

A spectral operational matrix method using shifted fractional Legendre functions for nonlinear fractional Volterra integro-differential equations

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Abstract. This paper proposes an operational matrix method based on shifted fractional Legendre functions for solving nonlinear fractional Volterra integro-differential equations. The existence and uniqueness of the solution are established via the contraction mapping theorem. The method begins by constructing shifted Legendre functions of fractional order, followed by deriving a fractional integration operational matrix via the Laplace transform. These tools are then employed to convert the original nonlinear equation into an equivalent algebraic system, thereby enhancing computational efficiency. This formulation effectively addresses equations with weakly singular kernels and solutions exhibiting endpoint singularities or low regularity. The nonlinear term is handled directly through fixed-point iteration without requiring linearization or decomposition. An error analysis is conducted, and an upper bound in the L_w^2 -norm is derived. The results show that the error tends to zero as the number of basis functions increases, under appropriate smoothness conditions. Compared with traditional integer-order basis functions, fractional functions better accommodate the endpoint singularities and low-regularity features of the solution. The superior adaptability yields improved accuracy and more efficient convergence.

Keywords: fractional calculus; operational matrix method; fractional Legendre functions; Laplace transform applications; error analysis.

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1 Introduction

Fractional calculus extends classical integral calculus and has gained widespread attention for its ability to model complex systems with memory and hereditary effects. It has been widely applied to model various real-world phenomena, including soft-magnetic elastomer rheology, financial volatility, pollutant transport in hydrology, motion in heterogeneous media, and anomalous diffusion in physics [17].

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Fractional integro-differential equations (IDEs) form an important subclass of fractional models and provide a powerful tool for representing dynamic systems and natural physical processes with memory and nonlocal behavior. These equations have been widely applied in various fields, including viscoelasticity, heat transfer, and nonlinear dynamical systems [7, 9]. Fractional IDEs are particularly effective in capturing long-range dependencies and hereditary effects—phenomena that traditional differential equations are often inadequate to describe. As such, they play a crucial role in addressing challenges related to spatiotemporal dynamics, physical and biological systems, epidemic modeling, and boundary value problems.

As exact analytical solutions for fractional IDEs are rarely obtainable, numerical approximation methods have become an essential tool for analyzing their behavior. In recent years, a variety of computational techniques have been proposed. These methods aim to provide effective solutions to the challenges associated with this problem. Among the most prominent are spectral and semi-analytical techniques, such as the Chebyshev wavelet method [30] and the Adomian decomposition method [15]. Other notable approaches include collocation-based techniques [3], as well as related polynomial and Galerkin-type methods. Hybrid-function based schemes are also available; see [14].

Volterra integro-differential equations (VIDEs), which involve integral operators with time-dependent upper limits, play a fundamental role in modeling systems with memory and nonlocal effects. These equations appear in a wide array of mathematical and applied contexts, including computer algorithms, engineering analysis, physical modeling, and the theory of fractional calculus [12]. Their structure has also attracted attention in studies involving generating functions and combinatorial formulations arising in discrete and continuous models.

Numerical techniques for solving fractional VIDEs have received considerable attention, leading to the development of various schemes tailored to different equation types and kernel structures. Behiry and Mohamed [11] employed the differential transformation method to address high-order nonlinear equations with separable kernels, achieving highly accurate results. Ordokhani et al. [22] introduced a hybrid approach that integrates the tau method with Bernstein multi-scaling functions for solving Hammerstein-type VIDEs. To handle mixed boundary conditions, Zhou and Xu [29] applied a fourth-kind Chebyshev wavelet collocation method combined with Gaussian quadrature. Saemi et al. [23] transformed the original integral equations into solvable systems of equations using Müntz–Legendre wavelets, thereby enhancing computational efficiency. Ajileye et al. [4] developed a collocation framework specifically for fractional-order VIDEs based on the Caputo derivative. More recently, Alam et al. [5] proposed an iterative method for approximating fixed points in uniformly convex Banach spaces and demonstrated its applicability to solving such equations. These efforts collectively reflect the growing emphasis on designing efficient and accurate numerical schemes capable of addressing the challenges posed by weakly singular kernels and nonlinear dynamics inherent in fractional VIDEs.

This work investigates initial value problems (IVPs) for a class of nonlinear fractional VIDEs with weakly singular kernels, given by

$$\begin{cases} \mathcal{D}^{\mu_1} y(x) = f(x) + c(x)y(x) + \int_0^x (x-s)^{-\mu_2} K(x,s)F(y(s)) ds, \\ y^{(k)}(0) = y_k, \quad k = 0, 1, \dots, m-1, \end{cases} \quad (1.1)$$

where $m = \lceil \mu_1 \rceil$, $\mathcal{D}^{\mu_1}$ denotes the Caputo fractional derivative of order $\mu_1 > 0$, and $0 < \mu_2 < 1$. The solution $y(x)$ is defined on the interval $[0, M]$, with initial data $\{y_k\}_{k=0}^{m-1} \subset \mathbb{R}$. The functions $f(x)$, $c(x)$, and $K(x, s)$ are supposed to be sufficiently smooth over their respective domains. The nonlinear term $F(y(s))$ is assumed to satisfy a Lipschitz condition, ensuring well-posedness of the problem.

In recent years, considerable effort has been devoted to developing efficient spectral and algebraic techniques for solving fractional VIDEs. Zhao et al. [28] reformulated the equation as a second-kind VFIDE and applied a piecewise polynomial collocation method, establishing its optimal convergence rate. Nemati et al. [20] employed a spectral method based on second-kind Chebyshev polynomials, reducing the original problem to a system of linear algebraic equations through an equivalent Volterra formulation. Building upon this approach, Nemati and Lima [19] used a modified hat function basis in conjunction with an operational matrix technique to obtain a bivariate polynomial system. Shi [24] proposed a spectral collocation scheme using Jacobi polynomials, incorporating smoothing transformations to address kernel singularities. Behera et al. [10] proposed a Taylor wavelet-based numerical method for solving linear and nonlinear fractional VIDEs with weakly singular kernels. Chen et al. [13] introduced a spectral Galerkin method employing generalized Jacobi basis functions to approximate fractional Volterra equations with weakly singular kernels. In another direction, Afiatdoust et al. [2] developed a block-by-block extrapolation-based procedure, generalizing the classical technique of Young [27] for sequential solution of such equations. More recently, fractional-order hybrid and wavelet bases have been employed to treat similar classes of fractional integro-differential equations with weakly singular kernels and distributed-order derivatives, see for instance [1, 18, 25].

Despite the progress made, existing numerical methods still encounter several challenges when applied to fractional VIDEs, particularly those involving weakly singular kernels. The piecewise polynomial collocation method, for instance, is computationally intensive and requires intricate treatment of singular integrals. Spectral methods based on second-kind Chebyshev polynomials tend to suffer from stability issues and significant numerical errors in the presence of singular behavior. The modified hat basis offers limited accuracy and is not well-suited for high-precision applications, while Jacobi-based spectral methods, although theoretically sound, often exhibit high algorithmic complexity and limited flexibility. A fundamental limitation of most classical orthogonal polynomial approaches is their reliance on integer-order smoothness assumptions, which makes them inadequate for capturing endpoint singularities or weakly regular solutions. In contrast, fractional-order basis functions are nat-

urally aligned with the structure of fractional differential operators and provide improved approximation near singularities. Moreover, techniques based on Taylor wavelets may introduce additional approximation errors, whereas block-by-block methods are highly sensitive to parameter tuning and implementation details. These limitations underscore the need for a numerical scheme that is not only accurate and efficient, but also scalable and well-adapted to the intrinsic singularity structure of fractional VIDEs. To address these issues, the present study proposes a novel spectral framework based on fractional shifted Legendre functions, which effectively overcomes the aforementioned drawbacks.

The existence and uniqueness of the solution are established using the Banach fixed-point theorem. Next, a novel class of fractional shifted Legendre functions is constructed, and the Laplace transform is employed to derive an operational matrix for fractional integration in the spectral domain. This matrix facilitates the accurate and efficient treatment of weakly singular integral operators. The original model is approximated using the proposed basis functions. After substitution, this yields a system of algebraic equations. Solving for the expansion coefficients gives the approximate solution.

Unlike many existing methods that either linearize nonlinear terms or employ decomposition strategies often at the cost of accuracy or implementation complexity the proposed method retains the nonlinear term in its original form and solves the resulting system iteratively using a fixed-point approach. This direct treatment enhances both computational precision and structural simplicity.

The main contributions of this work are summarized below:

1. A rigorous existence and uniqueness result is established using the Banach fixed-point theorem.
2. A novel class of shifted fractional-order Legendre functions are constructed and analyzed, which are particularly effective in approximating solutions with endpoint singularities or low regularity.
3. A comprehensive error estimate is derived, accompanied by a convergence analysis in the weighted L_w^2 -norm, demonstrating the method's robustness even for weakly singular problems.
4. The nonlinear term is retained in its original form and incorporated directly into the spectral scheme via fixed-point iteration. This avoids linearization and decomposition-based approximations while maintaining accuracy for nonsmooth solutions.

The structure of the paper is as follows. Section 2 introduces essential concepts from fractional calculus and the necessary mathematical preliminaries. Section 3 provides a rigorous proof of existence and uniqueness for the fractional VIDE (1.1), supported by detailed theoretical analysis. Section 4 discusses the forms of fractional Legendre functions, while Section 5 describes the construction of their operational matrix. Section 6 demonstrates the application of the proposed method to the target equation and elaborates on each stage of the computational procedure. In Section 7, we focus on analyzing

the error, where a rigorous proof of the method's convergence is provided. In Section 8, the proposed method is validated through several numerical experiments that demonstrate its accuracy and effectiveness. Finally, the conclusion is presented in the last section.

2 Preliminaries

This section introduces essential definitions and mathematical tools from fractional calculus that will be used throughout the paper.

DEFINITION 1. [8] Let $\alpha > 0$ and $f \in L^1[a, b]$. The Riemann–Liouville fractional integral of order α is defined as

$$(\mathcal{I}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x \in (a, b],$$

where $\Gamma(\cdot)$ denotes the Gamma function.

DEFINITION 2. [8] Let $\alpha > 0$ and define $n = \lceil \alpha \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. Assume that $f \in C^n[a, b]$. The Caputo fractional derivative of order α is given by

$$\mathcal{D}^\alpha f(x) := \mathcal{I}^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt, \quad x \in [a, b],$$

where $f^{(n)}$ denotes the n -th derivative of f .

DEFINITION 3. Let $\alpha > 0$ be a constant, and define the weight function as $w(x) = \alpha x^{\alpha-1}$, where $x \in [0, M]$. The weighted L^2 space $L_w^2[0, M]$ is defined as

$$L_w^2[0, M] := \left\{ y : [0, M] \rightarrow \mathbb{R} \mid \int_0^M |y(x)|^2 w(x) dx < \infty \right\}.$$

The inner product is defined as

$$\langle u, v \rangle_w := \int_0^M u(x)v(x)w(x) dx = \int_0^M u(x)v(x)\alpha x^{\alpha-1} dx,$$

and the corresponding norm is

$$\|y\|_w := \left(\int_0^M |y(x)|^2 \alpha x^{\alpha-1} dx \right)^{1/2}.$$

Theorem 1. [21] Suppose that the function $g(x)$ satisfies $\mathcal{D}^{k\gamma} g(x) \in C[0, M]$ for all integers $k = 0, 1, \dots, n+1$, where the order satisfies $0 < \gamma \leq 1$. The function $g(x)$ can accordingly be represented through a generalized Taylor-type series expansion.

$$g(x) = \sum_{k=0}^n \frac{\mathcal{D}^{k\gamma} g(0^+)}{\Gamma(k\gamma + 1)} x^{k\gamma} + R_{n+1}(x),$$

where the remainder term is given by

$$R_{n+1}(x) = \frac{\mathcal{D}^{(n+1)\gamma} g(\xi)}{\Gamma((n+1)\gamma + 1)} x^{(n+1)\gamma}, \quad \text{for some } \xi \in (0, x].$$

Theorem 2. [6] Let $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ be vectors in an inner product space. Define the Gram matrix $\mathbf{G} = [\langle \varphi_i, \varphi_j \rangle]_{i,j=1}^n$. Then,

1. $\mathbf{G}$ is symmetric and positive semi-definite.
2. $\det(\mathbf{G}) = 0$ if and only if the vectors $\{\varphi_1, \dots, \varphi_n\}$ are linearly dependent.
3. If the vectors are linearly independent, then $\det(\mathbf{G}) > 0$.

3 Existence and uniqueness of the solution

By invoking the Banach fixed-point theorem [26], we establish that the fractional IVP (1.1) admits a unique continuous solution in an appropriate function space.

Theorem 3. Let $m = \lceil \mu_1 \rceil$ and consider the Banach space $(C[0, M], \|\cdot\|_\infty)$. Suppose that

$$\frac{(1 - \mu_2)l_1 M^{\mu_1} + l_2 k_1 M^{1-\mu_2+\mu_1}}{(1 - \mu_2)\Gamma(\mu_1 + 1)} < 1,$$

where $l_1 = \|c\|_\infty$, $l_2 = \sup_{(x,s) \in [0,M]^2} |K(x,s)|$, and k_1 is the Lipschitz constant of F . Then the problem (1.1) admits a unique solution $y \in C[0, M]$.

Proof. Let $m = \lceil \mu_1 \rceil$ and assume that $y^{(r)}(0) = y_r$ for $r = 0, 1, \dots, m-1$. By applying the fractional integral operator $\mathcal{I}^{\mu_1}$ to both sides of (1.1) and using the standard identity for the Caputo derivative (see, e.g., [8])

$$\mathcal{I}^{\mu_1} \mathcal{D}^{\mu_1} y(x) = y(x) - \sum_{r=0}^{m-1} \frac{y_r x^r}{r!},$$

the differential equation can be transformed into the equivalent integral formulation

$$\begin{aligned} y(x) &= \sum_{r=0}^{m-1} \frac{y_r x^r}{r!} + \mathcal{I}^{\mu_1} f(x) + \mathcal{I}^{\mu_1} (c(x)y(x)) \\ &\quad + \mathcal{I}^{\mu_1} \left(\int_0^x (x-s)^{-\mu_2} K(x,s) F(y(s)) ds \right). \end{aligned}$$

Define the operator $\mathcal{T} : C[0, M] \rightarrow C[0, M]$ as

$$\begin{aligned} (\mathcal{T}y)(x) &:= \sum_{r=0}^{m-1} \frac{y_r x^r}{r!} + \frac{1}{\Gamma(\mu_1)} \int_0^x (x-t)^{\mu_1-1} [f(t) + c(t)y(t)] dt \\ &\quad + \frac{1}{\Gamma(\mu_1)} \int_0^x (x-t)^{\mu_1-1} \left(\int_0^t (t-s)^{-\mu_2} K(t,s) F(y(s)) ds \right) dt. \end{aligned}$$

To show that $\mathcal{T}$ is a contraction, let $y_1, y_2 \in C[0, M]$. Then, for all $x \in [0, M]$,

$$\begin{aligned} |(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x)| &\leq \frac{1}{\Gamma(\mu_1)} \int_0^x (x-t)^{\mu_1-1} |c(t)| |y_1(t) - y_2(t)| dt \\ &+ \frac{1}{\Gamma(\mu_1)} \int_0^x (x-t)^{\mu_1-1} \left(\int_0^t (t-s)^{-\mu_2} |K(t, s)| |F(y_1(s)) - F(y_2(s))| ds \right) dt \\ &\leq \frac{1}{\Gamma(\mu_1)} \int_0^x (x-t)^{\mu_1-1} \left[l_1 \|y_1 - y_2\|_\infty + l_2 k_1 \|y_1 - y_2\|_\infty \int_0^t (t-s)^{-\mu_2} ds \right] dt \\ &= \frac{1}{\Gamma(\mu_1)} \left(l_1 + \frac{l_2 k_1}{1 - \mu_2} M^{1-\mu_2} \right) \|y_1 - y_2\|_\infty \int_0^x (x-t)^{\mu_1-1} dt \\ &= \frac{(1 - \mu_2) l_1 M^{\mu_1} + l_2 k_1 M^{1-\mu_2+\mu_1}}{(1 - \mu_2) \Gamma(\mu_1 + 1)} \|y_1 - y_2\|_\infty. \end{aligned}$$

By the assumption of the theorem, the constant in front of $\|y_1 - y_2\|_\infty$ is strictly less than 1. Hence, $\mathcal{T}$ is a strict contraction on the Banach space $C[0, M]$. By the Banach fixed-point theorem, there exists a unique function $y \in C[0, M]$ such that $\mathcal{T}y = y$, which solves the original problem (1.1). $\square$

Remark 1. The above inequality is a conservative global sufficient condition for contraction; it is not necessary, and specific examples may fall outside this bound while still admitting a unique solution and a convergent fixed-point iteration under sharper local estimates.

4 The shifted Legendre functions of fractional order

Fractional-order Legendre functions offer greater flexibility than their classical integer-order counterparts, enabling more accurate approximation of functions with weak regularity or endpoint singularities. This section constructs a family of shifted fractional Legendre functions tailored for approximating functions defined on the interval $[0, M]$, particularly within the weighted Hilbert space $L_w^2[0, M]$, where the weight is $w(x) = \alpha x^{\alpha-1}$, with $0 < \alpha \leq 1$.

We begin by shifting the standard Legendre polynomials $\{G_n(t)\}$, originally defined on $[-1, 1]$, to the interval $[0, M^\alpha]$ via the linear transformation

$$t = \frac{2q - M^\alpha}{M^\alpha}, \quad \text{with } q \in [0, M^\alpha], \quad t \in [-1, 1].$$

This yields the shifted Legendre polynomials

$$\tilde{G}_n(q) := \sqrt{\frac{2n+1}{M^\alpha}} G_n\left(\frac{2q - M^\alpha}{M^\alpha}\right),$$

which form an orthonormal system in $L^2[0, M^\alpha]$ with respect to the standard Lebesgue measure.

To define the fractional-order polynomials on $[0, M]$, we introduce the variable substitution $q = x^\alpha$, leading to

$$\varphi_n(x) := \tilde{G}_n(x^\alpha), \quad x \in [0, M].$$

The corresponding weight function becomes $w(x) = \alpha x^{\alpha-1}$, so the inner product in $L_w^2[0, M]$ is defined as

$$\langle u, v \rangle_w := \int_0^M u(x)v(x)w(x) dx = \int_0^M u(x)v(x)\alpha x^{\alpha-1} dx.$$

It follows from the change-of-variable argument that

$$\langle \varphi_n, \varphi_m \rangle_w = \delta_{nm},$$

i.e., the system $\{\varphi_n(x)\}_{n=0}^\infty$ is orthonormal in $L_w^2[0, M]$.

The first few fractional shifted Legendre functions are given by

$$\begin{aligned}\varphi_0(x) &= \sqrt{\frac{1}{M^\alpha}}, \quad \varphi_1(x) = \sqrt{3} \left(\frac{1}{M^\alpha} \right)^{3/2} (2x^\alpha - M^\alpha), \\ \varphi_2(x) &= \frac{1}{2} \sqrt{5} \sqrt{\frac{1}{M^\alpha}} \left(\frac{3(2x^\alpha - M^\alpha)^2}{(M^\alpha)^2} - 1 \right), \\ \varphi_3(x) &= \frac{1}{3} \sqrt{7} \sqrt{\frac{1}{M^\alpha}} \left(\frac{5(2x^\alpha - M^\alpha)}{2M^\alpha} \left(\frac{3(2x^\alpha - M^\alpha)^2}{(M^\alpha)^2} - 1 \right) - \frac{2(2x^\alpha - M^\alpha)}{M^\alpha} \right).\end{aligned}$$

We now establish the completeness of $\{\varphi_n(x)\}$ in $L_w^2[0, M]$.

Proposition 1. *Let $\varphi_n(x) := \tilde{G}_n(x^\alpha)$, where $\tilde{G}_n(q)$ denotes the shifted Legendre polynomials on $[0, M^\alpha]$, and let the weight function be $w(x) = \alpha x^{\alpha-1}$, with $0 < \alpha \leq 1$. Then the system $\{\varphi_n(x)\}_{n=0}^\infty$ forms a complete orthonormal basis in the weighted space $L_w^2[0, M]$.*

Proof. It is well known that the Legendre polynomials $\{G_n(t)\}$ form a complete orthonormal system in $L^2[-1, 1]$. After applying a linear transformation to shift the domain to $[0, M^\alpha]$, the resulting polynomials

$$\tilde{G}_n(q) := \sqrt{\frac{2n+1}{M^\alpha}} G_n\left(\frac{2q - M^\alpha}{M^\alpha}\right)$$

form a complete orthonormal system in $L^2[0, M^\alpha]$.

Now, consider the change of variable $q = x^\alpha$, which is a bijection from $x \in [0, M]$ to $q \in [0, M^\alpha]$, with differential $dq = \alpha x^{\alpha-1} dx$. Then, for any $f \in L_w^2[0, M]$, where $w(x) = \alpha x^{\alpha-1}$, the weighted norm satisfies

$$\|f\|_w^2 = \int_0^M |f(x)|^2 \alpha x^{\alpha-1} dx = \int_0^{M^\alpha} |f(q^{1/\alpha})|^2 dq.$$

This identity defines a linear isometric isomorphism between the weighted space $L_w^2[0, M]$ and the standard space $L^2[0, M^\alpha]$. As such, it preserves norms, inner products, orthogonality, and completeness of function systems under the transformation $f(x) \mapsto f(x^\alpha)$. Therefore, the completeness of the system $\{\tilde{G}_n(q)\}$ in $L^2[0, M^\alpha]$ implies that the set $\{\varphi_n(x) = \tilde{G}_n(x^\alpha)\}$ forms a complete orthonormal system in $L_w^2[0, M]$. $\square$

5 Fractional Legendre functions and the construction of operational matrices for fractional integration

Let $h(x)$ be a function defined on the interval $[0, M]$. It can be approximated by a finite series expansion in terms of shifted fractional Legendre functions as follows

$$h(x) \approx \sum_{i=0}^n a_i \varphi_i(x) = \mathbf{C}^T \mathbf{\Phi}(x), \quad (5.1)$$

where $\mathbf{C} = [a_0, a_1, \dots, a_n]^T \in \mathbb{R}^{n+1}$ is the coefficient vector, and $\mathbf{\Phi}(x) = [\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)]^T$ is the basis vector of fractional shifted Legendre functions. Due to orthonormality in the weighted space $L_w^2([0, M])$, the coefficients can be computed as

$$\langle h(x), \varphi_j(x) \rangle_w = \sum_{i=0}^n a_i \langle \varphi_i(x), \varphi_j(x) \rangle_w = a_j, \quad j = 0, 1, \dots, n. \quad (5.2)$$

This establishes the framework for representing general functions in terms of fractional Legendre bases. We now proceed to construct an operational matrix for the fractional integral operator $\mathcal{I}^{\mu_1}$ applied to the basis functions $\{\varphi_i(x)\}$. According to the definition of the Riemann–Liouville fractional integral, the following identity holds

$$\mathcal{I}^{\mu_1} \varphi_i(x) = \frac{1}{\Gamma(\mu_1)} \int_0^x (x-t)^{\mu_1-1} \varphi_i(t) dt = \left(\frac{x^{\mu_1-1}}{\Gamma(\mu_1)} \right) * \varphi_i(x), \quad (5.3)$$

where “ $*$ ” denotes the convolution operator over $[0, x]$, defined as

$$(f * g)(x) := \int_0^x f(x-t)g(t) dt.$$

Taking the Laplace transform of both sides of Equation (5.3) and utilizing the convolution theorem, we obtain

$$\mathcal{L} [\mathcal{I}^{\mu_1} \varphi_i(x)](s) = \mathcal{L} \left[\frac{x^{\mu_1-1}}{\Gamma(\mu_1)} \right](s) \cdot \mathcal{L}[\varphi_i(x)](s) = s^{-\mu_1} \cdot \mathcal{L}[\varphi_i(x)](s). \quad (5.4)$$

Denoting the right-hand side as $R_i(s, \alpha)$, the inverse Laplace transform yields the fractional integral of φ_i

$$\mathcal{I}^{\mu_1} \varphi_i(x) = \mathcal{L}^{-1} [R_i(s, \alpha)] = \tilde{R}_i(x, \alpha).$$

To approximate $\tilde{R}_i(x, \alpha)$ in the weighted space $L_w^2([0, M])$, we project it onto the same fractional Legendre basis

$$\tilde{R}_i(x, \alpha) \approx \sum_{j=0}^n c_{ij} \varphi_j(x), \quad i = 0, 1, \dots, n,$$

where $c_{ij} = \langle \tilde{R}_i(x, \alpha), \varphi_j(x) \rangle_w$ for $i, j = 0, 1, \dots, n$. Collecting all coefficients gives the operational matrix of the fractional integral operator

$$\mathbf{A}_{\mu_1, \alpha} = [c_{ij}] \in \mathbb{R}^{(n+1) \times (n+1)},$$

which satisfies the matrix approximation

$$\mathcal{I}^{\mu_1} \Phi(x) \approx \mathbf{A}_{\mu_1, \alpha} \Phi(x).$$

This matrix allows for efficient spectral approximation of fractional integral terms in the model equations using the fractional Legendre basis.

6 Solution of the nonlinear fractional VIDEs

In this section, we focus on the numerical solution of the nonlinear fractional IVP presented in (1.1). In order to accomplish this, we utilize the fractional integration operational matrices obtained in the preceding section. To solve the given IVP, we use fractional Legendre functions to approximate $\mathcal{D}^{\mu_1} y(x)$, with the approximation expressed as

$$\mathcal{D}^{\mu_1} y(x) \approx \mathbf{Q}^T \Phi(x), \quad (6.1)$$

which implies

$$\mathcal{I}^{\mu_1} \mathcal{D}^{\mu_1} y(x) \approx \mathbf{Q}^T \mathcal{I}^{\mu_1} \Phi(x) \approx \mathbf{Q}^T \mathbf{A}_{\mu_1, \alpha} \Phi(x). \quad (6.2)$$

Let $m = \lceil \mu_1 \rceil$. Using the definition of the Caputo derivative of order $\mu_1 > 0$ with initial conditions $y^{(r)}(0) = y_r$ for $r = 0, 1, \dots, m-1$ [8], Equation (6.2) can be rewritten as

$$y_n(x) := \mathbf{Q}^T \mathbf{A}_{\mu_1, \alpha} \Phi(x) + \sum_{r=0}^{m-1} \frac{y_r x^r}{r!}, \quad (6.3)$$

where $y_n(x)$ denotes the approximate solution to problem (1.1). The Equation (6.3) ensures that the approximate solution satisfies the boundary conditions and no additional processing is required. Furthermore, the functions $f(x)$, $c(x)y_n(x)$, and the nonlinear integral term $\int_0^x (x-s)^{-\mu_2} K(x, s) F(y_n(s)) ds$ are approximated similarly using fractional Legendre functions

$$\begin{aligned} f(x) &\approx \mathbf{E}^T \Phi(x), \quad c(x)y_n(x) \approx \mathbf{N}(\mathbf{Q})^T \Phi(x), \\ \int_0^x (x-s)^{-\mu_2} K(x, s) F(y_n(s)) ds &\approx \mathbf{W}(\mathbf{Q})^T \Phi(x), \end{aligned} \quad (6.4)$$

where $\mathbf{E}, \mathbf{N}, \mathbf{W} \in \mathbb{R}^{n+1}$, $\Phi(x) = [\varphi_0(x), \varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)]^T$ denotes the vector of fractional Legendre basis functions, and F is a given nonlinear function.

Finally, by substituting the approximations of the functions into equation (1.1)

and expanding everything in the orthonormal basis $\{\varphi_j\}_{j=0}^n$. The resulting system is as follows:

$$\mathbf{Q}^T \Phi(x) = \mathbf{E}^T \Phi(x) + \mathbf{N}(\mathbf{Q})^T \Phi(x) + \mathbf{W}(\mathbf{Q})^T \Phi(x). \quad (6.5)$$

Here $\mathbf{E}$ collects the coefficients of the projection of $f(x)$ onto the fractional Legendre basis, so it is independent of $\mathbf{Q}$. The vector $\mathbf{N}(\mathbf{Q})$ comes from the projection of $c(x)y_n(x)$, and $\mathbf{W}(\mathbf{Q})$ from the projection of the Volterra term with $F(y_n(s))$; both depend on $\mathbf{Q}$ through the approximate solution $y_n(x)$. Since $\{\varphi_j\}_{j=0}^n$ is orthonormal in $L_w^2[0, M]$, equating the coefficients of each φ_j in (6.5) is equivalent to the nonlinear algebraic system

$$\mathbf{Q} = \mathbf{E} + \mathbf{N}(\mathbf{Q}) + \mathbf{W}(\mathbf{Q}).$$

The unknown components of the vector $\mathbf{Q}$ can be determined by solving the nonlinear system of equations represented by (6.5). To solve for $\mathbf{Q}$, we employ a fixed-point iteration scheme. Starting from an initial guess $\mathbf{Q}^{(0)}$, we update

$$\mathbf{Q}^{(k+1)} = \mathbf{E} + \mathbf{N}(\mathbf{Q}^{(k)}) + \mathbf{W}(\mathbf{Q}^{(k)}),$$

and stop when, for example,

$$\|\mathbf{Q}^{(k+1)} - \mathbf{Q}^{(k)}\|_2 < \varepsilon.$$

This iteration can be written as $\mathbf{Q}^{(k+1)} = \mathcal{T}_n(\mathbf{Q}^{(k)})$, with $\mathcal{T}_n(\mathbf{Q}) := \mathbf{E} + \mathbf{N}(\mathbf{Q}) + \mathbf{W}(\mathbf{Q})$, and is the discrete analogue of the fixed-point map used in the existence and uniqueness analysis in Section 3. Under the same Lipschitz assumptions on F and for sufficiently small M , $\mathcal{T}_n$ is a contraction on a suitable ball in $\mathbb{R}^{n+1}$, so that $\mathbf{Q}^{(k)} \rightarrow \mathbf{Q}$ as $k \rightarrow \infty$, where $\mathbf{Q}$ is the unique solution of (6.5). Once convergence is reached, the limit vector $\mathbf{Q}$ determines the approximate solution to the nonlinear fractional IVP.

7 Convergence analysis and error estimation

This section focuses on the theoretical verification of the proposed operational matrix method, which is employed to solve the fractional IDE (1.1) and to derive error estimates in the L_w^2 -norm. The function $h(x)$, assumed to be in $L_w^2[0, M]$, is approximated using fractional Legendre functions. Define the pointwise error function as

$$e_n(x) := h(x) - h_n(x), \quad x \in [0, M].$$

The following theorem provides a rigorous proof that the error sequence $e_n(x)$ converges to zero in the weighted L_w^2 -norm as $n \rightarrow \infty$. In other words,

$$\lim_{n \rightarrow \infty} \|e_n\|_w = \lim_{n \rightarrow \infty} \|h - h_n\|_w = \lim \left(\int_0^M (h(x) - h_n(x))^2 \alpha x^{\alpha-1} dx \right)^{1/2} = 0.$$

Theorem 4. Let $\mathcal{D}^{k\alpha}h(x) \in C[0, M]$ for $k = 0, 1, \dots, n+1$, where $0 < \alpha \leq 1$, and let $\mathbb{Y} = \text{span}\{\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)\}$. For any $h \in L_w^2[0, M]$, let $h_n(x)$ denote its best approximation within $\mathbb{Y}$. Then, the following error estimate in the L_w^2 -norm holds

$$\|h - h_n\|_w \leq \frac{M^{\frac{(2n+3)\alpha}{2}} M_\alpha}{\Gamma((n+1)\alpha + 1) \sqrt{(2n+3)}}, \quad (7.1)$$

where $M_\alpha = \sup_{x \in [0, M]} |\mathcal{D}^{(n+1)\alpha}h(x)|$.

Proof. Using the generalized Taylor series expansion (Theorem 1), the function $h(x)$ can be written as

$$h(x) = \sum_{k=0}^n \frac{\mathcal{D}^{k\alpha}h(0^+)}{\Gamma(k\alpha + 1)} x^{k\alpha} + R_{n+1}(x),$$

where the remainder term satisfies $R_{n+1}(x) = \frac{\mathcal{D}^{(n+1)\alpha}h(\xi)}{\Gamma((n+1)\alpha + 1)} x^{(n+1)\alpha}$, $\xi \in (0, x)$.

Let $\hat{h}(x) := \sum_{k=0}^n \frac{\mathcal{D}^{k\alpha}h(0^+)}{\Gamma(k\alpha + 1)} x^{k\alpha}$ be the fractional Taylor polynomial of degree n . Then the approximation error satisfies

$$|h(x) - \hat{h}(x)| \leq \frac{x^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} \sup_{\xi \in [0, M]} |\mathcal{D}^{(n+1)\alpha}h(\xi)|.$$

Given that $h_n(x)$ represents the optimal approximation of $h(x)$ within the space $\mathbb{Y}$, we can consequently derive the following inequality for the error

$$\begin{aligned} \|h - h_n\|_w^2 &\leq \|h - \hat{h}\|_w^2 = \int_0^M |h(x) - \hat{h}(x)|^2 \alpha x^{\alpha-1} dx \\ &\leq \int_0^M \frac{x^{(2n+2)\alpha}}{\Gamma((n+1)\alpha + 1)^2} M_\alpha^2 \cdot \alpha x^{\alpha-1} dx \\ &= \frac{M_\alpha^2 \alpha}{\Gamma((n+1)\alpha + 1)^2} \int_0^M x^{(2n+3)\alpha-1} dx = \frac{M^{(2n+3)\alpha} M_\alpha^2}{\Gamma((n+1)\alpha + 1)^2 \cdot (2n+3)}. \end{aligned}$$

Thus, we obtain the following bound:

$$\|h - h_n\|_w \leq \frac{M^{\frac{(2n+3)\alpha}{2}} M_\alpha}{\Gamma((n+1)\alpha + 1) \sqrt{(2n+3)}}.$$

□

At this stage, we establish a theoretical estimate for the error vector arising from the use of the fractional integration operational matrix $\mathbf{A}_{\mu_1, \alpha}$. Specifically, we show that this error becomes arbitrarily small as the number of basis functions increases. The convergence of the operational matrix approximation is rigorously demonstrated in the theorem that follows.

Theorem 5. Let $\mathbb{H}$ be a Hilbert space, and let $\mathbb{Y} \subset \mathbb{H}$ be a finite-dimensional closed subspace with basis $\{y_1, y_2, \dots, y_n\}$. For any given element $z \in \mathbb{H}$, denote by $y_0 \in \mathbb{Y}$ the unique element that minimizes the norm $\|z - y\|$ over all $y \in \mathbb{Y}$; that is, y_0 is the best approximation of z in $\mathbb{Y}$. Then, we have

$$\|z - y_0\|_{\mathbb{H}}^2 = \frac{\text{Gram}(z, y_1, y_2, y_3, \dots, y_n)}{\text{Gram}(y_1, y_2, y_3, \dots, y_n)}, \quad (7.2)$$

where

$$\text{Gram}(y_1, y_2, y_3, \dots, y_n) = \det[\langle y_i, y_j \rangle_{\mathbb{H}}]_{i,j=1}^n. \quad (7.3)$$

Proof. For the proof, refer to [16]. $\square$

Theorem 6. Let the error vector $\mathbf{E}_{\mu_1, \alpha}$ associated with the fractional integration operational matrix $\mathbf{A}_{\mu_1, \alpha}$ be defined by

$$\mathbf{E}_{\mu_1, \alpha}(x) = \mathcal{I}^{\mu_1} \Phi(x) - \mathbf{A}_{\mu_1, \alpha} \Phi(x),$$

where $\mathbf{E}_{\mu_1, \alpha} = (e_{\mu_1, \alpha}^0, e_{\mu_1, \alpha}^1, \dots, e_{\mu_1, \alpha}^n)^T$. Then, for each $i = 0, 1, \dots, n$, the component $e_{\mu_1, \alpha}^i$ satisfies

$$\|e_{\mu_1, \alpha}^i\|_w \leq \left(\text{Gram}(\tilde{R}_i(x, \alpha), \varphi_0, \dots, \varphi_n) / \text{Gram}(\varphi_0, \dots, \varphi_n) \right)^{1/2},$$

where $\tilde{R}_i(x, \alpha) := \mathcal{I}^{\mu_1} \varphi_i(x)$ and $\text{Gram}(\cdot)$ denotes the Gram determinant with respect to the L_w^2 inner product.

Proof. Fix $i \in \{0, 1, \dots, n\}$ and consider the Hilbert space $\mathbb{H} = L_w^2[0, M]$ with inner product $\langle \cdot, \cdot \rangle_w$. Let $\mathbb{Y}_n := \text{span}\{\varphi_0, \varphi_1, \dots, \varphi_n\} \subset \mathbb{H}$. By construction of the operational matrix, for each i we approximate $\tilde{R}_i(x, \alpha) = \mathcal{I}^{\mu_1} \varphi_i(x)$ in $\mathbb{Y}_n$ by

$$\tilde{R}_{i,n}(x, \alpha) := \sum_{j=0}^n c_{ij} \varphi_j(x), \quad c_{ij} = \langle \tilde{R}_i, \varphi_j \rangle_w.$$

Since $\{\varphi_0, \dots, \varphi_n\}$ is an orthonormal system in $L_w^2[0, M]$, the coefficients c_{ij} are exactly those of the orthogonal projection of $\tilde{R}_i$ onto $\mathbb{Y}_n$. Hence $\tilde{R}_{i,n}$ is the unique best approximation of $\tilde{R}_i$ in $\mathbb{Y}_n$ in the sense of the L_w^2 norm. By the definition of $\mathbf{E}_{\mu_1, \alpha}$, the i th component is

$$e_{\mu_1, \alpha}^i(x) = \mathcal{I}^{\mu_1} \varphi_i(x) - \sum_{j=0}^n c_{ij} \varphi_j(x) = \tilde{R}_i(x, \alpha) - \tilde{R}_{i,n}(x, \alpha),$$

so $e_{\mu_1, \alpha}^i$ is precisely the best-approximation error $\tilde{R}_i - y_0$ in the notation of Theorem 5, with $z = \tilde{R}_i$ and $\mathbb{Y} = \mathbb{Y}_n$. Therefore, applying Theorem 5 gives

$$\|e_{\mu_1, \alpha}^i\|_w^2 = \text{Gram}(\tilde{R}_i, \varphi_0, \dots, \varphi_n) / \text{Gram}(\varphi_0, \dots, \varphi_n),$$

and hence

$$\|e_{\mu_1, \alpha}^i\|_w = \left(\text{Gram}(\tilde{R}_i, \varphi_0, \dots, \varphi_n) / \text{Gram}(\varphi_0, \dots, \varphi_n) \right)^{1/2}.$$

In particular, the stated inequality holds. $\square$

This bound implies that the accuracy of the operational matrix approximation improves as the number of basis functions increases, leading to convergence of the error vector $\mathbf{E}_{\mu_1, \alpha}$ to zero in the L_w^2 -norm.

Theorem 7. Let $y(x)$ be the exact solution to the fractional IVP (1.1) and set $g(x) := \mathcal{D}^{\mu_1} y(x)$. Let g_n be its approximation in the fractional Legendre basis and define

$$\tilde{y}_n(x) := \mathcal{I}^{\mu_1} g_n(x) + \sum_{r=0}^{\lceil \mu_1 \rceil - 1} \frac{y^{(r)}(0)}{r!} x^r.$$

Then there exists a constant $C > 0$, independent of n , such that

$$\|y - \tilde{y}_n\|_w \leq C \|g - g_n\|_w. \quad (7.4)$$

In particular, by the approximation estimate in Section 7, $\|y - \tilde{y}_n\|_w \rightarrow 0$ as $n \rightarrow \infty$.

Proof. By the definition of the Caputo derivative and the identity $\mathcal{I}^{\mu_1} \mathcal{D}^{\mu_1} y(x) = y(x) - \sum_{r=0}^{\lceil \mu_1 \rceil - 1} y^{(r)}(0) x^r / r!$, we have

$$y(x) = \mathcal{I}^{\mu_1} g(x) + \sum_{r=0}^{\lceil \mu_1 \rceil - 1} \frac{y^{(r)}(0)}{r!} x^r.$$

Hence

$$y(x) - \tilde{y}_n(x) = \mathcal{I}^{\mu_1} (g(x) - g_n(x)),$$

and therefore

$$\|y - \tilde{y}_n\|_w = \|\mathcal{I}^{\mu_1} (g - g_n)\|_w \leq C \|g - g_n\|_w,$$

where we used the boundedness of $\mathcal{I}^{\mu_1}$ on $L_w^2[0, M]$ (see, e.g., [8]). This proves the claim. $\square$

Theorem 8. Let $y(x)$ be the exact solution to the fractional IVP (1.1), and let $y_n(x)$ be the numerical approximation obtained by the proposed operational matrix method. Then the following estimate holds in the weighted norm $\|\cdot\|_w$:

$$\|y - y_n\|_w \leq \|y - \tilde{y}_n\|_w + \|\mathbf{Q}^T\|_2 \|\mathbf{E}_{\mu_1, \alpha}\|_w, \quad (7.5)$$

where $\tilde{y}_n$ is defined as in Theorem 7 and $\mathbf{E}_{\mu_1, \alpha}$ is the error vector associated with the operational matrix $\mathbf{A}_{\mu_1, \alpha}$.

Proof. From (6.2)–(6.3) we have

$$y_n(x) = \mathbf{Q}^T \mathbf{A}_{\mu_1, \alpha} \Phi(x) + \sum_{r=0}^{\lceil \mu_1 \rceil - 1} \frac{y^{(r)}(0)}{r!} x^r,$$

whereas the semi-discrete approximation based on the exact integral operator is

$$\tilde{y}_n(x) = \mathbf{Q}^T \mathcal{I}^{\mu_1} \Phi(x) + \sum_{r=0}^{\lceil \mu_1 \rceil - 1} \frac{y^{(r)}(0)}{r!} x^r.$$

Thus

$$\tilde{y}_n(x) - y_n(x) = \mathbf{Q}^T (\mathcal{I}^{\mu_1} \Phi(x) - \mathbf{A}_{\mu_1, \alpha} \Phi(x)) = \mathbf{Q}^T \mathbf{E}_{\mu_1, \alpha}(x),$$

and the total error can be decomposed as

$$y(x) - y_n(x) = (y(x) - \tilde{y}_n(x)) + (\tilde{y}_n(x) - y_n(x)).$$

Taking the L_w^2 -norm and using the triangle inequality yields

$$\|y - y_n\|_w \leq \|y - \tilde{y}_n\|_w + \|\mathbf{Q}^T \mathbf{E}_{\mu_1, \alpha}(x)\|_w \leq \|y - \tilde{y}_n\|_w + \|\mathbf{Q}^T\|_2 \|\mathbf{E}_{\mu_1, \alpha}\|_w,$$

which is the desired estimate. Combining Theorem 7 with Theorem 6 then shows that $\|y - y_n\|_w \rightarrow 0$ as $n \rightarrow \infty$. $\square$

8 Numerical examples

To demonstrate the effectiveness of the proposed method, we apply it to four test problems using Mathematica 14.0. For each example, we compare the maximum numerical error with results obtained from other established methods. We define the pointwise maximum absolute error as

$$e_n := \max_{x \in [0, M]} |y(x) - y_n(x)|.$$

Example 1. Consider the following equation [2, 10, 19, 20, 28]

$$\mathcal{D}^{\frac{1}{3}} y(x) = f(x) + c(x)y(x) + \int_0^x (x-s)^{-\frac{1}{2}} y(s) ds, \quad x \in [0, 1], \quad (8.1)$$

where $f(x) = \frac{6x^{8/3}}{\Gamma(11/3)} + \left(\frac{32}{35} - \frac{\Gamma(1/2)\Gamma(7/3)}{\Gamma(17/6)}\right)x^{11/6} + \Gamma\left(\frac{7}{3}\right)x$, $c(x) = -\frac{32}{35}x^{\frac{1}{2}}$ with initial value $y(0) = 0$. The exact solution to this problem is given by $y(x) = x^3 + x^{4/3}$.

Table 1. Comparison of maximum errors for different methods for Example 1.

n	e_n of [19]	e_n of [20]	e_n of our method ($\alpha = \frac{2}{3}$)
4	1.14967e-03	5.13797e-04	4.66462e-03
8	3.15569e-04	6.46120e-05	4.41544e-07
16	8.90424e-05	—	5.08585e-10
32	1.93665e-05	—	7.35725e-11
1024	1.07676e-08	—	—

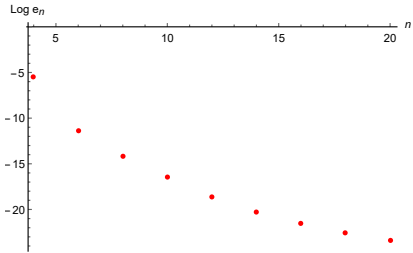


Figure 1. Maximum absolute error on a logarithmic scale for Example 1.

Table 2. Comparison of maximum errors for different methods for Example 1.

	[28] ($r = 5, s = 3$)	[10] ($h = \frac{1}{128}$)	our meth. ($\alpha=\frac{2}{3}, n=20$)
e_n	1.21136e-04	2.02310e-04	7.35725e-11

Table 3. Error comparison at different points under different methods for Example 1.

x	Method of [19]		Method of [10]		Method of [2]		our met. ($\alpha=\frac{2}{3}$)	
	$n = 8$	$n = 16$	$n = 8$	$n = 16$	$n = 8$	$n = 16$	$n = 8$	$n = 16$
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.25	5.70e-4	1.84e-4	1.67e-4	1.15e-5	8.05e-6	9.04e-7	1.25e-7	1.26e-11
0.50	8.57e-4	2.20e-4	1.73e-4	1.62e-5	3.33e-6	4.73e-7	1.22e-7	1.63e-10
0.75	1.43e-3	3.68e-4	2.09e-4	2.54e-5	1.76e-6	2.61e-6	9.41e-8	2.70e-11
1.00	3.31e-3	8.85e-4	6.46e-4	6.84e-5	1.66e-5	1.09e-5	1.69e-7	5.82e-11

For Example 1, Tables 1 and 2 compare the maximum errors of different methods. In Table 1, as n increases, our method’s error decreases rapidly and remains much smaller than the other two methods for multiple values of n . Table 2 shows our method has the smallest error, reaffirming its accuracy advantage. Table 3 presents error values at different x -points for the comparison of methods in references [2, 10, 19] and ours when $n = 8$ and $n = 16$. Figure 1 shows the plot of the maximum absolute error on a logarithmic scale.

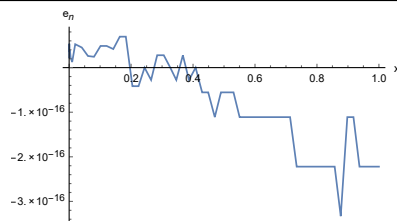
Example 2. Consider the following equation [10, 19]

$$\mathcal{D}^{\frac{3}{2}}y(x) = f(x) + c(x)y(x) + \int_0^x (x - s)^{-\frac{1}{2}}y^2(s)ds, \quad x \in [0, 1], \tag{8.2}$$

where $f(x) = \frac{3\Gamma(\frac{1}{2})}{4\Gamma(\frac{11}{6})}x^{\frac{5}{6}} - x^{\frac{1}{2}} - \frac{32}{35}x^{\frac{7}{2}}$, $c(x) = x$ with initial value $y(0) = 0, y'(0) = 0$. The exact solution to this problem is given by $y(x) = x^{\frac{3}{2}}$. Since the order of the derivative here is greater than 1, we use a polynomial to represent the first derivative of $y(x)$, after solving the integral to obtain an approximate solution.

Table 4. Comparison of maximum errors for different methods for Example 2.

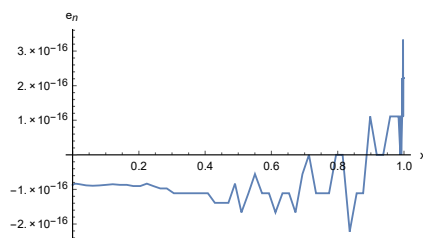
n	e_n of [19]	e_n of our method ($\alpha = \frac{1}{2}$)
4	1.39991e-02	1.86533e-11
8	3.30803e-03	1.11022e-16
16	8.84780e-04	—
32	2.46509e-04	—
1024	1.07144e-06	—

**Figure 2.** Maximum absolute error for $n = 8$ in Example 2 with $\alpha = \frac{1}{2}$.

Example 3. Consider the following equation [2]

$$\mathcal{D}^{\frac{1}{3}}y(x) = f(x) + c(x)y(x) + \frac{1}{\Gamma(\frac{2}{3})} \int_0^x (x-s)^{-\frac{1}{3}} y^2(s) ds, \quad x \in [0, 1], \quad (8.3)$$

where $f(x) = \frac{2}{\Gamma(\frac{8}{3})}x^{\frac{5}{3}} - \frac{24}{\Gamma(\frac{17}{3})}x^{\frac{14}{3}}$, $c(x) = 0$ with initial value $y(0) = 0$. The exact solution to this problem is given by $y(x) = x^2$.

**Figure 3.** Maximum absolute error for $n = 8$ in Example 3 with $\alpha = 1$.

For Examples 2 and 3, Figure 2 and Figure 3 present the maximum error plots for each method, respectively. Table 4 shows that for Example 2, our results outperform those in [19], with much smaller maximum errors at $n = 4$ and $n = 8$.

Example 4. Consider the following equation

$$\mathcal{D}^{\mu_1}y(x) = f(x) + c(x)y(x) + \int_0^x (x-s)^{-\frac{1}{2}} y^2(s) ds, \quad x \in [0, 1], \quad (8.4)$$

where $f(x)$ is the corresponding function, $c(x) = 1$ with initial value $y(0) = 0$. The exact solution to this problem is given by $y(x) = x^2 + x^\beta$.

Table 5. Maximum absolute error under different values of β , μ_1 , and α for Example 4.

α		$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
$\beta = 0.50$ $\mu_1 = 0.25$	4	0.0962495	0.0194347	6.04832e-11	0.0207438	0.0345699
	8	2.22045e-16	0.000514327	4.44089e-16	0.00404472	0.0103244
	16	2.22045e-15	2.00394e-6	6.66134e-16	0.0015563	0.00097812
$\beta = 0.75$ $\mu_1 = 0.5$	4	0.0553683	0.0189931	0.00927306	0.00241515	0.00035896
	8	4.44089e-16	0.0000525436	0.00132227	0.00141194	0.000010401
	16	9.08995e-16	3.85328e-8	0.000105817	0.00102421	2.19719e-6
α		1				
$\beta = 0.50$ $\mu_1 = 0.25$	4	0.0813707				
	8	0.0396741				
	16	0.00906009				
$\beta = 0.75$ $\mu_1 = 0.5$	4	0.0156173				
	8	0.00624135				
	16	0.0137052				

For Example 4, Table 5 shows the maximum absolute errors corresponding to different values of α and n under two sets of parameters: $\beta = 0.50$, $\mu_1 = 0.25$ and $\beta = 0.75$, $\mu_1 = 0.5$, allowing analysis of parameter effects on error. Notably, the table also reveals the comparative performance of fractional-order ($\alpha < 1$) versus integer-order ($\alpha = 1$) basis functions across different solution regularities. When the solution exhibits reduced smoothness (smaller β), fractional-order approximations significantly outperform their integer-order counterparts. Figure 4 presents the corresponding maximum absolute errors for $n = 20$, highlighting the variation of accuracy with respect to α , and further demonstrating the reliability and flexibility of the presented method.

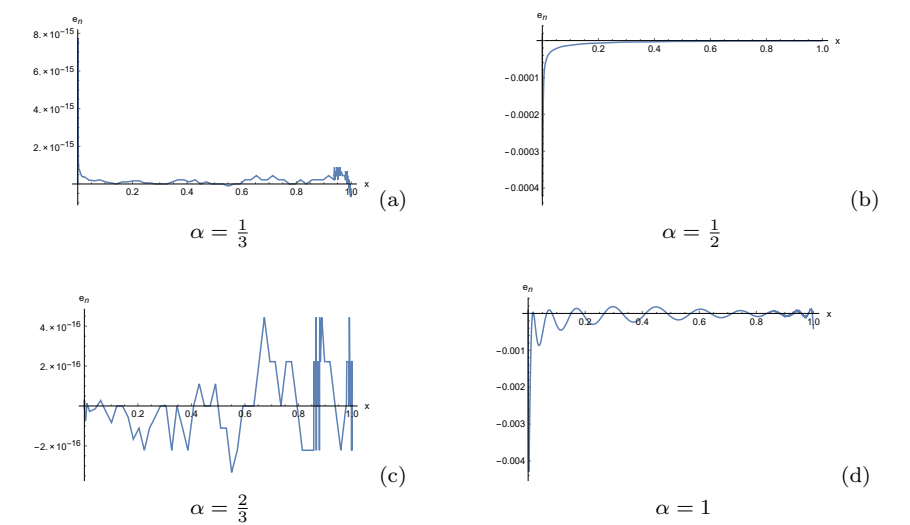


Figure 4. The maximum absolute error with $n = 20$ under different α for Example 4.

Example 5. Consider the following equation

$$\mathcal{D}^{\frac{1}{3}}y(x) = f(x) + c(x)y(x) + \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-s)^{-\frac{1}{2}} e^{y(s)} ds, \quad x \in [0, 2], \quad (8.5)$$

where $c(x) = 0$, $f(x) = \frac{\sqrt{\frac{\pi}{2}} \sqrt{x} {}_1F_2\left(1; \frac{3}{4}, \frac{5}{4}; -\frac{x^2}{4}\right)}{\Gamma(\frac{3}{4})\Gamma(\frac{5}{4})} - 2\sqrt{x} \sin x$, with the initial condition $y(0) = 0$. The exact solution to this problem is $y(x) = \sin(x)$.

Example 5 considers a nonlinear fractional integro-differential equation on $[0, 2]$ with exponential nonlinearity $e^{y(s)}$ and exact solution $y(x) = \sin x$. Figure 5 shows the error $e_n(x) = y_n(x) - y(x)$ for $n = 6, 8, 10, 12$; the errors decrease as n increases, which confirms convergence on the larger interval and for a non-quadratic nonlinearity. Table 6 compares the maximum absolute errors for different choices of α and n , indicating a fast decay with respect to n and demonstrating that selecting a fractional-order basis (i.e., $\alpha < 1$) can improve accuracy for this example.

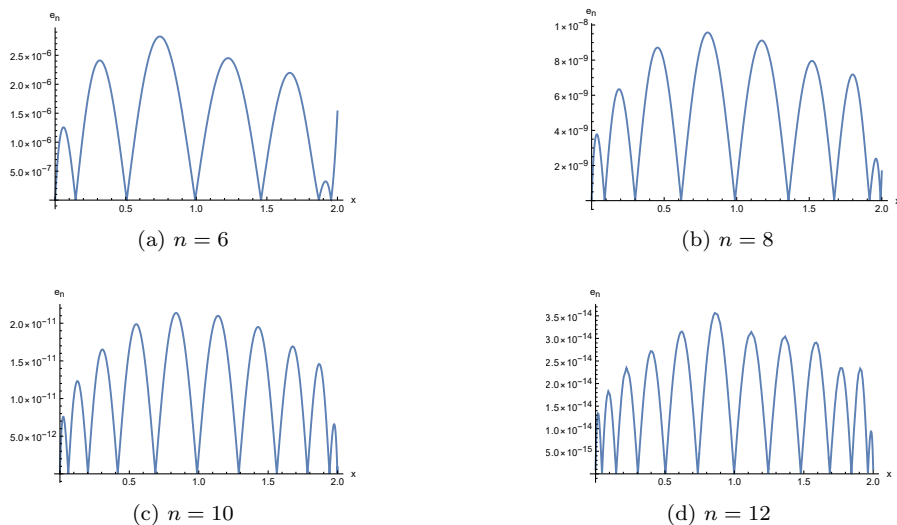


Figure 5. Error curves for different truncation orders $n = 6, 8, 10, 12$ for Example 5 with $\alpha = 1$.

Table 6. Maximum absolute error for different α and n in Example 5.

n	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = 1$
4	0.0924608	0.00294668	0.00325401	0.000478163
6	0.00895037	0.00294668	0.000993728	2.82428e-6
8	0.0087469	0.000565681	0.0000107914	9.57287e-9
10	0.000298661	8.5704e-6	4.20421e-7	2.13503e-11
12	0.0000127924	0.0000117496	8.99935e-9	3.57492e-14

9 Conclusions

In this study, we develop a numerical scheme to solve nonlinear fractional VIDEs. The proposed approach is constructed using shifted fractional Legendre functions in conjunction with an operational matrix technique. The method leverages the Laplace transform to derive an operational matrix for fractional integration, enabling efficient and accurate approximation of weakly singular integral operators. To effectively handle solutions with endpoint singularities or low regularity, fractional-order basis functions are employed. These functions offer better approximation properties than their integer-order counterparts. The nonlinear term is retained within the spectral approximation and handled through a fixed-point iteration scheme. This eliminates the need for linearization or decomposition techniques commonly used in existing methods. The existence and uniqueness of the solution are established via the Banach fixed-point theorem. Furthermore, a comprehensive error analysis is performed, showing that under appropriate regularity conditions, the error measured in the L_w^2 -norm diminishes as the number of basis functions grows. Numerical experiments confirm the accuracy and consistency of the proposed approach across a variety of examples. These include problems with singular kernels, nonlinearities, and non-smooth solutions. Overall, the method offers a flexible and effective tool for solving fractional integro-differential models, especially when facing low-regularity behavior. Future work may focus on extending this framework to coupled systems of fractional equations, higher-dimensional domains, or fractional PDEs with variable-order derivatives.

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