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On the put-call duality of the game options

Tsvetelin Zaevski[a,b]

[a] *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria*
[b] *Centre of Excellence in Informatics and Information and Communication Technologies, Sofia, Bulgaria*

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Abstract. The purpose of this paper is to obtain a duality between the game put and call options assuming three component penalties – proportion of the usual option payoff, shares of the underlying asset, and a fixed amount. We examine separately the cases of finite and infinite maturities. For the perpetual options, we need to derive a polynomial-style equations for the optimal boundaries. We prove the existence and uniqueness of their solutions as well as provide a method for their deriving. This result is important in itself since the current literature in the field is based on inverting of several functions or on solving of non-linear systems which may lead to some computational difficulties and significant errors for extreme parameter values. Furthermore, the duality is established under a finite time horizon too. It is important to note that this duality does not hold under the classical assumption of a fixed penalty.

Keywords: put-call duality; game options; optimal boundaries; polynomial-style formula; pricing.

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✉ Corresponding author. E-mail: `t_s_zaevski@math.bas.bg`, `t_s_zaevski@abv.bg`

1 Introduction

The options are one of the most popular financial securities. There are mainly two kinds – call and put. A call option provides its holder the right to buy an underlying asset for a predefined price named strike price. Alternatively, the put option gives the holder the right to sell. Another classification divides the options w.r.t. their time of maturity. The European options expire at a fixed future moment whereas the American-style derivatives give their holder the right to choose when to exercise. In such a way an optimal stopping problem arises. It turns out that the time-price space can be divided into two parts – a set in which the immediate exercise is the optimal strategy for the option's holder, and the so-called continuation set – keeping the option is preferable. For the regular American call options, the optimal set contains all points above some boundary, whereas the points below are optimal for the puts. This boundary is known as the optimal or early exercise boundary. Some important results in the field can be found in [12, 14].

In addition to the specifics of the American options, the game options, first studied in [13], provide a canceling right to the option's writer. He owes some amount above the usual payoff to use it. The time-price space can be divided now into three parts – the continuation set and the optimal sets for the holder and the writer. Typically, the holder's optimal points of a call-styled option are again above some boundary. On the other hand, the writer's optimal set can be a strip above the strike (for small enough penalties), the singleton formed by the strike (for the medium penalties), or the empty set (for the large enough penalties). The continuation set is of two parts – the points below the strike and the points between both optimal sets (if the writer's one exists). The shape of these sets for the put-styled options is similar but inverse w.r.t. the strike. Although the options without maturity constraints are rarely traded, they have outstanding importance since they give the model asymptotics. The call style such options are considered in [7, 25], whereas the put variants are examined in [5, 17, 24], see also [9]. The more realistic assumption, that the options are written on the finite maturity horizon, is considered in [16, 25] (call options) and [15] (put options). All theses studies are stated under the classical assumption of [2] – the underlying asset is driven by a geometric Brownian motion. Some alternatives based on the Lévy processes can be found in [20]. Several exotic game options are considered in [1, 11]. Generally said, pricing of such instruments falls in the field of the Dynkin's stochastic games [3] – for some important results see also [6, 21].

Following the classical approach, we base our investigation under log-normal assumptions. Also, the dividends are introduced through an additional discount factor in a way similar to one used in [23]. The traditional assumption is that the penalty which the writer owes for his cancelling right is a constant during the option life. Alternatively, we consider three-component penalties in this paper – a proportion of the usual payoff, some shares of the underlying asset, and a fixed amount. The main result of the paper is in obtaining of a put-call duality for these options. Roughly said, it states that a call option can be viewed as a put one after a suitable change of parameters. Some analogies can be found in the options for exchanging two different asset – see [18]. Our results have some predecessors for the European and regular American options under different assumptions in [4, 8, 10, 19, 22]. We first obtain the duality under the infinite time horizon. In this case, we have two flat optimal boundaries – one for the writer and another for the holder. To establish the desired result, we derive polynomial-style equations for the optimal boundaries and later prove the existence and uniqueness of their solutions. These results have their own significance since all of the above-referred papers use an inversion of several functions or non-linear systems. Furthermore, we obtain the same duality for the options with finite maturities. We do this by a measure change with the discounted asset for the Radon-Nikodym derivative which is a positive martingale under the risk neutral market. Thus, using Girsanov theorem, we transform the stochastic part of a call, presented by the asset, into a constant (strike) part for a put and vice versa. Note that this construction is possible only under the three component penalties, but not when the overprice is a constant during the option life.

The findings of this paper are important in several aspects. First, the duality allows the use of the same framework for both calls and puts. Given the pricing method for one of them, the other can be evaluated by changing only the inputs. Second, the duality property is much stronger than the important and well-known *put-call parity.* And third, as noted in [22], the parity property cannot be extended from the European options to the more complex instruments such as American ones. The duality principle fills in this gap. Furthermore, the proposed duality is applicable to even more complex class of cancellable options. In addition, the results of [22] have a clear geometric interpretation that can be easily extended to a financial context. While in [22] the geometric meaning is the symmetry (as a mirror image), our results are related rather to the inversion. If the strike is unit valued both for a call and a put, then the call boundaries are the reciprocals of the put boundaries. This result is closer to [4, 8, 10]. Thus the inversion $x \to \frac{1}{x}$ maps the call behavior that is important above the strike into the put behavior, which is interesting below the strike.

The paper is organized as follows. In Section 2, we derive the polynomial style equations for the optimal boundaries of the game options. The main result of the paper, namely the put-call duality, is formulated and proven in Section 3. The existence and uniqueness results for the solution of the equation that describes the perpetual case as well as a method for its deriving are provided in Section 4.

2 Polynomial-style formulas for the optimal boundaries

Let the underlying asset be modeled by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t$$

under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$. The constant r is the risk-free rate and thus the measure $\mathbb{Q}$ is risk neutral. Let λ be an additional discount factor which can be also viewed as a dividend rate. We impose $\lambda \geq 0$ and $r + \lambda > 0$, but we admit negative values for the risk-free rate r. Following [23], we conclude that a model continuously paying dividends can be viewed as a non-dividend model but with an additional discount factor. More precisely, if the triple (r, λ, δ) gives the risk-free, discount, and dividend rates, then a (r, λ, δ)-model is equivalent to a $(r - \delta, \lambda + \delta, 0)$-one – for the proof see Proposition 2.2 from [27]. We shall use hereafter the following notations:

$$\begin{aligned} p &= 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}, \\ q &= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \end{aligned} \tag{2.1}$$

These constants are closely related to the roots of the quadratic equation that characterizes the ordinary differential equation related to the first exit of a Brownian motion from a strip. We have that $p \geq q + 1$ – the equality holds only in the undiscounted case $\lambda = 0$.

As we mentioned above, before to establish the put-call duality for the perpetual options, we need to derive polynomial-style equations for the related optimal boundaries.

2.1 Call options

Let us consider a game call option with strike K. The option's writer has to pay the amount of

$$N_1(t,x) = e^{-\lambda t}(x-K)^+, \tag{2.2}$$

if the holder exercises at a moment t at a spot price $S_t = x$. Let the penalty for the writer's canceling right consists of an $\eta_1 \geq 1$ proportion of the usual option payment, $\eta_2 \geq 0$ shares of the underlying asset, and a fixed amount of $\eta_3 \geq 0$. Thus, the writer owes the amount of

$$N_2(t,x) = e^{-\lambda t}\left[\eta_1(x-K)^+ + \eta_2 x + \eta_3\right], \tag{2.3}$$

if he cancels the contract. It is proven in [26] that the points that make the immediate exercise optimal for the holder, form an interval (B,∞), whereas the writer's optimal region can be an interval $[K,A]$, $K < A \leq B \leq \infty$, the singleton $\{K\}$, or the empty set. It is proven in [26] that if the writer's exercise boundary A exists, then it solves the equation

$$b(y)\,a\big(yb(y)\big) = 1. \tag{2.4}$$

The functions $a(y)$ and $b(y)$ are defined as the roots of the functions

$$\begin{aligned} f_1(a;y) =& a^{p+1}(\eta_1+\eta_2)(p-q-1) - \frac{a^p}{y}(\eta_1 K - \eta_3)(p-q) \\ &- a^{p-q}p\left(1-\frac{K}{y}\right) + a(q+1)(\eta_1+\eta_2) - \frac{q}{y}(\eta_1 K - \eta_3), \\ f_2(b;y) =& -b^{p+1}(p-q-1) + b^p\frac{K}{y}(p-q) \\ &+ b^{p-q}p\left(\eta_1+\eta_2-\eta_1\frac{K}{y}+\frac{\eta_3}{y}\right) - b(q+1) + q\frac{K}{y} \end{aligned} \tag{2.5}$$

in the interval $(0,1)$ or $(1,\infty)$, respectively. The root $a(y)$ of the function $f_1(a;y)$ is w.r.t. to the variable a for a fixed y, whereas $b(y)$ is the root of the function $f_2(b;y)$ w.r.t. b. It is also proven in the same paper that these roots exist and they are unique. If the root of Equation (2.4) is lower than the strike, then the writer's optimal boundary is the strike or does not exist. Note that Proposition 3.7. from [26] shows that this is the case when $r \leq 0$. We distinguish both alternatives by comparing the price of the perpetual American at-the-money option with the value $\eta_2 K + \eta_3$. The strike is the optimal boundary when $\eta_2 K + \eta_3$ is the lower one. The idea that stands beyond this is the fact that if the option is at-the-money positioned and the opposite relation holds, then the writer will prefer to not use the option instead of canceling it paying too much money, namely $\eta_2 K + \eta_3$. If the root of the Equation (2.4) is larger than the strike, then it is the writer's optimal boundary. Hence, the important and more difficult case is $r > 0$. We can easily check that this is equivalent to $p < 2q + 1$ – see formulas (2.1). So, let us assume $q + 1 \leq p < 2q + 1$.

Suppose that $B \geq A > K$ are the optimal boundaries. Therefore, if a and b are defined as $a = \frac{A}{B}$ and $b = \frac{B}{A}$, then $f_1(a; B) = 0$ and $f_2(b; A) = 0$, where the functions $f_1(\cdot;\cdot)$ and $f_2(\cdot;\cdot)$ are given by formulas (2.5). From now on, we explicitly denote the dependence on variables a and b, i.e., we have $f_1\big(a; B(a)\big) = 0$ and $f_2\big(b; A(b)\big) = 0$. The first equation can be rewritten as

$$B(a) = \frac{a^p(\eta_1 K - \eta_3)(p-q) - a^{p-q}pK + q(\eta_1 K - \eta_3)}{a^{p+1}(\eta_1 + \eta_2)(p-q-1) - a^{p-q}p + a(q+1)(\eta_1 + \eta_2)}. \tag{2.6}$$

Let the constants c_1 and c_2 be defined as

$$c_1 = \eta_1 - \eta_3/K, \quad c_2 = \eta_1 + \eta_2. \tag{2.7}$$

Note that $c_2 \geq 1$ and $0 < c_1 \leq c_2$ due to Proposition 3.2 from [26]. Using the relation $A(a) = aB(a)$ and formulas (2.6) and (2.7) we obtain the first presentation of $A(a)$ indexed by a subscript:

$$\begin{aligned} A_1(a) &= \frac{a^p(\eta_1 K - \eta_3)(p-q) - a^{p-q}pK + q(\eta_1 K - \eta_3)}{a^p(\eta_1 + \eta_2)(p-q-1) - a^{p-q-1}p + (q+1)(\eta_1 + \eta_2)} \\ &= \frac{p-q}{p-q-1} a \frac{a^q(\eta_1 K - \eta_3) - K + \frac{q}{p-q}\left(-K + \frac{\eta_1 K - \eta_3}{a^{p-q}}\right)}{a^{q+1}(\eta_1 + \eta_2) - 1 + \frac{q+1}{p-q-1}\left(-1 + \frac{\eta_1 + \eta_2}{a^{p-q-1}}\right)} \\ &= \frac{p-q}{p-q-1} \frac{X_1(a) + X_2(a)}{X_3(a) + X_4(a)}, \end{aligned}$$

where the functions $X_1(\cdot)$, $X_2(\cdot)$, $X_3(\cdot)$, and $X_4(\cdot)$ are defined as

$$\begin{aligned} X_1(a) &= Ka(c_1 a^q - 1), \quad X_2(a) = Ka\frac{q}{p-q}\left(\frac{c_1}{a^{p-q}} - 1\right), \\ X_3(a) &= a^{q+1}c_2 - 1, \quad X_4(a) = \frac{q+1}{p-q-1}\left(\frac{c_2}{a^{p-q-1}} - 1\right). \end{aligned}$$

On the other hand, the equation $f_2(b; A) = 0$ leads to the second form of $A(a)$ marked again by a subscript:

$$A_2(b) = \frac{b^p k(p-q) - b^{p-q}p(\eta_1 K - \eta_3) + qK}{b^{p+1}(p-q-1) - b^{p-q}p(\eta_1 + \eta_2) + b(q+1)}. \tag{2.8}$$

Having in mind $ab = 1$, we rewrite Equation (2.8) as

$$A_2(a) = \frac{p-q}{p-q-1} \frac{X_1(a) + a^p X_2(a)}{X_3(a) + a^p X_4(a)}.$$

Hence, a solution $\overline{a}$ of the equation $A_1(a) = A_2(a)$ leads to $X_1(\overline{a}) X_4(\overline{a}) = X_2(\overline{a}) X_3(\overline{a})$. We need now the following simple lemma:

Lemma 1. *If $X_1X_4 = X_2X_3$ for some constants X_1, X_2, X_3, and X_4, then,*

$$\frac{X_1+X_2}{X_3+X_4} = \frac{X_1}{X_3} = \frac{X_2}{X_4}.$$

Lemma 1 shows that $\overline{a}$ can be derived as a solution of the equation

$$h(a) = g(a), \tag{2.9}$$

where the functions $h(\cdot)$ and $g(\cdot)$ are defined as

$$\begin{aligned} h(a) &= \frac{p-q}{p-q-1}\frac{1}{K}\frac{X_1}{X_3} = \frac{p-q}{p-q-1}\frac{c_1a^{q+1}-a}{c_2a^{q+1}-1}, \\ g(a) &= \frac{p-q}{p-q-1}\frac{1}{K}\frac{X_2}{X_4} = \frac{q}{q+1}\frac{a^{p-q}-c_1}{a^{p-q-1}-c_2}. \end{aligned} \tag{2.10}$$

Equation (2.9) leads to our first main result which is presented in the following theorem:

Theorem 1. *The writer's and holder's exercise boundaries can be derived as $A = Kh(\overline{a}) \equiv Kg(\overline{a})$ and $B = \frac{A}{\overline{a}}$, respectively, where $\overline{a} \in (0,1)$ is a solution of the following equation*

$$\begin{aligned} H(a) :=& a^{p+1}q(p-q-1)c_2 - a^p(p-q)(q+1)c_1 + a^{p-q}p + a^{q+1}c_1c_2p \\ &- a(p-q)(q+1)c_2 + q(p-q-1)c_1 = 0. \end{aligned} \tag{2.11}$$

Proof. The conclusions above show that the optimal boundaries can be derived as $A = Kh(\overline{a}) \equiv Kg(\overline{a})$ and $B = \frac{A}{\overline{a}}$, where $\overline{a}$ solves Equation (2.9). We have only to transform it having in mind formulas (2.10). The existence and uniqueness of the solution of Equation (2.11) are discussed later in Section 4. We provide also an algorithm for deriving this solution therein. □

2.2 Put style options

Let us consider a put style game option under the same assumptions. Thus the payoff structures (2.2)–(2.3) turn into

$$N_1(t,x) = e^{-\lambda t}(K-x)^+, \quad N_2(t,x) = e^{-\lambda t}\Big[\eta_1(K-x)^+ + \eta_2 x + \eta_3\Big].$$

It is proven in [26] that the holder's optimal set is the interval $(0,B)$ for some constant $B < K$, whereas the writer's set is empty, the singleton $\{K\}$, or an interval $(A,K]$. Note that the order now is $B < A < K$ and the constants c_1 and c_2 must be defined as

$$c_1 = \eta_1 - \eta_2, \quad c_2 = \eta_1 + \frac{\eta_3}{K}. \tag{2.12}$$

Propositions 4.2 and 4.5 from [26] show that we need to examine the case $c_1 > 0$ and $r \leq 0$, or equivalently $p \geq 2q + 1$ – the constants p and q are again defined by formulas (2.1). The analogues of functions (2.5) are

$$\begin{aligned} f_1(a, y) =& a^{p+1}(\eta_1 - \eta_2)(p - q - 1) - \frac{a^p}{y}(\eta_1 K + \eta_3)(p - q) \\ & + a^{p-q} p \left(\frac{K}{y} - 1 \right) + a(q+1)(\eta_1 - \eta_2) - \frac{q}{y}(\eta_1 K + \eta_3), \\ f_2(b, y) =& - b^{p+1}(p - q - 1) + b^p \frac{K}{y}(p - q) \\ & + b^{p-q} p \left(\eta_1 - \eta_2 - \eta_1 \frac{K}{y} - \frac{\eta_3}{y} \right) - b(q+1) + q \frac{K}{y}. \end{aligned}$$

We search for the solutions of the equations $f_1(a, B) = 0$ and $f_2(b, A) = 0$ w.r.t. the variables a and b in the intervals $(1, \infty)$ and $(0, 1)$, respectively. The same reasons as above lead to the following dual presentation of the writer's boundary

$$\frac{A}{K} = \frac{p - q}{p - q - 1} \frac{b^q - c_2}{b^{q+1} - c_1} = \frac{q}{q + 1} \frac{b^{p-q} c_2 - 1}{b^{p-q} c_1 - b}. \tag{2.13}$$

Let us change the variable q to $\overline{q} = p - q - 1$. It leads to $\overline{q} + 1 \leq p < 2\overline{q} + 1$ and thus the pair $(p, \overline{q})$ satisfies the condition for the call style options. Substituting $\overline{q}$ into equation (2.13), we obtain following theorem:

Theorem 2. *Let we change the variable q to $\overline{q} = p - q - 1$. The writer's and holder's exercise boundaries can be derived as*

$$A = K/h\left(\overline{b}\right) \equiv K/g\left(\overline{b}\right), \quad B = A\overline{b},$$

where the functions $h(\cdot)$ and $g(\cdot)$ are defined by formulas (2.10), *and $\overline{b} \in (0, 1)$ is the solution of Equation* (2.11).

3 Duality

We are ready now to establish the main result of the paper, namely the put-call duality. We first investigate the perpetual case.

Theorem 3. *Let a perpetual put style game option with parameter set $\{r, \lambda, \eta_1, \eta_2, \eta_3\}$ has optimal boundaries A and B, and x be such that*

$$0 \leq x \leq \min\{\eta_1 - 1, \eta_2\} + \eta_3/K.$$

Under these assumptions, the optimal boundaries of a call option with parameters

$$\left\{-r, r + \lambda, \eta_1 + \eta_3/K - x, x, K(\eta_2 - x) + \eta_3\right\}$$

are

$$A_{call} = K^2/A, \quad B_{call} = K^2/B.$$

Particularly, if we want to keep the first penalty coefficient η_1*, then we have a call-model with parameters*

$$\left\{-r, r+\lambda, \eta_1, \eta_3/K, K\eta_2\right\}.$$

If we want to keep the second and third penalty parts η_2 *and* η_3*, then the parameters for the call-model are*

$$\left\{-r, r+\lambda, \eta_1+\eta_3/K-\eta_2, \eta_2, \eta_3\right\}.$$

The last is possible only when $\min\{0, \eta_1-\eta_2-1\}+\eta_3/K>0$.

Furthermore, the put and call prices, $P(S_0;\cdot)$ *and* $C(S_0;\cdot)$*, are related as*

$$\frac{S_0}{K} C\left(\frac{K^2}{S_0}; -r, r+\lambda, \eta_1+\frac{\eta_3}{K}-x, x, K(\eta_2-x)+\eta_3\right) = P(S_0; r, \lambda, \eta_1, \eta_2, \eta_3). \tag{3.1}$$

Proof. The proof of the first part is a consequence of Theorems 1 and 2, and the form of c_1 and c_2 for the put and call options used in Sections 2.1 and 2.2 – formulas (2.7) and (2.12), respectively. We have to mention only that the change of variable from q to $\overline{q}=p-q-1$ leads to $\overline{r}=-r$ and $\overline{\lambda}=r+\lambda$. The conditions for x guarantee the consistency of the call penalty coefficients.

We have to consider several cases to prove formula (3.1). Note that we have the following change of parameters

$$\begin{aligned} &\overline{q}=p-q-1, \quad \overline{A}=\frac{K^2}{A}, \quad \overline{B}=\frac{K^2}{B}, \\ &\overline{S}_0=\frac{K^2}{S_0}, \quad \overline{\eta}_1=\eta_1+\frac{\eta_3}{K}-x, \\ &\overline{\eta}_2=x, \quad \overline{\eta}_3=K(\eta_2-x)+\eta_3. \end{aligned} \tag{3.2}$$

1. Suppose first that we have a real game put option, i.e., $A<K$.

    (a) If $S_0 \geq K$, then we prove formula (3.1) changing the parameters as in (3.2) in the option prices that are obtained in equations (24) and (43) from [26]:

    $$C=(\overline{\eta}_2 K+\overline{\eta}_3)\left(\frac{\overline{S}_0}{K}\right)^{p-\overline{q}}, \quad P=(\eta_2 K+\eta_3)\left(\frac{K}{S_0}\right)^{q}.$$

    (b) If $S_0 \in (A, K)$, then the option prices are

    $$\begin{aligned} C &= \overline{\eta}_1\left(\overline{S}_0-K\right)+\overline{\eta}_2\overline{S}_0+\overline{\eta}_3, \\ P &= \eta_1(K-S_0)+\eta_2 S_0+\eta_3. \end{aligned}$$

    Formula (3.1) follows from the change (3.2).

   (c) Suppose that $S_0 \in (B, A)$. The option prices are presented in formulas (16) and (38) of [26]:

$$\begin{aligned} C = & \left((\overline{\eta}_1 + \overline{\eta}_2)\,\overline{A} - \overline{\eta}_1 K + \overline{\eta}_3 \right) \left(\frac{\overline{A}}{\overline{S}_0} \right)^{\overline{q}} \frac{\overline{B}^p - \overline{S}_0^p}{\overline{B}^p - \overline{A}^p} \\ & + \left(\overline{B} - K \right) \left(\frac{\overline{B}}{\overline{S}_0} \right)^{\overline{q}} \frac{\overline{S}_0^p - \overline{A}^p}{\overline{B}^p - \overline{A}^p}, \\ P = & (K - B) \left(\frac{B}{S_0} \right)^{q} \frac{A^p - S_0^p}{A^p - B^p} \\ & + \left(\eta_1 K - (\eta_1 - \eta_2)\, A + \eta_3 \right) \left(\frac{A}{S_0} \right)^{q} \frac{S_0^p - B^p}{A^p - B^p}. \end{aligned}$$

   We have only to check that the change (3.2) leads to formula (3.1).

   (d) If $S_0 \in (0, B]$, then the option prices are $C = \overline{S}_0 - K$ and $P = K - S_0$. Thus formula (3.1) is an immediate consequence of change (3.2).

2. If the writer's boundary is the strike, then the desired result can be obtained in the same manner, keeping in mind that the writer would cancel the option only if the underlying asset hits the strike.

3. Suppose that the penalties are large enough and the option is ordinary American. The put and call optimal boundaries are $B = \frac{q}{q+1} K$ and $\overline{B} = \frac{p - \overline{q}}{p - \overline{q} - 1} K$. The prices can be written as

$$C = \left(\frac{\overline{S}_0}{p - \overline{q}} \right)^{p - \overline{q}} \left(\frac{p - \overline{q} - 1}{K} \right)^{p - \overline{q} - 1}, \quad P = \left(\frac{K}{q + 1} \right)^{q+1} \left(\frac{q}{S_0} \right)^{q}$$

   due to Theorems 6.1 and 6.2 from [28]. The change (3.2) proves again formula (3.1).

□

Theorem 3 inspires us to check whether this duality holds under the finite maturity horizon too. It turns out that this is true and we prove this result in the following theorem:

Theorem 4. *Theorem 3 holds when the maturity is finite, $T < \infty$, too. Note that now the exercise boundaries are time-functions, $A(t)$ and $B(t)$.*

Proof. Let τ_1 and τ_2 be two stopping times and $\tau = \tau_1 \wedge \tau_2 \wedge T$. We shall associate the holder's strategy of a put option with τ_1 and the writer's one with τ_2. We shall use the superscripts p and c to mark the put and call models. We can think without loss of generality that $S^p_{\tau_1} \leq K$ and $S^p_{\tau_2} \leq K$. Let the Brownian motions B^p_t and B^c_t w.r.t. the measures $\mathbb{Q}^p$ and $\mathbb{Q}^c$ be related through the Radon–Nikodym derivative

$$\left. \frac{d\mathbb{Q}^p}{d\mathbb{Q}^c} \right|_t = e^{-\frac{\sigma^2}{2} t + \sigma B^c_t}. \tag{3.3}$$

We restrict it till the maturity T. Note that (3.3) is a $\mathbb{Q}^c$-martingale. The Girsanov theorem and the symmetry of the Brownian motion show that

$$B_t^c = -B_t^p + \sigma t. \tag{3.4}$$

Therefore, the financial result of the strategies τ_1 and τ_2 is

$$\begin{aligned}
&P\left(S_0; r, \lambda, \eta_1, \eta_2, \eta_3, \tau_1, \tau_2\right) \\
&= \mathbb{E}^p \left[\begin{array}{l}
e^{-(r+\lambda)\tau_1}\left(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)\tau_1 + \sigma B_{\tau_1}^p}\right) I_{\tau=\tau_1} \\
e^{-(r+\lambda)\tau_2}\left[\eta_1\left(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)\tau_2 + \sigma B_{\tau_2}^p}\right) + A\right] I_{\tau=\tau_2} \\
A = \eta_2 S_0 e^{\left(r-\frac{\sigma^2}{2}\right)\tau_2 + \sigma B_{\tau_2}^p} + \eta_3 \\
+e^{-(r+\lambda)T}\left(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T + \sigma B_T^p}\right)^+ I_{\tau=T}
\end{array}\right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}^c \left[\begin{array}{l}
e^{-\frac{\sigma^2}{2}\tau_1 + \sigma B_{\tau_1}^c} e^{-(r+\lambda)\tau_1}\left(K - S_0 e^{\left(r+\frac{\sigma^2}{2}\right)\tau_1 - \sigma B_{\tau_1}^c}\right) I_{\tau=\tau_1} \\
e^{-\frac{\sigma^2}{2}\tau_2 + \sigma B_{\tau_2}^c} e^{-(r+\lambda)\tau_2}\left[\begin{array}{l} \eta_1\left(K - S_0 e^{\left(r+\frac{\sigma^2}{2}\right)\tau_2 - \sigma B_{\tau_2}^c}\right) \\ +\eta_2 S_0 e^{\left(r+\frac{\sigma^2}{2}\right)\tau_2 - \sigma B_{\tau_2}^c} + \eta_3 \end{array}\right] I_{\tau=\tau_2} \\
+e^{-\frac{\sigma^2}{2}T + \sigma B_T^c} e^{-(r+\lambda)T}\left(K - S_0 e^{\left(r+\frac{\sigma^2}{2}\right)T - \sigma B_T^c}\right)^+ I_{\tau=T}
\end{array}\right] \\
&= \frac{S_0}{K}\mathbb{E}^c \left[\begin{array}{l}
e^{-\lambda\tau_1}\left(\frac{K^2}{S_0} e^{-\left(r+\frac{\sigma^2}{2}\right)\tau_1 + \sigma B_{\tau_1}^c} - K\right) I_{\tau=\tau_1} \\
e^{-\lambda\tau_2}\left[\begin{array}{l} \left(\eta_1 + \frac{\eta_3}{K} - x\right)\left(\frac{K^2}{S_0} e^{-\left(r+\frac{\sigma^2}{2}\right)\tau_2 + \sigma B_{\tau_2}^c} - K\right) \\ +x\frac{K^2}{S_0} e^{-\left(r+\frac{\sigma^2}{2}\right)\tau_2 + \sigma B_{\tau_2}^c} + K\left(\eta_2 - x\right) + \eta_3 \end{array}\right] I_{\tau=\tau_2} \\
+e^{-\lambda T}\left(\frac{K^2}{S_0} e^{-\left(r+\frac{\sigma^2}{2}\right)T + \sigma B_T^c} - K\right)^+ I_{\tau=T}
\end{array}\right] \\
&= \frac{S_0}{K} C\left(\frac{K^2}{S^0}; -r, r+\lambda, \eta_1 + \frac{\eta_3}{K} - x, x, K\left(\eta_2 - x\right) + \eta_3, \tau_1, \tau_2\right).
\end{aligned}$$

Note that $S_{\tau_1}^c \geq K$ and $S_{\tau_2}^c \geq K$ due to $S_{\tau_1}^p \leq K$, $S_{\tau_2}^p \leq K$, and relation (3.4). Thus the price relation (3.1) is established.

The next step is to prove the connection between the optimal boundaries $A^c\left(t\right) = \frac{K^2}{A^p(t)}$ and $B^c\left(t\right) = \frac{K^2}{B^p(t)}$. Suppose that τ_1 is the first hitting moment of B_t^c to the function $l\left(t\right)$. Therefore, it can be also viewed as first hitting of $S_t^c \equiv \frac{K^2}{S_0}\exp\left(-\left(r + \frac{\sigma^2}{2}\right)t + \sigma B_t^c\right)$ to the function

$$l^c\left(t\right) = \frac{K^2}{S_0} e^{-\left(r+\frac{\sigma^2}{2}\right)t + \sigma l(t)}.$$

Having in mind the relation between the Brownian motions (3.4), we conclude that τ_1 is the first hitting time of $S_t^p \equiv S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t^p\right)$ to the function

$$l^p(t) = S_0 e^{\left(r + \frac{\sigma^2}{2}\right) t - \sigma l(t)}.$$

The fact $l^c(t)\, l^p(t) = K^2$ finishes the proof. $\square$

Remark 1. We can analogously prove the duality between regular American call and put options. In the proof, we have to use only one stopping time τ instead of τ_1 and τ_2. On the other hand, a cancellable option turns into regular American for large enough values of the penalties.

Remark 2. If instead of the additional discount factor λ, we have a continuously paying dividends at rate δ underlying asset, then a $\{r, \delta, \eta_1, \eta_2, \eta_3\}$-put model is dual to a $\left\{\delta, r, \eta_1 + \frac{\eta_3}{K} - x, x, K(\eta_2 - x) + \eta_3\right\}$-call one. This is due to Proposition 2.2 from [27].

4 Existence and uniqueness of the solution of Equation (2.11)

We need several auxiliary results about some important functions before to establish the existence and uniqueness of the solution of Equation (2.11) that describe the optimal boundaries.

4.1 Behavior of some important functions

Let us define the constants α, β, γ_1, and γ_2 as

$$\alpha = c_2^{-\frac{1}{q+1}}, \quad \beta = \frac{c_1}{c_2}, \quad \gamma_1 = c_2^{\frac{q}{q+1}}, \quad \gamma_2 = \frac{qc_2 + 1}{q+1}, \tag{4.1}$$

where c_1 and c_2 are defined by formulas (2.7). Remind that $q+1 < p < 2q+1$, $c_2 \geq 1$, and $0 < c_1 \leq c_2$. The function $h(a)$, defined in formulas (2.10), has a singularity in the point α, the constant γ_1 is related to the sign of its numerator, and the constant γ_2 is related to the existence of a local minimum. We shall prove now a series of propositions.

Proposition 1. *We have the order $\gamma_1 < \gamma_2 \leq c_2$.*

Proof. The function

$$f(c) = c^{\frac{q}{q+1}} - \frac{qc+1}{q+1}$$

is decreasing for $c \geq 1$ since

$$f'(c) = \frac{q}{q+1}\left(c^{-\frac{1}{q+1}} - 1\right) < 0.$$

Hence, $f(c_2) < f(1) = 0$, which is equivalent to $\gamma_1 < \gamma_2$. The second inequality holds since $c_2 \geq 1$. $\square$

Proposition 2. *The function $H(a)$ given in Equation (2.11) is convex in the interval $a \in (0,1)$.*

Proof. We have for the second derivative $H''(a) = a^{p-q-2} H_1(a)$ for

$$\begin{aligned} H_1(a) =& a^{q+1} q(p-q-1)(p+1)c_2 - a^q (p-q)(p-1)(q+1)c_1 \\ &+ a^{2q+1-p} c_1 c_2 q(q+1) + (p-q)(p-q-1). \end{aligned}$$

Its derivative can be presented as $H_1'(a) = a^{2q-p} q(q+1) H_2(a)$ for

$$H_2(a) = a^{p-q}(p-q-1)(p+1)c_2 - a^{p-q-1}(p-q)(p-1)c_1 + (2q+1-p)c_1c_2.$$

Obviously, $H_2(0) > 0$ since $p < 2q+1$. We shall prove that $H_2(1)$ is also non-negative. Marking the dependence on c_2, we derive

$$H_2(1;c_2) = (p-q-1)(p+1)c_2 - (p-q)(p-1)c_1 + (2q+1-p)c_1c_2.$$

If $c_1 \geq 1$, then,

$$H_2(1;c_2) \geq H_2(1;c_1) = (2q+1-p)c_1(c_1-1) \geq 0.$$

Otherwise, if $c_1 < 1$, then,

$$H_2(1;c_2) \geq H_2(1;1) = (p+1)(p-q-1)(1-c_1) \geq 0.$$

The derivative of the function $H_2(a)$ is

$$H_2'(a) = a^{p-q-2}(p-q)(p-q-1)\left[a(p+1)c_2 - (p-1)c_1\right].$$

Hence, the function $H_2(a)$ starts from a positive value decreases to a minimum, and then increases to another positive value. If the minimum is positive, then $H_2(a) \geq 0$ for all $a \in (0,1)$. Hence, the functions $H_1(a)$ and $H''(a)$ are increasing functions. Therefore $H''(a) > 0$ since $H''(0) \geq 0$. Otherwise, if the minimum of the function $H_2(a)$ is negative, then it has two roots. We shall prove that if $H_2(\tilde{a}) = 0$, then $H_1(\tilde{a}) > 0$. Note that we have the following presentation of $H_1(a)$

$$H_1(a) = a^{2q+1-p} q H_2(a) + (p-q) f(a),$$

where the function $f(\cdot)$ is

$$f(a) = -a^q(p-1)c_1 + a^{2q+1-p}c_1c_2q + p - q - 1.$$

Let us examine the function $f(a)$. We have $f(0) > 0$. We shall prove that

$$f(1) = -(p-1)c_1 + c_1c_2q + p - q - 1 > 0$$

too. Marking again the dependence on c_2 we see that if $c_1 \leq 1$, then,

$$f(1,c_2) > f(1,1) = (p-q-1)(1-c_1) > 0.$$

Otherwise, if $c_1 > 1$, then,

$$f(1, c_2) > f(1, c_1) = qc_1^2 - (p-1)c_1 + p - q - 1.$$

Let us consider the quadratic function $l(\cdot)$ defined as

$$l(c) = qc^2 - (p-1)c + p - q - 1.$$

The value $c = 1$ is a root. Also, the vertex is for $c = \frac{p-1}{2q} < 1$. Hence, $l(c) > 0$ for $c > 1$. Therefore, $l(c_1) > 0$ which finish the proof of $f(1) > 0$.
Having in mind the form of the derivative $f'(a)$,

$$f'(a) = a^{2q-p} c_1 q \left[-a^{p-q-1}(p-1) + (2q+1-p) c_1 c_2 \right],$$

we see that the function $f(a)$ may increase between two positive values or may exhibit an inverted U-shape with positive endpoints. In both cases, $f(a) > 0$ for $a \in (0,1)$. This confirms that $H_1(\tilde{a}) > 0$ when $H_2(\tilde{a}) = 0$. This means that the function $H_1(a)$ is positive in its extrema which is enough to finish the proof. □

Proposition 3. *The function* $g(a)$*, defined in Equations* (2.10)*, has the following behavior in the interval* $a \in (0,1)$*. It starts from the value* $g(0)$*, increases to a maximum, and decreases to* $g(1)$*. Its endpoints are lower than one and* $g(0) > g(1)$*. Something more, the point* β *belongs to the decreasing domain –* β *is defined in formulas* (4.1)*.*

Proof. The derivative of the function $g(a)$ is

$$g'(a) = \frac{a^{p-q-2} g_1(a)}{\left((q+1)\left(a^{p-q-1} - c_2\right)\right)^2},$$

where $g_1(a)$ is defined as

$$g_1(a) = a^{p-q} - a(p-q)c_2 + (p-q-1)c_1.$$

Its derivative is negative since

$$g_1'(a) = (p-q)\left(a^{p-q-1} - c_2\right) < 0.$$

Hence $g_1(a)$ is a decreasing function. Having in mind

$$\begin{aligned} g_1(0) &= (p-q-1)c_1 > 0, \\ g_1(1) &= (p-q-1)(c_1 - c_2) + 1 - c_2 < 0, \end{aligned}$$

we describe the function $g(0)$. We have to observe that $g(\beta) = g(0)$ to finish the proof. □

Proposition 4. *Remind that we have the order* $\gamma_1 < \gamma_2 \le c_2$ *due to Proposition 1. The behavior in the interval* $a \in (0,1)$ *of the function* $h(a)$*, defined in equations* (2.10)*, can be characterized through the following statements:*

1. *If* $c_1 < \gamma_1$*, then the function* $h(a)$ *is increasing,* $\lim h(a) \to +\infty$ *for* $a \to \alpha^-$*, and* $\lim h(a) \to -\infty$ *for* $a \to \alpha^+$*.*
2. *If* $c_1 = \gamma_1$*, then* $h(a)$ *is an increasing continuous function.*
3. *If* $c_1 \in (\gamma_1, \gamma_2)$*, then the function* $h(a)$ *starts from the zero, increases to a local maximum, decreases to the minus infinity when* $a \to \alpha^-$*, has the plus infinity value for* $a = \alpha^+$*, decreases to a local minimum, and increases to the value* $h(1)$*.*
4. *The case* $c_1 \in [\gamma_2, c_2]$ *is similar to the previous one. The difference is that the function* $h(a)$ *is decreasing for* $a > \alpha$*.*

Proof. The derivative of the function $h(a)$ can be presented as

$$h'(a) = \frac{h_1(a)}{\big((p-q-1)(c_2 a^{q+1} - 1)\big)^2},$$

where

$$h_1(a) = a^{q+1} q c_2 - a^q (q+1) c_1 + 1.$$

We have $h_1(0) > 0$, $h_1(1) = qc_2 + 1 - (q+1)c_1$, and

$$h_1'(a) = a^{q-1} q (q+1)(ac_2 - c_1).$$

Note that $h_1'(\beta) = 0$ and $h_1(\beta) = 1 - c_1^{q+1} c_2^{-q}$. Thus we see that the function $h_1(a)$ starts from a positive value, decreases to a minimum for $a = \beta$, and increases. Also,

$$h_1(\alpha) = (q+1)\left(1 - c_1 c_2^{-\frac{q}{q+1}}\right),$$

which means that the signs of $h_1(\alpha)$ and $h_1(\beta)$ coincide.

Suppose first that $c_1 < \gamma_1$. Therefore, $h_1(\beta) > 0$, and thus the whole function is positive. This means that the function $h(a)$ is increasing. To finish the first case, we use the fact that the numerator changes its sign in the singular point α.

The case $c_1 = \gamma_1$ differs only by the vanishing singularity. Suppose now $\gamma_1 < c_1 < \gamma_2$. The desired behavior of the function $h(a)$ is valid due to $h_1(\beta) < 0$, $h_1(\alpha) < 0$, and $h_1(1) > 0$.

The case $\gamma_2 \le c_1 \le c_2$ is similar and differs only by $h_1(1) \le 0$, which removes the local minimum after the singular point α. □

We also need to examine the roots of the equations $h(a) = 1$ and $g(a) = 1$. Formulas (2.10) leads to $m(a) = 0$ and $n(a) = 0$ where the functions $m(\cdot)$ and $n(\cdot)$ are defined as

$$\begin{aligned} m(a) &= a^{q+1}\big(c_1(p-q) - c_2(p-q-1)\big) - a(p-q) + p - q - 1, \\ n(a) &= a^{p-q} q - a^{p-q-1}(q+1) + c_2(q+1) - qc_1. \end{aligned} \tag{4.2}$$

We immediately obtain the following relation:

Corollary 1. The inequalities $g(a) > 1$ and $n(a) < 0$ are equivalent.

The form of the derivatives of functions (4.2),

$$\begin{aligned} m'(a) &= a^q (q+1) \big(c_1 (p-q) - c_2 (p-q-1)\big) - (p-q), \\ n'(a) &= a^{p-q-2} \left[a (p-q) q - (p-q-1)(q+1)\right], \end{aligned}$$

leads to the following statements:

Proposition 5. *The equations $m(a) = 0$ and $n(a) = 0$ have no more than two solutions in the interval $a \in (0,1)$. Something more, the function $n(a)$ starts from a positive value decreases to a minimum a^*,*

$$a^* = \frac{(p-q-1)(q+1)}{(p-q)q}, \tag{4.3}$$

and increases to another positive value. Note that $a^* < 1$.

We also need the following proposition:

Proposition 6. *If $n(a_2) = 0$ and $a_2 > a^*$, then $m(a_2) < 0$. Also, $n(\alpha) > 0$ or equivalently $g(\alpha) < 1$.*

Proof. The equality $n(a_2) = 0$ leads to

$$(q+1) c_2 - q c_1 = -a_2^{p-q} q + a_2^{p-q-1} (q+1).$$

Therefore,

$$m(a_2) = -a_2^{q+1} p (c_2 - c_1) + f(a_2)$$

for

$$f(a) = -a^{p+1} q + a^p (q+1) - a(p-q) + p - q - 1.$$

We shall prove that $f(a) < 0$ for $a > a^*$. We have for the derivatives of the function $f(a)$:

$$\begin{aligned} f'(a) &= -a^p q (p+1) + a^{p-1} p (q+1) - (p-q), \\ f''(a) &= a^{p-2} p \overline{f}(a), \end{aligned}$$

where $\overline{f}(a) = -a(p+1) q + (p-1)(q+1)$. Note that $f(1) = f'(1) = 0$. The root of the function $\overline{f}(a)$, let us denote it by ξ, is

$$\xi = \frac{(p-1)(q+1)}{(p+1)q} < 1.$$

Hence, the derivative $f'(a)$ starts from a negative value, increases to a positive maximum, and decreases to zero. Therefore, the function $f(a)$ starts from a positive value, decreases to a negative minimum, and increases to zero. It left to be proven that $f(a^*) < 0$, which inequality can be written as

$$f(a^*) = \left[\frac{(p-q-1)(q+1)}{q(p-q)}\right]^p \frac{q+1}{p-q} - \frac{p-q-1}{q} < 0. \tag{4.4}$$

Inequality (4.4) is equivalent to

$$\left(\frac{q+1}{p-q}\right)^{p+1} < \left(\frac{q}{p-q-1}\right)^{p-1}.$$

Let us examine the function $F(x)$, defined as

$$F(x) = \left(\frac{q+x}{p-q-1+x}\right)^{p+2x-1}.$$

We need to prove $F(1) < F(0)$. We shall show that $F(x)$ is a decreasing function in the interval $(0,1)$. Its derivative can be written as $F'(x) = F(x)\,G(x)$ for

$$G(x) = -\frac{(2q+1-p)(p+2x-1)}{(q+x)(p-q-1+x)} + 2\ln\frac{q+x}{p-q-1+x}.$$

The derivative of the function $G(x)$,

$$G'(x) = \frac{(2q+1-p)^3}{(q+x)^2(p-q-1+x)^2},$$

is positive and therefore, $G(x) < G(1)$. Let us examine $G(1)$ as a function of p,

$$G(1) = G_1(p) = -\frac{(2q+1-p)(p+1)}{(q+1)(p-q)} + 2\ln\frac{q+1}{p-q}.$$

Its derivative is positive –

$$G_1'(p) = \frac{(2q+1-p)^2}{(q+1)(p-q)^2} > 0.$$

Therefore, $G_1(p) < G_1(2q+1) = 0$. This finishes the first part of the proof. Let us turn to the second part. Note that

$$n(\alpha) > \alpha^{p-q-1}\left[q\alpha - (q+1) + c_2\right] = \alpha^{p-q-1} l(c_2),$$

where the function $l(\cdot)$ is defined as

$$l(c) = qc^{-\frac{1}{q+1}} - (q+1) + c.$$

Its derivative is positive for $c > 1$, since

$$l'(c) = 1 - \frac{q}{q+1}c^{-\frac{1}{q+1}-1}.$$

Therefore, $l(c_2) > l(1) = 0$. □

4.2 Uniqueness of the solution

First, let us note that we are interested only in solutions of Equation (2.11) for which $h(a) \equiv g(a) \geq 1$. They lead to the writer's boundary above the strike. The desired result is proven in the following theorem:

Theorem 5. *Equation* (2.11) *has no more than two roots. Furthermore, if this equation has two roots, then one of them leads to* $h(a) \equiv g(a) \leq 1$.
Also, if Equation (2.11) *has a root that leads to a value larger than one, then it is between the roots of the equation* $n(a) = 0$ *which exist in this case.*

Proof. The first statement is a consequence of Proposition 2.
We shall separate the second task into several cases. First, suppose that $c_1 \leq \gamma_1$. Note that function (2.11) can be presented as

$$H(a) = \big(g(a) - h(a)\big)(p-q-1)(q+1)\left(c_2 a^{q+1} - 1\right)\left(a^{p-q-1} - c_2\right).$$

Therefore, $H(\beta) < 0$. Hence, if Equation (2.11) has two roots, say $\zeta_1 < \zeta_2$, then $\zeta_2 > \beta$. Proposition 3 leads $g(\zeta_2) < g(\beta) < 1$ since the point β belongs to the decreasing part of the function $g(a)$.

Suppose now that $c_1 > \gamma_1$. Having in mind Corollary 1 and Proposition 5, we conclude that if the function $n(a)$ has no roots, then Equation (2.11) has not a root larger than one. Suppose now, that the function $n(a)$ has two roots – say $a_1 \leq a_2$. Proposition 6 gives $m(a_2) < 0$. We shall examine separately the cases $a_2 \leq \gamma_1$ and $a_2 > \gamma_1$. If $a_2 \leq \gamma_1$, then $m(a_2) < 0$ leads to $h(a_2) > 1$ and therefore the equation $h(a) = g(a)$ (or equivalently $H(a) = 0$) has a root in the interval (a_2, γ_1) that leads to a value lower than one. Otherwise, if $a_2 > \gamma_1$, then $m(a_2) < 0$ leads to $h(a_2) < 1$ and hence the equation $h(a) = g(a)$ has a root larger than a_2 and it leads to a value lower than one. □

Having in mind Corollary 1, Proposition 5, and Theorem 5, we construct the following algorithm for deriving the desired solution of Equation (2.11), if it exists:

Algorithm 1.

1. If $n(a^*) \geq 0$, then Equation (2.11) has not a root that leads to $h(a) > 1$. The function $n(\cdot)$ and the constant a^* are defined by formulas (4.2) and (4.3). Thus the writer's optimal boundary is the strike or does not exist.
2. Suppose that $n(a^*) < 0$. We derive the roots a_1 and a_2 of the equation $n(a) = 0$. We look for them in the intervals $(0, a^*)$ and $(a^*, 1)$, respectively.
3. We search the root of Equation (2.11) in the interval (a_1, a_2).
4. We obtain the optimal boundaries using this root and Theorem 1.

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