

Boundary feedback stabilization of conformable linear systems

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Abstract. In this paper, we study the stabilization of infinite-dimensional boundary control systems governed by the conformable derivative of order $\alpha \in (0, 1]$. To address this problem, we reformulate the boundary stabilization problem as that of an associated linear internal control system governed by the conformable derivative, thereby enabling the use of techniques originally developed for integer-order dynamics. A stabilizing feedback law is then designed by extending classical results on linear internal control systems to the conformable setting. Exponential stabilization is established, and strong stabilization is also obtained for a broad class of Riesz-spectral boundary control systems. The effectiveness of the proposed approach is demonstrated through its application to the heat equation governed by the conformable derivative, supported by numerical simulations that confirm the theoretical findings.

Keywords: conformable derivative; stabilization; boundary control; Dirichlet operator; fractional semi-groups.

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1 Introduction

The field of control theory has long focused on the design and manipulation of dynamical systems using traditional integer-order derivatives. In recent years, however, growing interest has emerged in extending control theory to fractional derivatives. Fractional calculus, which deals with derivatives and integrals of non-integer orders, provides a robust framework for modeling and analyzing complex systems with memory and hereditary properties. This has given rise to a promising research area: control with fractional derivatives, applied across various domains [14].

Among fractional derivative definitions, the conformable derivative, introduced in [13], has attracted significant attention in the control community. It

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satisfies desirable properties and enables the description of fractional system dynamics [1]. Unlike Caputo or Riemann-Liouville definitions, it offers a more straightforward mathematical treatment, facilitating the analysis and design of control strategies for fractional systems.

In this context, the stabilization of infinite-dimensional fractional boundary control systems using the conformable derivative poses a major challenge. These systems involve controlling a dynamical process by acting on inputs at the domain boundary, rather than throughout the entire domain. Their stabilization requires a deep understanding of the interplay between fractional dynamics, control inputs, and boundary conditions.

Currently, the conformable derivative is under intensive study in both finite and infinite dimensions, particularly for the Cauchy problem, as well as controllability, observability, stability, and stabilization [4,10,21]. Other fractional derivative definitions are also considered [23,28]. For instance, [23] establishes stability conditions and exponential stabilization for a semilinear system with Caputo derivative.

Stabilization of fractional boundary control systems has also been explored, notably using the backstepping method for fractional reaction-diffusion equations [19,27]. In the integer-order case, recent works have addressed feedback stabilization of quasilinear hyperbolic systems [25] or null controllability and stabilization via transformation of unbounded to bounded control operators [16,17]. This paper extends the latter technique to the stabilization of systems with conformable derivatives.

We investigate the stabilization of abstract conformable boundary control systems. To address inherent difficulties, we transform the problem into an associated conformable linear internal control system, thereby leveraging existing results.

Exponential stability is achieved via spectral decomposition, extending methods developed for integer-order linear systems [7]. Strong stabilization is also established for a broad class of boundary control problems governed by Riesz-spectral operators.

The paper is structured as follows: Section 2 presents preliminaries on the conformable derivative and α -semigroups. Section 3 addresses the existence of mild solutions for conformable linear systems of order $\alpha \in (0, 1]$ with boundary control in Banach spaces, along with explicit solution expressions. Section 4 introduces an equivalent conformable linear internal control system and designs a stabilizing feedback based on results from integer-order internal control systems. Section 5 establishes exponential stability via spectral decomposition in Hilbert space and strong stabilization for a class of Riesz-spectral operators. The conclusion applies the abstract results to a specific class of heat equations and provides supporting numerical simulations.

2 Preliminaries

In this section, we introduce fundamental definitions and key results from the literature that are crucial for the development of this work.

DEFINITION 1. [1, 13] Let $\alpha \in (0, 1]$ and X represent a Banach space. For a function $f : [0, \infty[\rightarrow X$, the conformal derivative of order α is defined as follows:

$$D_t^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \text{ for every } t > 0.$$

Moreover, we call f α -differentiable if $D_t^\alpha f(t)$ exists for all $t > 0$. In addition, if $D_t^\alpha f(t)$ exists on $]0, \tau[$, for some $\tau > 0$, then $D_t^\alpha f(0) = \lim_{t \rightarrow 0} D_t^\alpha f(t)$.

The definition of the conformable integral, with order $\alpha \in (0, 1]$, applies to a function $f : [0, \infty[\rightarrow X$ as follows:

$$I_\alpha f(t) = \int_0^t s^{\alpha-1} f(s) ds.$$

The C_0 - α -semigroup corresponding to the conformable derivative is introduced in [2].

DEFINITION 2. Let X be a Banach space and $0 < \alpha \leq 1$. A family of bounded linear operators $(T_\alpha(t))_{t \geq 0}$ on X is called a *conformable C_0 - α -semigroup* if it satisfies:

- $T_\alpha(0) = I$,
- $T_\alpha\left((t+s)^{\frac{1}{\alpha}}\right) = T_\alpha\left(t^{\frac{1}{\alpha}}\right) T_\alpha\left(s^{\frac{1}{\alpha}}\right)$, for all $t, s \geq 0$,
- $\lim_{t \rightarrow 0^+} T_\alpha(t)x = x$, for all $x \in X$.

The definition for the infinitesimal generator of the above C_0 - α -semigroup is as follows:

DEFINITION 3. We call the α -infinitesimal generator of the C_0 - α -semigroup $(T_\alpha(t))_{t \geq 0}$ the operator $(A, D(A))$ given by

$$Ax = \lim_{t \rightarrow 0^+} D_t^\alpha T_\alpha(t)x, \text{ for all } x \in D(A),$$

where

$$D(A) = \left\{ x \in X / \lim_{t \rightarrow 0^+} D_t^\alpha T_\alpha(t)x \text{ exists in } X \right\}.$$

Some basic properties of the conformable C_0 - α -semigroup on a Banach space, which we will use, either in a direct or indirect way, are provided by the following theorems for $\alpha \in (0, 1]$.

Theorem 1. [20] Suppose $(T_\alpha(t))_{t \geq 0}$ is a C_0 - α -semigroup on X , and let A denote its α -infinitesimal generator. Then,

1. For $x \in X$, $\int_0^t s^{\alpha-1} T_\alpha(s)x ds \in D(A)$ and

$$A \left(\int_0^t s^{\alpha-1} T_\alpha(s)x ds \right) = T_\alpha(t)x - x.$$

2. For $x \in D(A)$, $T_\alpha(t)x \in D(A)$ and

$$D_t^\alpha T_\alpha(t)x = AT_\alpha(t)x = T_\alpha(t)Ax.$$

3. For $x \in D(A)$,

$$T_\alpha(t)x - T_\alpha(s)x = \int_s^t r^{\alpha-1} T_\alpha(r) A x dr = \int_s^t r^{\alpha-1} A T_\alpha(r) x dr.$$

Theorem 2. [20] Consider a C_0 - α -semigroup $(T_\alpha(t))_{t \geq 0}$ on the Banach space X . There are constants $M \geq 1$ and $\omega \geq 0$ for which the following holds:

$$\|T_\alpha(t)\| \leq M e^{\omega t^\alpha} \text{ for every } t \in]0, +\infty[.$$

We can observe that, for a given C_0 - α -semigroup $(T_\alpha(t))_{t \geq 0}$, for any x in X , the function $t \mapsto T_\alpha(t)x$ is continuously defined on $[0, +\infty[$ with values in X .

If $\omega < 0$, the C_0 - α -semigroup $(T_\alpha(t))_{t \geq 0}$ is said to be β -exponentially stable. Where $\beta = -\omega$.

Remark 1. If A is the infinitesimal generator of C_0 - α -semigroup $(T_\alpha(t))_{t \geq 0}$ and $B \in \mathcal{L}(L^p(\mathbb{R}_+, U), X)$. Then, the system

$$D_t^\alpha x(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

has a unique mild solution $x(t)$ given by

$$x(t) = T_\alpha(t)x_0 + \int_0^t s^{\alpha-1} T_\alpha \left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}} \right) Bu(s) x_0 ds. \quad (2.1)$$

The expression (2.1) is said to be conformable variation of parameters formula.

Remark 2. If A is an infinitesimal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$, then A also functions as the α -infinitesimal generator of the C_0 - α -semigroup $(T_\alpha(t))_{t \geq 0}$ given by $T_\alpha(t)x = T(\frac{t^\alpha}{\alpha})x$, for every $x \in X$ and $t \geq 0$. Furthermore, (2.1) can be reformulated as

$$x(t) = T \left(\frac{t^\alpha}{\alpha} \right) x_0 + \int_0^t s^{\alpha-1} T \left(\frac{t^\alpha - s^\alpha}{\alpha} \right) Bu(s) x_0 ds.$$

In the sequel of this paper, the following notations will be useful: the C_0 - α -semigroup $(T_\alpha(t))_{t \geq 0}$ will be denoted by $T_\alpha := (T_\alpha(t^\alpha))_{t \geq 0}$. We define $R(\lambda, A) := (\lambda - A)^{-1}$ for any $\lambda \in \rho(A)$. The space obtained by completing X with respect to the norm $\|x\|_{-1, \lambda} := \|R(\lambda, A)x\|$, for a fixed $\lambda \in \rho(A)$, is called the extrapolation space associated with X and T_α . This space is denoted as X_{-1} . Based on references [9, 24], it can be established that the unique extension of T_α onto X_{-1} forms a C_0 - α -semigroup, denoted as $T_{-1}^\alpha := (T_{-1}^\alpha(t^\alpha))_{t \geq 0}$, and its generator is indicated as A_{-1} .

3 Position problem

In this paper, the following conformable linear boundary control problem

$$(FBCP^\alpha) \begin{cases} D_t^\alpha x(t) = A_m x(t), & t > 0, \\ Qx(t) = Bu(t), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

is considered where D_t^α is the conformable derivative of order α , $0 < \alpha \leq 1$, $x(\cdot) \in X$ is the state, $u(\cdot) \in L_{loc}^p(\mathbb{R}_+, U)$ ($p \geq 1$) is the boundary control with Banach spaces X , $Z \subset X$, ∂X , and U are called the state space, the solution space, the boundary and the control space, respectively. On its domain $D(A_m)$, the linear operator $A_m : Z := D(A_m) \subset X \rightarrow X$ is both closed and densely defined, Q is a boundary operator in $\mathcal{L}([D(A_m)], \partial X)$, with $[D(A_m)]$ is the subspace $D(A_m)$ equipped with the graph norm, and $B \in \mathcal{L}(U, \partial X)$ is a control operator.

Let's define the concept of a classical solution before examining the well-posedness of the system.

DEFINITION 4. Let $u \in L_{loc}^p(\mathbb{R}_+, U)$ where $p \geq 1$. A classical solution of $(FBCP^\alpha)$ is a function $x(\cdot) = x(\cdot, u, x_0)$ continuously α -differentiable such that $x(t) \in Z$ for all $t \geq 0$ and satisfying $(FBCP^\alpha)$.

To explore the well-posedness of $(FBCP^\alpha)$, we require the following assumptions, which establish the well-posedness of the uncontrolled abstract Cauchy problem $(FBCP^\alpha)$, that is, $(FBCP^\alpha)$ with $B = 0$.

Assumptions 1.

(H1) The restricted operator $A \subset A_m$ with domain $D(A) := \ker Q$ generates a C_0 -semigroup $T := (T(t))_{t \geq 0}$ on X .

(H2) The operator $Q : D(A_m) \rightarrow \partial X$ is surjective.

Greiner [11] established the following lemma under the above assumptions.

Lemma 1. For every $\lambda \in \rho(A)$, we have

- i) $D(A_m) = D(A) \oplus \ker(\lambda - A_m)$;
- ii) $Q|_{\ker(\lambda - A_m)}$ is invertible and the operator $Q_\lambda := (Q|_{\ker(\lambda - A_m)})^{-1} : \partial X \rightarrow \ker(\lambda - A_m) \subset X$ is bounded;
- iii) $P_\lambda := Q_\lambda Q \in \mathcal{L}([D(A_m)])$ is a projection on $\ker(\lambda - A_m)$ along $D(A) = \ker Q$;
- iv) $R(\mu, A)Q_\lambda = \frac{1}{\lambda - \mu}(Q_\mu - Q_\lambda) = R(\lambda, A)Q_\mu$ for all $\lambda, \mu \in \rho(A)$, $\lambda \neq \mu$.

Consider $x(\cdot)$ as a classical solution of $(FBCP^\alpha)$ and let $\lambda \in \rho(A)$. Then, from (i)–(iii) in Lemma 1, we get $x(t) - P_\lambda x(t) \in D(A)$, for all $t > 0$. Thus,

$$x(t) - B_\lambda u(t) \in D(A), \text{ for all } t > 0,$$

where the control operator $B_\lambda := Q_\lambda B$ is in $\mathcal{L}(L^p(\mathbb{R}_+, U), \ker(\lambda - A_m))$. Moreover, we have

$$\begin{aligned} D_t^\alpha x(t) &= A_m x(t) = A_m \left(x(t) - B_\lambda u(t) \right) + A_m B_\lambda u(t) \\ &= A \left(x(t) - B_\lambda u(t) \right) + \lambda B_\lambda u(t) = A_{-1} x(t) + B_A u(t), \end{aligned}$$

where the control operator $B_A := (\lambda - A_{-1})B_\lambda$ is unbounded.

As a result, the solutions of $(FBCP^\alpha)$ can be described using the following extrapolated version of the conformable variation of parameters formula.

$$\begin{aligned} x(t) &= T_{-1}^\alpha(t^\alpha)x_0 + \int_0^t s^{\alpha-1} T_{-1}^\alpha(t^\alpha - s^\alpha)(\lambda - A_{-1})B_\lambda u(s) ds \\ &= T_\alpha(t^\alpha)x_0 + (\lambda - A_{-1}) \int_0^t s^{\alpha-1} T_{-1}^\alpha(t^\alpha - s^\alpha)B_\lambda u(s) ds \\ &= T_\alpha(t^\alpha)x_0 + (\lambda - A_{-1}) \int_0^t s^{\alpha-1} T(t^\alpha - s^\alpha)B_\lambda u(s) ds, \quad t \geq 0. \end{aligned}$$

In general, a solution of this kind doesn't exist within the space X . Consequently, the challenge of ensuring solution existence poses a handicap for both system analysis and control. To address this, we will use the next definition.

DEFINITION 5. Under Assumptions 1 and for $p \geq 1$, the operator $B_A \in \mathcal{L}(U, X_{-1})$ (or $B \in \mathcal{L}(U, \partial X)$) is p -admissible for A or T_α if there exists $\tau > 0$ such that, for all $u \in L_{\text{loc}}^p(\mathbb{R}_+, U)$,

$$(\lambda - A_{-1}) \int_0^\tau s^{\alpha-1} T_\alpha(\tau^\alpha - s^\alpha)B_\lambda u(s) ds \in X.$$

The existence of solutions in X for linear and bilinear systems with unbounded control operators has been studied in [12, 26]. As noted in [9, 24], $A_{-1}x \in X$ if and only if $x \in D(A)$. Thus, B is p -admissible for T_α if and only if

$$(AC) \quad \int_0^t s^{\alpha-1} T_\alpha(t^\alpha - s^\alpha)B_\lambda u(s) ds \in D(A),$$

for some $p \geq 1$, all $u \in L_{\text{loc}}^p(\mathbb{R}_+, U)$, $\lambda \in \rho(A)$, and $t > 0$. This admissibility condition (AC) is necessary for $(FBCP^\alpha)$ to admit solutions in X . By Lemma 1(iv), (AC) holds for one $\lambda \in \rho(A)$ if and only if it holds for all $\mu \in \rho(A)$.

When $B = I$, p -admissibility depends only on T_α , Q , and $u(\cdot)$. For $x_0 \in X$ and $u \in W_{\text{loc}}^{2,p}(\mathbb{R}^+, U)$ with $Qx_0 = Bu(0)$, $(FBCP^\alpha)$ has a unique classical solution, implying B is admissible. Conversely, if B is p -admissible, then for $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^+, U)$ with $Qx_0 = Bu(0)$, the solution is classical [8, 24].

In non-reflexive spaces, [8, Lemma A.2–A.3] shows (AC) holds if $\|Qx\| \geq \gamma\lambda\|x\|$ for $\lambda > \lambda_0$ and $x \in \ker(\lambda - A_m)$, with $\gamma > 0$, $\lambda_0 \in \mathbb{R}$. If $(T(t))_{t \geq 0}$ is analytic, (AC) holds [3, Subsection 6.2.4].

In this work, we assume B is p -admissible for T_α , i.e., (AC) holds. Thus, under (H1)–(H2), for some $p \geq 1$ and $\lambda \in \rho(A)$, the mild solution of $(FBCP^\alpha)$ is

$$x(t) = x(t, u, x_0) = T_\alpha(t^\alpha)x_0 + (\lambda - A) \int_0^t s^{\alpha-1} T_\alpha(t^\alpha - s^\alpha) B_\lambda u(s) ds, \quad t \geq 0. \quad (3.1)$$

4 Stabilization results

In this section, our idea is to study the stabilization of $(FBCP^\alpha)$ by associating it with the following standard conformable control system for some $\lambda \in \rho(A)$.

$$(A, B_\lambda)_\alpha \begin{cases} D_t^\alpha y(t) = Ay(t) + B_\lambda u(t), & t > 0, \\ y(0) = y_0 := R(\lambda, A)x_0, \end{cases}$$

where $\alpha \in (0, 1]$ and the operators A , B_λ and $R(\lambda, A)$ are defined in the previous section. Here $y_\lambda(\cdot) = y(\cdot, u, y_0, \lambda)$ is called a classical solution of $(A, B_\lambda)_\alpha$ if $y_\lambda(\cdot)$ is continuously α -differentiable on $[0, \infty[$, $y_\lambda(t) \in D(A)$ for all $t \geq 0$ and $y_\lambda(\cdot)$ satisfied $(A, B_\lambda)_\alpha$.

As B_λ is bounded and from the Assumption (H1), the system $(A, B_\lambda)_\alpha$ has a unique mild solution on X and is given by the conformable variation of parameters formula

$$y_\lambda(t) = T_\alpha(t)y_0 + \int_0^t s^{\alpha-1} T_\alpha(t^\alpha - s^\alpha) B_\lambda u(s) ds, \quad (4.1)$$

where $u \in L_{loc}^p(\mathbb{R}_+, U)$.

Moreover, since B is p -admissible control operator for T_α and $y_0 \in D(A)$, then the solution given by (4.1) is a classical solution of the system $(A, B_\lambda)_\alpha$.

Remark 3. Under assumptions (H1)–(H2), by part (iv) of Lemma 1, it can be shown that the mild solution $x(t)$ of $(FBCP^\alpha)$, as defined in (3.1), does not depend on the value of λ in $\rho(A)$; for more details, see [18]. Moreover, for any sufficiently large $\lambda \in \rho(A)$ and for all $t \geq 0$, we have

$$\lambda R(\lambda, A)x(t) = \lambda \left(R(\lambda, A)T_\alpha(t)x_0 + \int_0^t s^{\alpha-1} T_\alpha(t^\alpha - s^\alpha) B_\lambda u(s) ds \right) = \lambda y_\lambda(t),$$

where $y_\lambda(\cdot)$ is the mild solution of $(A, B_\lambda)_\alpha$. It's know that

$\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A)x = x$, for any $x \in X$. This implies that the solution (3.1) is given by

$$x(t) = \lim_{\lambda \rightarrow \infty} \lambda y_\lambda(t), \quad \text{for all } t \geq 0.$$

We now introduce key definitions for the stabilization of system $(FBCP^\alpha)$. The conformable exponential function is defined for $s \geq 0$, $\alpha \in (0, 1]$, and $\omega \in \mathbb{R}$ as

$$E_\alpha(\omega, s) = \exp(\omega s^\alpha / \alpha).$$

The system $(FBCP^\alpha)$ is exponentially stable in the fractional sense if there exist $M, \omega > 0$ such that $\|x(t)\| \leq M\|x_0\|E_\alpha(-\omega, t)$, $t \geq 0$.

The system $(FBCP^\alpha)$ (resp. $(A, B_\lambda)_\alpha$) is strongly stabilizable (for $p \geq 1$, $\lambda \in \rho(A)$) if there exists a control $u \in L^p_{\text{loc}}(\mathbb{R}_+, U)$ such that, for any $x_0 \in X$, the solution $x(t, u, x_0)$ (resp. $y(t, u, y_0, \lambda)$) satisfies

$$x(t, u, x_0) \rightarrow 0 \quad (\text{resp. } y(t, u, y_0, \lambda) \rightarrow 0) \quad \text{as } t \rightarrow \infty.$$

4.1 Characterizations for the stabilization of $(FBCP^\alpha)$

We begin by demonstrating that the stabilization of the system $(A, B_\lambda)_\alpha$ remains unaffected by the selection of $\lambda \in \rho(A)$.

Proposition 1. *Let λ and μ in $\rho(A)$. The system $(A, B_\lambda)_\alpha$ is strongly (resp. exponentially) stabilizable if and only if the system (A, B_μ) is as well.*

Proof. Let $\lambda, \mu \in \rho(A)$. For any $y_0 \in X$, $y(t, u, y_0, \mu)$ represents the solution of (A, B_μ) . By *iv*) of Lemma 1, we obtain

$$R(\mu, A)y(t, u, y_0, \lambda) = R(\lambda, A)y(t, u, \tilde{y}_0, \mu),$$

where $\tilde{y}_0 = (\lambda - A)R(\mu, A)y_0$. Thus, we get

$$y(t, u, y_0, \lambda) = (\mu - \lambda)R(\lambda, A)y(t, u, \tilde{y}_0, \mu) + y(t, u, \tilde{y}_0, \mu),$$

it follows that

$$\|y(t, u, y_0, \lambda)\| \leq (\|\mu - \lambda\| \|R(\lambda, A)\| + 1)\|y(t, u, \tilde{y}_0, \mu)\|,$$

therefore, which completes the proof. $\square$

In the sequel, for any $\lambda \in \rho(A)$, the mild solution of $(A, B_\lambda)_\alpha$ will be denoted $y(t, u, y_0)$.

Remark 4. [9] For each $\lambda \in \rho(A)$ and $x \in D(A)$, we define the 1-norm $\|\cdot\|_\lambda$ as follows

$$\|x\|_\lambda := \|(\lambda - A)x\|.$$

The space $X_\lambda := (D(A), \|\cdot\|_\lambda)$ is referred to as the first-order Sobolev space associated with $(T(t))_{t \geq 0}$.

Now, under Assumptions (H1)–(H2), we establish the first main result of this section:

Theorem 3. *The following properties hold:*

1. *Let $(T_\alpha(t))_{t \geq 0}$ be a holomorphic C_0 -semigroup on X . The system $(FBCP^\alpha)$ is strongly (resp. exponentially) stabilizable if and only if the system $(A, B_\lambda)_\alpha$ is strongly (resp. exponentially) stabilizable.*
2. *The system $(FBCP^\alpha)$ is exponentially stabilizable for the norm $\|\cdot\|_{-1, \lambda}$, if and only if the system $(A, B_\lambda)_\alpha$ is exponentially stabilizable.*

3. The system $(FBCP^\alpha)$ is exponentially stabilizable for the usual norm, if and only if the system $(A, B_\lambda)_\alpha$ is exponentially stabilizable for the norm $\|\cdot\|_\lambda$.

Proof. From the Remark 3, we have

$$y(t, u, y_0) = R(\lambda, A)x(t, u, x_0) \text{ for all } t \geq 0, \quad (4.2)$$

where $y(t, u, y_0)$ and $x(t, u, x_0)$ are the solutions of systems $(A, B_\lambda)_\alpha$ and $(FBCP^\alpha)$ respectively. Now from (4.2), we can check the properties of Theorem 3. Indeed,

1. If $(FBCP^\alpha)$ is strongly stabilizable by a control $u \in L^p_{loc}(\mathbb{R}_+, U)$, we have

$$\begin{aligned} \|y(t, u, y_0)\| &= \|R(\lambda, A)x(t, u, x_0)\| \\ &\leq \|R(\lambda, A)\| \|x(t, u, x_0)\| \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus, $(A, B_\lambda)_\alpha$ is strong stabilizable. Conversely, assume that $(A, B_\lambda)_\alpha$ is strongly stabilizable. Then,

$$\lim_{t \rightarrow \infty} R(\lambda, A)x(t, u, x_0) = \lim_{t \rightarrow \infty} y(t, u, y_0) = 0.$$

As (T_α) is a holomorphic C_0 -semigroup, there exist constants $c > 0$ and $\eta > 0$ such that:

$$\|AR(\lambda, A)\| \leq c/\lambda, \quad \text{for all } \lambda > \eta,$$

which implies $1 - c/\lambda > 0$ for sufficiently large λ .

From the system dynamics, we have:

$$x(t) = \lambda y(t) - Ay(t).$$

Taking the norm on both sides:

$$\begin{aligned} \|x(t)\| &\leq \lambda \|y(t)\| \\ &+ \|Ay(t)\| \leq \lambda \|y(t)\| + \|AR(\lambda, A)x(t)\| \leq \lambda \|y(t)\| + \frac{c}{\lambda} \|x(t)\|. \end{aligned}$$

Hence, we get

$$\|x(t)\| \leq \lambda (1 - c/\lambda)^{-1} \|y(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Therefore, the system $(FBCP^\alpha)$ is strong stabilizable.

2. The system $(FBCP^\alpha)$ is exponentially stabilizable for the norm $\|\cdot\|_{-1, \lambda}$, if and only if there exist a control $u \in L^p_{loc}(\mathbb{R}_+, U)$, $\omega > 0$ and $M > 0$ such that

$$\|R(\lambda, A)x(t, u, x_0)\| \leq M \|R(\lambda, A)x_0\| E_\alpha(-\omega, t), \text{ for all } x_0 \in X,$$

if and only if (by the Equation (4.2)) there exist a control $u \in L^p_{loc}(\mathbb{R}_+, U)$, $\omega > 0$ and $M > 0$ such that

$$\|y(t, u, y_0)\| \leq M\|y_0\|E_\alpha(-\omega, t), \text{ for all } y_0 \in D(A),$$

is equivalent to the system $(A, B_\lambda)_\alpha$ is exponentially stabilizable.

3. The result in the assertion 3) follows from the assertion 2).

□

Remark 5. • If the system $(FBCP^\alpha)$ is exponentially stabilizable for the norm $\|\cdot\|_{-1, \lambda}$, it implies that it is also strongly stabilizable.

- The results of Theorem 3 may also hold true when considering Caputo's fractional derivative, where conformable exponential stability is replaced by Mittag-Leffler stability.

In the general context of feedback stabilization, the control law $u(\cdot)$ that stabilizes the conformable system $(A, B_\lambda)_\alpha$ is expressed as a feedback of the form

$$u(t) = f(t, y(t)), \quad (4.3)$$

where $f: \mathbb{R}_+ \times X \rightarrow U$ is a suitably chosen function, and $y(t)$ denotes the solution of $(A, B_\lambda)_\alpha$.

According to the proof of Theorem 3, we can deduce our second main result in this section.

Corollary 1. Assume that $(A, B_\lambda)_\alpha$ is strongly (resp. exponentially) stabilizable and a stabilizing control law is provided by (4.3). Then, the problem $(FBCP^\alpha)$ is strongly (resp. exponentially for the $\|\cdot\|_{-1, \lambda}$ norm) stabilizable, and a stabilizing feedback control $u(t)$ is given by $u(t) = f(t, R(\lambda, A)x(t))$, where $x(t)$ is the solution of $(FBCP^\alpha)$.

In the context of internal control systems, the following theorem establishes the stabilizability of conformable systems via time-scaled feedback.

Theorem 4. If the integer-order system $(A, B_\lambda)_1$ is stabilizable by a feedback $f(t, y(t))$, then for every $\alpha \in (0, 1]$, the conformable system $(A, B_\lambda)_\alpha$ is stabilizable by the control law $f(t^\alpha/\alpha, y(t^\alpha/\alpha))$.

Proof. For $u(t) = v(t) := f(t, z(t))$, the mild solution $z(\cdot) := z(\cdot, y_0, v(\cdot))$ of the system $(A, B_\lambda)_1$ is given by

$$z(t) = T(t)y_0 + \int_0^t T(t-s)B_\lambda f(s, z(s))ds, \text{ for all } t \geq 0 \text{ and } y_0 \in X. \quad (4.4)$$

And for $\alpha \in (0, 1]$ and $u(t) = f(\frac{t^\alpha}{\alpha}, y(\frac{t^\alpha}{\alpha}))$, according to Remark 2, $y(\cdot)$ the solution of $(A, B_\lambda)_\alpha$ is given by

$$y(t) = T(\frac{t^\alpha}{\alpha})y_0 + \int_0^t s^{\alpha-1}T(\frac{t^\alpha - s^\alpha}{\alpha})B_\lambda f(\frac{s^\alpha}{\alpha}, y(\frac{s^\alpha}{\alpha}))ds, \text{ for all } t \geq 0 \text{ and } y_0 \in X.$$

However, we examine the change of variable $r = \frac{s^\alpha}{\alpha}$. Consequently, we derive

$$y(t) = T\left(\frac{t^\alpha}{\alpha}\right)y_0 + \int_0^{\frac{t^\alpha}{\alpha}} T\left(\frac{t^\alpha}{\alpha} - r\right) B_\lambda f(r, y(r)) dr,$$

and, using the variation of parameters formula (4.4), we have

$$y(t) = z\left(t^\alpha/\alpha, y_0, v(\cdot)\right), \text{ for all } t \geq 0 \text{ and } y_0 \in X.$$

Thus, the proof is complete. $\square$

In the preceding sections, we studied the stabilization of abstract conformable boundary control problems in a Banach space setting. To derive concrete stabilization results, we now restrict our analysis to Hilbert spaces. This choice leverages key structural properties, such as spectral decomposition and the analyticity of the contraction semigroup T_α , that are essential to our approach. Moreover, we focus on systems where the input operator B_λ is finite-rank. This assumption, combined with the Hilbert space framework, enables precise spectral analysis and explicit feedback design.

In the following sections, we assume X and ∂X are Hilbert spaces.

4.2 Spectral decomposition method

Using Theorem 3 and Corollary 1, we derive sufficient conditions for the stabilization of the conformable boundary control problem $(FBCP^\alpha)$. When B_λ is finite-rank and A satisfies the spectral decomposition assumption, we follow the approach of [7] to characterize exponential stabilization of $(FBCP^\alpha)$.

Remark 6. We have $\text{range}(B_\lambda) \subset \ker(\lambda - A_m)$, with equality if B is surjective. If $\dim(\partial X) < \infty$, then $\text{rank}(B_\lambda) < \infty$.

For $\delta > 0$, we decompose the spectrum of A as

$$\begin{aligned}\sigma^+(A) &= \sigma(A) \cap \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > -\delta\}, \\ \sigma^-(A) &= \sigma(A) \cap \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leq -\delta\},\end{aligned}$$

with $\sigma^+(A)$ finite. The spectral projection is

$$P_\delta x = \frac{1}{2\pi i} \oint_{\Gamma_\delta} (\lambda I - A)^{-1} x d\lambda,$$

where Γ_δ is a Jordan contour enclosing $\sigma^+(A)$. This yields the decomposition

$$X = X^+ \oplus X^-, \quad X^+ := P_\delta X, \quad X^- := (I - P_\delta)X.$$

The operators decompose as

$$A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}, \quad B_\lambda = \begin{pmatrix} B_\lambda^+ \\ B_\lambda^- \end{pmatrix}, \quad T(t) = \begin{pmatrix} T^+(t) & 0 \\ 0 & T^-(t) \end{pmatrix},$$

with $B_\lambda^+ = P_\delta B_\lambda \in \mathcal{L}(U, X^+)$ and $B_\lambda^- = (I - P_\delta)B_\lambda \in \mathcal{L}(U, X^-)$. See [7, Chapter V] for details.

By Remark 2, the fractional semigroup $T_\alpha^-(t^\alpha)$ is δ -exponentially stable. Adapting [7, Chapter V] via Theorem 4, Theorem 3, and Corollary 1, we obtain the following result.

Theorem 5. *Let $\lambda \in \rho(A)$ and $\alpha \in (0, 1]$. If $\dim(\partial X) < \infty$, X^+ is finite-dimensional, and there exists an operator $F_0 \in \mathcal{L}(X^+, U)$ such that the spectrum of $A^+ + B_\lambda^+ F_0$ lies in $\mathbb{C}_-^\circ$. Then, $(FBCP^\alpha)$ is δ -exponentially stabilizable for the norm $\|\cdot\|_{-1, \lambda}$.*

Moreover, $x(t)$ the state of $(FBCP^\alpha)$ is δ -exponentially stabilizable for the norm $\|\cdot\|_{-1, \lambda}$ by the feedback $u(t) = F_0 P_\delta R(\lambda, A)x(t^\alpha/\alpha)$.

Proof. Form [7], if for some $F_0 \in \mathcal{L}(X^+, U)$ we have the spectrum of $A^+ + B_\lambda^+ F_0$ lies in $\mathbb{C}_-^\circ$. Then, $A^+ + B_\lambda^+ F_0$ generates an exponentially stable C_0 - α -semigroup. Therefore, the perturbed operator

$$A + B_\lambda F = \begin{pmatrix} A^+ + B_\lambda^+ F_0 & 0 \\ B_\lambda^- F_0 & A^- \end{pmatrix}$$

generates a C_0 -semigroup δ -exponentially stable. Where $F = (F_0, 0) \in \mathcal{L}(X, U)$. Hence, from Theorem 4, $y(t)$ the state of $(A, B_\lambda)_\alpha$ is δ -exponentially stabilizable and the δ -stabilizing feedback operator is given by $u(t) = F_0 P_\delta y(t^\alpha/\alpha)$. It follows that, by Theorem 3 and Corollary 1, the proof is complete. $\square$

4.3 Stabilization of Riesz-spectral boundary control problems

This subsection provides a framework for stabilizing linear PDEs with boundary control, using operators with orthogonal eigenvectors forming a Riesz basis (see [7] for details).

DEFINITION 6. The system $(FBCP^\alpha)$ is a *Riesz-spectral boundary control problem* if (H1)–(H2) hold, and A is Riesz-spectral.

Strong stabilization of integer-order systems $(A, B_\lambda)_1$ has been studied in [6, 22]. Bounit [6] showed that if A generates a contraction C_0 -semigroup T with compact resolvent, the saturated feedback

$$u(t) = -\rho \frac{B_\lambda^* y(t)}{1 + \|B_\lambda^* y(t)\|}, \quad \rho > 0,$$

globally asymptotically stabilizes $(A, B_\lambda)_1$ provided

$$M := \{y \in X : B_\lambda^* T(t)y = 0 \ \forall t \geq 0\} = \{0\}.$$

Combining this with Theorems 3, 4, and Corollary 1, we obtain:

Theorem 6. *If T is a contraction C_0 -semigroup with compact resolvent, then the feedback*

$$u(t) = -\rho \frac{B_\lambda^* R(\lambda, A)x(t^\alpha/\alpha)}{1 + \|B_\lambda^* R(\lambda, A)x(t^\alpha/\alpha)\|}, \quad \rho > 0, \quad (4.5)$$

strongly stabilizes $(FBCP^\alpha)$, provided

$$M_\alpha := \{y \in X : B_\lambda^* T_\alpha(t^\alpha)y = 0 \ \forall t \geq 0\} = \{0\}. \quad (4.6)$$

Remark 7 [9]. If $X = L^q(\Omega)$ ($1 \leq q < \infty$), $\Omega \subset \mathbb{R}^n$ bounded with smooth boundary, and $D(A) \hookrightarrow W^{1,q}(\Omega)$ with $\rho(A) \neq \emptyset$, then A has compact resolvent.

Now let A be self-adjoint:

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n, \quad (4.7)$$

with $\lambda_1 > \lambda_2 > \dots$, $\{\phi_n\}_{n \geq 1}$ orthonormal basis of X . If $\dim(\partial X) < \infty$, then B_λ is finite-rank:

$$B_\lambda u = \sum_{i=1}^m b_i u_i, \quad b_i \in \ker(\lambda - A_m), \quad u_i \in L^p([0, \infty), U). \quad (4.8)$$

As a result of the Theorem 6, we deduce the next result.

Corollary 2. Under the assumptions of Theorem 6, and let the operator A be defined by (4.7). The control law (4.5) stabilizes strongly system $(FBCP^\alpha)$, provided the following assumption is satisfied:

$$B_\lambda^* \phi_n \neq 0, \text{ for all } n \geq 1. \quad (4.9)$$

Moreover, if B_λ is given by (4.8), then the condition (4.9) becomes

$$\text{rank} \begin{pmatrix} \langle b_1, \phi_n \rangle \\ \vdots \\ \langle b_m, \phi_n \rangle \end{pmatrix} = 1, \text{ for all } n \geq 1. \quad (4.10)$$

Proof. According to Theorem 6 and Corollary 2, in order to guarantee the strong stabilization of $(FBCP^\alpha)$, it is sufficient to show that condition (4.9) to imply the condition (4.6). Moreover, we will establish the equivalence between these two conditions. So, for every $y \in X$, we get $y = \sum_{n=1}^{\infty} \langle y, \phi_n \rangle \phi_n$ and

$$T_\alpha(t^\alpha)y = \sum_{n=1}^{\infty} e^{\frac{\lambda_n t^\alpha}{\alpha}} \langle y, \phi_n \rangle \phi_n,$$

as the functions $\{e^{\lambda_n \cdot}\}_{n \geq 1}$ form a free family in $L^2(0, \infty)$, we obtain

$$B_\lambda^* T_\alpha(t^\alpha)y = 0, \forall t \geq 0 \iff \langle y, \phi_n \rangle B_\lambda^* \phi_n = 0, \forall n \geq 1.$$

But $y = 0$ if and only if $\langle y, \phi_n \rangle = 0$, for all $n \geq 1$. It's follows that $M = \{0\}$ if and only if $B_\lambda^* \phi_n \neq 0$, for all $n \geq 1$. $\square$

5 Applications

5.1 Strong stabilization of heat equation with Neumann boundary conditions

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded, connected open set with smooth boundary $\partial\Omega$, and $\Gamma \subset \partial\Omega$ a nonempty open subset. We consider the fractional heat equation with boundary control on Γ :

$$\begin{cases} D_t^\alpha x(z, t) = \Delta x(z, t), & (z, t) \in \Omega \times (0, \infty), \\ \frac{\partial x}{\partial \eta} = u(t), & (\xi, t) \in \Gamma \times (0, \infty), \\ \frac{\partial x}{\partial \eta} = 0, & (\xi, t) \in (\partial\Omega \setminus \Gamma) \times (0, \infty), \\ x(z, 0) = x_0(z), & z \in \Omega. \end{cases} \quad (5.1)$$

This is formulated as $(FBCP^\alpha)$ in $X = L^2(\Omega)$, $U = H^{1/2}(\Gamma)$, $\partial X = H^{1/2}(\Gamma)$. Define $A_m f = \Delta f$ on

$$D(A_m) = \{f \in H^2(\Omega) : f = 0 \text{ on } \partial\Omega \setminus \Gamma\},$$

the boundary operator $Qf = \frac{\partial f}{\partial \eta}|_\Gamma$, and $B = I_U$. The operator $Q : D(A_m) \rightarrow \partial X$ is bounded and surjective, so (H2) holds. The restriction $A = A_m|_{\ker(Q)}$ is Riesz-spectral, generates an analytic contraction C_0 -semigroup $(T(t))_{t \geq 0}$, and satisfies (H1), (AC). Thus, B is admissible and (5.1) is well-posed.

By Lemma 1, for $\lambda \in \rho(A)$, there exists a unique $Q_\lambda \in \mathcal{L}(L^2(\Gamma), \ker(\lambda - A_m))$ such that $QQ_\lambda = I$. For $g \in L^2(\Gamma)$,

$$Q_\lambda g = x \iff \begin{cases} \Delta x = \lambda x, & \text{on } \Omega, \\ \frac{\partial x}{\partial \eta} = 0, & \text{on } \partial\Omega \setminus \Gamma, \\ \frac{\partial x}{\partial \eta} = g, & \text{on } \Gamma. \end{cases}$$

Using Green's formula for $h \in D(A)$,

$$\begin{aligned} \langle Q_\lambda^*(\lambda - A)h, g \rangle_{L^2(\Gamma)} &= \langle (\lambda - A)h, Q_\lambda g \rangle_{L^2(\Omega)} = \lambda \int_\Omega h x \, dz - \int_\Omega \Delta h x \, dz \\ &= \lambda \int_\Omega h x \, dz - \int_{\partial\Omega} \frac{\partial h}{\partial \eta} x \, d\xi + \int_{\partial\Omega} h \frac{\partial x}{\partial \eta} \, d\xi - \int_\Omega \Delta x h \, dz \\ &= \int_\Gamma h g \, d\xi = \langle h, g \rangle_{L^2(\Gamma)}. \end{aligned}$$

Thus, the adjoint is $Q_\lambda^* f = R(\lambda, A)f \cdot 1_\Gamma$ for all $f \in L^2(\Omega)$.

Corollary 2 applies: (5.1) is Riesz-spectral, A has compact resolvent (Remark 7), and for $n \geq n_0$, there exists $\xi \in \Gamma$ such that $R(\lambda, A)\phi_n(\xi) \neq 0$, satisfying (4.9) for eigenvectors $\{\phi_n\}_{n \geq n_0}$ of $A = \Delta$.

The saturated control law

$$u(t) = -\rho \frac{R(\lambda, A)x(\cdot, t^\alpha/\alpha)}{1 + \|R(\lambda, A)x(\cdot, t^\alpha/\alpha)\|} 1_\Gamma, \quad \rho > 0,$$

strongly stabilizes (5.1). Numerical illustrations for $n = 1$ follow. For these illustrations, the spatial discretization is performed using a finite difference scheme on a uniform grid, while the time integration is carried out using an explicit Euler method adapted for conformable fractional derivatives. All computations are implemented in MATLAB (version R2025b). These choices ensure accurate and reliable simulations of both the uncontrolled and controlled systems, as well as the computation of the corresponding L^2 -norms of the solutions.

5.2 1D strong and exponential stabilization

Consider the 1D heat equation with Neumann boundary control at $z = 1$:

$$\begin{cases} D_t^\alpha x(z, t) = x_{zz}(z, t), & t > 0, z \in (0, 1), \\ x_z(0, t) = 0, x_z(1, t) = u(t), \\ x(z, 0) = x_0(z) \in L^2(0, 1). \end{cases} \quad (5.2)$$

This is formulated as $(FBCP^\alpha)$ with $A_m f = f''$, $D(A_m) = \{f \in H^2(0, 1) : f'(0) = 0\}$, $Qf = f'(1)$, $B = I$. The operator $A = A_m|_{\ker(Q)}$ with $D(A) = \{f \in H^2(0, 1) : f'(0) = f'(1) = 0\}$ generates an analytic contraction C_0 -semigroup $(T(t))_{t \geq 0}$. Assumptions (H1) and (AC) hold. Moreover, (H2) holds: Q is surjective and A_m -bounded, since

$$\|Qf\| = |f'(1)| = \left| \int_0^1 f''(s) ds \right| \leq \|f''\|_{L^2} \leq \|f\|_{D(A_m)}.$$

The spectrum consists of simple eigenvalues $\lambda_n = -n^2\pi^2$ ($n \geq 0$), with eigenvectors $\phi_0(z) = 1$, $\phi_n(z) = \sqrt{2} \cos(n\pi z)$ ($n \geq 1$), forming an orthonormal basis of $L^2(0, 1)$. The resolvent is

$$R(\lambda, A)f = \sum_{n=0}^{\infty} \frac{\langle f, \phi_n \rangle}{\lambda - \lambda_n} \phi_n, \quad \lambda \in \rho(A).$$

For $\lambda \in \rho(A)$, $Q_\lambda b = h_\lambda(z)b$ with

$$h_\lambda(z) = \begin{cases} \frac{-\cos(z\sqrt{-\lambda})}{\sqrt{-\lambda} \sin(\sqrt{-\lambda})}, & \lambda < 0, \\ \frac{e^{z\sqrt{\lambda}} + e^{-z\sqrt{\lambda}}}{\sqrt{\lambda}(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})}, & \lambda > 0. \end{cases}$$

Thus, $B_\lambda u = u h_\lambda(z)$ satisfies (4.8) and (4.10). By Corollary 2, the control

$$u(t) = -\rho \frac{\int_0^1 h_\lambda(z) R(\lambda, A)x(z, t^\alpha/\alpha) dz}{1 + \left| \int_0^1 h_\lambda(z) R(\lambda, A)x(z, t^\alpha/\alpha) dz \right|} \quad (5.3)$$

strongly stabilizes (5.2) for all $\rho > 0$.

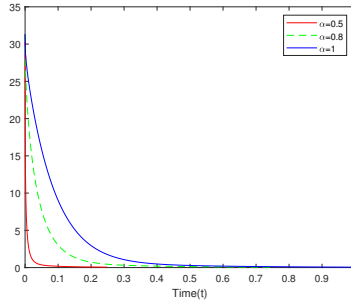


Figure 1. Evolution of $\|x(t)\|$ under strong stabilization (5.3) for $\alpha = 0.5, 0.8, 1$.

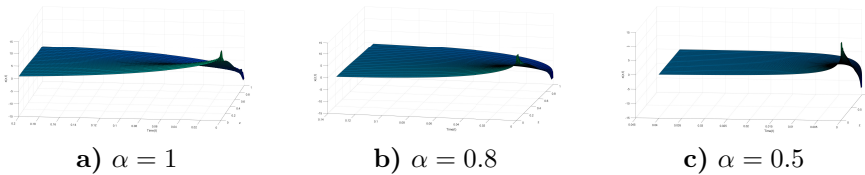


Figure 2. State evolution under strong stabilization (5.3).

With $\lambda = -\pi^2/4$, $\rho = 1$, $x_0(z) = \frac{3\pi^2}{4} \cos(\pi z)$, Figures 1 and 2 show $\|x(t)\| \rightarrow 0$ and $x(z, t) \rightarrow 0$ for $\alpha = 0.5, 0.8, 1$, suggesting enhanced performance of fractional-order systems.

Figure 3 illustrates the comparison between the uncontrolled system and the system under strong stabilization for the initial state $x_0(z) = 2$ and $\alpha = 0.3$. In the absence of control in system (5.2), the state converges to 1, indicating that the open-loop system is not strongly stable. In contrast, when the strong stabilizing feedback (5.3) is applied to (5.2), the state converges to zero, demonstrating the effectiveness of the proposed control strategy. This comparison clearly highlights the stabilizing effect of the boundary feedback law and confirms the theoretical strong stabilization results established in Section 4. For exponential stabilization, consider the internal system $(A, B_\lambda)_\alpha$:

$$\begin{cases} D_t^\alpha y = y_{zz} + h_\lambda(z)u(t), \\ y_z(0, t) = y_z(1, t) = 0, \\ y(z, 0) = R(\lambda, A)x_0(z). \end{cases}$$

The unstable spectrum is $\sigma^+(A) = \{0\}$ (multiplicity 1). For circle $I_\delta = \mathcal{C}(0, 1)$, $P_\delta f = \langle f, 1 \rangle$, so $B_\lambda^+ u(t) = \langle h_\lambda, 1 \rangle u(t)$.

Decompose $A = \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix}$: $X^+ = \text{span}\{\phi_0\}$ (unstable: $T_\alpha^+(t^\alpha) = 1$), $X^- = \overline{\text{span}}\{\phi_n : n \geq 1\}$ (stable):

$$T_\alpha^-(t^\alpha)y^- = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t^\alpha/\alpha} \langle y^-, \phi_n \rangle \phi_n.$$

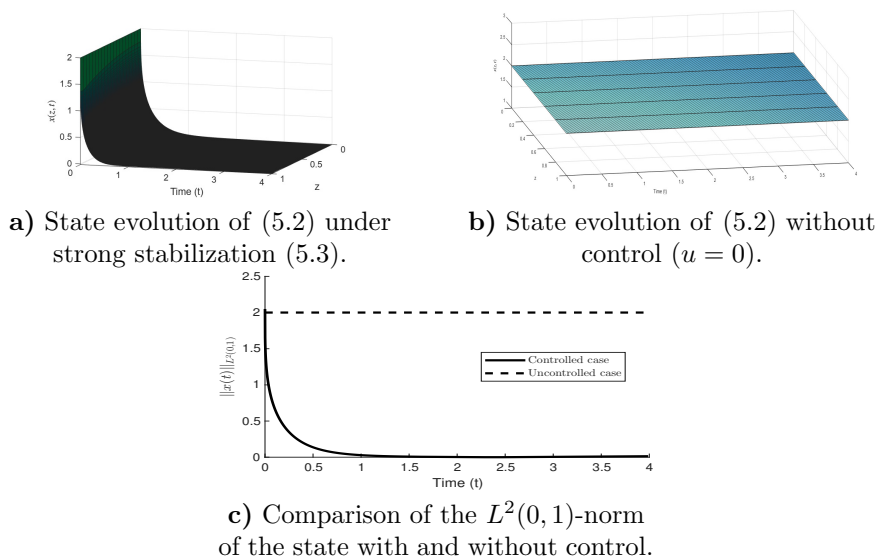


Figure 3. Comparison of the state evolution without control and under strong stabilization (5.3) for the initial state $x_0(z) = 2$ and $\alpha = 0.3$.

The system (A^+, B_λ^+) is exponentially stabilizable by

$$u(t) = -\rho \langle h_\lambda, 1 \rangle y^+(t^\alpha/\alpha), \quad y^+ = \langle y, 1 \rangle.$$

By Theorem 5, the feedback

$$u(t) = -\rho \langle h_\lambda, 1 \rangle \langle R(\lambda, A)x(\cdot, t^\alpha/\alpha), 1 \rangle \quad (5.4)$$

exponentially stabilizes (5.2). With $\lambda = -\pi^2/4$, $\rho = 1$ and $x_0(z) = \frac{-\pi^2}{4}(z -$

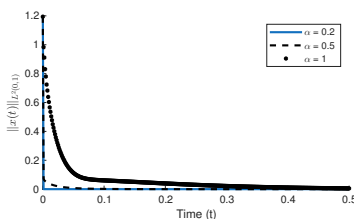
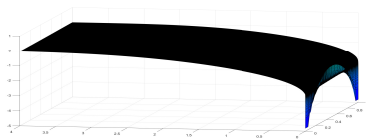
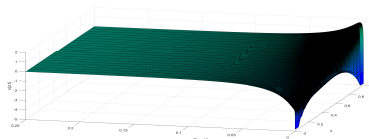


Figure 4. Time evolution of $\|x(t)\|$ under the exponentially stabilizing feedback (5.4) for $\alpha = 0.2, 0.5, 1$.

$z^2)^2 + 12(z - z^2) - 2$, Figure 4 shows the time evolution of the state norm $\|x(t)\|$ under the exponentially stabilizing feedback (5.4) for $\alpha = 0.2, 0.5$, and 1 . In all cases, exponential convergence to zero is observed. Lower values of α lead to faster decay rates, highlighting the influence of the conformable derivative order on the stabilization dynamics.



a) Strong stabilization.



b) Exponential stabilization.

Figure 5. State evolution for $\alpha = 0.5$, $x_0(z) = \frac{-\pi^2}{4}(z - z^2)^2 + 12(z - z^2) - 2$.

Figure 5 further compares the state evolution for $\alpha = 0.5$, showing that the exponentially stabilizing control achieves faster convergence than strong stabilization.

5.3 Exponential stabilization of perturbed 1D heat equation

Consider the unstable system ($\mu > \pi^2/4$):

$$\begin{cases} D_t^\alpha x = x_{zz} + \mu x, & z \in (0, 1), t > 0, \\ x_z(0, t) = 0, \quad x(1, t) = u(t), \\ x(z, 0) = x_0(z) \in L^2(0, 1). \end{cases} \quad (5.5)$$

Formulated as $(FBCP^\alpha)$ with $A_m f = f'' + \mu f$, $D(A_m) = \{f \in H^2 : f'(0) = 0\}$, $Qf = f(1)$, $B = I$. Assumptions (H1) and (H2) hold. Eigenvalues: $\lambda_n = \mu - (\pi/2 + n\pi)^2$, eigenvectors $\varphi_n(z) = \sqrt{2} \cos((\pi/2 + n\pi)z)$, orthonormal basis. The semigroup is

$$T_\alpha(t^\alpha)x_0(z) = \sum_{n=0}^{\infty} 2e^{\lambda_n t^\alpha / \alpha} \cos((\pi/2 + n\pi)z) \int_0^1 \cos((\pi/2 + n\pi)s) x_0(s) ds.$$

For $\lambda \in \rho(A)$, $Q_\lambda u = u h_\lambda(z)$ with

$$h_\lambda(z) = \begin{cases} \frac{\cos(z\sqrt{\mu - \lambda})}{\cos(\sqrt{\mu - \lambda})}, & \lambda < \mu, \\ 1, & \lambda = \mu, \\ \frac{e^{z\sqrt{\lambda - \mu}} + e^{-z\sqrt{\lambda - \mu}}}{e^{\sqrt{\lambda - \mu}} + e^{-\sqrt{\lambda - \mu}}}, & \lambda > \mu. \end{cases}$$

The internal system is

$$\begin{cases} D_t^\alpha y = y_{zz} + \mu y + u(t)h_\lambda(z), \\ y_z(0, t) = 0, \quad y(1, t) = 0, \\ y(z, 0) = R(\lambda, A)x_0(z). \end{cases}$$

Let N be such that $\lambda_n < -\delta$ for $n > N$. For circle $\Gamma_\delta = \mathcal{C}(O, R)$ enclosing $\sigma^+(A) = \{\lambda_n : n \leq N\}$, the projection is $P_\delta f = \sum_{n=0}^N \langle f, \varphi_n \rangle \varphi_n$. Then,

$X^+ = \text{span}\{\varphi_0, \dots, \varphi_N\}$, X^- stable:

$$\begin{aligned} T_\alpha^-(t^\alpha)x^- &= \sum_{n=N+1}^{\infty} 2e^{\lambda_n t^\alpha/\alpha} \langle x^-, \varphi_n \rangle \varphi_n, & A^+ f &= \sum_{n=0}^N \lambda_n \langle f, \varphi_n \rangle \varphi_n, \\ T_\alpha^+(t^\alpha)x^+ &= \sum_{n=0}^N 2e^{\lambda_n t^\alpha/\alpha} \langle x^+, \varphi_n \rangle \varphi_n, & B_\lambda^+ u &= u \sum_{n=0}^N \langle h_\lambda, \varphi_n \rangle \varphi_n. \end{aligned}$$

We stabilize the finite-dimensional system (A^+, B_λ^+) using the feedback

$$u(t) = -\rho \sum_{n=0}^N f_n \langle y(\cdot, t^\alpha/\alpha), \varphi_n \rangle, \quad F = (f_0 \cdots f_N).$$

Let $h_n = \langle h_\lambda, \varphi_n \rangle$. The closed-loop matrix is

$$A^+ - \rho B_\lambda^+ F = \begin{pmatrix} \lambda_0 - \rho h_0 f_0 & -\rho h_0 f_1 & \cdots & -\rho h_0 f_N \\ -\rho h_1 f_0 & \lambda_1 - \rho h_1 f_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\rho h_{N-1} f_N \\ -\rho h_N f_0 & \cdots & -\rho h_N f_{N-1} & \lambda_N - \rho h_N f_N \end{pmatrix}.$$

Choose $\lambda = \mu$, so $h_\mu(z) = 1$, $h_n = \frac{2(-1)^n \sqrt{2}}{(2n+1)\pi}$. If $\sigma(A^+ - \rho B_\mu^+ F) \subset \mathbb{C}_-^\circ$, Theorem 5 yields exponential stabilization via

$$u(t) = -\rho \sum_{n=0}^N f_n \frac{\langle x(\cdot, t^\alpha/\alpha), \varphi_n \rangle}{(\pi/2 + n\pi)^2}.$$

Example 1. For $\mu = 25$, $N = 1$, $\rho = \pi^2$,

$$A^+ - \rho B_\mu^+ F = \begin{pmatrix} 25 - \pi^2/4 - 2\pi\sqrt{2}f_0 & -2\pi\sqrt{2}f_1 \\ \frac{2\pi\sqrt{2}}{3}f_0 & 25 - 9\pi^2/4 + \frac{2\pi\sqrt{2}}{3}f_1 \end{pmatrix}.$$

With $f_0 = \pi$, $f_1 = 1$, eigenvalues are in $\mathbb{C}_-^\circ$. The feedback

$$u(t) = 4\pi \langle x(\cdot, t^\alpha/\alpha), \varphi_0 \rangle + \frac{4}{9} \langle x(\cdot, t^\alpha/\alpha), \varphi_1 \rangle$$

exponentially stabilizes (5.5).

This extends [5] ($\alpha = 1$, smoother x_0) to all $x_0 \in L^2(0, 1)$ and $\alpha \in (0, 1]$.

6 Conclusions

This paper aims to explore the stabilization of conformable boundary control systems. We establish the correlation between the stabilization of these boundary control systems and their corresponding internal control systems.

Furthermore, we establish that both systems can be stabilized using the same control law. Although several topics of interest remain open, we are currently working on extending this research to semilinear systems. Specifically, we intend to investigate the stabilization of conformable systems with time-varying delay and the bilinear boundary control systems discussed in [15] utilizing the conformable derivative.

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