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Generalized dynamic inequalities similar to Hardy's inequality involving a convex function on time scales

Ahmed I. Saied[a,b] and Mario Krnić[c] ✉

[a] *Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 04001 Košice, Slovakia*
[b] *Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt*
[c] *Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia*

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Abstract. In this paper, we establish some new generalizations of dynamic inequalities similar to Hardy's inequality on a time scale $\mathbb{T}$, by applying Jensen's inequality, integration by parts and chain rule on time scales. In particular, when $\mathbb{T} = \mathbb{R}$, we get the classical inequalities known from the literature, while in the discrete case $\mathbb{T} = \mathbb{N}$, the obtained inequalities are essentially new. In addition, we show that our results are more accurate than some recent dynamic inequalities known from the literature. Finally, we establish the corresponding relations in quantum calculus, when $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$.

Keywords: Hardy-type inequalities; time scales; Jensen's inequality; chain rule.

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✉ Corresponding author. E-mail: mario.krnic@fer.hr

1 Introduction

A famous discrete inequality, introduced by Hardy [10] in 1920, states that if $p > 1$ and $\{a(n)\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p} \leq \left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n). \tag{1.1}$$

The integral analogue of (1.1) was stated by Hardy [10] without proof but in [11, Theorem A], he gave its proof and showed that if $p > 1$ and $\psi \geq 0$ is a p-integrable function on $(0, \infty)$, then ψ is integrable over any finite interval $(0, \tau)$, for each $\tau \in (0, \infty)$ and

$$\int_0^{\infty}\left(\frac{1}{\tau}\int_0^{\tau}\psi(\xi)d\xi\right)^{p}\mathrm{d}\tau \leq \left(\frac{p}{p-1}\right)^{p}\int_0^{\infty}\psi^{p}(\tau)\mathrm{d}\tau. \tag{1.2}$$

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The constant $(p/(p-1))^p$ in both relations (1.1) and (1.2) is sharp, i.e., it cannot be replaced with a smaller one without affecting the validity of (1.1) and (1.2) for all possible sequences and functions.

In recent years, a natural research problem investigated by many authors is how to estimate the left-hand side of (1.2) when ψ is a positive and integrable function on $[a,b] \subseteq (0,\infty)$.

This was firstly answered by Sulaiman [21], who proved that if ψ is a positive and integrable function on $[a,b] \subseteq (0,\infty)$ and $p \geq 1$, then

$$p\int_a^b \frac{F^p(\tau)}{\tau^p}\mathrm{d}\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\tau^p}\mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p\,\psi^p(\tau)}{\tau^p}\mathrm{d}\tau, \tag{1.3}$$

and if $0 < p < 1$, then

$$p\int_a^b \frac{F^p(\tau)}{\tau^p}\mathrm{d}\tau \geq \frac{(b-a)^p}{b^p} \int_a^b \psi^p(\tau)\mathrm{d}\tau - \frac{1}{b^p}\int_a^b (\tau-a)^p\,\psi^p(\tau)\mathrm{d}\tau, \tag{1.4}$$

where $F(\tau) = \int_a^\tau \psi(t)\mathrm{d}t$, $\tau \in [a,b]$.

In 2013, Sroysang [20], added a new parameter $\eta > 0$ to the previous inequalities (1.3) and (1.4), which are obtained as a special case when $\eta = p$. For clarification, he proved that if ψ is a positive and integrable function on $[a,b] \subseteq (0,\infty)$ and $\eta > 0$, then

$$p\int_a^b \frac{F^p(\tau)}{\tau^\eta}\mathrm{d}\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\tau^\eta}\mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p\,\psi^p(\tau)}{\tau^\eta}\mathrm{d}\tau, \tag{1.5}$$

provided that $p \geq 1$, (see [20, Theorem 2.1]) and

$$p\int_a^b \frac{F^p(\tau)}{\tau^\eta}\mathrm{d}\tau \geq \frac{(b-a)^p}{b^\eta} \int_a^b \psi^p(\tau)\mathrm{d}\tau - \frac{1}{b^\eta}\int_a^b (\tau-a)^p\,\psi^p(\tau)\mathrm{d}\tau, \tag{1.6}$$

provided that $0 < p < 1$.

In 2016, Wu et. al. [22], generalized inequalities (1.5), (1.6) by replacing τ^η, b^η by $\lambda^\eta(\tau)$, $\lambda^\eta(b)$, respectively, with the nondecreasing function λ and showed that if ψ is a positive and integrable function on $[a,b] \subseteq (0,\infty)$, $\eta > 0$ and λ is positive, then

$$p\int_a^b \frac{F^p(\tau)}{\lambda^\eta(\tau)}\mathrm{d}\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\tau)}\mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p\,\psi^p(\tau)}{\lambda^\eta(\tau)}\mathrm{d}\tau, \tag{1.7}$$

provided that $p \geq 1$ and

$$p\int_a^b \frac{F^p(\tau)}{\lambda^\eta(\tau)}\mathrm{d}\tau \geq \frac{(b-a)^p}{\lambda^\eta(b)} \int_a^b \psi^p(\tau)\mathrm{d}\tau - \frac{1}{\lambda^\eta(b)}\int_a^b (\tau-a)^p\,\psi^p(\tau)\mathrm{d}\tau, \tag{1.8}$$

provided that $0 < p < 1$.

In 2022, Bendaoud and Senouci [7], studied the inequalities (1.7) and (1.8) for the nonincreasing function λ and showed that if $\eta > 0$, ψ is positive p-integrable, $F(\tau) = \int_a^\tau \psi(t)\mathrm{d}t$, $x > a$, and $\lambda > 0$ on $[a,b] \subseteq (0,\infty)$ is nonincreasing, then

$$p\int_a^b \frac{F^p(\tau)}{\lambda^\eta(\tau)}\mathrm{d}\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(b)}\mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p\,\psi^p(\tau)}{\lambda^\eta(b)}\mathrm{d}\tau, \tag{1.9}$$

provided that $p \geq 1$ and

$$p \int_a^b \frac{F^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau \geq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p\, \psi^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau, \tag{1.10}$$

provided that $0 < p < 1$.

Also, they proved that if $1 \leq p < \infty$, then

$$p \int_a^b \frac{G^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau - \int_a^b \frac{(b-\tau)^p\, \psi^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau, \tag{1.11}$$

provided that λ is nonincreasing and

$$p \int_a^b \frac{G^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau \leq \frac{(b-a)^p}{\lambda^\eta(a)} \int_a^b \psi^p(\tau) \mathrm{d}\tau - \frac{1}{\lambda^\eta(a)} \int_a^b (b-\tau)^p\, \psi^p(\tau) \mathrm{d}\tau, \tag{1.12}$$

provided that λ is nondecreasing, where $G(\tau) = \int_\tau^b \psi(t)\mathrm{d}t$, $a < x < b$. In addition, for $0 < p < 1$, the inequality

$$p \int_a^b \frac{G^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau \geq \lambda^{-\eta}(a) \int_a^b \psi^p(\tau) \left[(b-a)^p - (b-\tau)^p\right] \mathrm{d}\tau \tag{1.13}$$

holds provided that λ is nonincreasing and

$$p \int_a^b \frac{G^p(\tau)}{\lambda^\eta(\tau)} \mathrm{d}\tau \geq \int_a^b \psi^p(\tau) \lambda^{-\eta}(\tau) \left[(b-a)^p - (b-\tau)^p\right] \mathrm{d}\tau, \tag{1.14}$$

provided that λ is nondecreasing.

On the other hand, another general question explored by many authors working in the field of inequalities, and which we are now also addressing, is how continuous and discrete inequalities can be unified into a single new relation to avoid the need for proving them separately.

The answer to this question was presented by Hilger [14], when he introduced the time scale theory and established a new inequality with a general domain called a time scale $\mathbb{T}$ which is understood as an arbitrary closed subset of the real numbers $\mathbb{R}$. As special cases of any dynamic inequality, when $\mathbb{T} = \mathbb{R}$, we get the continuous inequalities, while for $\mathbb{T} = \mathbb{N}$, we have the discrete analogues. For more details about the dynamic inequalities on time scales, we refer the reader to the papers [5,6,8,16,17,18,19,23] and for the dynamic inequalities of Hardy-type, see monograph [1] by Agarwal et al..

In particular, Rehak [16], unified inequalities (1.1) and (1.2) in a new time scale version and proved that if $a \in \mathbb{T}$, $p > 1$, a function ψ is nonnegative such that the integral $\int_a^\infty \psi^p(s)\Delta s$ exists as a finite number and $F(\xi) = \int_a^\xi \psi(s)\Delta s$,

$$\int_a^\infty \left(\frac{F^\sigma(\xi)}{\sigma(\xi) - a}\right)^p \Delta\xi < \left(\frac{p}{p-1}\right)^p \int_a^\infty \psi^p(\xi)\Delta\xi, \tag{1.15}$$

unless $\psi \equiv 0$. In addition, if $\mu(\xi)/\xi \to 0$ as $\xi \to \infty$, then the constant $(p/(p-1))^p$ in (1.15) is best possible.

In addition, Hasan et al. [12], established the time scale version of classical integral inequalities (1.11), (1.12) and proved that for $a, b \in \mathbb{T}$, $p \geq 1$, $\eta > 0$ and $\psi, \lambda \in C_{\mathrm{rd}}\left([a,b]_{\mathbb{T}}, \mathbb{R}^+\right)$,

$$p\int_a^b \frac{\left(\int_{\sigma(\tau)}^b \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\sigma(\tau))}\Delta\tau - \int_a^b \frac{\psi^p(\tau)(b-\sigma(\tau))^p}{\lambda^\eta(\sigma(\tau))}\Delta\tau, \tag{1.16}$$

provided that λ is nonincreasing and

$$p\int_a^b \frac{\left(\int_{\sigma(\tau)}^b \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(a)}\Delta\tau - \int_a^b \frac{\psi^p(\tau)(b-\sigma(\tau))^p}{\lambda^\eta(a)}\Delta\tau, \tag{1.17}$$

provided that λ is nondecreasing. Furthermore, Hasan et al. [13], proved the time scale version of inequalities (1.7), (1.9) which states that if $p \geq 1$ and $F(\tau) = \int_a^\tau \psi(t)\Delta t$, then

$$p\int_a^b \frac{F^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau - \int_a^b \frac{(\tau-a)^p\,\psi^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau, \tag{1.18}$$

provided that λ is nondecreasing and

$$p\int_a^b \frac{F^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(b)}\Delta\tau - \int_a^b \frac{(\tau-a)^p\,\psi^p(\tau)}{\lambda^\eta(b)}\Delta\tau, \tag{1.19}$$

provided that λ is nonincreasing.

Fractional Hardy-type integral inequalities also play an important role in time scales. Let H_β and $\tilde{H}_\beta$ be the fractional Hardy operator and the corresponding adjoint on $(0,\infty)$ respectively, i.e.

$$H_\beta\psi(t) = \frac{1}{t^{1-\beta}}\int_0^t \psi(s)\mathrm{d}s \quad \text{and}$$

where $0 < \beta < 1$. The case of $\beta = 0$ provides the Hardy-type inequality which states that if $p > 1$ and ψ is a positive real function, then

$$\int_0^\infty \left(\frac{1}{\tau}\int_\tau^\infty \psi(\xi)d\xi\right)^p \mathrm{d}\tau \leq p^p \int_0^\infty \psi^p(\tau)\mathrm{d}\tau.$$

The constant p^p is the best possible (see, e.g. [15]). For more details about fractional Hardy-type inequalities on time scales the reader is referred to [2,3,4].

In the present paper, we introduce several new generalized dynamic inequalities similar to Hardy's inequality involving convex and concave functions on time scales. In particular, the results established by Bendaoud and Senouci in [7] turn to be special cases of our main results. In addition, we improve the recent dynamic relations established in [12,13], by deriving sharper inequalities.

In this towards, we establish the time scales analogues of the classical inequalities (1.9)–(1.14), proved by Bendaoud and Senouci [7], as well as the corresponding analogues of (1.7) and (1.8), derived by Wu et al. [22]. Furthermore, we show that our results, in particular settings, are sharper (qualitatively better) than inequalities (1.16)–(1.19) presented in [12, 13]. The improvement will be attained by using a different technique, based on the use of Jensen's inequality, dynamic integration by parts and chain rule formula on time scales, which we present in Section 3. In addition, we obtain new inequalities in difference and quantum calculi, which are considered to be essentially new.

The paper is organized as follows: Section 2 involves some basic lemmas and theorems on time scales needed for establishing our main results. Section 3 includes the main results, the corresponding consequences, as well as several numerical examples.

2 Auxiliary lemmas

In order to summarize our further discussion, for the basic facts of time scales and the corresponding notation, we refer the reader to the monograph by Bohner and Peterson [9], which summarizes and organizes fundamental theory on time scales. Further, to establish our main results, we give the following auxiliary lemmas.

Lemma 1 [Keller's Chain Rule, [9, Theorem 1.90]]. *Assume $v:\mathbb{R}\to\mathbb{R}$ is continuous, $v:\mathbb{T}\to\mathbb{R}$ is delta differentiable on $\mathbb{T}$, and $u:\mathbb{R}\to\mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[\xi,\sigma(\xi)]$ such that*

$$(u\circ v)^{\Delta}(\xi)=u'(v(c)v^{\Delta}(\xi). \tag{2.1}$$

Lemma 2 [[9, Theorem 1.75]]. *If $\psi\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and $\xi\in\mathbb{T}$, then*

$$\int_{\xi}^{\sigma(\xi)}\psi(\tau)\,\Delta\tau=\mu(\xi)\,\psi(\xi)\,. \tag{2.2}$$

Lemma 3 [[9, Theorem 1.77]]. *If $a,b\in\mathbb{T}$ and $u,v\in \mathrm{C}_{\mathrm{rd}}([a,\,b]_{\mathbb{T}},\mathbb{R})$, then*

$$\int_a^b u(\xi)v^{\Delta}(\xi)\Delta\xi=[u(\xi)v(\xi)]_a^b-\int_a^b u^{\Delta}(\xi)v^{\sigma}(\xi)\Delta\xi. \tag{2.3}$$

Theorem 1 [Jensen's inequality, [9, Theorem 6.17]]. *Let $\alpha,\beta\in\mathbb{T}$, $c,d\in\mathbb{R}$ and $\lambda\in C_{\mathrm{rd}}\left([\alpha,\beta]_{\mathbb{T}},(c,d)\right)$. Then*

$$\phi\left(\frac{\int_\alpha^\beta\lambda(\xi)\Delta\xi}{\beta-\alpha}\right)\le\frac{\int_\alpha^\beta\phi(\lambda(\xi))\Delta\xi}{\beta-\alpha}, \tag{2.4}$$

provided that $\phi\in C\left((c,d),\mathbb{R}\right)$ is a convex function and

$$\phi\left(\frac{\int_\alpha^\beta\lambda(\xi)\Delta\xi}{\beta-\alpha}\right)\ge\frac{\int_\alpha^\beta\phi(\lambda(\xi))\Delta\xi}{\beta-\alpha}, \tag{2.5}$$

provided that $\phi\in C\left((c,d),\mathbb{R}\right)$ is a concave function.

3 Main results

Throughout this paper, all the integrals considered are assumed to exist. Now, we are prepared to establish our main results. More precisely, we start with the time scale version of relations (1.7)–(1.10).

Theorem 2. *Let $a, b \in \mathbb{T}$, $\eta > 0$ and $\psi, \lambda \in C_{\mathrm{rd}}\left([a,b]_{\mathbb{T}}, \mathbb{R}^{+}\right)$. Then,*

(a) for $p > 1$ and a convex function $\phi \in C((c,d),\mathbb{R})$,

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(b)}\left[(b-a)^p-(\sigma(\tau)-a)^p\right]\Delta\tau, \tag{3.1}$$

provided that λ is nonincreasing and

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p-(\sigma(\tau)-a)^p\right]\Delta\tau, \tag{3.2}$$

provided that λ is nondecreasing.

(b) for $0 < p < 1$and concave function $\phi \in C((c,d),\mathbb{R})$, then

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \geq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p-(\sigma(\tau)-a)^p\right]\Delta\tau, \tag{3.3}$$

provided that λ is nonincreasing and

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \geq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(b)}\left[(b-a)^p-(\sigma(\tau)-a)^p\right]\Delta\tau, \tag{3.4}$$

provided that λ is nondecreasing.

Proof. Case (a): Applying Jensen's inequality (2.4), where ϕ is convex, we get

$$\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \leq \int_a^b (\tau-a)^{p-1}\lambda^{-\eta}(\tau)\left(\int_a^\tau \phi(\psi(t))\Delta t\right)\Delta\tau. \tag{3.5}$$

Utilizing the integration by parts formula (2.3), we see that

$$\int_a^b \frac{(\tau-a)^{p-1}}{\lambda^\eta(\tau)}\left(\int_a^\tau \phi(\psi(t))\Delta t\right)\Delta\tau = \int_a^b \phi(\psi(\tau))\left(\int_{\sigma(\tau)}^b \frac{(t-a)^{p-1}}{\lambda^\eta(t)}\Delta t\right)\Delta\tau. \tag{3.6}$$

Now, we can prove (3.1).

Since $\eta > 0$ and λ is a nonincreasing function, we have that $\lambda^{-\eta}(t) \leq \lambda^{-\eta}(b)$, for $t \leq b$, thus by combining (3.5) and (3.6), we can write

$$\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \leq \lambda^{-\eta}(b)\int_a^b \phi(\psi(\tau))\left(\int_{\sigma(\tau)}^b (t-a)^{p-1}\Delta t\right)\Delta\tau. \tag{3.7}$$

Using the chain rule formula (2.1), we see that

$$((t-a)^p)^\Delta = p(c-a)^{p-1} \quad \text{for some} \quad c \in [t, \sigma(t)].$$

For $p > 1$ and $c \geq t$, we deduce that $(c-a)^{p-1} \geq (t-a)^{p-1}$, that is, $\frac{1}{p}((t-a)^p)^\Delta \geq (t-a)^{p-1}$ and

$$\int_{\sigma(\tau)}^b (t-a)^{p-1}\Delta t \leq \frac{1}{p}\int_{\sigma(\tau)}^b ((t-a)^p)^\Delta \Delta t = \frac{1}{p}\left[(b-a)^p - (\sigma(\tau)-a)^p\right], \tag{3.8}$$

so that (3.7) becomes

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(b)}\left[(b-a)^p - (\sigma(\tau)-a)^p\right]\Delta\tau,$$

which is the desired inequality (3.1).

To prove (3.2), we have several steps. Using (2.2), as well as, noting that $\eta > 0$ and λ is nondecreasing, we see that

$$\int_{\sigma(\tau)}^b \frac{[t-a]^{p-1}}{\lambda^\eta(t)}\Delta t = \int_\tau^b \frac{[t-a]^{p-1}}{\lambda^\eta(t)}\Delta t - \mu(\tau)\frac{[\tau-a]^{p-1}}{\lambda^\eta(\tau)} \leq \lambda^{-\eta}(\tau)$$
$$\left[\int_\tau^b [t-a]^{p-1}\Delta t - \int_\tau^{\sigma(\tau)} [t-a]^{p-1}\,\Delta t\right] = \lambda^{-\eta}(\tau)\int_{\sigma(\tau)}^b [t-a]^{p-1}\Delta t,$$

so (3.6) becomes

$$\int_a^b \frac{(\tau-a)^{p-1}}{\lambda^\eta(\tau)}\left(\int_a^\tau \phi(\psi(t))\Delta t\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left(\int_{\sigma(\tau)}^b (t-a)^{p-1}\Delta t\right)\Delta\tau. \tag{3.9}$$

From (3.8) and (3.9), we observe that

$$\int_a^b \frac{(\tau-a)^{p-1}}{\lambda^\eta(\tau)}\left(\int_a^\tau \phi(\psi(t))\Delta t\right)\Delta\tau \leq \frac{1}{p}\int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p - (\sigma(\tau)-a)^p\right]\Delta\tau. \tag{3.10}$$

Substituting (3.10) into (3.5), we have

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p - (\sigma(\tau)-a)^p\right]\Delta\tau,$$

which is (3.2).

Case b): Again by applying Jensen's inequality (2.5) where ϕ is concave, we see that

$$\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \geq \int_a^b (\tau-a)^{p-1}\lambda^{-\eta}(\tau)\left(\int_a^\tau \phi(\psi(t))\Delta t\right)\Delta\tau. \tag{3.11}$$

Applying the integration by parts formula (2.3) on the right-hand side of (3.11), we observe that

$$\int_a^b \frac{(\tau-a)^{p-1}}{\lambda^\eta(\tau)}\left(\int_a^\tau \phi(\psi(t))\Delta t\right)\Delta\tau = \int_a^b \phi(\psi(\tau))\left(\int_{\sigma(\tau)}^b [t-a]^{p-1}\lambda^{-\eta}(t)\Delta t\right)\Delta\tau. \tag{3.12}$$

Now, we are able to prove (3.3). Using (2.2), we see that

$$\int_{\sigma(\tau)}^b [t-a]^{p-1}\lambda^{-\eta}(t)\Delta t = \int_\tau^b [t-a]^{p-1}\lambda^{-\eta}(t)\Delta t - \mu(\tau)[\tau-a]^{p-1}\lambda^{-\eta}(\tau). \tag{3.13}$$

Since $\eta > 0$ and λ is a nonincreasing function, it follows that for $t \geq \tau$ holds $\lambda^{-\eta}(t) \geq \lambda^{-\eta}(\tau)$, thus (3.13) becomes

$$\int_{\sigma(\tau)}^b [t-a]^{p-1}\lambda^{-\eta}(t)\Delta t \geq \lambda^{-\eta}(\tau)\int_{\sigma(\tau)}^b [t-a]^{p-1}\Delta t. \tag{3.14}$$

Substituting (3.14) into (3.12), we get

$$\int_a^b \frac{(\tau-a)^{p-1}}{\lambda^\eta(\tau)}\left(\int_a^\tau \phi(\psi(t))\Delta t\right)\Delta\tau \geq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left(\int_{\sigma(\tau)}^b (t-a)^{p-1}\Delta t\right)\Delta\tau,$$

and then (3.11) becomes

$$\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \geq \int_a^b \phi(\psi(\tau))\lambda^{-\eta}(\tau)\left(\int_{\sigma(\tau)}^b (t-a)^{p-1}\Delta t\right)\Delta\tau. \tag{3.15}$$

Using the chain rule formula (2.1) for the term $((t-a)^p)^\Delta$, $0<p<1$, we find that $((t-a)^p)^\Delta \leq p(t-a)^{p-1}$ and thus

$$p\int_{\sigma(\tau)}^b (t-a)^{p-1}\Delta t \geq (b-a)^p - (\sigma(\tau)-a)^p, \tag{3.16}$$

so the inequality (3.15) gives

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \geq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p - (\sigma(\tau)-a)^p\right]\Delta\tau,$$

which satisfies (3.3).

To conclude the proof, we need to establish (3.4). Since $\eta > 0$ and λ is nondecreasing, we have

$$\int_{\sigma(\tau)}^{b} [t-a]^{p-1} \lambda^{-\eta}(t)\Delta t \geq \lambda^{-\eta}(b) \int_{\sigma(\tau)}^{b} [t-a]^{p-1} \Delta t. \tag{3.17}$$

Substituting (3.17) into (3.12), we have

$$\int_a^b \frac{(\tau-a)^{p-1}}{\lambda^{\eta}(\tau)} \left(\int_a^{\tau} \phi(\psi(t))\Delta t \right) \Delta\tau \geq \lambda^{-\eta}(b) \int_a^b \phi(\psi(\tau)) \left(\int_{\sigma(\tau)}^{b} [t-a]^{p-1} \Delta t \right) \Delta\tau. \tag{3.18}$$

From (3.16) and (3.18), we obtain

$$\begin{aligned} &p \int_a^b (\tau-a)^{p-1} \lambda^{-\eta}(\tau) \left(\int_a^{\tau} \phi(\psi(t))\Delta t \right) \Delta\tau \\ &\geq \lambda^{-\eta}(b) \int_a^b \phi(\psi(\tau)) \left[(b-a)^p - (\sigma(\tau)-a)^p \right] \Delta\tau. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.11), we get

$$p \int_a^b \frac{(\tau-a)^p}{\lambda^{\eta}(\tau)} \phi \left(\frac{\int_a^{\tau} \psi(t)\Delta t}{\tau - a} \right) \Delta\tau \geq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^{\eta}(b)} \left[(b-a)^p - (\sigma(\tau)-a)^p \right] \Delta\tau,$$

which is desired inequality (3.4). The proof is now complete. □

Theorem 2 represents further generalization of the corresponding relations established by Bendaoud, Senouci, Sroysang and Hasan [7, 13, 20], as discused in the following three remarks.

Remark 1. If $\mathbb{T} = \mathbb{R}$ and $\phi(y) = y^p$, then inequalities (3.1) and (3.2) reduce to (1.7) and (1.8), respectively, proved by Wu et al. [22]. Moreover, relations (3.3) and (3.4) reduce to (1.9) and (1.10), respectively, established by Bendaoud and Senouci [7].

Remark 2. Taking into account the previous remark with $\mathbb{T} = \mathbb{R}$, $\lambda(y) = y$ and $\phi(y) = y^p$, we notice that inequalities (3.1), (3.2) represent generalization of (1.5), (1.6), respectively, proved by Sroysang [20].

Remark 3. If $p > 1$ and $\phi(\tau) = \tau^p$, then it follows that

$$p \int_a^b \frac{\left(\int_a^{\tau} \psi(t)\Delta t \right)^p}{\lambda^{\eta}(\tau)} \Delta\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^{\eta}(b)} \Delta\tau - \int_a^b \frac{(\sigma(\tau)-a)^p \psi^p(\tau)}{\lambda^{\eta}(b)} \Delta\tau,$$

provided that λ is nonincreasing. In addition, since $\tau \leq \sigma(\tau)$, we obtain

$$p \int_a^b \frac{\left(\int_a^{\tau} \psi(t)\Delta t \right)^p}{\lambda^{\eta}(\tau)} \Delta\tau \leq (b-a)^p \int_a^b \frac{\psi^p(\tau)}{\lambda^{\eta}(b)} \Delta\tau - \int_a^b \frac{(\tau-a)^p \psi^p(\tau)}{\lambda^{\eta}(b)} \Delta\tau,$$

which is (1.18), proved by Hasan et al. [13]. Consequently, our result is sharper than the latter one. Similarly, if λ is nondecreasing, our general result implies

$$p\int_a^b \frac{\left(\int_a^\tau \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \le (b-a)^p\int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau - \int_a^b \frac{(\sigma(\tau)-a)^p\psi^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau,$$

and utilizing $\tau \le \sigma(\tau)$ yields

$$p\int_a^b \frac{\left(\int_a^\tau \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \le (b-a)^p\int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau - \int_a^b \frac{(\tau-a)^p\psi^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau,$$

which represents (1.19). Consequently, our relation is more accurate than the latter one.

The next examples illustrate the rules of differentiation and integration on time scales.

Example 1. For the validity of inequality (3.1), we choose the time scale $\mathbb{T} = 3\mathbb{N}_0$, where $\sigma(t) = t+3$ and $\mu(t) = 3$. Further, let $a = 0$, $b = 6$, $\psi(t) = 2t+3$, $p = 2$, $\eta = 1$, $\phi(\tau) = \tau^2$, and $\lambda(\tau) = \frac{1}{\tau+1}$. Then the left hand side of (3.1) becomes

$$\begin{aligned} p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau &= 2\int_0^6 \tau^2(\tau+1)\left(\frac{\int_0^\tau (2t+3)\Delta t}{\tau}\right)^2\Delta\tau \\ &= 2\int_0^6 \tau^2(\tau+1)\left(\frac{\int_0^\tau (t+\sigma(t))\Delta t}{\tau}\right)^2\Delta\tau = 2\int_0^6 \tau^2(\tau+1)\left(\frac{\int_0^\tau (t^2)^\Delta\Delta t}{\tau}\right)^2\Delta\tau \\ &= 2\int_0^6 [\tau^5+\tau^4]\Delta\tau. \end{aligned}$$

We have

$$(\tau^6)^\Delta = \frac{[\sigma(\tau)]^6-\tau^6}{\sigma(\tau)-\tau} = \frac{(\tau+3)^6-\tau^6}{3} = 6\tau^5+45\tau^4+180\tau^3+405\tau^2+486\tau+243,$$

$$(\tau^5)^\Delta = \frac{[\sigma(\tau)]^5-\tau^5}{\sigma(\tau)-\tau} = \frac{(\tau+3)^5-\tau^5}{3} = 5\tau^4+30\tau^3+90\tau^2+135\tau+81,$$

and similarly, $(\tau^4)^\Delta = 4\tau^3+18\tau^2+36\tau+27$, $(\tau^3)^\Delta = 3\tau^2+9\tau+9$, $(\tau^2)^\Delta = 2\tau+3$. Hence, we have

$$\tau^4+\tau^5 = \frac{1}{6}(\tau^6)^\Delta - \frac{13}{10}(\tau^5)^\Delta + \frac{9}{4}(\tau^4)^\Delta + 3(\tau^3)^\Delta - \frac{27}{4}(\tau^2)^\Delta - \frac{378}{10},$$

and consequently,

$$\begin{aligned} &p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^\tau \psi(t)\Delta t}{\tau-a}\right)\Delta\tau \\ &= 2\int_0^6 \left[\frac{1}{6}(\tau^6)^\Delta - \frac{13}{10}(\tau^5)^\Delta + \frac{9}{4}(\tau^4)^\Delta + 3(\tau^3)^\Delta - \frac{27}{4}(\tau^2)^\Delta - \frac{378}{10}\right]\Delta\tau \\ &= 1522.8. \end{aligned} \tag{3.20}$$

Finally, the right-hand side of (3.1) becomes

$$\int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(b)} \left[(b-a)^p - (\sigma(\tau)-a)^p\right] \Delta\tau = 7\int_0^6 (2\tau+3)^2 \left[36-(\tau+3)^2\right] \Delta\tau$$
$$= 7\int_0^6 [-4\tau^4 - 36\tau^3 + 27\tau^2 + 270\tau + 243]\Delta\tau$$
$$= 7\int_0^6 \left[-\frac{4}{5}(\tau^5)^\Delta - 3(\tau^4)^\Delta + 51(\tau^3)^\Delta + \frac{135}{10}(\tau^2)^\Delta - \frac{1323}{10}\right] \Delta\tau = 4195.8. \tag{3.21}$$

Clearly, relations (3.20) and (3.21) show validity of (3.1) in this particular setting.

Example 2. Consider Theorem 2, with $\mathbb{T} = \mathbb{R}$, $\sigma(\tau) = \tau$, $a = 1$, $b = 2$, $\eta = 1$, $\phi(\tau) = \tau^p$ and $\psi(\tau) = \tau - 1$.
To check (3.1), we put $p = 2$ and $\lambda(\tau) = 1/\tau$, so in this setting we have

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)} \phi\left(\frac{\int_a^\tau \psi(t)\mathrm{d}t}{\tau-a}\right) \mathrm{d}\tau = \frac{1}{2}\int_1^2 \tau(\tau-1)^4 \mathrm{d}\tau = \frac{1}{6}$$
$$(b-a)^p \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(b)} \mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p \phi(\psi(\tau))}{\lambda^\eta(b)} \mathrm{d}\tau = \frac{4}{15}.$$

To check (3.2) in particular setting, let $p = 2$ and $\lambda(\tau) = \tau$. Then,

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)} \phi\left(\frac{\int_a^\tau \psi(t)\mathrm{d}t}{\tau-a}\right) \mathrm{d}\tau = 2\int_1^2 \frac{\left(\int_1^\tau (t-1)\mathrm{d}t\right)^2}{\tau} \mathrm{d}\tau = \frac{12\ln 2 - 7}{24}$$

and

$$(b-a)^p \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)} \mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p \phi(\psi(\tau))}{\lambda^\eta(\tau)} \mathrm{d}\tau = \frac{37}{12},$$

thus (3.2) is satisfied.

In addition, putting $p = \frac{1}{2}$ and $\lambda(\tau) = \frac{1}{\tau}$ in (3.3), we have

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)} \phi\left(\frac{\int_a^\tau \psi(t)\mathrm{d}t}{\tau-a}\right) \mathrm{d}\tau = \frac{1}{2}\int_1^2 \tau\left(\int_1^\tau (t-1)\mathrm{d}t\right)^{\frac{1}{2}} \mathrm{d}\tau = \frac{4+3\sqrt{2}}{12}$$

and

$$(b-a)^p \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)} \mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p \phi(\psi(\tau))}{\lambda^\eta(\tau)} \mathrm{d}\tau = \frac{7}{30},$$

so inequality (3.3) is fulfilled.

Finally, considering (3.4) with $p = \frac{1}{2}$, $\psi(\tau) = 1$ and $\lambda(\tau) = \tau$, it follows that

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)} \phi\left(\frac{\int_a^\tau \psi(t)\mathrm{d}t}{\tau-a}\right) \mathrm{d}\tau = \frac{1}{2}\int_1^2 \frac{(\tau-1)^{\frac{1}{2}}}{\tau} \mathrm{d}\tau = 2 - \sqrt{2}$$

and

$$(b-a)^p \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(b)} \mathrm{d}\tau - \int_a^b \frac{\phi(\psi(\tau))(\tau-a)^p}{\lambda^\eta(b)} \mathrm{d}\tau = \frac{1}{6}.$$

Example 3. Let $\mathbb{T}=\mathbb{R}$, $\sigma(\tau)=\tau$, $a=1$, $b=2$, $\eta=1$ and $\psi(\tau)=\tau-1$. Let us consider (3.1) with $\phi(\tau)=\tau^p$, $p>1$ and $\lambda(\tau)=1/\tau$. Then,

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)} \phi\left(\frac{\int_a^\tau \psi(t)\mathrm{d}t}{\tau-a}\right) \mathrm{d}\tau = \frac{p}{2^{p+1}}\left(\frac{1}{p+1}+\frac{2}{2p+1}\right)$$

and

$$(b-a)^p \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(b)} \mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p\phi(\psi(\tau))}{\lambda^\eta(b)} \mathrm{d}\tau = 2\left(\frac{1}{p+1}-\frac{1}{2p+1}\right),$$

so (3.1) holds in this particular setting.

Moreover, substituting $\phi(\tau)=\tau^p$, $0<p<1$ and $\lambda(\tau)=\frac{1}{\tau}$ in (3.3) we have that

$$p\int_a^b \frac{(\tau-a)^p}{\lambda^\eta(\tau)} \phi\left(\frac{\int_a^\tau \psi(t)\mathrm{d}t}{\tau-a}\right) \mathrm{d}\tau = p2^{-p-1}\left(\frac{1}{p+1}+\frac{2}{2p+1}\right)$$

and

$$(b-a)^p \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)} \mathrm{d}\tau - \int_a^b \frac{(\tau-a)^p\phi(\psi(\tau))}{\lambda^\eta(\tau)} \mathrm{d}\tau = \frac{1}{p+2}+\frac{1}{2p+2}-\frac{1}{2p+1},$$

so (3.3) is fulfilled in this setting.

Considering Theorem 2 with $\mathbb{T}=\mathbb{N}$, $a,b\in\mathbb{N}$, $\eta>0$, and $\phi(\tau)=\tau^p$, we arrive at the corresponding discrete case.

Corollary 1. Let $a,b\in\mathbb{N}$, $\eta>0$, and let ψ,λ be positive sequences such that $F(n)=\sum_{s=a}^{n-1}\psi(s)$.

(a) If $p>1$, then

$$p\sum_{n=a}^{b-1} \frac{F^p(n)}{\lambda^\eta(n)} \le (b-a)^p \sum_{n=a}^{b-1} \frac{\psi^p(n)}{\lambda^\eta(b)} - \sum_{n=a}^{b-1} \frac{(n+1-a)^p\psi^p(n)}{\lambda^\eta(b)}, \tag{3.22}$$

provided that λ is nonincreasing and

$$p\sum_{n=a}^{b-1} \frac{F^p(n)}{\lambda^\eta(n)} \le (b-a)^p \sum_{n=a}^{b-1} \frac{\psi^p(n)}{\lambda^\eta(n)} - \sum_{n=a}^{b-1} \frac{(n+1-a)^p\psi^p(n)}{\lambda^\eta(n)}, \tag{3.23}$$

provided that λ is nondecreasing.

(b) If $0 < p < 1$, then

$$p\sum_{n=a}^{b-1}\frac{F^p(n)}{\lambda^\eta(n)} \geq (b-a)^p\sum_{n=a}^{b-1}\frac{\psi^p(n)}{\lambda^\eta(n)} - \sum_{n=a}^{b-1}\frac{(n+1-a)^p\psi^p(n)}{\lambda^\eta(n)}, \tag{3.24}$$

provided that λ is nonincreasing and

$$p\sum_{n=a}^{b-1}\frac{F^p(n)}{\lambda^\eta(n)} \geq \sum_{n=a}^{b-1}\frac{\psi^p(n)(b-a)^p}{\lambda^\eta(b)} - \sum_{n=a}^{b-1}\frac{\psi^p(n)(n+1-a)^p}{\lambda^\eta(b)}, \tag{3.25}$$

provided that λ is nondecreasing.

Example 4. Let $\mathbb{T} = \mathbb{N}$, $a = 1, b = 6$, $\eta = 1$ and $\psi(n) = n$. Then, $F(n) = \sum_{s=1}^{n-1}\psi(s) = \frac{1}{2}n(n-1)$. Now, considering (3.22) for $p = 2$ and $\lambda(n) = \frac{1}{n}$, we have

$$p\sum_{n=a}^{b-1}\frac{F^p(n)}{\lambda^\eta(n)} = \frac{1}{2}\sum_{n=1}^{5} n^3(n-1)^2 = 1346$$

and

$$(b-a)^p\sum_{n=a}^{b-1}\frac{\psi^p(n)}{\lambda^\eta(b)} - \sum_{n=a}^{b-1}\frac{(n+1-a)^p\psi^p(n)}{\lambda^\eta(b)} = 150\sum_{n=1}^{5}n^2 - 6\sum_{n=1}^{5}n^4 = 2376,$$

which shows correctness of (3.22) in this case. Similarly, to check (3.23), we put $p = 4$ and $\lambda(n) = n^2$, so we have

$$p\sum_{n=a}^{b-1}\frac{F^p(n)}{\lambda^\eta(n)} = 4\sum_{n=1}^{5}\frac{\left(\frac{1}{2}n(n-1)\right)^4}{n^2} = 1961$$

and

$$(b-a)^p\sum_{n=a}^{b-1}\frac{\psi^p(n)}{\lambda^\eta(n)} - \sum_{n=a}^{b-1}\frac{(n+1-a)^p\psi^p(n)}{\lambda^\eta(n)} = 5^4\sum_{n=1}^{5}n^2 - \sum_{n=1}^{5}n^6 = 13960.$$

Further, considering (3.24) with $p = \frac{1}{2}$ and $\lambda(n) = \frac{1}{n^2}$, we have that

$$p\sum_{n=a}^{b-1}\frac{F^p(n)}{\lambda^\eta(n)} = \frac{1}{2}\sum_{n=1}^{5}n^2\sqrt{\frac{1}{2}n(n-1)} = 68.918617$$

and

$$(b-a)^p\sum_{n=a}^{b-1}\frac{\psi^p(n)}{\lambda^\eta(n)} - \sum_{n=a}^{b-1}\frac{(n+1-a)^p\psi^p(n)}{\lambda^\eta(n)} = \sqrt{5}\sum_{n=1}^{5}n^2\sqrt{n} - \sum_{n=1}^{5}n^3 = 21.2962,$$

which shows correctness of (3.24). Finally, we check (3.25) for $p = \frac{1}{3}$ and $\lambda(n) = n + 5$. In this case, we have

$$p\sum_{n=a}^{b-1}\frac{F^p(n)}{\lambda^\eta(n)} = \frac{1}{3}\sum_{n=1}^{5}\frac{\left[\frac{1}{2}n(n-1)\right]^{\frac{1}{3}}}{n+5} = 0.2468$$

and

$$\sum_{n=a}^{b-1} \frac{\psi^p(n)(b-a)^p}{\lambda^\eta(b)} - \sum_{n=a}^{b-1} \frac{\psi^p(n)(n+1-a)^p}{\lambda^\eta(b)} = 5^{\frac{1}{3}} \sum_{n=1}^{5} \frac{n^{\frac{1}{3}}}{11} - \sum_{n=1}^{5} \frac{n^{\frac{2}{3}}}{11} = 0.171.$$

Our next consequence of Theorem 2 refers to the time scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, where $a = q^m$, $b = q^n$.

Corollary 2. Let $q > 1$, $m, n \in \mathbb{N}_0$, $\eta > 0$ and let ψ, λ be positive sequences.

(a) If $p > 1$ and ϕ is a convex function, then

$$p \sum_{k=m}^{n-1} \frac{q^k (q^k - q^m)^p}{\lambda^\eta(q^k)} \phi\left(\frac{\sum_{s=m}^{k-1} (q-1) q^s \psi(q^s)}{q^k - q^m} \right)$$
$$\leq (q^n - q^m)^p \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))}{\lambda^\eta(q^n)} - \sum_{k=m}^{n-1} \frac{q^k (q^{k+1} - q^m)^p \phi(\psi(q^k))}{\lambda^\eta(q^n)},$$

provided that λ is nonincreasing and

$$p \sum_{k=m}^{n-1} \frac{q^k (q^k - q^m)^p}{\lambda^\eta(q^k)} \phi\left(\frac{\sum_{s=m}^{k-1} (q-1) q^s \psi(q^s)}{q^k - q^m} \right)$$
$$\leq \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))}{\lambda^\eta(q^k)} \left[(q^n - q^m)^p - (q^{k+1} - q^m)^p \right],$$

provided that λ is nondecreasing.

(b) If $0 < p < 1$ and ϕ is a concave function, then

$$p \sum_{k=m}^{n-1} \frac{q^k (q^k - q^m)^p}{\lambda^\eta(q^k)} \phi\left(\frac{\sum_{s=m}^{k-1} (q-1) q^s \psi(q^s)}{q^k - q^m} \right)$$
$$\geq (q^n - q^m)^p \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))}{\lambda^\eta(q^k)} - \sum_{k=m}^{n-1} \frac{q^k (q^{k+1} - q^m)^p \phi(\psi(q^k))}{\lambda^\eta(q^k)},$$

provided that λ is nonincreasing and

$$p \sum_{k=m}^{n-1} \frac{q^k (q^k - q^m)^p}{\lambda^\eta(q^k)} \phi\left(\frac{\sum_{s=m}^{k-1} (q-1) q^s \psi(q^s)}{q^k - q^m} \right)$$
$$\geq (q^n - q^m)^p \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))}{\lambda^\eta(q^n)} - \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k)) (q^{k+1} - q^m)^p}{\lambda^\eta(q^n)},$$

provided that λ is nondecreasing.

Our next intention is to establish the corresponding time scale versions of inequalities (1.11) and (1.12).

Theorem 3. *Let $a, b \in \mathbb{T}$, $p > 1$, $\eta > 0$, $c, d \in \mathbb{R}$, let $\phi \in C((c,d),\mathbb{R})$ be convex and let $\psi, \lambda \in C_{\mathrm{rd}}\left([a,b]_{\mathbb{T}}, \mathbb{R}^+\right)$. Then,*

$$p\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b \psi(t)\Delta t}{b-\sigma(\tau)}\right)\Delta\tau \le \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p-(b-\tau)^p\right]\Delta\tau, \tag{3.26}$$

provided that λ is nonincreasing and

$$p\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b \psi(t)\Delta t}{b-\sigma(\tau)}\right)\Delta\tau \le \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(a)}\left[(b-a)^p-(b-\tau)^p\right]\Delta\tau, \tag{3.27}$$

provided that λ is nondecreasing.

Proof. By applying Jensen's inequality (2.4), it follows that

$$\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b \psi(t)\Delta t}{b-\sigma(\tau)}\right)\Delta\tau \le \int_a^b \frac{(b-\sigma(\tau))^{p-1}}{\lambda^\eta(\tau)}\left(\int_{\sigma(\tau)}^b \phi(\psi(t))\Delta t\right)\Delta\tau. \tag{3.28}$$

Applying (2.3) to the right-hand side of (3.28), we have that

$$\int_a^b \frac{(b-\sigma(\tau))^{p-1}}{\lambda^\eta(\tau)}\left(\int_{\sigma(\tau)}^b \phi(\psi(t))\Delta t\right)\Delta\tau = \int_a^b \phi(\psi(\tau))\left(\int_a^\tau \frac{(b-\sigma(t))^{p-1}}{\lambda^\eta(t)}\Delta t\right)\Delta\tau. \tag{3.29}$$

Now, we can split the proof into two cases. We start with proving (3.26). Since $\eta > 0$ and λ is a nonincreasing function, we have that $\lambda^{-\eta}(t) \le \lambda^{-\eta}(\tau)$, for $t \le \tau$, so (3.29) becomes

$$\int_a^b \frac{(b-\sigma(\tau))^{p-1}}{\lambda^\eta(\tau)}\left(\int_{\sigma(\tau)}^b \phi(\psi(t))\Delta t\right)\Delta\tau \le \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left(\int_a^\tau (b-\sigma(t))^{p-1}\Delta t\right)\Delta\tau.$$

Combining the last inequality and (3.28) gives

$$\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b \psi(t)\Delta t}{b-\sigma(\tau)}\right)\Delta\tau \le \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left(\int_a^\tau (b-\sigma(t))^{p-1}\Delta t\right)\Delta\tau. \tag{3.30}$$

Taking into account the chain rule formula (2.1), it follows that

$$\int_a^\tau (b-\sigma(t))^{p-1}\Delta t \le \frac{1}{p}\left[(b-a)^p-(b-\tau)^p\right], \tag{3.31}$$

so (3.30) becomes

$$p\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b \psi(t)\Delta t}{b-\sigma(\tau)}\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p-(b-\tau)^p\right]\Delta\tau,$$

which is (3.26).

Now, we prove (3.27). Since $\eta > 0$ and λ is a nondecreasing function, we have that $\lambda^{-\eta}(t) \leq \lambda^{-\eta}(a)$, for $t \geq a$, thus (3.29) becomes

$$\int_a^b \frac{(b-\sigma(\tau))^{p-1}}{\lambda^\eta(\tau)}\left(\int_{\sigma(\tau)}^b \phi(\psi(t))\Delta t\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(a)}\left(\int_a^\tau (b-\sigma(t))^{p-1}\Delta t\right)\Delta\tau,$$

and then (3.28) gives

$$\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b \psi(t)\Delta t}{b-\sigma(\tau)}\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(a)}\left(\int_a^\tau (b-\sigma(t))^{p-1}\Delta t\right)\Delta\tau. \tag{3.32}$$

From (3.31) and (3.32), we get

$$p\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b \psi(t)\Delta t}{b-\sigma(\tau)}\right)\Delta\tau \leq \int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(a)}\left[(b-a)^p-(b-\tau)^p\right]\Delta\tau,$$

which is (3.27), as claimed. □

In the following remark, we show that our general result is more accurate than the corresponding result of Hasan et. al. [12].

Remark 4. If $a, b \in \mathbb{T}$, $p > 1$, $\eta > 0$ and $\phi(\tau) = \tau^p$, then it follows that

$$p\int_a^b \frac{\left(\int_{\sigma(\tau)}^b \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \leq (b-a)^p\int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\tau)}\Delta\tau - \int_a^b \frac{\psi^p(\tau)(b-\tau)^p}{\lambda^\eta(\tau)}\Delta\tau,$$

provided that λ is nonincreasing. Moreover, since $\tau \leq \sigma(\tau)$, one can obtain

$$p\int_a^b \frac{\left(\int_{\sigma(\tau)}^b \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \leq (b-a)^p\int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(\sigma(\tau))}\Delta\tau - \int_a^b \frac{\psi^p(\tau)(b-\sigma(\tau))^p}{\lambda^\eta(\sigma(\tau))}\Delta\tau,$$

which represents inequality (1.16) proved in [12]. This means that our result is more accurate. Following the same lines as above, for a nondecreasing function λ, we have

$$p\int_a^b \frac{\left(\int_{\sigma(\tau)}^b \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \leq (b-a)^p\int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(a)}\Delta\tau - \int_a^b \frac{\psi^p(\tau)(b-\tau)^p}{\lambda^\eta(a)}\Delta\tau.$$

Furthermore, since $\tau \le \sigma(\tau)$, it follows that

$$p\int_a^b \frac{\left(\int_{\sigma(\tau)}^b \psi(t)\Delta t\right)^p}{\lambda^\eta(\tau)}\Delta\tau \le (b-a)^p\int_a^b \frac{\psi^p(\tau)}{\lambda^\eta(a)}\Delta\tau - \int_a^b \frac{\psi^p(\tau)(b-\sigma(\tau))^p}{\lambda^\eta(a)}\Delta\tau,$$

which represents (1.17). Of course, we again get more precise estimate.

Remark 5. If $\mathbb{T}=\mathbb{R}$ and $\phi(y)=y^p$ (i.e. $\sigma(t)=t$, $t\in\mathbb{R}$), then inequalities (3.26), (3.27) reduce to (1.11), (1.12), respectively, proved by Bendaoud and Senouci [7].

Example 5. Let $\mathbb{T}=\mathbb{R}$, $\sigma(\tau)=\tau$, $a=1, b=2$, $p=2$, $\eta=1$, $\phi(\tau)=\tau^p$ and $\psi(\tau)=1$, so that $G(\tau)=\int_\tau^2\psi(t)\mathrm{d}t = 2-\tau$.

(a) If $\lambda(\tau)=1/\tau$, then

$$p\int_a^b \frac{(b-\tau)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_\tau^b\psi(t)\mathrm{d}t}{b-\tau}\right)\mathrm{d}\tau = 2\int_1^2 x(2-\tau)^2\mathrm{d}\tau = \frac{5}{6}$$

and

$$(b-a)^p\int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\mathrm{d}\tau - \int_a^b \frac{\phi(\psi(\tau))(b-\tau)^p}{\lambda^\eta(\tau)}\mathrm{d}\tau = \frac{13}{12},$$

which shows correctness of (3.26) in this particular setting.

(b) If $\lambda(\tau)=\tau$, we have that

$$p\int_a^b \frac{(b-\tau)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_\tau^b\psi(t)\mathrm{d}t}{b-\tau}\right)\mathrm{d}\tau = 2\int_1^2 \frac{(2-\tau)^2}{\tau}\mathrm{d}\tau = 8\ln 2 - 5$$

and

$$(b-a)^p\int_a^b \frac{\phi(\psi(\tau))}{\lambda^\eta(a)}\mathrm{d}\tau - \int_a^b \frac{\phi(\psi(\tau))(b-\tau)^p}{\lambda^\eta(a)}\mathrm{d}\tau = \frac{2}{3}.$$

Our next result is a discrete version of Theorem 3, where $\mathbb{T}=\mathbb{N}$ and $\phi(\tau)=\tau^p$.

Corollary 3. Let $a,b\in\mathbb{N}$, $p>1$, $\eta>0$, and let ψ,λ be positive sequences such that $G(n)=\sum_{s=n}^{b-1}\psi(s)$. Then,

$$p\sum_{n=a}^{b-1}\frac{(G(n+1))^p}{\lambda^\eta(\tau)} \le (b-a)^p\sum_{n=a}^{b-1}\frac{\psi^p(n)}{\lambda^\eta(n)} - \sum_{n=a}^{b-1}\frac{\psi^p(n)(b-n)^p}{\lambda^\eta(n)}, \tag{3.33}$$

provided that λ is nonincreasing and

$$p\sum_{n=a}^{b-1}\frac{(G(n+1))^p}{\lambda^\eta(n)} \le \sum_{n=a}^{b-1}\frac{\psi^p(n)(b-a)^p}{\lambda^\eta(a)} - \sum_{n=a}^{b-1}\frac{\psi^p(n)(b-n)^p}{\lambda^\eta(a)},$$

provided that λ is nondecreasing.

Example 6. Let $\mathbb{T} = \mathbb{N}$, $a = 1, b = 6$, $p = 2$, $\eta = 1$ and $\psi(\tau) = 1$, so that $G(n) = \sum_{s=n}^{5} \psi(s) = 6 - n$. Then,

(a) for $\lambda(n) = 1/n$, it follows that

$$p \sum_{n=a}^{b-1} \frac{(G(n+1))^p}{\lambda^\eta(\tau)} = 2 \sum_{n=1}^{5} n(5-n)^2 = 100$$

and

$$(b-a)^p \sum_{n=a}^{b-1} \frac{\psi^p(n)}{\lambda^\eta(n)} - \sum_{n=a}^{b-1} \frac{\psi^p(n)(b-n)^p}{\lambda^\eta(n)} = \sum_{n=1}^{5} n \left[25 - (6-n)^2\right] = 270,$$

thus (3.33) is correct in this setting.

(b) Similarly, for $\lambda(n) = n$ we have that

$$p \sum_{n=a}^{b-1} \frac{(G(n+1))^p}{\lambda^\eta(n)} = 2 \sum_{n=1}^{5} \frac{(5-n)^2}{n} = 44.1666,$$

$$\sum_{n=a}^{b-1} \frac{\psi^p(n)(b-a)^p}{\lambda^\eta(a)} - \sum_{n=a}^{b-1} \frac{\psi^p(n)(b-n)^p}{\lambda^\eta(a)} = \sum_{n=1}^{5} \left[25 - (6-n)^2\right] = 70,$$

that is, the left-hand side is smaller than the right-hand side of the corresponding inequality.

Our next consequence of Theorem 3 refers to the time scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, and $a = q^m$, $b = q^n$.

Corollary 4. Let $p, q > 1$, $m, n \in \mathbb{N}_0$, $\eta > 0$ and let ψ, λ be positive sequences. Then,

$$p \sum_{k=m}^{n-1} \frac{q^k (q^n - q^{k+1})^p}{\lambda^\eta(q^k)} \phi \left(\frac{\sum_{s=k+1}^{n-1} (q-1) q^s \psi(q^s)}{q^n - q^{k+1}} \right)$$
$$\leq (q^n - q^m)^p \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))}{\lambda^\eta(q^k)} - \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))(q^n - q^k)^p}{\lambda^\eta(q^k)},$$

provided that λ is nonincreasing and

$$p \sum_{k=m}^{n-1} \frac{q^k (q^n - q^{k+1})^p}{\lambda^\eta(q^k)} \phi \left(\frac{\sum_{s=k+1}^{n-1} (q-1) q^s \psi(q^s)}{q^n - q^{k+1}} \right)$$
$$\leq (q^n - q^m)^p \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))}{\lambda^\eta(q^m)} - \sum_{k=m}^{n-1} \frac{q^k \phi(\psi(q^k))(q^n - q^k)^p}{\lambda^\eta(q^m)},$$

provided that λ is a nondecreasing.

Utilizing the same techniques as in the proofs of the previous theorems, we easily establish the corresponding time scale versions of relations (1.13) and (1.14).

Theorem 4. *Let $a,b\in\mathbb{T}$, $0<p<1$, $\eta>0$, $c,d\in\mathbb{R}$, and let $\phi\in C((c,d),\mathbb{R})$ be a concave function. If $\psi,\lambda\in C_{\mathrm{rd}}\left([a,b]_{\mathbb{T}},\mathbb{R}^{+}\right)$, then*

$$p\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b\psi(\xi)\,\Delta\xi}{b-\sigma(\tau)}\right)\Delta\tau \geq \frac{(b-a)^p\int_a^b\phi(\psi(\tau))\,\Delta\tau-\int_a^b\phi(\psi(\tau))(b-\tau)^p\Delta\tau}{\lambda^\eta(a)}, \tag{3.34}$$

provided that λ is nonincreasing and

$$p\int_a^b \frac{(b-\sigma(\tau))^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_{\sigma(\tau)}^b\psi(\xi)\,\Delta\xi}{b-\sigma(\tau)}\right)\Delta\tau \geq \int_a^b\frac{\phi(\psi(\tau))\left[(b-a)^p-(b-\tau)^p\right]}{\lambda^\eta(\tau)}\Delta\tau, \tag{3.35}$$

provided that λ is nondecreasing.

Remark 6. If $\mathbb{T}=\mathbb{R}$ and $\phi(y)=y^p$, then inequalities (3.34) and (3.35) reduce to relations (1.13) and (1.14), established in [7].

The following theorem covers the case $p<0$, which is a new result in both continuous and discrete calculus.

Theorem 5. *Let $a,b\in\mathbb{T}$, $\eta>0$, $p<0$, $c,d\in\mathbb{R}$, and let $\phi\in C((c,d),\mathbb{R})$ be a convex function. If $\psi,\lambda\in C_{\mathrm{rd}}\left([a,b]_{\mathbb{T}},\mathbb{R}^{+}\right)$, then*

$$p\int_a^b \frac{(\sigma(\tau)-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^{\sigma(\tau)}\psi(t)\Delta t}{\sigma(\tau)-a}\right)\Delta\tau \geq \int_a^b\frac{\phi(\psi(\tau))}{\lambda^\eta(b)}\left[(b-a)^p-(\tau-a)^p\right]\Delta\tau,$$

provided that λ is nonincreasing and

$$p\int_a^b \frac{(\sigma(\tau)-a)^p}{\lambda^\eta(\tau)}\phi\left(\frac{\int_a^{\sigma(\tau)}\psi(t)\Delta t}{\sigma(\tau)-a}\right)\Delta\tau \geq \int_a^b\frac{\phi(\psi(\tau))}{\lambda^\eta(\tau)}\left[(b-a)^p-(\tau-a)^p\right]\Delta\tau,$$

provided that λ is nondecreasing.

Remark 7. The techniques of proving Theorems 4 and 5 are similar to the proofs of Theorems 2 and 3. Hence, they are omitted here and left to the interested reader.

4 Conclusions and discussion

In this paper, we have established a class of generalized inequalities similar to Hardy's inequality involving a convex function on time scales. Our results are extensions of the corresponding classical real inequalities known from the literature. In particular, we have extended several integral relations established by Bendaoud, Senouci, Sroysang and Hasan [7, 13, 20]. Furthermore, we have established new dynamic inequalities with negative parameters on time scales. In addition, the corresponding discrete and quantum relations are essentially new. For the reader's convenience, some numerical examples are also discussed.

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