

General stability of memory-type model for double-wall carbon nanotubes

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Abstract. We study the general stability of solutions for a double-wall carbon nanotubes (DWCNTs), modeled as coupled Timoshenko beams with viscoelastic damping acting on the transverse displacements. First, we establish the well-posedness via the Faedo-Galerkin approximation method. Then, with general decay assumption on the memory functions, the Multiplier method is used to prove a general stability of the solution energy. The viscoelastic dissipation on the transverse displacements and linear frictional damping on the longitudinal displacements are strong enough to achieve our results without imposing the equal-wave speed condition on the coefficient parameters.

Keywords: double-wall carbon nanotubes; coupled Timoshenko beam; memory dissipation; general stability.

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1 Introduction

In the past decades, carbon nanotubes (CNTs) have been considered as one of the most promising materials in nanotechnology, due to their unique electronic and mechanical properties, see [8, 13]. Notably, they show great potential as super fibers in nanocomposites, thus attracting a great number of studies from scientists. Researching the behaviour of waves and vibrations in carbon nanotubes has become an important field and focus within current studies. Out of all the approaches, the continuum elastic-beam models have been successful in the analysis of vibrations and the propagation of sound waves through these structures, see [18]. In these works, based on various physical considerations, a number of models of carbon nanotubes have been described and classified, including single-wall carbon nanotubes (SWCNTs), double-wall carbon nanotubes (DWCNTs), and multi-wall carbon nanotubes (MWCNTs), see Figure 1. Using the well-known Timoshenko-beam theory [12], Yoon et

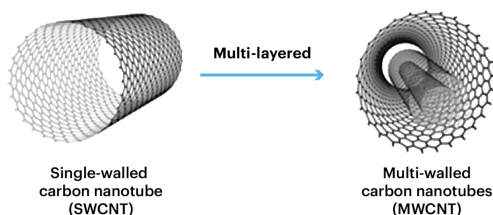


Figure 1. Layers of carbon nanotubes.

al. [20] introduced a coupled Timoshenko beam model that characterizes the wave propagation in double-wall carbon nanotubes (DWCNTs) structures, see Figure 2 and system (1.1)

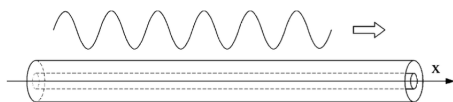


Figure 2. Propagation of transverse waves in a single DWCNT.

$$\left\{ \begin{array}{l} \rho A_1 \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial S_1}{\partial x} - P = 0, \quad \text{in }]0, L[\times]0, +\infty[, \\ \rho I_1 \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial M_1}{\partial x} + S_1 = 0, \quad \text{in }]0, L[\times]0, +\infty[, \\ \rho A_2 \frac{\partial^2 w}{\partial t^2} - \frac{\partial S_2}{\partial x} + P = 0, \quad \text{in }]0, L[\times]0, +\infty[, \\ \rho I_2 \frac{\partial^2 z}{\partial t^2} - \frac{\partial M_2}{\partial x} + S_2 = 0, \quad \text{in }]0, L[\times]0, +\infty[, \end{array} \right. \quad (1.1)$$

where the constitutive laws S_i, M_i , $i = 1, 2$ and P are defined by

$$\begin{aligned} S_1 &= \kappa G A_1 (\phi_x + \psi), \quad M_1 = E I_1 \psi_x, \quad S_2 = \kappa G A_2 (w_x + z), \\ M_2 &= E I_2 z_x, \quad P = \kappa_0 (\phi - w), \end{aligned}$$

with $\phi = \phi(x, t)$ and $w = w(x, t)$ representing the transverse displacement of the inner and outer tubes, respectively, while $\psi = \psi(x, t)$ and $z = z(x, t)$ denote the associated bending deformation of the inner and outer nanotubes. The physical parameters: $A_j, j = 1, 2$ and $I_j, j = 1, 2$ are, respectively: cross-sectional area and moment of inertia of the inner and outer tubes, P is known as the Van der Waals force acting on the interaction between the two nanotubes and κ_0 represents the Van der Waals interaction coefficient for the interaction pressure between the nanotubes. As remarked in [20], the model (1.1) remains valid if the Van der Waals force $P = \kappa_0(\psi - z)$ is on the second and fourth equations in (1.1). Finally, ρ, E, G and κ are mass density, the Young modulus, shear modulus and shear correction coefficient, respectively. There are limited

amount of results in literature concerning system (1.1). Recently, Guesmia [11] studied (1.1) with linear damping on all four equations and proved the existence and uniqueness result, as well as asymptotic analysis for various variation of the damping coefficients. Subsequently, other authors have used thermoelastic coupling to investigate system (1.1), and well-posedness and exponential decay results have been established, see the works in Aparala et al. [4] and Mukiawa [15]. Now, viscoelasticity for materials containing history properties have been extensively studied in literature, see for instance Drozdov et al. [9] and Prüss [17]. In the case of CNTs materials, Suhr et al. [19] performed a direct shear testing of epoxy thin films containing dense packing of MWCNTs fillers and established strong viscoelastic behaviour. The viscoelastic behaviour CNTs confirmed the inherent ability to exhibit both elastic (stiff, energy-storing) and viscous (dissipative, energy-losing), interlayer sliding, and interfacial slippage in composites. Recently, Chang et al. [6] characterized the viscoelasticity of CNTs between laden air and water interface through two different experimental methods. They observed linear viscoelastic regime for a normal strain. Systems with viscoelastic damping have been extensively studied in literature, for instance, Conti and Pata [7] investigated a volterra integro-differential equation without imposing the usual differentiability decay condition on the kernel function and proved uniform decay estimate. Buriol et al. [5] studied a coupled Volterra integro-differential equations with fading memories and established a uniform general stability result depending on the kernel functions. We refer the reader to the results in [10, 16] and the references within for a rich literature on memory-type Timoshenko beam results.

Motivated by the works above, the present work is devoted to the investigation of the asymptotic behaviour of viscoelasticity mechanism in DWCNTs. In particular, we consider a DWCNT modeled as a coupled-Timoshenko beam with viscoelastic damping acting on the shear force. More specifically, the analysis is carried out for system (1.1) with the following constitutive laws:

$$\begin{aligned} S_1 &= \kappa G A_1 (\phi_x + \psi) - \int_0^t g_1(t-s)(\phi_x + \psi)(x, s) ds, \\ S_2 &= \kappa G A_2 (w_x + z) - \int_0^t g_2(t-s)(w_x + z)(x, s) ds, \\ M_1 &= E I_1 \psi_x, \quad M_2 = E I_2 z_x, \quad P = -\kappa_0 (\psi - z). \end{aligned} \quad (1.2)$$

Thus, substituting (1.2) into (1.1) and setting

$$\begin{aligned} \rho_1 &= \rho A_1, \quad \rho_2 = \rho I_1, \quad \rho_3 = \rho A_2, \quad \rho_4 = \rho I_2, \\ \kappa_1 &= \kappa G A_1, \quad \kappa_2 = \kappa G A_2, \quad b_1 = E I_1, \quad b_2 = E I_2, \end{aligned}$$

we obtain the following memory-type double-wall carbon nanotubes (DWCNTs) beam system:

$$\left\{ \begin{array}{l} \rho_1 \phi_{tt} - \kappa_1 (\phi_x + \psi)_x + \int_0^t g_1(t-s)(\phi_x + \psi)_x(x, s) ds = 0, \quad \text{in }]0, L[\times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b_1 \psi_{xx} + \kappa_1 (\phi_x + \psi) \\ \quad - \int_0^t g_1(t-s)(\phi_x + \psi)(x, s) ds + \kappa_0 (\psi - z) + \gamma_1 \psi_t = 0, \quad \text{in }]0, L[\times \mathbb{R}_+, \\ \rho_3 w_{tt} - \kappa_2 (w_x + z)_x + \int_0^t g_2(t-s)(w_x + z)_x(x, s) ds = 0, \quad \text{in }]0, L[\times \mathbb{R}_+, \\ \rho_4 z_{tt} - b_2 z_{xx} + \kappa_2 (w_x + z) \\ \quad - \int_0^t g_2(t-s)(w_x + z)(x, s) ds - \kappa_0 (\psi - z) + \gamma_2 z_t = 0, \quad \text{in }]0, L[\times \mathbb{R}_+, \end{array} \right. \quad (1.3)$$

where $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$ are constant damping coefficients, and the memory kernels g_1 and g_2 are specified in Section 2. We observe that when $\kappa_0 = 0$ and $\gamma_1 = \gamma_2 = 0$, then (1.3)₁–(1.3)₂ and (1.3)₃–(1.3)₄ are exactly the memory-type Timoshenko beam model studied in [3]. To study system (1.3), we endow it with Dirichlet-Neumann boundary conditions:

$$\left\{ \begin{array}{l} \phi_x(0, t) = \psi(0, t) = w_x(0, t) = z(0, t) = 0, \quad t \geq 0, \\ \phi_x(L, t) = \psi(L, t) = w_x(L, t) = z(L, t) = 0, \quad t \geq 0 \end{array} \right. \quad (1.4)$$

and initial data

$$\left\{ \begin{array}{ll} \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), & x \in [0, L], \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), & x \in [0, L], \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), & x \in [0, L], \\ z(x, 0) = z_0(x), \quad z_t(x, 0) = z_1(x), & x \in [0, L]. \end{array} \right. \quad (1.5)$$

The novelty of the present work is to address the general stability result for memory-type double-wall carbon nanotubes (DWCNTs) beam system (1.3)–(1.5) under general decay conditions on the kernels g_1 and g_2 . The rest of this work is organized as follows: In Section 2, we recall a few tools and state assumptions on the relaxation functions g_1 and g_2 . In Section 3, we establish the existence result for system (1.3)–(1.5). In Section 4, a few technical lemmas will be established. In Section 5, we state and prove a general decay result for the associated energy functional of system (1.3)–(1.5).

2 Conditions on memory functions and functional setting

For the rest of this work $c > 0$, and $c_i > 0$ are constants that may change from a line to another, or within the same line, and $c_p > 0$ the Poincaré constant. We denote by $\|\cdot\|$ the usual norm in $L^2(]0, L[)$. For the relaxation functions g_i , $i = 1, 2$, we impose the following conditions:

(C1) $g_i : [0, +\infty) \rightarrow (0, +\infty)$, $i = 1, 2$ is a $\mathcal{C}^1$ -function that is decreasing and

$$\widehat{\kappa}_i = \kappa_i - \int_0^\infty g_i(s)ds = \beta_i > 0.$$

(C2) For $i = 1, 2$, there exist a differentiable and non-increasing function $\xi_i : [0, +\infty) \rightarrow (0, +\infty)$, such that

$$g'_i(t) \leq -\xi_i(t)g_i(t), \quad \forall t \geq 0.$$

Now, from the boundary conditions in (1.4), we observe that the Poincaré inequality cannot be directly apply to ϕ and w . Thus, integrating (1.3)₁ and (1.3)₃ over $]0, L[$, we obtain

$$\frac{d}{dt^2} \left(\int_0^L \phi(x, t) dx \right) = 0 \text{ and } \frac{d}{dt^2} \left(\int_0^L w(x, t) dx \right) = 0.$$

It follows that

$$\begin{aligned} \int_0^L \phi(x, t) dx &= \int_0^L \phi_0(x) dx + t \int_0^L \phi_1(x) dx, \\ \int_0^L w(x, t) dx &= \int_0^L w_0(x) dx + t \int_0^L w_1(x) dx. \end{aligned}$$

So, by setting

$$\begin{aligned} \bar{\phi}(x, t) &= \phi(x, t) - \int_0^L \phi_0(x) dx - t \int_0^L \phi_1(x) dx, \\ \bar{w}(x, t) &= w(x, t) - \int_0^L w_0(x) dx - t \int_0^L w_1(x) dx, \end{aligned}$$

we obtain

$$\int_0^L \bar{\phi}(x, t) dx = \int_0^L \bar{w}(x, t) dx = 0. \quad (2.1)$$

Therefore, we can apply the Poincaré inequality to $\bar{\phi}$ and $\bar{w}$. Going forward, we work with $(\bar{\phi}, \psi, \bar{w}, z)$ which satisfies (1.3)–(1.5) with initial and boundary conditions for $\bar{\phi}$ and $\bar{w}$ given by

$$\begin{cases} \bar{\phi}(x, 0) = \phi_0(x) - \int_0^L \phi_0(x) dx, \quad \bar{\phi}_t(x, 0) = \phi_1(x) - \int_0^L \phi_1(x) dx, \\ \bar{w}(x, 0) = w_0(x) - \int_0^L w_0(x) dx, \quad \bar{w}_t(x, 0) = w_1(x) - \int_0^L w_1(x) dx \end{cases}$$

and

$$\bar{\phi}_x(0, t) = \bar{\phi}_x(L, t) = \bar{w}_x(0, t) = \bar{w}_x(L, t) = 0, \quad \forall t \geq 0,$$

respectively. For the purpose of convenience, we shall perform our computations with (ϕ, ψ, w, z) . Also, the following notations shall use throughout this work:

$$(g \diamond f)(t) = \int_0^t g(t-s) \|f(t) - f(s)\|^2 ds,$$

Next, we introduce the following Hilbert spaces:

$$\begin{aligned} L^2 &:= L^2([0, L]), \|f\|^2 = \int_0^L |f(x)|^2 dx, \\ H^1 &:= H^1([0, L]), \|f\|_{H^1}^2 = \|f_x\|^2 + \|f\|^2, \\ L_*^2 &:= L_*^2([0, L]) = \{f \in L^2([0, L]) : \frac{1}{L} \int_0^L f(x) dx = 0\}, \\ H_0^1 &:= H_0^1([0, L]) = \{u \in H^1([0, L]) : u(0) = u(L) = 0\}, \\ H_*^1 &:= H_*^1([0, L]) = \{f \in H^1([0, L]) : \frac{1}{L} \int_0^L f(x) dx = 0\}. \end{aligned}$$

The phase space of our problem is given by

$$H = H_*^1 \times L_*^2 \times H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_0^1 \times L^2$$

and the domain of the evolution operator is given by

$$H_1 = (H^2 \cap H_*^1) \times H_*^1 \times (H^2 \cap H_0^1) \times H_0^1 \times (H^2 \cap H_*^1) \times H_*^1 \times (H^2 \cap H_0^1) \times H_0^1.$$

3 Existence and uniqueness result

In this section, we present the well-posedness result for system (1.3)–(1.5). The result is obtainable via the Faedo-Galerkin approximation method, see [14].

Theorem 1. *Suppose assumptions (C1) and (C2) hold.*

- (i) *Let $U_0 = (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1, z_0, z_1) \in H$, be given, then system (1.3)–(1.5) has a global unique weak solution (ϕ, ψ, w, z) such that*

$$\begin{aligned} \phi, w &\in L^\infty(0, T; H_*^1), \phi_t, w_t \in L^\infty(0, T; L_*^2), \\ \psi, z &\in L^\infty(0, T; H_0^1), \psi_t, z_t \in L^\infty(0, T; L^2). \end{aligned}$$

- (ii) *Furthermore, if $U_0 = (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1, z_0, z_1) \in H_1$, then, the global unique weak solution (ϕ, ψ, w, z) has more regularity in the class*

$$\begin{aligned} \phi, w &\in L^\infty(0, T; H^2 \cap H_*^1), \phi_t, w_t \in L^\infty(0, T; H_*^1), \phi_{tt}, w_{tt} \in L^\infty(0, T; L_*^2), \\ \psi, z &\in L^\infty(0, T; H^2 \cap H_0^1), \psi_t, z_t \in L^\infty(0, T; H_0^1), \psi_{tt}, z_{tt} \in L^\infty(0, T; L^2). \end{aligned}$$

Proof. Step 1: Approximate system. For the linear unbounded operator $-\partial_{xx}$; we define the following:

$$-\partial_{xx} : \mathcal{D}(-\partial_{xx}) \rightarrow L^2 \text{ and } -\partial_{xx} : \mathcal{D}^*(-\partial_{xx}) \rightarrow L_*^2$$

having domains $\mathcal{D}(-\partial_{xx}) = H^2 \cap H_0^1$ and $\mathcal{D}^*(-\partial_{xx}) = H^2 \cap H_\star^1$, respectively. We know that the operator $-\partial_{xx}$ is self-adjoint and nonnegative, thus, we can find a complete orthonormal family of bases $\{p_i\}$ and $\{q_i\}$ on $L^2(0, 1)$ for $H^2 \cap H_0^1$ and $H^2 \cap H_\star^1$, respectively. Next, we consider finite dimensional subspaces $\Psi_n = \text{span}\{p_1, p_2, \dots, p_n\}$ and $\Phi_n = \text{span}\{q_1, q_2, \dots, q_n\}$. Thus, for each n , we search an approximate solutions $(\phi^n, \psi^n, w^n, z^n)$ of the form

$$(\phi^n(x, t), \psi^n(x, t), w^n(x, t), z^n(x, t)) \\ = \left(\sum_{i=1}^n a_i, n(t) q_i, \sum_{i=1}^n b_i, n(t) p_j, \sum_{i=1}^n c_i, n(t) q_j, \sum_{i=1}^n d_i, n(t) p_j \right)$$

satisfying the approximate system

$$\left\{ \begin{array}{l} \rho_1 \int_0^L \phi_{tt}^n u dx + \kappa_1 \int_0^L (\phi_x^n + \psi^n) u_x dx \\ \quad - \int_0^L u_x \int_0^t g_1(t-s)(\phi_x^n + \psi^n)(x, s) ds dx = 0, \\ \rho_2 \int_0^L \psi_{tt}^n v dx + b_1 \int_0^L \psi_x^n v_x dx + \kappa_1 \int_0^L (\phi_x^n + \psi^n) v dx + \gamma_1 \int_0^L \psi_t^n v dx \\ \quad - \int_0^L v \int_0^t g_1(t-s)(\phi_x^n + \psi^n)(x, s) ds dx + \kappa_0 \int_0^L (\psi^n - z^n) v dx = 0, \\ \rho_3 \int_0^L w_{tt}^n \sigma dx + \kappa_2 \int_0^L (w_x^n + z^n) \sigma_x dx \\ \quad - \int_0^L \sigma_x \int_0^t g_2(t-s)(w_x^n + z^n)_x(x, s) ds dx = 0, \\ \rho_4 \int_0^L z_{tt}^n \varphi dx + b_2 \int_0^L z_x^n \varphi_x dx + \kappa_2 \int_0^L (w_x^n + z^n) \varphi dx + \gamma_2 \int_0^L z_t^n \varphi dx \\ \quad - \int_0^L \varphi \int_0^t g_2(t-s)(w_x^n + z^n)_x(x, s) ds dx - \kappa_0 \int_0^L (\psi^n - z^n) \varphi = 0 \end{array} \right. \quad (3.1)$$

for all $(u, v, \sigma, \varphi) \in \Phi_n \times \Psi_n \times \Phi_n \times \Psi_n$ with initial conditions satisfying

$$(\phi^n(\cdot, 0), \phi_t^n(\cdot, 0), \psi^n(\cdot, 0), \psi_t^n(\cdot, 0), w^n(\cdot, 0), w_t^n(\cdot, 0), z^n(\cdot, 0), z_t^n(\cdot, 0)) \\ = (\phi_0^n, \phi_1^n, \psi_0^n, \psi_1^n, w_0^n, w_1^n, z_0^n, z_1^n) \longrightarrow (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1, z_0, z_1), \quad (3.2)$$

strongly in H .

Noting that (3.1)–(3.2) reduces to is finite dimensional system of ordinary differential equations (ODE), then standard ODE theory guarantees the existence of a unique local solution $(\phi^n(t), \psi^n(t), w^n(t), z^n(t))$ on $[0, t_n)$, where $0 \leq t_n < T$, for all $n \in \mathbb{N}$.

Step 2: Prior estimate. Setting $u = \phi_t^n, v = \psi_t^n, \sigma = w_t^n$ and $\varphi = z_t^n$ in system (3.1), integration by parts and adding the equations, leads to

$$\frac{d}{dt} E^n(t) + \gamma_1 \|\psi_t^n\|^2 + \gamma_2 \|z_t^n\|^2 \quad (3.3)$$

$$\begin{aligned}
&= -\frac{1}{2}g_1(t)\|\phi_x^n + \psi^n\|^2 + \frac{1}{2}(g_1' \diamond (\phi_x^n + \psi^n))(t) \\
&\quad - \frac{1}{2}g_2(t)\|w_x^n + z^n\|^2 + \frac{1}{2}(g_2' \diamond (w_x^n + z^n))(t) \leq 0, \quad \forall t \geq 0,
\end{aligned}$$

where

$$\begin{aligned}
E^n(t) &= \frac{1}{2} \left[\rho_1 \|\phi_t^n\|^2 + \rho_2 \|\psi_t^n\|^2 + \rho_3 \|w_t^n\|^2 + \rho_4 \|z_t^n\|^2 + \widehat{\kappa}_1 \|\phi_x^n + \psi^n\|^2 \right] \\
&\quad + \frac{1}{2} \left[b_1 \|\psi_x^n\|^2 + \widehat{\kappa}_2 \|w_x^n + z^n\|^2 + b_2 \|z_x^n\|^2 + \kappa_0 \|\psi^n - z^n\|^2 \right] \\
&\quad + \frac{1}{2} \left[(g_1 \diamond (\phi_x^n + \psi^n))(t) + (g_2 \diamond (w_x^n + z^n))(t) \right], \quad \forall t \geq 0.
\end{aligned}$$

Integration of (3.3) over $0 \leq t < t_n$ and taking into account (3.2), we get for any $n \in \mathbb{N}$

$$E^n(t) + \gamma_1 \int_0^t \|\psi_t^n(s)\|^2 ds + \gamma_2 \int_0^t \|z_t^n(s)\|^2 ds \leq E^n(0) \leq c,$$

where c is a positive constant independent of t and n . Thus, taking the supremum over $t \in (0, t_n)$, we arrive at

$$E^n(t) + \gamma_1 \int_0^T \|\psi_t^n(t)\|^2 dt + \gamma_2 \int_0^T \|z_t^n(t)\|^2 dt \leq c_1, \quad (3.4)$$

with c_1 been a positive constant independent of t and n . Therefore, the approximate solution for system(3.1) exist $\forall t \in [0, T]$.

Step 3: Passage to Limit. On account of (3.4) and recalling the definition of $E^n(t)$, we obtain

$$\begin{cases} \{\phi^n\} \text{ is bounded in } L^\infty(0, T; H_*^1) \text{ and } \{\phi_t^n\} \text{ is bounded in } L^\infty(0, T; L_*^2), \\ \{\psi^n\} \text{ is bounded in } L^\infty(0, T; H_0^1) \text{ and } \{\psi_t^n\} \text{ is bounded in } L^\infty(0, T; L^2), \\ \{w^n\} \text{ is bounded in } L^\infty(0, T; H_*^1) \text{ and } \{w_t^n\} \text{ is bounded in } L^\infty(0, T; L_*^2), \\ \{z^n\} \text{ is bounded in } L^\infty(0, T; H_0^1) \text{ and } \{z_t^n\} \text{ is bounded in } L^\infty(0, T; L^2). \end{cases}$$

Therefore, we can extract subsequences $\{\phi^m\}, \{\psi^m\}, \{w^m\}$ and $\{z^m\}$ of $\{\phi^n\}, \{\psi^n\}, \{w^n\}$ and $\{z^n\}$, respectively, such that

$$\begin{cases} \phi^m \rightharpoonup^* \phi \text{ in } L^\infty(0, T; H_*^1) \text{ and } \phi_t^m \rightharpoonup^* \phi_t \text{ in } L^\infty(0, T; L_*^2), \\ \psi^m \rightharpoonup^* \psi \text{ in } L^\infty(0, T; H_0^1) \text{ and } \psi_t^m \rightharpoonup^* \psi_t \text{ in } L^\infty(0, T; L^2), \\ w^m \rightharpoonup^* w \text{ in } L^\infty(0, T; H_*^1) \text{ and } w_t^m \rightharpoonup^* w_t \text{ in } L^\infty(0, T; L_*^2), \\ z^m \rightharpoonup^* z \text{ in } L^\infty(0, T; H_0^1) \text{ and } z_t^m \rightharpoonup^* z_t \text{ in } L^\infty(0, T; L^2). \end{cases} \quad (3.5)$$

On account of the limits above and replacing $\{\phi^n\}, \{\psi^n\}, \{w^n\}, \{z^n\}$ with $\{\phi^m\}, \{\psi^m\}, \{w^m\}, \{z^m\}$ in the approximate system (3.1)–(3.2), then passing to limit, we obtain a weak solution satisfying

$$\begin{aligned}
&\phi, w \in L^\infty(0, T; H_*^1), \psi, z \in L^\infty(0, T; H_0^1), \\
&\phi_t, w_t \in L^\infty(0, T; L_*^2), \psi_t, z_t \in L^\infty(0, T; L^2).
\end{aligned}$$

Step 4: Regular solution. Next, we assume the initial data for the approximate system (3.1)–(3.2) $(\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1, z_0, z_1) \in H_1$ satisfying

$$(\phi_0^n, \phi_1^n, \psi_0^n, \psi_1^n, w_0^n, w_1^n, z_0^n, z_1^n) \rightarrow (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1, z_0, z_1) \text{ strongly in } \mathcal{H}_1. \quad (3.6)$$

Now, setting $u = -\phi_{xxt}^n, v = -\psi_{xxt}^n, \sigma = -w_{xxt}^n$ and $\varphi = -z_{xxt}^n$ in (3.1), integration by parts and adding the equations, we arrive at

$$\begin{aligned} & \frac{d}{dt} E_1^n(t) + \gamma_1 \|\psi_{xt}^n\|^2 + \gamma_2 \|z_{xt}^n\|^2 \\ &= -\frac{1}{2} g_1(t) \|\phi_{xx}^n + \psi_x^n\|^2 + \frac{1}{2} (g_1' \diamond (\phi_{xx}^n + \psi_x^n))(t) \\ & - \frac{1}{2} g_2(t) \|w_{xx}^n + z_x^n\|^2 + \frac{1}{2} (g_2' \diamond (w_{xx}^n + z_x^n))(t) \leq 0, \quad \forall t \geq 0, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} E_1^n(t) &= \frac{1}{2} \left[\rho_1 \|\phi_{xt}^n\|^2 + \rho_2 \|\psi_{xt}^n\|^2 + \rho_3 \|w_{xt}^n\|^2 + \rho_4 \|z_{xt}^n\|^2 + \widehat{\kappa}_1 \|\phi_{xx}^n + \psi_x^n\|^2 \right] \\ &+ \frac{1}{2} \left[b_1 \|\psi_{xx}^n\|^2 + \widehat{\kappa}_1 \|w_{xx}^n + z_x^n\|^2 + b_2 \|z_{xx}^n\|^2 + \kappa_0 \|\psi_x^n - z_x^n\|^2 \right] \\ &+ \frac{1}{2} \left[(g_1 \diamond (\phi_{xx}^n + \psi_x^n))(t) + (g_2 \diamond (w_{xx}^n + z_x^n))(t) \right], \quad \forall t \geq 0. \end{aligned}$$

Integration of (3.7) and recalling (3.6) gives for all $t \in [0, T]$, $n \in \mathbb{N}$,

$$E_1^n(t) + \gamma_1 \int_0^t \|\psi_{xt}^n(s)\|^2 ds + \gamma_2 \int_0^t \|z_{xt}^n(s)\|^2 ds \leq E_1^n(0) \leq c_2, \quad (3.8)$$

for some positive constant c_2 independent of t and n . Also, setting $u = \phi_{tt}^n, v = \psi_{tt}^n, \sigma = w_{tt}^n$ and $\varphi = z_{tt}^n$ in (3.1), we have for any $\eta > 0$,

$$\begin{aligned} & \rho_1 \|\phi_{tt}^n\|^2 + \|\psi_{tt}^n\|^2 + \rho_3 \|w_{tt}^n\|^2 + \rho_4 \|z_{tt}^n\|^2 \\ & \leq \eta \left(\rho_1 \|\phi_{tt}^n\|^2 + \|\psi_{tt}^n\|^2 + \rho_3 \|w_{tt}^n\|^2 + \rho_4 \|z_{tt}^n\|^2 \right) \\ & + c_\eta \left(\beta_1 \|\phi_{xx}^n + \psi_x^n\|^2 + b_1 \|\psi_{xx}^n\|^2 + \beta_2 \|w_{xx}^n + z_x^n\|^2 + b_2 \|z_{xx}^n\|^2 \right) \\ & + c_\eta \left(\kappa_0 \|\psi_x^n - z_x^n\|^2 + \gamma_1 \|\psi_t^n\|^2 + \gamma_2 \|z_t^n\|^2 \right) \\ & + c_\eta \left((g_1 \diamond (\phi_{xx}^n + \psi_x^n))(t) + (g_2 \diamond (w_{xx}^n + z_x^n))(t) \right). \end{aligned} \quad (3.9)$$

Consequently, choosing η small enough such that $(1 - \eta) > 0$, and combining with (3.4) and (3.8), it follows from (3.9) that

$$\rho_1 \|\phi_{tt}^n\|^2 + \|\psi_{tt}^n\|^2 + \rho_3 \|w_{tt}^n\|^2 + \rho_4 \|z_{tt}^n\|^2 \leq c_3, \quad (3.10)$$

for some positive constant c_3 independent of t and n . From (3.8) and (3.10) we

conclude that

$$\left\{ \begin{array}{l} \{\phi^n\} \text{ is bounded in } L^\infty(0, T; H^2 \cap H_*^1), \\ \{\phi_t^n\} \text{ is bounded in } L^\infty(0, T; H_*^1) \text{ and } \{\phi_{tt}^n\} \text{ is bounded in } L^\infty(0, T; L_*^2), \\ \{\psi^n\} \text{ is bounded in } L^\infty(0, T; H^2 \cap H_0^1), \\ \{\psi_t^n\} \text{ is bounded in } L^\infty(0, T; H_0^1) \text{ and } \{\psi_{tt}^n\} \text{ is bounded in } L^\infty(0, T; L^2), \\ \{w^n\} \text{ is bounded in } L^\infty(0, T; H^2 \cap H_*^1), \\ \{w_t^n\} \text{ is bounded in } L^\infty(0, T; H_*^1) \text{ and } \{w_{tt}^n\} \text{ is bounded in } L^\infty(0, T; L_*^2), \\ \{z^n\} \text{ is bounded in } L^\infty(0, T; H^2 \cap H_0^1), \\ \{z_t^n\} \text{ is bounded in } L^\infty(0, T; H_0^1) \text{ and } \{z_{tt}^n\} \text{ is bounded in } L^\infty(0, T; L^2). \end{array} \right.$$

Therefore, we can extract subsequences $\{\phi^m\}, \{\psi^m\}, \{w^m\}$ and $\{z^m\}$ such that

$$\left\{ \begin{array}{l} \phi^m \rightharpoonup^* \phi \text{ in } L^\infty(0, T; H^2 \cap H_*^1), \\ \phi_t^m \rightharpoonup^* \phi_t \text{ in } L^\infty(0, T; H_*^1) \text{ and } \phi_{tt}^m \rightharpoonup^* \phi_{tt} \text{ in } L^\infty(0, T; L_*^2), \\ \psi^m \rightharpoonup^* \psi \text{ in } L^\infty(0, T; H^2 \cap H_0^1), \\ \psi_t^m \rightharpoonup^* \psi_t \text{ in } L^\infty(0, T; H_0^1) \text{ and } \psi_{tt}^m \rightharpoonup^* \psi_{tt} \text{ in } L^\infty(0, T; L^2), \\ w^m \rightharpoonup^* w \text{ in } L^\infty(0, T; H^2 \cap H_*^1), \\ w_t^m \rightharpoonup^* w_t \text{ in } L^\infty(0, T; H_*^1) \text{ and } w_{tt}^m \rightharpoonup^* w_{tt} \text{ in } L^\infty(0, T; L_*^2), \\ z^m \rightharpoonup^* z \text{ in } L^\infty(0, T; H^2 \cap H_0^1), \\ z_t^m \rightharpoonup^* z_t \text{ in } L^\infty(0, T; H_0^1) \text{ and } z_{tt}^m \rightharpoonup^* z_{tt} \text{ in } L^\infty(0, T; L^2). \end{array} \right. \quad (3.11)$$

Thus, we conclude that (ϕ, ψ, w, z) is a more regular solution of system (3.1)–(3.2) satisfying

$$\begin{aligned} \phi, w &\in L^\infty(0, T; H^2 \cap H_*^1), \phi_t, w_t \in L^\infty(0, T; H_*^1), \phi_{tt}, w_{tt} \in L^\infty(0, T; L_*^2), \\ \psi, z &\in L^\infty(0, T; H^2 \cap H_0^1), \psi_t, z_t \in L^\infty(0, T; H_0^1), \psi_{tt}, z_{tt} \in L^\infty(0, T; L^2). \end{aligned}$$

Step 5: Initial conditions: From (3.5) and (3.11), (see Aubin-Lions lemma [14]), we deduce that up to subsequences

$$\begin{aligned} \phi^m &\rightarrow \phi \text{ strongly in } C([0, T]; H_*^1) \text{ and } \phi_t^m \rightarrow \phi_t \text{ strongly in } C([0, T]; L_*^2), \\ \psi^m &\rightarrow \psi \text{ strongly in } C([0, T]; H_0^1) \text{ and } \psi_t^m \rightarrow \psi_t \text{ strongly in } C([0, T]; L^2), \\ w^m &\rightarrow w \text{ strongly in } C([0, T]; H_*^1) \text{ and } w_t^m \rightarrow w_t \text{ strongly in } C([0, T]; L_*^2), \\ z^m &\rightarrow z \text{ strongly in } C([0, T]; H_0^1) \text{ and } z_t^m \rightarrow z_t \text{ strongly in } C([0, T]; L^2). \end{aligned}$$

It follows that

$$(\phi^m(x, 0), \phi_t^m(x, 0), \psi^m(x, 0), \psi_t^m(x, 0), w^m(x, 0), w_t^m(x, 0), z^m(x, 0), z_t^m(x, 0))$$

all makes sense and

$$\begin{aligned} &(\phi^m(x, 0), \phi_t^m(x, 0), \psi^m(x, 0), \psi_t^m(x, 0), w^m(x, 0), w_t^m(x, 0), z^m(x, 0), z_t^m(x, 0)) \\ &\longrightarrow (\phi(x, 0), \phi_t(x, 0), \psi(x, 0), \psi_t(x, 0), w(x, 0), w_t(x, 0), z(x, 0), z_t(x, 0)) \\ &= (\phi_0(x), \phi_1(x), \psi_0(x), \psi_1(x), w_0(x), w_1(x, 0), z_0(x), z_1(x)) \in H, \end{aligned} \quad (3.12)$$

Recalling (3.2) and (3.12), and the uniqueness of limit, we obtain the initial data.

Step 6: Uniqueness Suppose system (1.3)–(1.5) has two $(\phi_1, \psi_1, w_1, z_1)$ and $(\phi_2, \psi_2, w_2, z_2)$ Then, the solution

$$(\Phi, \Psi, W, Z) = (\phi_1 - \phi_2, \psi_1 - \psi_2, w_1 - w_2, z_1 - z_2)$$

satisfies the system

$$\left\{ \begin{array}{l} \rho_1 \Phi_{tt} - \kappa_1 (\Phi_x + \Psi)_x + \int_0^t g_1(t-s)(\Phi_x + \Psi)_x(x, s) ds = 0, \\ \rho_2 \Psi_{tt} - b_1 \Psi_{xx} + \kappa_1 (\Phi_x + \Psi) \\ \quad - \int_0^t g_1(t-s)(\Phi_x + \Psi)(x, s) ds + \kappa_0 (\Psi - Z) + \gamma_1 \Psi_t = 0, \\ \rho_3 W_{tt} - \kappa_2 (W_x + Z)_x + \int_0^t g_2(t-s)(W_x + Z)_x(x, s) ds = 0, \\ \rho_4 Z_{tt} - b_2 Z_{xx} + \kappa_2 (W_x + Z) \\ \quad - \int_0^t g_2(t-s)(W_x + Z)(x, s) ds - \kappa_0 (\Psi - Z) + \gamma_2 Z_t = 0, \end{array} \right. \quad (3.13)$$

with boundary conditions

$$\left\{ \begin{array}{l} \Phi_x(0, t) = \Psi(0, t) = W_x(0, t) = Z(0, t) = 0, \quad t \geq 0, \\ \Phi_x(L, t) = \Psi(L, t) = W_x(L, t) = Z(L, t) = 0, \quad t \geq 0 \end{array} \right.$$

and initial data

$$\left\{ \begin{array}{l} \Phi(x, 0) = \Phi_t(x, 0) = \Psi(x, 0) = \Psi_t(x, 0) = 0, \quad x \in [0, L] \\ W(x, 0) = W_t(x, 0) = w_1(x) = Z(x, 0) = Z_t(x, 0) = 0, \quad x \in [0, L]. \end{array} \right. \quad (3.14)$$

Multiplying the equations in (3.13) by Φ_t, Ψ_t, W_t and Z_t , respectively, we arrive dissipative law

$$\begin{aligned} \frac{d}{dt} \widehat{E}(t) &= -\gamma_1 \|\Psi_t\|^2 - \frac{1}{2} g_1(t) \|\Phi_x + \Psi\|^2 + \frac{1}{2} (g'_1 \diamond (\Phi_x + \Psi))(t) \\ &\quad - \gamma_2 \|Z_t\|^2 - \frac{1}{2} g_2(t) \|W_x + Z\|^2 + \frac{1}{2} (g'_2 \diamond (W_x + Z))(t) \leq 0, \quad \forall t \geq 0, \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} \widehat{E}(t) &= \frac{1}{2} \left[\rho_1 \|\Phi_t\|^2 + \rho_2 \|\Psi_t\|^2 + \rho_3 \|W_t\|^2 + \rho_4 \|Z_t\|^2 + \widehat{\kappa}_1 \|\Phi_x + \Psi\|^2 \right] \\ &\quad + \frac{1}{2} \left[b_1 \|\Psi_x\|^2 + \widehat{\kappa}_2 \|W_x + Z\|^2 + b_2 \|Z_x\|^2 + \kappa_0 \|\Psi - Z\|^2 \right] \\ &\quad + \frac{1}{2} \left[(g_1 \diamond (\Phi_x + \Psi))(t) + (g_2 \diamond (W_x + Z))(t) \right], \quad \forall t \geq 0. \end{aligned}$$

Integrating (3.15) over $(0, t)$ and recalling (3.14), we have

$$\widehat{E}(t) \leq \widehat{E}(0) = 0,$$

which gives $\Phi = \Psi = W = Z = 0$. Hence, $(\phi_1, \psi_1, w_1, z_1) = (\phi_2, \psi_2, w_2, z_2)$. Therefore, system (1.3)–(1.5) has a unique solution. This completes the proof of Theorem 1. $\square$

4 Needed lemmas

In this section, we present some important lemmas that will use to establish the stability result. The following Lemma will be applicable for the rest of the paper.

Lemma 1. For any $f, h \in L^2(0, L)$ and $\varphi \in L^2([0, \infty); L^2(0, L))$, we have

(A)

$$\int_0^L \left(\int_0^t g_i(t-s)[\varphi(t) - \varphi(s)] ds \right)^2 dx \leq (\kappa_i - \beta_i)(g_i \diamond \varphi)(t).$$

(B) Given $\sigma > 0$, we have

$$\int_0^L f(x) \int_0^x h(y) dy dx \leq \frac{\sigma}{2} \|f\|_2^2 + \frac{L^2}{2\sigma} \|h\|_2^2.$$

(C)

$$\left(\int_0^x f(y) dy \right)^2 \leq x \int_0^x f^2(y) dy \leq L \|f\|_2^2. \quad (4.1)$$

Proof. Using Cauchy-Schwarz and Young inequalities we obtain the results. $\square$

Lemma 2. Let (ϕ, ψ, w, z) be the solution of Problem (1.3)–(1.5). The energy functional of system (1.3)–(1.5) is given by

$$\begin{aligned} E(t) = & \frac{1}{2} \left[\rho_1 \|\phi_t\|^2 + \rho_2 \|\psi_t\|^2 + \rho_3 \|w_t\|^2 + \rho_4 \|z_t\|^2 + \widehat{\kappa}_1 \|\phi_x + \psi\|^2 + b_1 \|\psi_x\|^2 \right] \\ & + \frac{1}{2} \left[\widehat{\kappa}_2 \|w_x + z\|^2 + b_2 \|z_x\|^2 + \kappa_0 \|\psi - z\|^2 \right] \\ & + \frac{1}{2} \left[(g_1 \diamond (\phi_x + \psi))(t) + (g_2 \diamond (w_x + z))(t) \right], \quad \forall t \geq 0 \end{aligned} \quad (4.2)$$

and satisfies the dissipative inequality

$$\begin{aligned} E'(t) = & -\gamma_1 \|\psi_t\|^2 - \frac{1}{2} g_1(t) \|\phi_x + \psi\|^2 + \frac{1}{2} (g'_1 \diamond (\phi_x + \psi))(t) \\ & - \gamma_2 \|z_t\|^2 - \frac{1}{2} g_2(t) \|w_x + z\|^2 + \frac{1}{2} (g'_2 \diamond (w_x + z))(t) \leq 0, \quad \forall t \geq 0. \end{aligned} \quad (4.3)$$

Proof. Multiplying (1.3)₁, (1.3)₂, (1.3)₃ and (1.3)₄ by ϕ_t , ψ_t , w_t and z_t respectively, follow by integrating over $]0, L[$, using integration by parts and the

boundary condition (1.4), then summing the outcome yields

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left[\rho_1 \|\phi_t\|^2 + \kappa_1 \|\phi_x + \psi\|^2 + \rho_2 \|\psi_t\|^2 + b_1 \|\psi_x\|^2 + \rho_3 \|w_t\|^2 \right] \\
 & + \frac{1}{2} \frac{d}{dt} \left[\kappa_2 \|w_x + z\|^2 + \rho_4 \|z_t\|^2 + b_2 \|z_x\|^2 + \kappa_0 \|\psi - z\|^2 \right] \\
 & - \underbrace{\int_0^L (\phi_x + \psi)_t(t) \int_0^t g_1(t-s)(\phi_x + \psi)(s) ds dx}_{J_1} + \gamma_1 \|\psi_t\|^2 \\
 & - \underbrace{\int_0^L (w_x + z)_t(t) \int_0^t g_2(t-s)(w_x + z)(s) ds dx}_{J_2} + \gamma_2 \|z_t\|^2 = 0.
 \end{aligned} \tag{4.4}$$

We express J_1 and J_2 as follows:

$$\begin{aligned}
 J_1 &= \int_0^L (\phi_x + \psi)_t(t) \int_0^t g_1(t-s) [(\phi_x + \psi)(s) - (\phi_x + \psi)(t) + (\phi_x + \psi)(t)] ds dx \\
 &= \int_0^L (\phi_x + \psi)_t(t) \int_0^t g_1(t-s)(\phi_x + \psi)(t) ds dx \\
 &+ \int_0^L \int_0^t g_1(t-s) [(\phi_x + \psi)(s) - (\phi_x + \psi)(t)] (\phi_x + \psi)_t(t) ds dx \\
 &= \frac{1}{2} \frac{d}{dt} \left[\left(\int_0^t g_1(s) ds \right) \|\phi_x + \psi\|^2 \right] - \frac{1}{2} g_1(t) \|\phi_x + \psi\|^2 \\
 &- \frac{1}{2} \frac{d}{dt} (g_1 \diamond (\phi_x + \psi))(t) + \frac{1}{2} (g_1' \diamond (\phi_x + \psi))(t).
 \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 J_2 &= \frac{1}{2} \frac{d}{dt} \left[\left(\int_0^t g_2(s) ds \right) \|w_x + z\|^2 \right] - \frac{1}{2} g_2(t) \|w_x + z\|^2 \\
 &- \frac{1}{2} \frac{d}{dt} (g_2 \diamond (w_x + z))(t) + \frac{1}{2} (g_2' \diamond (w_x + z))(t).
 \end{aligned}$$

On substituting the expressions of J_1 and J_2 into (4.4), we arrive at (4.3). Thus, it follows from (4.3) that the energy is decreasing and bounded above by $E(0)$. $\square$

Lemma 3. The functionals M_1 and M_2 defined as

$$\begin{aligned}
 M_1(t) &= -\rho_1 \int_0^L \phi_t \int_0^x \int_0^t g_1(t-s) [(\phi_y + \psi)(t) - (\phi_y + \psi)(s)] ds dy dx, \\
 M_2(t) &= -\rho_3 \int_0^L w_t \int_0^x \int_0^t g_2(t-s) [(w_y + z)(t) - (w_y + z)(s)] ds dy dx
 \end{aligned}$$

along the the solution of system (1.3)–(1.5), satisfies respectively, for any $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $t_0 > 0$, the estimates

$$M'_1(t) \leq -\frac{\rho_1 g_1^0}{2} \|\phi_t\|^2 + \epsilon_1 \|\phi_x + \psi\|^2 + c \|\psi_t\|^2 + c \left(1 + \frac{1}{\epsilon_1}\right) (g_1 \diamond (\phi_x + \psi))(t) - c(g'_1 \diamond (\phi_x + \psi))(t), \quad \forall t \geq t_0, \quad (4.5)$$

$$M'_2(t) \leq -\frac{\rho_3 g_2^0}{2} \|w_t\|^2 + \epsilon_2 \|w_x + z\|^2 + c \|z_t\|^2 + c \left(1 + \frac{1}{\epsilon_2}\right) (g_2 \diamond (w_x + z))(t) - c(g'_2 \diamond (w_x + z))(t), \quad \forall t \geq t_0, \quad (4.6)$$

where $g_1^0 = \int_0^{t_0} g_1(\tau) d\tau > 0$, $g_2^0 = \int_0^{t_0} g_2(\tau) d\tau > 0$.

Proof. Direct differentiation of M_1 with respect to t gives

$$\begin{aligned} M'_1(t) = & \underbrace{-\rho_1 \int_0^L \phi_{tt} \int_0^x \int_0^t g_1(t-s) [(\phi_y + \psi)(t) - (\phi_y + \psi)(s)] ds dy dx}_{Q_1} \\ & \underbrace{-\rho_1 \int_0^L \phi_t \int_0^x \int_0^t g'_1(t-s) [(\phi_y + \psi)(t) - (\phi_y + \psi)(s)] ds dy dx}_{Q_2} \\ & \underbrace{-\rho_1 \int_0^L \phi_t \int_0^x \int_0^t g_1(t-s) (\phi_y + \psi)_t ds dy dx}_{Q_3}. \end{aligned} \quad (4.7)$$

On account of continuity of g_1 , we can fixed $t_0 > 0$ such that

$$g_1^0 = \int_0^{t_0} g_1(\tau) d\tau \leq \int_0^t g_1(\tau) d\tau, \quad \forall t \geq t_0. \quad (4.8)$$

Now, we estimate the terms $Q_1 - Q_3$ as follows: On account of $(1.3)_1$ and using integration by parts, we arrive at

$$\begin{aligned} Q_1 = & \kappa_1 \int_0^L (\phi_x + \psi)(t) \int_0^t g_1(t-s) [(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds dx \\ & - \int_0^L \int_0^t g_1(t-s) (\phi_x + \psi)(t) ds \int_0^t g_1(t-s) [(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds dx \\ & + \int_0^L \left(\int_0^t g_1(t-s) [(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds \right)^2 dx. \end{aligned}$$

Recalling Lemma 1 and (4.8) and making use of Cauchy-Schwarz and Young's

inequalities, we have for any positive ϵ_1

$$\begin{aligned}
 Q_1 &= \widehat{\kappa}_1 \int_0^L (\phi_x + \psi)(t) \int_0^t g_1(t-s) [(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds dx \\
 &\quad + \int_0^L \left(\int_0^t g_1(t-s) [(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds \right)^2 dx \\
 &\leq \epsilon_1 \|\phi_x + \psi\|^2 + \frac{c(\kappa_1 - \beta_1)(g_1^0)^2}{\epsilon_1} (g_1 \diamond (\phi_x + \psi))(t) \\
 &\quad + \kappa_1(\kappa_1 - \beta_1)(g_1 \diamond (\phi_x + \psi))(t) \\
 &\leq \epsilon_1 \|\phi_x + \psi\|^2 + c \left(1 + \frac{1}{\epsilon_1} \right) (g_1 \diamond (\phi_x + \psi))(t).
 \end{aligned}$$

Applying Cauchy-Schwarz and Young's inequality and recalling (4.1), we get

$$\begin{aligned}
 Q_2 &\leq \frac{\delta_1}{2} \|\phi_t\|^2 + \frac{c}{\delta_1} \int_0^L \left(\int_0^x \int_0^t g_1'(t-s) [(\phi_y + \psi)(t) - (\phi_y + \psi)(s)] ds dy \right)^2 dx \\
 &\leq \frac{\delta_1}{2} \|\phi_t\|^2 + \frac{c}{\delta_1} \int_0^L \left(\int_0^t \sqrt{-g_1'(t-s)} \sqrt{-g_1'(t-s)} [(\phi_x + \psi)(t) \right. \\
 &\quad \left. - (\phi_x + \psi)(s)] ds \right)^2 dx \\
 &\leq \frac{\delta_1}{2} \|\phi_t\|^2 - \frac{c g_1(0)}{\delta_1} \int_0^L \int_0^t g_1'(t-s) [(\phi_x + \psi)(t) - (\phi_x + \psi)(s)]^2 ds dx \\
 &\leq \frac{\delta_1}{2} \|\phi_t\|^2 - \frac{c}{\delta_1} (g_1' \diamond (\phi_x + \psi))(t), \quad \text{for any } \delta_1 > 0.
 \end{aligned}$$

For the term Q_3 , recalling (2.1), we obtain

$$\begin{aligned}
 Q_3 &= -\rho_1 \left(\int_0^t g_1(s) ds \right) \int_0^L \phi_t \int_0^x (\phi_y + \psi)_t(t) dy dx \\
 &= -\rho_1 \left(\int_0^t g_1(s) ds \right) \left[\int_0^L \phi_t^2 dx - \phi_t(0, t) \int_0^L \phi_t dx + \int_0^L \phi_t \int_0^x \psi_t(y) dy dx \right] \\
 &\leq -\rho_1 g_1^0 \|\phi_t\|^2 + \frac{\delta_1}{2} \|\phi_t\|^2 + \frac{c}{\delta_1} \|\psi_t\|^2, \quad \text{for any } \delta_1 > 0.
 \end{aligned}$$

Now, substitution of the estimates for Q_1 – Q_3 into (4.7) and choosing $\delta_1 = \frac{\rho_1 g_1^0}{2}$ yield (4.5). Following the exactly thesame steps, we obtain the estimate for M_2 in (4.6). $\square$

Lemma 4. The functionals M_3 and M_4 defined by

$$M_3(t) = -\rho_1 \int_0^L (\phi_x + \psi)(t) \int_0^x \phi_t(y) dy dx$$

and

$$M_4(t) = -\rho_3 \int_0^L (w_x + z)(t) \int_0^x w_t(y) dy dx$$

along the solution of system (1.3)–(1.5), satisfies the estimates, respectively

$$M'_3(t) \leq -\frac{\beta_1}{2} \|\phi_x + \psi\|^2 + c \|\phi_t\|^2 + c \|\psi_t\|^2 + c(g_1 \diamond (\phi_x + \psi))(t), \quad \forall t \geq 0 \quad \text{and} \quad (4.9)$$

$$M'_4(t) \leq -\frac{\beta_2}{2} \|w_x + z\|^2 + c \|w_t\|^2 + c \|z_t\|^2 + c(g_2 \diamond (w_x + z))(t), \quad \forall t \geq 0. \quad (4.10)$$

Proof. Differentiating M_3 with respect to t , using (1.3)₁, integration by parts, and keeping mind (2.1), we have

$$\begin{aligned} M'_3(t) &= -\rho_1 \int_0^L (\phi_{xt} + \psi_t)(t) \int_0^x \phi_t(y) dy dx - \rho_1 \int_0^L (\phi_x + \psi)(t) \int_0^x \phi_{tt}(y) dy dx \\ &= -\rho_1 \left[\phi_t \int_0^x \phi_t(y) dy \Big|_{x=0}^{x=L} - \|\phi_t\|^2 \right] - \rho_1 \int_0^L \psi_t \int_0^x \phi_t(y) dy dx - \kappa_1 \|(\phi_x + \psi)\|^2 \\ &\quad + \int_0^L (\phi_x + \psi)(t) \int_0^t g_1(t-s)(\phi_x + \psi)(s) ds dx \\ &= -\hat{\kappa}_1 \|(\phi_x + \psi)\|^2 + \rho_1 \|\phi_t\|^2 - \rho_1 \int_0^L \psi_t \int_0^x \phi_t(y) dy dx \\ &\quad - \int_0^L (\phi_x + \psi)(t) \int_0^t g_1(t-s)[(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds dx. \end{aligned}$$

Applying Lemma 1 and Young's inequality, we obtain

$$M'_3(t) \leq -\frac{\beta_1}{2} \|\phi_x + \psi\|^2 + \frac{\rho_1}{2} \|\psi_t\|^2 + \frac{3\rho_1}{2} \|\phi_t\|^2 + \frac{2(\kappa_1 - \beta_1)}{\beta_1} (g_1 \diamond (\phi_x + \psi))(t).$$

Following exactly the same steps, we obtain the estimate for M_4 in (4.10). $\square$

Lemma 5. The functional M_5 defined as

$$M_5(t) = \rho_2 \int_0^L \psi \psi_t dx + \frac{\gamma_1}{2} \int_0^L \psi_t^2 dx + \rho_4 \int_0^L z z_t dx + \frac{\gamma_2}{2} \int_0^L z_t^2 dx \quad (4.11)$$

along the solution of system (1.3)–(1.5) satisfies, the estimate

$$\begin{aligned} M'_5(t) &\leq -\frac{b_1}{2} \|\psi_x\|^2 - \frac{b_2}{2} \|z_x\|^2 - \kappa_0 \|\psi - z\|^2 + \rho_2 \|\psi_t\|^2 + \rho_4 \|z_t\|^2 \\ &\quad + c \|\phi_x + \psi\|^2 + c \|w_x + z\|^2 + c(\kappa_1 - \beta_1)(g_1 \diamond (\phi_x + \psi))(t) \\ &\quad + c(\kappa_2 - \beta_2)(g_2 \diamond (w_x + z))(t), \quad \forall t \geq 0. \end{aligned} \quad (4.12)$$

Proof. Differentiating (4.11) and using (1.3)₂ and (1.3)₄, we get

$$M'_5(t) = -b_1 \|\psi_x\|^2 - b_2 \|z_x\|^2 - \kappa_0 \|\psi - z\|^2 + \rho_2 \|\psi_t\|^2 + \rho_4 \|z_t\|^2$$

$$\begin{aligned}
& -\kappa_1 \int_0^L \psi(\phi_x + \psi)(t) dx - \kappa_2 \int_0^L z(w_x + z)(t) dx \\
& + \int_0^L \psi \int_0^t g_1(t-s)(\phi_x + \psi)(s) ds dx + \int_0^L z \int_0^t g_2(t-s)(w_x + z)(s) ds dx \\
& = -b_1 \|\psi_x\|^2 - b_2 \|z_x\|^2 - \kappa_0 \|\psi - z\|^2 + \rho_2 \|\psi_t\|^2 + \rho_4 \|z_t\|^2 \\
& - \hat{\kappa}_1 \int_0^L \psi(\phi_x + \psi)(t) dx - \hat{\kappa}_2 \int_0^L z(w_x + z)(t) dx \\
& - \int_0^1 \psi \int_0^t g_1(t-s)[(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds dx \\
& - \int_0^L z \int_0^t g_2(t-s)[(w_x + z)(t) - (w_x + z)(s)] ds dx.
\end{aligned}$$

Thus, applying Lemma 1 and making use of Young's and Poincaré's inequalities, we have

$$\begin{aligned}
M'_5(t) & \leq -b_1 \|\psi_x\|^2 - b_2 \|z_x\|^2 - \kappa_0 \|\psi - z\|^2 + \rho_2 \|\psi_t\|^2 + \rho_4 \|z_t\|^2 \\
& + \frac{b_1}{4} \|\psi_x\|^2 + \frac{3\beta_1^2 c_p^2}{4b_1} \|\phi_x + \psi\|^2 + \frac{b_2}{4} \|z_x\|^2 + \frac{3\beta_2^2 c_p^2}{4b_2} \|w_x + z\|^2 \\
& + \frac{b_1}{4} \|\psi_x\|^2 + \frac{3c_p^2}{4b_1} \int_0^L \left(\int_0^t g_1(t-s)[(\phi_x + \psi)(t) - (\phi_x + \psi)(s)] ds \right)^2 dx \\
& + \frac{b_2}{4} \|z_x\|^2 + \frac{3c_p^2}{4b_2} \int_0^L \left(\int_0^t g_2(t-s)[(w_x + z)(t) - (w_x + z)(s)] ds \right)^2 dx \\
& \leq -\frac{b_1}{2} \|\psi_x\|^2 - \frac{b_2}{2} \|z_x\|^2 - \kappa_0 \|\psi - z\|^2 + \rho_2 \|\psi_t\|^2 + \rho_4 \|z_t\|^2 \\
& + \frac{3\beta_1^2 c_p^2}{4b_1} \|\phi_x + \psi\|^2 + \frac{3\beta_2^2 c_p^2}{4b_2} \|w_x + z\|^2 \\
& + \frac{3c_p^2(\kappa_1 - \beta_1)}{4b_1} (g_1 \diamond (\phi_x + \psi)(t)) + \frac{3c_p^2(\kappa_2 - \beta_2)}{4b_2} (g_2 \diamond (w_x + z)(t)).
\end{aligned}$$

Hence, we obtain (4.12). $\square$

Lemma 6. Let (ϕ, ψ, w, z) be the solution of system (1.3)–(1.5) in Theorem 1. Then the Lyapunov functional $\mathcal{G}$ defined by

$$\mathcal{G}(t) = nE(t) + n_1 M_1(t) + n_2 M_2(t) + n_3 M_3(t) + n_4 M_4(t) + n_5 M_5(t)$$

is equivalent to the energy functional (4.2) ($\mathcal{G} \equiv E$). That is, there exist some positive numbers a_1 and a_2 such that

$$a_1 E(t) \leq \mathcal{G}(t) \leq a_2 E(t), \quad \forall t \geq 0, \quad (4.13)$$

where $n, n_1, n_2, n_3, n_4, n_5$ are positive numbers to be specified later.

Proof. Applying Lemma 1, Cauchy-Schwarz, Young's and Poincaré's inequalities, routine estimations yield the result. $\square$

Lemma 7. Let (ϕ, ψ, w, z) be the solution of system (1.3)–(1.5) in Theorem 1. Then, the Lyapunov functional

$$\mathcal{G}(t) = nE(t) + n_1M_1(t) + n_2M_2(t) + n_3M_3(t) + n_4M_4(t) + n_5M_5(t) \quad (4.14)$$

satisfies for some positive numbers a_3 and a_4 the estimate

$$\mathcal{G}'(t) \leq -a_3E(t) + a_4 \left[(g_1 \diamond (\phi_x + \psi))(t) + (g_2 \diamond (w_x + z))(t) \right], \quad \forall t \geq t_0. \quad (4.15)$$

Proof. For $t_0 > 0$, let

$$g_0 = \min \left\{ g_1^0, g_2^0 \right\} = \min \left\{ \int_0^{t_0} g_1(s) ds, \int_0^{t_0} g_2(s) ds \right\} > 0.$$

Then, differentiating (4.14) bearing in mind (4.3), (4.5), (4.6), (4.9), (4.10), (4.12), and setting $n_5 = 1$, $\epsilon_1 = \frac{\beta_1 n_3}{4n_1}$ and $\epsilon_2 = \frac{\beta_2 n_4}{4n_2}$, we get

$$\begin{aligned} \mathcal{G}'(t) \leq & - \left[\frac{\rho_1 g_1^0}{2} n_1 - cn_3 \right] \|\phi_t\|^2 - \left[\gamma_1 n - cn_1 - cn_3 - \rho_2 \right] \|\psi_t\|^2 \\ & - \left[\frac{\rho_3 g_2^0}{2} n_2 - cn_4 \right] \|w_t\|^2 - \left[\gamma_2 n - cn_2 - cn_4 - \rho_4 \right] \|z_t\|^2 \\ & - \left[\frac{\beta_1}{4} n_3 - c \right] \|\phi_x + \psi\|^2 - \frac{b_1}{2} \|\psi_x\|^2 - \left[\frac{\beta_2}{4} n_4 - c \right] \|w_x + z\|^2 \\ & - \frac{b_2}{2} \|z_x\|^2 - \kappa_0 \|\psi - z\|^2 + \left[\frac{n}{2} - cn_1 \right] (g_1' \diamond (\phi_x + \psi))(t) \\ & + \left[c \left(1 + \frac{4n_1}{\beta_1 n_3} \right) n_1 + cn_3 + c(\kappa_1 - \beta_1) \right] (g_1 \diamond (\phi_x + \psi))(t) \\ & + \left[\frac{n}{2} - cn_2 \right] (g_2' \diamond (w_x + z))(t) \\ & + \left[c \left(1 + \frac{4n_2}{\beta_2 n_4} \right) n_2 + cn_4 + c(\kappa_2 - \beta_2) \right] (g_2 \diamond (w_x + z))(t). \end{aligned} \quad (4.16)$$

Now, we fixed the rest of the remaining parameters as follows:

First, we select n_3 and n_4 large such that

$$\frac{\beta_1}{4} n_3 - c > 0, \quad \frac{\beta_2}{4} n_4 - c > 0,$$

followed by choosing n_1 and n_2 even larger so that

$$\frac{\rho_1 g_1^0}{2} n_1 - cn_3 > 0, \quad \frac{\rho_3 g_2^0}{2} n_2 - cn_4 > 0.$$

Finally, we select n very large as possible such that (4.13) is always valid and

$$\begin{aligned} \gamma_1 n - cn_1 - cn_3 - \rho_2 &> 0, \quad \gamma_2 n - cn_2 - cn_4 - \rho_4 > 0, \\ \frac{n}{2} - cn_1 &> 0, \quad \frac{n}{2} - cn_2 > 0. \end{aligned}$$

It follows from (4.16) that for some positive numbers a_3 and a_4 , we arrive at

$$\mathcal{G}'(t) \leq -a_3 E(t) + a_4 [(g_1 \diamond (\phi_x + \psi))(t) + (g_2 \diamond (w_x + z))(t)], \quad \forall t \geq t_0.$$

□

Now, we are ready to give the main result of the paper. Using the energy dissipation in Lemma 2, and the estimates in Lemma 6 and Lemma 7.

5 General decay of energy

The main general decay of solution result is given below.

Theorem 2. Let (ϕ, ψ, w, z) be the solution of system (1.3)–(1.5) in Theorem 1. Suppose conditions (C_1) and (C_2) hold. Then, for any $t_0 > 0$ fixed, there exist positive numbers α_1 and α_2 such that the energy functional E defined in (4.2) satisfies

$$E(t) \leq \alpha_1 \exp \left\{ -\alpha_2 \int_{t_0}^t \xi(\tau) d\tau \right\}, \quad \forall t \geq t_0, \quad (5.1)$$

where $\xi(t) = \min \{ \xi_i(t), i = 1, 2 \}$.

Proof. We begin by multiplying (4.15) with $\xi(t)$, then on account of a assumption (C_2) and (4.3), we have for any $t_0 > 0$

$$\begin{aligned} \xi(t) \mathcal{G}'(t) &\leq -a_3 \xi(t) E(t) + a_4 \xi(t) [(g_1 \diamond (\phi_x + \psi))(t) + (g_2 \diamond (w_x + z))(t)] \\ &\quad - a_3 \xi(t) E(t) + a_4 \xi(t) \int_{t_0}^t g_1(t-s) \|(\phi_x + \psi)(t) - (\phi_x + \psi)(s)\|^2 ds \\ &\quad + a_4 \xi(t) \int_{t_0}^t g_2(t-s) \|(w_x + z)(t) - (w_x + z)(s)\|^2 ds \\ &\leq -a_3 \xi(t) E(t) + a_4 \xi_1(t) \int_{t_0}^t g_1(t-s) \|(\phi_x + \psi)(t) - (\phi_x + \psi)(s)\|^2 ds \\ &\quad + a_4 \xi_2(t) \int_{t_0}^t g_2(t-s) \|(w_x + z)(t) - (w_x + z)(s)\|^2 ds \\ &\leq -a_3 \xi(t) E(t) + a_4 \int_{t_0}^t \xi_1(s) g_1(t-s) \|(\phi_x + \psi)(t) - (\phi_x + \psi)(s)\|^2 ds \\ &\quad + a_4 \int_{t_0}^t \xi_2(s) g_2(t-s) \|(w_x + z)(t) - (w_x + z)(s)\|^2 ds \\ &\leq -a_3 \xi(t) E(t) - 4a_4 E'(t), \quad \forall t \geq t_0. \end{aligned}$$

It follows that

$$(\xi(t) \mathcal{G}(t) + 4a_4 E(t))' \leq -a_3 \xi(t) E(t), \quad \forall t \geq t_0.$$

since $\xi'(t) = \min \{\xi'_i(t), i = 1, 2\} \leq 0, \forall t \geq 0$, Using the relation (4.13), we obtain

$$\mathcal{F}(t) = \xi(t)\mathcal{G}(t) + 4a_4E(t) \sim E(t). \quad (5.2)$$

Therefore, for some positive number $\hat{\alpha}_2 > 0$, we arrive at

$$\mathcal{F}'(t) \leq -\hat{\alpha}_2\xi(t)\mathcal{F}(t), \quad \forall t \geq t_0. \quad (5.3)$$

Integration of (5.3) over (t_0, t) gives

$$\mathcal{F}(t) \leq \mathcal{F}(t_0) \exp \left(-\hat{\alpha}_2 \int_{t_0}^t \xi(s) ds \right), \quad \forall t \geq t_0. \quad (5.4)$$

On account of (5.2), we obtain (5.1) from (5.4). This completes the proof. $\square$

Corollary 1. Under the conditions of Theorem 2. There exist positive numbers $\bar{\alpha}_1$ and $\bar{\alpha}_2$ such that E the energy functional defined in (4.2) satisfies

$$E(t) \leq \bar{\alpha}_1 \exp \left\{ -\bar{\alpha}_2 \int_0^t \xi(\tau) d\tau \right\}, \quad \forall t \geq 0, \quad (5.5)$$

where $\xi(t) = \min \{\xi_i(t), i = 1, 2\}$.

Proof. On account of the continuity and boundedness of $\xi(t) = \min \{\xi_i(t), i = 1, 2\}$ and the energy functional E , we have that for $t \in [0, t_0]$

$$E(t) \leq E(0) \exp \left(\alpha_2 \int_0^{t_0} \xi(\tau) d\tau \right) \exp \left(-\alpha_2 \int_0^t \xi(\tau) d\tau \right).$$

Thus, set $\bar{\alpha}_1 = \alpha_1 E(0) \exp \left(\alpha_2 \int_0^{t_0} \xi(\tau) d\tau \right)$, then estimate (5.5) is valid for all $t \geq 0$. $\square$

6 Concluding remarks

In the present paper, we have studied a double-wall carbon nanotube model as new coupled Timoshenko beam model with viscoelastic dissipation effect acting on the transverse displacements and linear frictional damping on the longitudinal displacements. We showed that the energy functional of the system has a general decay estimate, from which the exponential and polynomial decay are particular cases. Of course, due to the nature of the damping on the system, no additional conditions were made on the coefficient parameters. We note the follow key points: If the damping coefficients $\gamma_1 = \gamma_2 = 0$, then the result in (5.1) holds provided the usual equal-wave speed propagation assumption holds. In this case, following the same steps in [3], with few modification to our case, Theorem 2 holds provided $\frac{\kappa_1}{\rho_1} = \frac{b_1}{\rho_2}$ and $\frac{\kappa_2}{\rho_3} = \frac{b_2}{\rho_4}$.

The assumption in (C_2) can be optimized to a more general smooth and convex function $G_j, j = 1, 2$ such that

$$g'_j(t) \leq -\xi_j(t)G_j(g_j(t)), j = 1, 2.$$

Also, if infinite memory is considered instead of the finite memory in this work, the exponential stability result is still obtainable, see [1, 2].

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