x, t) - u_N^2(x, t), \\ x = x_k, \quad 1 \leq k \leq N-1, t \in (0, T], \\ u_N(a, t) = g_1(t), u_N(b, t) = g_2(t), t \in [0, T], \\ u_N(x, 0) = g_0(x), x = x_k, 0 \leq k \leq N. \end{cases} \quad (3.5)$$

Now, let's consider the error analysis of the above equation using the Petrov-Galerkin method with numerical integration. The weak formulation of (1.1) is to find $u \in L^\infty(0, T; L^2(\mathbb{D})) \cap L^2(0, T; H^1(\mathbb{D}))$, such that

$$\begin{cases} (\partial_t u(t), v)_{\mathbb{D}} + (\partial_x u(t), \partial_x v)_{\mathbb{D}} \\ = (u(t), v)_{\mathbb{D}} - (u^2(t), v)_{\mathbb{D}}, \quad \forall v \in H_0^1(\mathbb{D}), t \in (0, T], \\ u(x, 0) = g_0(x), x \in \mathbb{D}. \end{cases} \quad (3.6)$$

With aid of (2.2), (3.5) is transformed into to find $u_N(z) \in \mathcal{P}_N(\mathbb{D})$

$$\begin{cases} (\partial_t u_N(t), \psi)_{N, \mathbb{D}} + (\partial_x u_N(t), \partial_x \psi)_{\mathbb{D}} \\ = (u_N(t), \psi)_{N, \mathbb{D}} - (u_N^2(t), \psi)_{N, \mathbb{D}}, \quad \forall \psi \in \mathcal{P}_N^0(\mathbb{D}), t \in (0, T], \\ u_N(x, 0) = \pi_{L, N, \mathbb{D}} g_0(x), \quad x \in \bar{\mathbb{D}}. \end{cases} \quad (3.7)$$

For error analysis, an auxiliary question is needed. Let $U_N = \pi_{N, \mathbb{D}}^{1,*} u$. By using (3.6), we obtain that

$$\begin{cases} (\partial_t U_N(t), \psi)_{N, \mathbb{D}} + (\partial_x U_N(t), \partial_x \psi)_{\mathbb{D}} = (U_N(t), \psi)_{N, \mathbb{D}} \\ - (U_N^2(t), \psi)_{N, \mathbb{D}} + \sum_{j=1}^4 K_j(\psi, t), \quad \forall \psi \in \mathcal{P}_N^0(\mathbb{D}), t \in (0, T], \\ u_N(x, 0) = \pi_{N, \mathbb{D}}^{1,*} g_0(x), \quad x \in \bar{\mathbb{D}}, \end{cases} \quad (3.8)$$

where

$$\begin{aligned} K_1(\psi, t) &= (\partial_t U_N(t), \psi)_{N, \mathbb{D}} - (\partial_t u(t), \psi)_{\mathbb{D}}, \\ K_2(\psi, t) &= (\partial_x U_N(t), \partial_x \psi)_{\mathbb{D}} - (\partial_x u(t), \partial_x \psi)_{\mathbb{D}}, \\ K_3(\psi, t) &= (u(t), \psi)_{\mathbb{D}} - (U_N(t), \psi)_{N, \mathbb{D}}, \quad K_4(\psi, t) = (U_N^2(t), \psi)_{N, \mathbb{D}} - (u^2(t), \psi)_{\mathbb{D}}. \end{aligned}$$

Taking $Q_N(t) = (u_N(t) - U_N(t)) \in \mathcal{P}_N^0(\mathbb{D})$, using (3.7) and (3.8) to give that

$$\begin{cases} (\partial_t Q_N(z), \psi)_{N, \mathbb{D}} + (\partial_x Q_N(t), \partial_x \psi)_{\mathbb{D}} \\ = (Q_N(t), \psi)_{N, \mathbb{D}} - \sum_{j=1}^5 K_j(\psi, t), \quad \forall \psi \in \mathcal{P}_N^0(\mathbb{D}), t \in (0, T], \\ Q_N(x, 0) = \pi_{L, N, \mathbb{D}} g_0(x) - \pi_{N, \mathbb{D}}^{1,*} g_0(x), \quad x \in \bar{\mathbb{D}}, \end{cases} \quad (3.9)$$

where

$$K_5(\psi, z) = (u_N^2(t), \psi)_{N, \mathbb{D}} - (U_N^2(t), \psi)_{N, \mathbb{D}}.$$

Take $\psi = 2Q_N(t)$ in (3.9) to obtain that

$$\begin{cases} \partial_t \|Q_N(t)\|_{N, \mathbb{D}}^2 + 2\|\partial_x Q_N(t)\|_{\mathbb{D}}^2 \\ = 2\|Q_N(t)\|_{N, \mathbb{D}}^2 - 2\sum_{j=1}^5 K_j(Q_N(t), t), & t \in (0, T], \\ Q_N(x, 0) = \pi_{L, N, \mathbb{D}} g_0(x) - P_{N, \mathbb{D}}^{1, *}(g_0(x)), & x \in \bar{\mathbb{D}}. \end{cases}$$

Let $\chi_{\mathbb{D}}^r = \chi_{\mathbb{D}}^{(r-1, r-1)}$, we use the following notations

$$\begin{aligned} E(w(t)) &= \|w(t)\|_{\mathbb{D}}^2 + \int_0^t \|w(z)\|_{1, \mathbb{D}}^2 dz, & \eta(u(z)) &= 10 + 12C\|u(t)\|_{1, \mathbb{D}}, \\ E_N(w(t)) &= \|w(t)\|_{N, \mathbb{D}}^2 + \int_0^t (\|w(z)\|_{N, \mathbb{D}}^2 + \|\partial_y w(z)\|_{\mathbb{D}}^2) dz, \\ \mathcal{A}_{\mathbb{D}}^r(u(t)) &= (\|u(t)\|_{1, \mathbb{D}}^2 + 1)\|\partial_x^r u\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2 + \|\partial_x^r \partial_t u\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2 + \|\partial_x^r u^2\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2. \end{aligned}$$

Using the above notations, the Cauchy inequality, and aid of (2.3), (2.4), (2.6), (2.7), (2.8), we have that

$$\begin{aligned} E_N(R_N(t)) &\leq \left(\frac{h}{N}\right)^{2(r-1)} \left(\int_0^t \mathcal{A}_{\mathbb{D}}^r(u(z)) dz + \|\partial_x^r g_0\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2 \right) \\ &\quad + \int_0^t \eta(u(z)) \|Q_N(z)\|_{\mathbb{D}}^2 dz + C_0 \int_0^t 6\|Q_N(z)\|_{1, \mathbb{D}}^2 \|Q_N(z)\|_{\mathbb{D}} dz. \end{aligned}$$

Let $\eta_0 = \max_{0 \leq z \leq t} \eta(u(z))$. By (2.3), $E(Q_N(z)) \leq E_N(Q_N(z))$, the above formulation can be written as

$$\begin{aligned} E(Q_N(t)) &\leq \left(\frac{h}{N}\right)^{2(r-1)} \left(\int_0^t \mathcal{A}_{\mathbb{D}}^r(u(z)) dz + \|\partial_x^r g_0\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2 \right) + \eta_0 \int_0^t \|Q_N(z)\|_{\mathbb{D}}^2 dz \\ &\quad + C_0 \int_0^t 6\|Q_N(z)\|_{1, \mathbb{D}}^2 \|Q_N(z)\|_{\mathbb{D}} dz. \end{aligned}$$

Thus, we have that

$$\begin{aligned} &\|Q_N(t)\|_{\mathbb{D}}^2 + \int_0^t (1 - 6C_0\|Q_N(z)\|_{\mathbb{D}}) \|Q_N(z)\|_{1, \mathbb{D}}^2 dz \\ &\leq \left(\frac{h}{N}\right)^{2(r-1)} \left(\int_0^t \mathcal{A}_{\mathbb{D}}^r(u(z)) dz + \|\partial_x^r g_0\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2 \right) + \eta_0 \int_0^t \|Q_N(z)\|_{\mathbb{D}}^2 dz. \end{aligned} \quad (3.10)$$

We need the following result (Lemma 3.1 of [6]).

Lemma 5. Assume that

- the constants $b_1 > 0, b_2 \geq 0, b_3 \geq 0$ and $d \geq 0$,
- $Z(t)$ and $A(t)$ are non-negative functions of t ,
- $\mathbf{d} \leq b_1^2/b_2^2 e^{-b_3 t_1}$ for $t_1 > 0$,
- for all $t \leq t_1$,

$$Z(t) + \int_0^t (b_1 - b_2 Z^{\frac{1}{2}}(\tau)) A(\tau) d\tau \leq \mathbf{d} + b_3 \int_0^t Z(\tau) d\tau.$$

Then for all $t \leq t_1$,

$$Z(t) \leq \mathbf{d} e^{b_3 t}.$$

Using the above lemma and (2.6), and following the same argument as in the proof of Theorem 3.1 in [24], we obtain the following convergence result:

Theorem 1. *If $u(z)$ and $u_N(z)$ are the solutions of (3.5) and (3.6), respectively, then for any integer $r \geq 1$ and $0 \leq t \leq T$,*

$$\begin{aligned} & \|u(t) - u_N(t)\|_{\mathbb{D}}^2 + \frac{1}{2} \int_0^t \|u(t) - u_N(t)\|_{1,\mathbb{D}}^2 dz \\ & \leq c \left(\frac{h}{N} \right)^{2(r-1)} \left(\int_0^t (\exp(\eta_0 T) \mathcal{A}_{\mathbb{D}}^r(u(z)) + \|\partial_x^r u(z)\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2) dz \right. \\ & \quad \left. + \exp(\eta_0 T) \|\partial_x^r g_0\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2 + \|\partial_x^r u(z)\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2 \right), \end{aligned} \quad (3.11)$$

provided that the terms

$$\|\partial_x^r u(z)\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2, \int_0^t (\exp(\eta_0 T) \mathcal{A}_{\mathbb{D}}^r(u(z)) + \|\partial_x^r u(z)\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2) dz, \exp(\eta_0 T) \|\partial_x^r g_0\|_{\chi_{\mathbb{D}}^r, \mathbb{D}}^2$$

are finite.

3.3 Single domain stability analysis

Next, we consider the generalized stability of Problem (3.7). Let $\tilde{u}_N(x, 0)$ represent a disturbance to $u_N(x, 0)$, leading to a disturbance in $u_N(x, t)$, denoted as $\tilde{u}_N(x, t) \in \mathcal{P}_N^0(\mathbb{D})$. Using Equation (3.7), $\tilde{u}_N(x, t)$ satisfies the equation,

$$\begin{cases} (\partial_t \tilde{u}_N(t), \psi)_{N, \mathbb{D}} + (\partial_x \tilde{u}_N(t), \partial_x \psi)_{\mathbb{D}} \\ = (\tilde{u}_N(t), \psi)_{N, \mathbb{D}} - (2\tilde{u}_N(t)u_N(t) + \tilde{u}_N^2(t), \psi)_{N, \mathbb{D}} \quad \forall \psi \in \mathcal{P}_N^0(\mathbb{D}), t \in (0, T], \\ \tilde{u}_N(x, 0) = \tilde{u}_{N,0}, \quad x \in \bar{\mathbb{D}}. \end{cases} \quad (3.12)$$

Taking $\psi = 2\tilde{u}_N(t)$ in (3.12), we obtain that

$$\begin{cases} \partial_t \|\tilde{u}_N(t)\|_{N, \mathbb{D}}^2 + 2\|\partial_x \tilde{u}_N(t)\|_{\mathbb{D}}^2 = 2\|\tilde{u}_N(t)\|_{N, \mathbb{D}}^2 \\ \quad - 2((2\tilde{u}_N(z)u_N(z) + \tilde{u}_N^2(z)), \tilde{u}_N(z))_{N, \mathbb{D}}, \quad z \in (0, T], \\ \tilde{u}_N(y, 0) = \tilde{u}_{N,0}, \quad y \in \bar{\mathbb{D}}. \end{cases}$$

Like (3.10), we have

$$\begin{aligned} & \|\tilde{u}_N(t)\|_{\mathbb{D}}^2 + \int_0^t (1 - 6C_0 \|\tilde{u}_N(t)\|_{\mathbb{D}}) \|\tilde{u}_N(t)\|_{1, \mathbb{D}}^2 dz \\ & \leq \|\tilde{u}_{N,0}\|_{\mathbb{D}}^2 + \eta_0 \int_0^t \|\tilde{u}_N(t)\|_{\mathbb{D}}^2 dz. \end{aligned} \quad (3.13)$$

Applying Lemma 5 to Equation (3.13), the following result is obtained:

Theorem 2. Define $\tilde{u}_N(t)$ as the solution of Equation (3.12). If there exists a certain $t_1 > 0$, $\|\tilde{u}_{N,0}\|_{\mathbb{D}}^2 \leq \frac{1}{144C_0^2} e^{-\eta_0 t_1}$, then for all $t \leq t_1$,

$$\|\tilde{u}_N(t)\|_{\mathbb{D}}^2 + \frac{1}{2} \int_0^t \|\tilde{u}_N(z)\|_{1,\mathbb{D}}^2 dz \leq \|\tilde{u}_{N,0}\|_{\mathbb{D}}^2 e^{\eta_0 t}. \quad (3.14)$$

This indicates that the error $\tilde{u}_N(t)$ is controlled by a finite $\tilde{u}_{N,0}$. In particular,

$$\text{while } \tilde{u}_{N,0} \rightarrow 0, \text{ one has } \tilde{u}_N(t) \rightarrow 0.$$

As in Equation (3.7), this is the generalized stability defined in [5].

4 Multi-domain collocation method

Comparison of Figures 1 and 2 in Subsection 3.2 shows that increasing interpolation nodes alone does not improve error accuracy when the spatial length is large. To address this, multi-domain approximation can be used. We first construct the algorithmic framework, present numerical results, and prove its convergence and stability.

4.1 Multi-domain algorithm scheme and numerical results

This subsection extends the single-domain collocation method to a multi-domain approach to improve error accuracy for large-domain problems.

We divide the domain $\mathbb{D} = (a, b)$ into two subdomains $\mathbb{D}_1 = (a, c)$ and $\mathbb{D}_2 = (c, b)$. Besides, $\Omega_1 = \mathbb{D}_1 \times (0, T)$, $\Omega_2 = \mathbb{D}_2 \times (0, T)$. Assume the variable transformation $y = y_j(x)$, $s = s(t)$, which transforms the domain Ω_j , $j = 1, 2$ into the domain Λ (cf. page 6 of this paper).

Let $\mathbb{N} = \{N_1, N_2\}$. Let $u_{M,\mathbb{N}}(x, t)$ (solution of Equation (1.1)) satisfy in each subdomain:

$$\begin{aligned} u_{M,\mathbb{N}}(y, s)|_{\Omega_j} &= u_{M,N_j}^j(y, s) \\ &= \sum_{n=0}^{N_j} \sum_{m=0}^M \hat{u}_{m,n}^{(j)} \varphi_n(y_j(x)) \varphi_m(s(t)), (x, t) \in \Omega_j, j = 1, 2. \end{aligned} \quad (4.1)$$

Now, let $y = y_1 = \frac{2x-c-a}{c-a}$, $s = \frac{2t-T}{T}$, $(x, t) \in \Omega_1$, which transforms the domain Ω_1 into the domain Λ . Equation (1.1) is changed into as follows:

$$\begin{cases} \frac{2}{T} \partial_s u - \frac{4}{(c-a)^2} \partial_y^2 u = -u^2 + u, & y \in I_{y_1} = (-1, 1), \quad s \in (-1, 1), \\ u(-1, s) = g_1(s), s \in [-1, 1], \\ u(y, -1) = \psi(y), y \in \bar{I}_{y_1}. \end{cases} \quad (4.2)$$

Next, let $y = y_2 = \frac{2x-c-b}{b-c}$, $s = \frac{2t-T}{T}$, $(x, t) \in \Omega_2$, which transforms the domain Ω_2 into the domain Λ . Equation (1.1) is changed into as follows:

$$\begin{cases} \frac{2}{T} \partial_s u - \frac{4}{(b-c)^2} \partial_y^2 u = -u^2 + u, & y \in I_{y_2} = (-1, 1), \quad s \in (-1, 1), \\ u(1, s) = g_2(s), s \in [-1, 1], \\ u(y, -1) = \psi(y), y \in \bar{I}_{y_2}. \end{cases} \quad (4.3)$$

Further, we consider the solution at $x = c$. Since the solution u of Equation (1.1) satisfies $u \in H^1(0, T; H^2(a, b))$, the numerical solution $u_{M, \mathbb{N}}$ also belongs to $H^1(0, T; H^2(a, b))$. This regularity ensures the continuity of both the function and its first derivative at $x = c$. Specifically, for $j = 0, 1$,

$$\sum_{n=0}^{N_1} \sum_{m=0}^M \hat{u}_{m,n}^{(1)} \partial_y^j \varphi_n(y_{N_1}) \varphi_m(s_l) = \sum_{n=0}^{N_2} \sum_{m=0}^M \hat{u}_{m,n}^{(2)} \partial_y^j \varphi_n(y_0) \varphi_m(s_l). \quad (4.4)$$

It can be seen from the definition of the differentiation matrix (2.1),

$$d_{l,m} = \partial_s \varphi_m(s_l), d_{k,n} = \partial_y \varphi_n(y_k), d_{k,n}^{(2)} = \partial_y^2 \varphi_n(y_k).$$

Let

$$\delta = \left(d_{N_1, N_1} - \frac{c-a}{b-c} d_{0,0} \right)^{-1}.$$

Combining the cases $j = 0, 1$ in (4.4), we obtain

$$u_{l, N_1}^{(1)} = u_{l, 0}^{(2)} = \delta \left(\frac{c-a}{b-c} \sum_{n=1}^{N_2} u_{l,n}^{(2)} d_{0,n} - \sum_{n=0}^{N_1-1} u_{l,n}^{(1)} d_{N_1,n} \right). \quad (4.5)$$

In the domain Ω_1 , approximate the exact solution within the domain Ω_1 using numerical methods. By substituting Equation (4.1) $u_{M, N_1}^1(y, s) \in \mathcal{P}_{M, N}(A)$ into Equation (4.2), using the differentiation matrix definition and Equation (4.5), we get:

$$\begin{aligned} & \frac{2}{T} \sum_{m=1}^M \hat{u}_{m,k}^{(1)} d_{l,m} - \left(\frac{2}{c-a} \right)^2 \sum_{n=1}^{N_1-1} \hat{u}_{l,n}^{(1)} d_{k,n}^{(2)} + \left(\hat{u}_{l,k}^{(1)} \right)^2 - \hat{u}_{l,k}^{(1)} + \frac{2}{T} \hat{u}_{0,k}^{(1)} d_{l,0} \quad (4.6) \\ &= \left(\frac{\frac{c-a}{b-c} \sum_{n=1}^{N_2} u_{l,n}^{(2)} d_{0,n} - \sum_{n=0}^{N_1-1} u_{l,n}^{(1)} d_{N_1,n}}{d_{N_1, N_1} - d_{0,0} (c-a)/(b-c)} \right) \left(\frac{2}{c-a} \right)^2 d_{k, N_1}^{(2)} + \left(\frac{2}{c-a} \right)^2 \hat{u}_{l,0}^{(1)} d_{k,0}^{(2)} \\ & \quad k = 1, 2, \dots, N_1 - 1, l = 1, 2, \dots, M. \end{aligned}$$

Equation (4.6) is first expanded for $l = 1, 2, \dots, M$, and then further expanded for $k = 1, 2, \dots, N_1 - 1$, resulting in the following matrix equation:

$$A_1 X_1 - X_1 B_1^T + D_1 - F_1 + X_1 \cdot * X_1 - X_1 = H_1 \otimes [\delta (X_2 E_2 - X_1 E_1) + \delta G_1], \quad (4.7)$$

where the matrices $A_1, B_1, \dots, X_1$ are similar to the $A, B, \dots, X$ in (3.4).

Similarly, following the approach used for domain Ω_1 , the exact solution in domain Ω_2 is approximated by a numerical solution. By substituting Equa-

tion (4.1) $u_{M,N_2}^2(y, s) \in \mathcal{P}_{M,N}(\Lambda)$ into Equation (4.3), we obtain:

$$\left\{ \begin{array}{l} \frac{2}{T} \sum_{n=0}^{N_2} \sum_{m=0}^M \hat{u}_{m,n}^{(2)} \varphi_n(y_k) \partial_s \varphi_m(s_l) - \left(\frac{2}{b-c} \right)^2 \sum_{n=0}^{N_2} \sum_{m=0}^M \hat{u}_{m,n}^{(2)} \partial_y^2 \varphi_n(y_k) \varphi_m(s_l) \\ = - \left[\sum_{n=0}^{N_2} \sum_{m=0}^M \hat{u}_{m,n}^{(2)} \varphi_n(y_k) \varphi_m(s_l) \right]^2 + \sum_{n=0}^{N_2} \sum_{m=0}^M \hat{u}_{m,n}^{(2)} \varphi_n(y_k) \varphi_m(s_l), \\ k = 1, 2, \dots, N_2 - 1, l = 1, 2, \dots, M, \\ \sum_{n=0}^{N_2} \sum_{m=0}^M \hat{u}_{m,n}^{(2)} \varphi_n(y_{N_2}) \varphi_m(s_l) = g_2(s_l), \\ \sum_{n=0}^{N_2} \sum_{m=0}^M \hat{u}_{m,n}^{(2)} \varphi_n(y_k) \phi_m(s_0) = \psi(y_k). \end{array} \right. \quad (4.8)$$

By proposing the endpoint values and using the differentiation matrix, we substitute Equation (4.5) into Equation (4.8), yielding:

$$A_2 X_2 - X_2 B_2^T + D_2 - F_2 + X_2 * X_2 - X_2 = H_2 \otimes [\delta(X_2 E_2 - X_1 E_1) + \delta G_2], \quad (4.9)$$

where the matrices $A_2, B_2, \dots, X_2$ are similar to the $A, B, \dots, X$ in (3.4).

Equation (1.1) is solved by integrating (4.7) and (4.9) with $T = 2$, $a = 0$, $c = 12.94255$, and $b = 55$. We measure the error via the L^∞ -norm:

$$E_{M,N} = \max\{E_{M,N_1}, E_{M,N_2}\},$$

where

$$E_{M,N_i} = \max_{1 \leq l \leq M, 1 \leq k \leq N_i - 1} |u(x_k, t_l) - u_{M,N_i}(x_k, t_l)|, \quad i = 1, 2,$$

with $N_1 = N_2 = N$ denoting the number of spatial interpolation points and $M = 12$ representing the number of time steps.

Figure 4 shows that the L^∞ -error $E_{M,N}$ decays as N increases. Compared to the single-domain results in Figure 2, the multi-domain method yields lower errors, validating the effectiveness of the multi-domain Legendre collocation scheme. Additionally, normalized L^2 -errors across multiple regions are presented. These values are lower than the corresponding L^∞ -errors, consistent with empirical observations.

Figure 5 (with the parameters shown in 4) shows CPU time consumption as a function of the spatial node count $N = N_1 + N_2$ (with fixed temporal nodes $M = 12$). The recorded times range from 0.1 s ($N = 24$) to 1.2 s ($N = 40$). This trend reflects the $O(N^3)$ complexity of the global space-time solver combined with the overhead of nonlinear iterations (≈ 15 steps). Despite this scaling, the exponential convergence of the spectral method ensures high accuracy with minimal absolute cost (< 1.5 s).

Figure 5 clearly indicates that the present space-time collocation method yields markedly better results than the method in [28], which combines pseudospectral spatial discretization with finite difference time stepping.

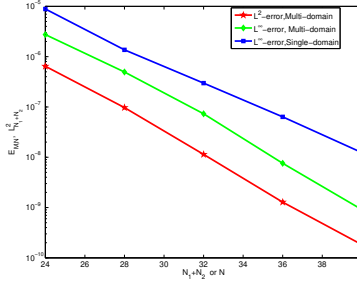


Figure 4. Multi-domain L^∞ – error.

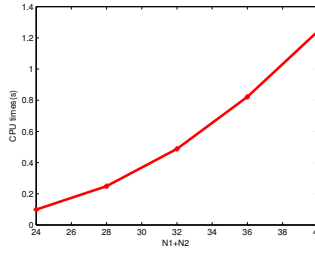


Figure 5. Multi-domain CPU time.

4.2 Multi-domain error analysis

In studying the error analysis for scheme of multi-domain, we need composite interpolation and quasi-orthogonal approximation.

Let $\mathbb{D}_i, i = 1, 2$ be the same as before. Moreover, let $\rho_{N_1, \mathbb{D}_1, j} = \frac{c-a}{2} \rho_{N_1, I_{y_1}, j}$ ($0 \leq j \leq N_1$) be the Christoffel number corresponding to node $x_j = \frac{c-a}{2} y_j + \frac{c+a}{2}, j = 1, 2, \dots, N_1$, and $\rho_{N_2, \mathbb{D}_2, j} = 0.5(b-c) \rho_{N_2, I_{y_2}, j}$ ($0 \leq j \leq N_2$) be the Christoffel number corresponding to node $x_j = \frac{b-c}{2} y_j + \frac{c+b}{2}, j = 1, 2, \dots, N_2$, respectively. In the same manner as Subsection 2.2, we define the space $H^r(\mathbb{D}_i)$ and their norm $\|u\|_{r, \mathbb{D}_i}$, the discrete inner product $(f, g)_{N_i, \mathbb{D}_i}$, the discrete norm $\|g\|_{N_i, \mathbb{D}_i}$ on the $L^2(\mathbb{D}_i)$, interpolation projection $\pi_{L, N_i, \mathbb{D}_i}$ and quasi-orthogonal projection $\pi_{N_i, \mathbb{D}_i}^{1,*}, i = 1, 2$. Let $\mathbb{N}$ be the same as in Subsection 4.1, $\mathcal{P}_{\mathbb{N}}(\mathbb{D}) = \{\psi \mid \psi|_{\mathbb{D}_i} \in \mathcal{P}_{N_i}(\mathbb{D}_i)\}$.

The discrete inner product $(f, g)_{\mathbb{N}, \mathbb{D}}$ is defined by

$$(f, g)_{\mathbb{N}, \mathbb{D}} = (f, g)_{N_1, \mathbb{D}_1} + (f, g)_{N_2, \mathbb{D}_2}, \|f\|_{\mathbb{N}, \mathbb{D}} = (f, f)_{\mathbb{N}, \mathbb{D}}^{\frac{1}{2}}.$$

The composite interpolation $\pi_{L, \mathbb{N}, \mathbb{D}}$ and quasi-orthogonal approximation $\pi_{\mathbb{N}, \mathbb{D}}^{1,*}$ is defined by

$$\pi_{L, \mathbb{N}, \mathbb{D}} u|_{\mathbb{D}_i} = \pi_{L, N_i, \mathbb{D}_i} u, \quad \pi_{\mathbb{N}, \mathbb{D}}^{1,*} u|_{\mathbb{D}_i} = \pi_{N_i, \mathbb{D}_i}^{1,*} u, \quad i = 1, 2.$$

Like (2.2)–(2.5) and (2.6), we can get:

$$\begin{aligned}(\phi, \psi)_{\mathbb{N}, \mathbb{D}} &= (\phi, \psi)_{\mathbb{D}}, \quad \forall \phi, \psi \in \mathcal{P}_{2N-1}(\mathbb{D}), \\ \|\psi\|_{\mathbb{D}} &\leq \|\psi\|_{\mathbb{N}, \mathbb{D}} \leq (\sqrt{2 + 1/N_1} + \sqrt{2 + 1/N_2}) \|\psi\|_{\mathbb{D}}, \quad \forall \psi \in \mathcal{P}_{\mathbb{N}}(\mathbb{D}).\end{aligned}$$

The Jacobi weight function $\chi^{(\gamma, \delta)}(y)$ is the same as before. Let $\chi_{\mathbb{D}_j}^{\gamma, \delta}(x) = \chi^{(\gamma, \delta)}(y)|_{y=(2x-h_j)/h_j}$ with $y = \frac{2x-h_j}{h_j} \in I_{y_j}$, $h_1 = c - a$, $h_2 = b - c$, $j = 1, 2$.

For $u|_{\mathbb{D}_i} \in C(\bar{\mathbb{D}}_i)$, the interpolation approximation $\pi_{L, \mathbb{N}, \mathbb{D}} u \in \mathcal{P}_{\mathbb{N}}(\mathbb{D})$, we have that if $u|_{\mathbb{D}_i} \in C(\bar{\mathbb{D}}_i)$, $(1 - y^2(x))^{-\frac{r-1}{2}} \partial_x^r u \in L^2(\mathbb{D}_i)$, $i = 1, 2$ and the integer $r \geq 1$, then:

$$\begin{aligned}\|\pi_{L, \mathbb{N}, \mathbb{D}} u - u\|_{\mathbb{D}} &\leq c \sum_{i=1}^2 \left(\frac{h_i}{N_i}\right)^r \|\partial_x^r u\|_{\chi_{\mathbb{D}_i}^{(r-1, r-1)}, \mathbb{D}_i}, \\ \left| (u, \psi)_{\mathbb{D}} - (u, \psi)_{\mathbb{N}, \mathbb{D}} \right| &\leq c \sum_{i=1}^2 \left(\frac{h_i}{N_i}\right)^r \|\partial_x^r u\|_{\chi_{\mathbb{D}_i}^{(r-1, r-1)}, \mathbb{D}_i} \|\psi\|_{\mathbb{D}_i}, \quad \psi \in \mathcal{P}_{N_i}(\mathbb{D}_i).\end{aligned}$$

Lemma 6. If $w|_{\mathbb{D}_i} \in H^1(\mathbb{D}_i)$, $\partial_y^r u \in L^2_{\chi_{\mathbb{D}_i}^{(r-1, r-1)}}(\mathbb{D}_i)$, $i = 1, 2$ and an integer $r \geq 1$, then

$$\|\partial_x^\mu (\pi_{\mathbb{N}, \mathbb{D}}^{1,*} w - w)\|_{\mathbb{D}} \leq c \sum_{i=1}^2 \left(\frac{h_i}{N_i}\right)^{r-\mu} \|\partial_x^r u\|_{\chi_{\mathbb{D}_i}^{(r-1, r-1)}, \mathbb{D}_i}.$$

Remark 2. By virtue of the definition of $\pi_{N_i, \mathbb{D}_i}^{1,*}$, $i = 1, 2$, the quasi-orthogonal approximation $\pi_{\mathbb{N}, \mathbb{D}}^{1,*}$ satisfies: $\pi_{\mathbb{N}, \mathbb{D}}^{1,*} u(x)|_{x=a, c, b} = u(x)|_{x=a, c, b}$, which not only accurately fits the boundary points of the domain but also precisely fits the points within the domain, playing a crucial role in error analysis and extending the existing quasi-orthogonal approximation results (see [7, 29]).

Note that common points of the two subdomains were not used as collocation points, complicating error analysis. To address this, we consider the Petrov-Galerkin spectral method with numerical integration.

We now consider the error analysis for scheme of multi-domain. We introduce the following spaces.

$$\begin{aligned}\mathcal{M}(\mathbb{D}) &= \{ u \mid u \in H^1(\mathbb{D}), u(x)|_{x=a, b, c} = 0 \} \quad \bar{\mathcal{P}}_{\mathbb{N}}(\mathbb{D}) = \mathcal{M}(\mathbb{D}) \cap \mathcal{P}_{\mathbb{N}}(\mathbb{D}). \\ u_{\mathbb{N}}(y, s)|_{\mathbb{D}_j} &= \sum_{n=0}^{N_j} (u_{\mathbb{N}}(y_n, s)|_{\mathbb{D}_j}) \varphi_n(y), \quad u_{\mathbb{N}}(y_n, s)|_{\mathbb{D}_j} = \sum_{m=0}^M \hat{u}_{m, n}^{(j)} \varphi_m(s), \quad j=1, 2.\end{aligned}$$

The weak formulation of (1.1) is to find $u \in L^\infty(0, T; L^2(\mathbb{D})) \cap L^2(0, T; H^1(\mathbb{D}))$, such that

$$\begin{cases} (\partial_t u(t), v)_{\mathbb{D}} + (\partial_x u(t), \partial_x v)_{\mathbb{D}} = (u(t) - u^2(t), v)_{\mathbb{D}}, & \forall v \in \mathcal{M}(\mathbb{D}), z \in (0, T], \\ u(x, 0) = \phi(x), x \in \mathbb{D}. \end{cases} \quad (4.10)$$

Let $u_{\mathbb{N}}(x, t) = u_{\mathbb{N}}(y(x), s(t))$. Like (3.7), with aid of (2.2), (4.2) and (4.3), the spectral collocation scheme is to find $u_{\mathbb{N}}(z) \in \mathcal{P}_{\mathbb{N}}(\mathbb{D})$,

$$\begin{cases} (\partial_t u_{\mathbb{N}}(t), \psi)_{\mathbb{N}, \mathbb{D}} + (\partial_x u_{\mathbb{N}}(t), \partial_x \psi)_{\mathbb{D}} \\ = (u_{\mathbb{N}}(t) - u_{\mathbb{N}}^2(t), \phi)_{\mathbb{N}, \mathbb{D}}, \quad \forall \phi \in \overline{\mathcal{P}}_{\mathbb{N}}(\mathbb{D}), t \in (0, T], \\ u_{\mathbb{N}}(x, 0) = \pi_{L, \mathbb{N}, \mathbb{D}} \phi(x), \quad x \in \bar{\mathbb{D}}. \end{cases} \quad (4.11)$$

For error analysis, an auxiliary question is needed. Let $U_{\mathbb{N}} = \pi_{\mathbb{N}, \mathbb{D}}^{1,*} u$. By virtue of (3.6), we have that

$$\begin{cases} (\partial_t U_{\mathbb{N}}(t), \psi)_{\mathbb{N}, \mathbb{D}} + (\partial_x U_{\mathbb{N}}(t), \partial_x \psi)_{\mathbb{D}} + (U_{\mathbb{N}}(z), \phi)_{\mathbb{N}, \mathbb{D}} \\ = (U_{\mathbb{N}}(t) - U_{\mathbb{N}}^2(t), \phi)_{\mathbb{N}, \mathbb{D}} + \sum_{j=1}^4 K_j(\phi, z), \quad \forall \phi \in \overline{\mathcal{P}}_{\mathbb{N}}(\mathbb{D}), z \in (0, T], \\ u_{\mathbb{N}}(x, 0) = P_{\mathbb{N}, \mathbb{D}}^{1,*} \varphi(x), \quad x \in \bar{\mathbb{D}}, \end{cases}$$

where

$$\begin{aligned} K_1(\phi, z) &= (\partial_z U_{\mathbb{N}}(z), \phi)_{\mathbb{N}, \mathbb{D}} - (\partial_z u(z), \phi)_{\mathbb{D}}, \\ K_2(\phi, z) &= (\partial_y U_{\mathbb{N}}(z), \partial_y \phi)_{\mathbb{D}} - (\partial_y u(z), \partial_y \phi)_{\mathbb{D}}, \\ K_3(\phi, z) &= (U_{\mathbb{N}}(z), \phi)_{\mathbb{N}, \mathbb{D}} - (u(z), \phi)_{\mathbb{D}}, \\ K_4(\phi, z) &= 2(u^2(z), \phi)_{\mathbb{D}} - 2(U_{\mathbb{N}}^2(z), \phi)_{\mathbb{N}, \mathbb{D}}, \end{aligned}$$

Following the same line as deriving (3.11), we have the following convergence result.

Theorem 3. *If $u(z)$ and $u_{\mathbb{N}}(z)$ are the solutions of (4.10) and (4.11), respectively, then for any integer $r \geq 1$ and $0 \leq t \leq T$,*

$$\begin{aligned} & \|u(t) - u_{\mathbb{N}}(t)\|_{\mathbb{D}}^2 + \int_0^t \|u(z) - u_{\mathbb{N}}(z)\|_{1, \mathbb{D}}^2 dz \\ & \leq c \sum_{j=1}^2 \left(\frac{h_j}{N_j} \right)^{2(r-1)} \left(\int_0^t \left(\exp(\eta_0 T) \mathcal{A}_{\mathbb{D}_j}^r(u(z)) + \|\partial_x^r u(z)\|_{\chi_{\mathbb{D}_j}^{(r-1, r-1), \mathbb{D}_j}}^2 \right) dz \right. \\ & \quad \left. + \exp(\eta_0 t) \|\partial_x^r g_0\|_{\chi_{\mathbb{D}_j}^{(r-1, r-1), \mathbb{D}_j}}^2 + \|\partial_x^r u(z)\|_{\chi_{\mathbb{D}_j}^{(r-1, r-1), \mathbb{D}_j}}^2 \right), \end{aligned}$$

provided that the terms

$$\begin{aligned} & \int_0^t \left(\exp(\eta_0 T) \mathcal{A}_{\mathbb{D}_j}^r(u(z)) + \|\partial_x^r u(z)\|_{\chi_{\mathbb{D}_j}^{(r-1, r-1), \mathbb{D}_j}}^2 \right) dz, \\ & \|\partial_x^r u(z)\|_{\chi_{\mathbb{D}_j}^{(r-1, r-1), \mathbb{D}_j}}^2, \quad \exp(\eta_0 t) \|\partial_x^r g_0\|_{\chi_{\mathbb{D}_j}^{(r-1, r-1), \mathbb{D}_j}}^2, \quad j = 1, 2 \end{aligned}$$

are finite.

4.3 Multi-domain stability analysis

Next, we will consider the generalized stability of Problem (4.11). Assuming that $\tilde{u}_{\mathbb{N}}(x, 0)$ is a disturbance from $u_{\mathbb{N}}(x, 0)$, resulting in $u_{\mathbb{N}}(x, t)$ having

a disturbance $\tilde{u}_{\mathbb{N}}(x, t) \in \overline{\mathcal{P}_{\mathbb{N}}}(\mathbb{D})$. Using Equation (4.11), $\tilde{u}_{\mathbb{N}}(x, t)$ satisfies the equation,

$$\begin{cases} (\partial_t \tilde{u}_{\mathbb{N}}(t), \psi)_{\mathbb{N}, \mathbb{D}} + (\partial_x \tilde{u}_{\mathbb{N}}(t), \partial_x \psi)_{\mathbb{D}} = (\tilde{u}_{\mathbb{N}}(t), \psi)_{\mathbb{N}, \mathbb{D}} \\ \quad - 2((\tilde{u}_{\mathbb{N}}(z)u_{\mathbb{N}}(z) + \tilde{u}_{\mathbb{N}}^2(z)), \psi)_{\mathbb{N}, \mathbb{D}} \quad \forall \psi \in \mathcal{P}_{\mathbb{N}}(\mathbb{D}), z \in (0, T], \\ \tilde{u}_{\mathbb{N}}(x, 0) = \tilde{u}_{\mathbb{N}, 0}, \quad x \in \bar{\mathbb{D}}. \end{cases} \quad (4.12)$$

Like (3.14), we obtain the result of Stability.

Theorem 4. *If $\tilde{u}_{\mathbb{N}}(t)$ is the solution of (4.12), if for certain $t_1 > 0$, $\|\tilde{u}_{\mathbb{N}, 0}\|_{\mathbb{D}}^2 \leq \frac{1}{144C_0^2}e^{-\eta_0 t_1}$, then for all $t \leq t_1$,*

$$\|\tilde{u}_{\mathbb{N}}(t)\|_{\mathbb{D}}^2 + \frac{1}{2} \int_0^t \|\tilde{u}_{\mathbb{N}}(z)\|_{1, \mathbb{D}}^2 dz \leq \|\tilde{u}_{\mathbb{N}, 0}\|_{\mathbb{D}}^2 \exp(\eta_0 T),$$

which indicates that the error $\tilde{u}_{\mathbb{N}}(t)$ is controlled by a finite $\tilde{u}_{\mathbb{N}, 0}$. In particular,

$$\text{while } \tilde{u}_{\mathbb{N}, 0} \rightarrow 0, \text{ one has } \tilde{u}_{\mathbb{N}}(t) \rightarrow 0.$$

5 Conclusions

In conclusion, this work presents a robust numerical framework for large-scale simulations of Fisher's equation, characterized by the following key contributions:

- **Simultaneous Space-Time Discretization:** The method employs Legendre-Gauss-Lobatto (LGL) nodes and Lagrange interpolation to discretize both spatial and temporal dimensions simultaneously. This transforms the governing partial differential equation directly into a nonlinear matrix system, bypassing sequential time-stepping.
- **Spatial Domain Decomposition:** The spatial domain is partitioned into smaller subdomains to enable parallel computation. Continuity across subdomain interfaces is rigorously enforced using constructed quasi-orthogonal approximation operators.
- **Theoretical Analysis:** we employ standard Petrov-Galerkin formulations and quasi-orthogonal approximations to verify the convergence of our method. A Petrov-Galerkin formulation is introduced to provide a rigorous proof of stability and convergence for the fully coupled space-time system, addressing the challenges of interface conditions in multi-domain settings.
- **Numerical Performance:** Extensive experiments demonstrate that localizing computational efforts within subdomains significantly enhances both accuracy and efficiency compared to conventional single-domain or sequential discretization methods.

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Appendix A

$$\begin{aligned}
 A &= \frac{2}{T} \begin{pmatrix} d_{1,1} & \cdots & d_{1,M} \\ \vdots & \ddots & \vdots \\ d_{M,1} & \cdots & d_{M,M} \end{pmatrix}, \quad B = \left(\frac{2}{b-a} \right)^2 \begin{pmatrix} d_{1,1}^{(2)} & \cdots & d_{1,N-1}^{(2)} \\ \vdots & \ddots & \vdots \\ d_{N-1,1}^{(2)} & \cdots & d_{N-1,N-1}^{(2)} \end{pmatrix}, \\
 X &= \begin{pmatrix} \hat{u}_{1,1} & \cdots & \hat{u}_{1,N-1} \\ \vdots & \ddots & \vdots \\ \hat{u}_{M,1} & \cdots & \hat{u}_{M,N-1} \end{pmatrix}, \quad D = \frac{2}{T} \begin{pmatrix} \hat{d}_{1,0} \\ \vdots \\ \hat{d}_{M,0} \end{pmatrix} \begin{pmatrix} \hat{u}_{0,1} & \cdots & \hat{u}_{0,N-1} \end{pmatrix}, \\
 F &= \frac{4}{(b-a)^2} \begin{pmatrix} \hat{u}_{1,0} \begin{pmatrix} d_{1,0}^{(2)} & \cdots & d_{N-1,0}^{(2)} \end{pmatrix} + \hat{u}_{1,N} \begin{pmatrix} d_{1,N}^{(2)} & \cdots & d_{N-1,N}^{(2)} \end{pmatrix} \\ \vdots \\ \hat{u}_{M,0} \begin{pmatrix} d_{1,0}^{(2)} & \cdots & d_{N-1,0}^{(2)} \end{pmatrix} + \hat{u}_{M,N} \begin{pmatrix} d_{1,N}^{(2)} & \cdots & d_{N-1,N}^{(2)} \end{pmatrix} \end{pmatrix}.
 \end{aligned}$$

Here, the ".*" indicates the multiplication of corresponding elements in the matrix.