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Second derivative two-step peer methods

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Abstract. This paper is devoted to extending two-step peer methods, for the numerical solution of ordinary differential equations, for the case where the second derivative of the solution is incorporated into the formula of the methods. The main features including consistency, zero-stability, and convergence of the proposed methods together with their order conditions and stability analysis are examined. Construction of explicit methods within the proposed class of the methods, possessing the Runge–Kutta stability property, is investigated, and examples of such methods up to order five are provided. The efficiency and accuracy of the constructed methods are validated through various numerical experiments conducted in both fixed and variable stepsize environments.

Keywords: two-step peer methods; second derivative methods; non-stiff or mildly stiff ODEs; Runge–Kutta stability.

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1 Introduction

Numerical methods for solving initial value problems (IVPs) of autonomous systems of ordinary differential equations (ODEs) in the form

$$\begin{cases} y'(x) = f\big(y(x)\big), & x \in [x_0, X], \\ y(x_0) = y_0, \end{cases} \tag{1.1}$$

where $y : \mathbb{R} \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$, are essential tools in various scientific and engineering fields. Among various classes of numerical methods, the family of two-step peer methods (TSPMs), which was first introduced in [22], is particularly popular due to its simplicity and efficiency. These methods combine desirable features of two main classes of the traditional methods: linear multistep methods (LMMs) and Runge–Kutta (RK) methods. Specifically, these methods are multistage methods similar to RK methods, except they do not have extraordinary solution variables—that is, their formulation requires no input/output vector and depends only on the stage values—and all of the stages have the same accuracy. In order to recall a TSPM for the numerical solution (1.1), we first take into consideration the nonuniform grid points

$$x_0 < x_1 < \cdots < x_N, \quad x_N = X, \tag{1.2}$$

with the stepsizes $h_n := x_n - x_{n-1}$, for $n = 1, 2, \ldots, N$, and the stepsize ratios $\delta_n = h_n/h_{n-1}$, for $n = 2, 3, \ldots, N$. Moreover, let $Y^{[n]} = \big[Y_i^{[n]}\big]_{i=1}^{s}$ stand for the vector of stage values as an approximation of the order p to the vector $y(x_{n-1}+ch_n)=\big[y(x_{n-1}+c_ih_n)\big]_{i=1}^{s}$ with the abscissa vector $c=[c_1 \ c_2 \ \cdots \ c_s]^T$, which specifies the position of internal stages within a step, and $F(Y^{[n]}) = \big[f(Y_i^{[n]})\big]_{i=1}^{s}$ be the vector of first derivative stage values. Then, an s-stage TSPM with the coefficient matrices $\mathbf{B} = \mathbf{B}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$, $\mathbf{A} = \mathbf{A}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$, and $\mathbf{R} = \mathbf{R}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$ for the numerical solution of (1.1) on the nonuniform grid (1.2) takes the form

$$Y^{[n]}=(\mathbf{B}\otimes I_m)Y^{[n-1]} + h_n(\mathbf{A} \otimes I_m)F\big(Y^{[n-1]}\big)+h_n(\mathbf{R} \otimes I_m)F\big(Y^{[n]}\big), \tag{1.3}$$

for $n = 1, 2, \ldots, N$, where I_m and $\otimes$ stand for the m-dimensional identity matrix and the Kronecker product of two matrices, respectively. Various studies have demonstrated the capability of the methods (1.3) in providing reliable and stable numerical results when solving non-stiff and stiff systems of ODEs (1.1). Numerous investigations have been conducted on these methods within various classes; for instance, two-step peer methods [28, 30], parallel peer two-step W-methods [29], Rosenbrock-type peer two-step methods [21], multi-implicit peer two-step W-methods [24], and implicit parallel peer methods for stiff systems [23]. Moreover, recent studies can be found in [2, 9, 17, 19, 20].

However, the pursuit of developing new and efficient methods, as well as improvement of the traditional methods, remains ongoing. Researchers have continued to use a variety of effective strategies to reach this goal. One of the most successful strategies to improve the properties of the numerical methods, such as high accuracy, extensive stability region, and efficiency, is incorporating the higher derivatives of the solution, particularly the second derivative, into the integration formulas. Employing this strategy has unveiled new opportunities for effectively solving both non-stiff and stiff ODEs. The utilization of second derivative terms in numerical methods has attracted significant interest among researchers, resulting in the development of diverse classes of the numerical methods; for instance, second derivative multistep methods (SDMMs) [7, 11, 13], second derivative multistep methods equipped with future-step points [3, 6], two-derivative Runge–Kutta (TDRK) methods [8, 27], second derivative multistep collocation methods [4, 10, 18], and second derivative general linear methods (SGLMs) [1, 25] as multivalue/multistage methods with high order and stage order including the traditional second derivative methods as special cases.

While the TSPMs have some advantageous features of both Runge–Kutta and multistep methods, we are particularly focused on expanding the TSPM framework to incorporate scenarios where the second derivative of the solution, along with the first derivative, can be computed at a moderate additional cost. More precisely, our aim is to introduce the class of *second derivative TSPMs (STSPMs)* and then investigate the derivation of explicit STSPMs with small error constants and desirable stability properties while demonstrating robust performance when dealing with non-stiff or mildly stiff problems.

The organization of the upcoming sections in this paper is as follows. In Section 2, we introduce a new class of two-step peer methods for solving (1.1) incorporating second derivative of the solution. The main features including consistency, zero-stability, and convergence of the methods are discussed in Section 3. The order conditions and stability analysis of the proposed methods are outlined in Section 4 and construction of explicit methods within this class of the methods is investigated in Section 5. Section 6 presents the results of some numerical experiments conducted in both fixed and variable stepsize environments to validate the theoretical results. The paper concludes in Section 7 with final remarks and ideas for future research directions.

2 The modification of TSPMs

In this section, we outline the modification made to TSPMs (1.3) for solving (1.1) by incorporating the second derivatives of the solution, $g(\cdot) := f_y(\cdot)f(\cdot)$, into the formula.

Once more, let $Y^{[n]} = \big[Y_i^{[n]}\big]_{i=1}^s$ and $F(Y^{[n]}) = \big[f\big(Y_i^{[n]}\big)\big]_{i=1}^s$ respectively denote the stage values and first derivative stage values. Along with these values, the second derivative stage values of the step number n are denoted by $G(Y^{[n]}) = \big[g\big(Y_i^{[n]}\big)\big]_{i=1}^s$. Then, utilizing the previous notations, an s-stage *STSPM* is defined by five coefficient matrices $\mathbf{B} = \mathbf{B}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$, $\mathbf{A} = \mathbf{A}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$, $\overline{\mathbf{A}} = \overline{\mathbf{A}}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$, $\mathbf{R} = \mathbf{R}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$, and $\overline{\mathbf{R}} = \overline{\mathbf{R}}\big(\delta_n\big) \in \mathbb{R}^{s\times s}$, and the abscissa vector $c \in \mathbb{R}^s$. In an STSPM on the nonuniform grid (1.2), the quantities computed in step number n and transferred for utilization in the subsequent step, are related by the formulae

$$\begin{aligned} Y^{[n]} =& (\mathbf{B}\otimes I_m)Y^{[n-1]} + h_n(\mathbf{A}\otimes I_m)F\big(Y^{[n-1]}\big) + h_n^2(\overline{\mathbf{A}}\otimes I_m)G\big(Y^{[n-1]}\big) \\ &+ h_n(\mathbf{R}\otimes I_m)F\big(Y^{[n]}\big) + h_n^2(\overline{\mathbf{R}}\otimes I_m)G\big(Y^{[n]}\big). \end{aligned} \tag{2.1}$$

One can see that the method (2.1) is *explicit* when both the coefficient matrices $\mathbf{R}$ and $\overline{\mathbf{R}}$ are *strictly lower triangular* matrices; otherwise, the method is *implicit*. In this paper, we concentrate specifically on the explicit STSPMs. A discussion of the implicit STSPMs can be found in [26].

3 Consistency, zero-stability and convergence of STSPMs

This section covers the basic concepts of pre-consistency, consistency, zero-stability, and convergence for an STSPM.

3.1 Consistency

The pre-consistency condition is determined by guaranteeing that the one-dimensional initial value problem $y'(x) = 0$, with $y(x_0) = 1$ can be solved exactly. To solve this problem by an STSPM, the computed value $Y^{[n]}$ at the step n, is given by

$$Y^{[n]} = \mathbf{B}Y^{[n-1]}.$$

Therefore, the pre-consistency condition is determined by satisfying $\mathbf{B}e = e$, in which the vector e stands for the s-dimensional all-ones vector. We state the following definition.

Definition 1. An STSPM is '*pre-consistent*' if $\mathbf{B}$ has an eigenvalue equal to 1, with a corresponding eigenvector consisting solely of ones.

The consistency condition is established by exactly solving the initial value problem $y'(x) = 1$ with $y(0) = 0$. To solve this problem by an STSPM, the computed value $Y^{[n]}$ at the step n, is given by

$$Y^{[n]} = \mathbf{B}Y^{[n-1]} + h_n\mathbf{A}e + h_n\mathbf{R}e. \tag{3.1}$$

Thus, the consistency conditions are determined by satisfying $\mathbf{B}e = e$ and

$$c = \frac{1}{\delta_n}\mathbf{B}(c - e) + \mathbf{A}e + \mathbf{R}e, \tag{3.2}$$

which are derived from the condition ensuring that, when exact input values are used, the formula (3.1) of the method for the numerical solution of $y'(x) = 1$ with $y(0) = 0$, reproduces the exact solution after a single step. This leads to the following definition.

Definition 2. An STSPM is '*consistent*' if it is pre-consistent and satisfies (3.2).

3.2 Zero-stability

Zero-stability for an STSPM is assessed by considering its behavior when applied to the trivial one-dimensional differential equation $y'(x) = 0$. The computed value $Y^{[n]}$ using an STSPM to solve $y'(x) = 0$ is given by

$$Y^{[n]} = \mathbf{B}Y^{[n-1]} = \Big(\mathbf{B}(\delta_n)\mathbf{B}(\delta_{n-1})\cdots\mathbf{B}(\delta_1)\Big)Y^{[0]}. \tag{3.3}$$

It should be noted that the matrices $\mathbf{B}(\delta_i)$, $n = 1, 2, \dots, n$, depending on the step size ratio present significant theoretical challenges related to the uniform boundedness of long matrix products [12], i.e.,

$$\exists C > 0,\ \ \forall n \geq 1 : \ \ \Big\|\mathbf{B}(\delta_n)\mathbf{B}(\delta_{n-1})\cdots\mathbf{B}(\delta_1)\Big\| \leq C.$$

Therefore, throughout the paper, the matrix $\mathbf{B}(\delta_n)$ is treated as a rank-one matrix, i.e., $\mathbf{B}(\delta_n) = eb^T(\delta_n)$, where $b(\delta_n) = \begin{bmatrix} b_1(\delta_n) & b_2(\delta_n) & \dots & b_s(\delta_n) \end{bmatrix}^T$, with $b^T(\delta_n)e = 1$. The latter guarantees that the method (2.1) is '*unconditionally zero-stable*' for any step size pattern. In fact, for such a case, the recursion (3.3) can be written in the form

$$Y^{[n]} = \Big(\prod_{j=0}^{n-1} \mathbf{B}(\delta_{n-j})\Big) Y^{[0]} = \Big(\prod_{j=0}^{n-1} eb^T(\delta_{n-j})\Big) Y^{[0]} = \big(eb^T(\delta_1)\big) Y^{[0]},$$

in which the matrix $eb^T(\delta_1)$ is a rank-one matrix with a nonzero eigenvalue equal to one. It is well known that the zero-stability condition is equivalent to the following criterion.

Theorem 1. *(cf. [15]) The STSPM* (2.1) *is zero-stable if no root of the minimal polynomial of the coefficient matrix* $\boldsymbol{B}$ *lies outside of the unit circle and any root on the unit circle is simple.*

The following property explains that an STSPM shares the same conditions associated with the same concepts as it does with a TSPM.

Property 1. If the TSPM with the coefficient matrices $\mathbf{B}$, $\mathbf{A}$, and $\mathbf{R}$ is consistent and zero-stable, then for those same coefficient matrices, the STSPM incorporating two arbitrary additional coefficient matrices $\overline{\mathbf{A}}$ and $\overline{\mathbf{R}}$ is consistent and zero-stable.

3.3 Convergence

The convergence of the class of STSPMs in the fixed stepsize environment is investigated under the assumption that there exists a starting procedure $S_h : \mathbb{R}^m \to \mathbb{R}^{ms}$ which associates with every stepsize h such that

$$\lim_{h\to 0} S_h(y_0) = \lim_{h\to 0} Y^{[0]} = (e \otimes I_m)y_0. \tag{3.4}$$

Our aim is to obtain a good approximation at some $\overline{x} \in [x_0, X]$, which should converge to $y(\overline{x})$ for any problem satisfying a Lipschitz conditions for the functions f and $g = f_y f$.

Definition 3. An STSPM (2.1) is '*convergent*' if for any initial value problem (1.1), subject to the Lipschitz conditions $\|f(y) - f(z)\| \le L\|y - z\|$ and $\|g(y) - g(z)\| \le K\|y - z\|$, there exists a starting procedure S_h satisfying (3.4), such that the sequence of the vectors $Y^{[n]}$, computed using n steps with stepsize $h = (\overline{x} - x_0)/n$ for any $\overline{x} \in [x_0, X]$ and using $S_h(y_0) = Y^{[0]}$, converges to $(e \otimes I_m)y(\overline{x})$.

The following theorem states that consistent and zero-stable STSPMs are convergent, which can be deduced by applying Theorem 2.3.4 in [15] related to general linear methods as a general framework of the traditional multivalue/multistage methods.

Theorem 2. *A consistent and zero-stable STSPM is convergent.*

4 Order conditions and stability analysis of STSPMs

In this section, we aim to derive the framework for the order conditions of STSPMs (2.1) in the general form. To accomplish this, we first introduce the residuals, denoted as Δ_{ni}, $i = 1, 2, \ldots, s$, corresponding to the methods (2.1), as

$$\begin{aligned}\Delta_{ni} &= y(x_{n-1} + c_i h_n) - \sum_{j=1}^{s} b_{ij} y(x_{n-1} + (c_j - 1)h_{n-1})\\ &- h_n \sum_{j=1}^{s} a_{ij} y'(x_{n-1} + (c_j - 1)h_{n-1}) - h_n^2 \sum_{j=1}^{s} \overline{a}_{ij} y''(x_{n-1} + (c_j - 1)h_{n-1})\\ &- h_n \sum_{j=1}^{i-1} r_{ij} y'(x_{n-1} + c_j h_n) - h_n^2 \sum_{j=1}^{i-1} \overline{r}_{ij} y''(x_{n-1} + c_j h_n),\end{aligned}$$

for $n = 1, 2, \ldots, N$. We have the following definition.

Definition 4. The STSPM (2.1) is said to be consistent of order p if the conditions

$$\Delta_{ni} = \mathcal{O}(h_n^{p+1}), \quad n = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, s,$$

are satisfied for all sufficiently smooth functions $y(x)$.

The conditions for the order of STSPMs are given in the following theorem.

Theorem 3. *The STSPMs* (2.1) *is of order p if and only if*

$$\begin{aligned}c_i^k &- \sum_{j=1}^{s} b_{ij} \frac{(c_j - 1)^k}{\delta_n^k} - \sum_{j=1}^{s} k a_{ij} \frac{(c_j - 1)^{k-1}}{\delta_n^{k-1}} - \sum_{j=1}^{s} k(k-1) \overline{a}_{ij} \frac{(c_j - 1)^{k-2}}{\delta_n^{k-2}}\\ &- \sum_{j=1}^{s} k r_{ij} c_j^{k-1} - \sum_{j=1}^{s} k(k-1) \overline{r}_{ij} c_j^{k-2} = 0, \quad k \geq 1. \qquad (4.1)\end{aligned}$$

It should be noted that by substituting $k = 0$ into the order conditions (4.1), one can derive the pre-consistency condition $\mathbf{B}e = e$ for the STSPMs (2.1). The following corollary, which provides an alternative and more compact representation of the order condition (4.1) for the STSPMs, follows directly from Theorem 3.

Corollary 1. The STSPMs (2.1) is of order p if and only if

$$\begin{aligned}\exp(c\delta_n z) &= \mathbf{B} \exp((c - e)z) + \mathbf{A}\delta_n z \exp((c - e)z) + \overline{\mathbf{A}}\delta_n^2 z^2 \exp((c - e)z)\\ &+ \mathbf{R}\delta_n z \exp(c\delta_n z) + \overline{\mathbf{R}}\delta_n^2 z^2 \exp(c\delta_n z) + \mathcal{O}(z^{p+1}), \qquad (4.2)\end{aligned}$$

where the function exp is applied componentwise to a vector.

Proof. Multiplying both sides of the order conditions (4.1) by $\delta_n^k z^k/k!$ and then summing over $k = 0, 1, \ldots, p$, we obtain the relation (4.2), which completes the proof. □

The next theorem presents a connection among the coefficient matrices of an s-stage STSPM of the order $p = s$ as an equivalent condition for the order conditions of these methods.

Theorem 4. *Suppose that the components of the abscissa vector c are distinct. Then the STSPMs (2.1) is of order $p = s$ if and only if $\boldsymbol{B}e = e$ and*

$$\mathbf{A} = \big(C_1 D - \delta_n^{-1}\mathbf{B}\widetilde{C}_1 - \delta_n \overline{\boldsymbol{A}}\widetilde{C}_0 K - \mathbf{R}C_0 D - \overline{\mathbf{R}}C_0 K D\big)\widetilde{C}_0^{-1}, \tag{4.3}$$

where

$$D = \operatorname{diag}\left\{1, \delta_n, \delta_n^2, \ldots, \delta_n^{s-1}\right\}, K = \begin{bmatrix} 0 & e_1 & e_2 & \ldots & e_{s-1} \end{bmatrix}, C_1 = \left[\frac{c_i^j}{j!}\right]_{i,j=1}^s,$$

$$\widetilde{C}_1 = \left[\frac{(c_i-1)^j}{j!}\right]_{i,j=1}^s, \quad C_0 = \left[\frac{c_i^{j-1}}{(j-1)!}\right]_{i,j=1}^s, \quad \widetilde{C}_0 = \left[\frac{(c_i-1)^{j-1}}{(j-1)!}\right]_{i,j=1}^s,$$

with e_j as the jth unit vector in $\mathbb{R}^s$.

Proof. Expressing the resulting equations from (4.1) as a system of linear equations gives the required result of the theorem. □

The rest of this section focuses on analyzing the linear stability behavior of STSPMs (2.1) in a fixed stepsize environment. In fact, we investigate the stability property of these methods with a fixed stepsize, i.e., $\delta_n = 1$, on the Dahlquist linear test problem $y'(x) = \xi y(x)$, wherein ξ represents a possibly complex number. By applying the method (2.1) to this test problem and setting $z = h\xi$, we derive $Y^{[n]} = M(z)Y^{[n-1]}$, where $M(z)$ refers to the "*stability matrix*" of the method, defined as

$$M(z) = \big(I_s - z\mathbf{R} - z^2\overline{\mathbf{R}}\big)^{-1}\big(\mathbf{B} + z\mathbf{A} + z^2\overline{\mathbf{A}}\big).$$

Therefore, the stability polynomial $\phi(w, z)$, representing the characteristic polynomial of $M(z)$ and governing the stability properties of the STSPMs (2.1), is defined by

$$\frac{\phi(w, z)}{\det(I_s - z\mathbf{R} - z^2\overline{\mathbf{R}})} = \det(wI_s - M(z)),$$

with $w \in \mathbb{C}$. These motivate us the standard definitions that follow.

Definition 5. STSPM (2.1) is said to be '*absolutely stable*' for given $z \in \mathbb{C}$ if for that z the stability matrix $M(z)$ is power bounded, i.e.,

$$\sup_{n=1}^{\infty}\big\|M(z)^n\big\| < \infty.$$

Equivalently, we say that the STSPM (2.1) is absolutely stable if for given $z \in \mathbb{C}$, all the roots $w_i = w_i(z)$, $i = 1, 2, \ldots, s$, of the stability function $\phi(w, z)$ lie inside the unit circle.

DEFINITION 6. Region $\mathcal{S}$ of absolute stability of (2.1) is the set of all $z \in \mathbb{C}$ such that the method is absolutely stable, i.e.,

$$\mathcal{S} = \left\{ z \in \mathbb{C} : \ |w_i(z)| < 1, \ \ i = 1, 2, \dots, s \right\}.$$

An interval $\mathcal{I}$ of absolute stability is the intersection of $\mathcal{S}$ with the negative real axis, i.e., $\mathcal{I} = \mathcal{S} \bigcap \mathbb{R}^-$.

Since, in the present article, the matrices $\mathbf{R}$ and $\overline{\mathbf{R}}$ are strictly lower triangular matrices, the matrix $(I - z\mathbf{R} - z^2\overline{\mathbf{R}})$ is triangular with unit diagonal entries, and therefore $\det(I_s - z\mathbf{R} - z^2\overline{\mathbf{R}}) = 1$. Here, we focus on the explicit methods with "Runge–Kutta stability" (RKS) property in the sense that the stability polynomial $\phi(w, z)$ possesses a specific structure as

$$\phi(w, z) = w^{s-1}(w - R(z)),$$

where $R(z)$ represents an approximation of the order p to the exponential function $\exp(z)$, that is

$$R(z) = \sum_{i=0}^{p} \frac{z^i}{i!} + \sum_{i=1}^{p} \gamma_{p+i} z^{p+i} + \mathcal{O}(z^{2p+1}), \tag{4.4}$$

and γ_{p+i}, $i = 1, \dots, p$, are expressed in terms of the method's coefficients. The conditions related to the RKS property, within the fixed step-size environment, will be outlined in the next section.

5 Construction of the explicit STSPMs

In this section, we are going to construct s-stage explicit STSPMs of order $p = s \le 5$ in the fixed stepsize environment equipped with the RKS property. One can see that the formula (4.3) in the case of fixed stepsize is simplified to

$$\mathbf{A} = \left(C_1 - \mathbf{B}\widetilde{C}_1 - \overline{\mathbf{A}}\widetilde{C}_0 K - \mathbf{R}C_0 - \overline{\mathbf{R}}C_0 K\right)\widetilde{C}_0^{-1}, \tag{5.1}$$

where the matrices C_0, $\widetilde{C}_0$, C_1, and $\widetilde{C}_1$ defined in Theorem 4. It should be noted that the developed methods in this section can be written in a variable stepsize environment by computing the matrix $\mathbf{A}$ using formula (4.3).

Here, we construct the methods with $c_s = 1$, allowing $Y_s^{[n]}$ to serve as an approximation of the solution y at the grid point x_n. We also consider that the remaining elements in the abscissa vector are presumed to be less than one. Furthermore, the matrices $\mathbf{B}$, $\overline{\mathbf{A}}$, $\mathbf{R}$, and $\overline{\mathbf{R}}$ will be considered as independent of the stepsize ratio δ_n, with a specific structure designated for the matrix $\mathbf{B}$, which will be represented as a rank-one matrix in the form

$$\mathbf{B} = e \cdot [1 - \mathbf{b} \quad b_1 \quad b_2 \quad \cdots \quad b_{s-1}],$$

where $\mathbf{b} = \sum_{i=1}^{s-1} b_i = 1$. This structure ensures the zero-stability property of the methods. For this class of STSPMs, the stability polynomial $\phi(w, z)$ takes the form

$$\phi(w, z) = w^s - \phi_{s-1}(z)w^{s-1} + \cdots + (-1)^{s-1}\phi_1(z)w + (-1)^s\phi_0(z),$$

where, ϕ_i, $i = 0, 1, \ldots, s-1$, are polynomials of degree less than or equal to $2s$ in terms of z. These polynomials take the form

$$\phi_k(z) = \sum_{l=s-k-1}^{2s} \phi_{k,l} z^l, \quad k = 0, 1, \ldots, s-1,$$

with $\phi_{s-1,0} = \operatorname{tr}(\mathbf{B}) = 1$, in which their coefficients ϕ_{ij} depend on the entries b_i, $i = 1, 2, \ldots, s-1$, $\overline{a}_{ij}$, $i, j = 1, 2, \ldots, s$, r_{ij} and $\overline{r}_{ij}$, $i = 2, 3, \ldots, s$, $j = 1, 2, \ldots, i-1$. The conditions for the STSPMs to possess RKS property are given by

$$\phi_{k,l} = 0, \quad k = 0, 1, \ldots, s-2, \quad l = s-k-1, s-k, \ldots, 2s, \tag{5.2}$$

which are referred to as RKS conditions. Therefore, these conditions result in a system of $(3s^2 - s - 2)/2$ nonlinear equations. Moreover, for the RKS methods of orders $p = s$, we have $R(z) = \phi_{s-1}(z)$ and

$$\exp(z) - R(z) = \sum_{i=1}^{s} E_{p+i} z^{p+i} + \mathcal{O}(z^{2s+1}), \tag{5.3}$$

where E_{p+i}, $i = 1, 2, \ldots, s$, are constant values given by

$$E_{p+i} = \frac{1}{(p+i)!} - \gamma_{p+i}. \tag{5.4}$$

Different choices of these constants yield different methods, and E_{p+1} is known as the error constant. Actually, to construct a method with some specific constants E_{p+i}, $i = 1, 2, \ldots, s$, we need to impose the conditions (5.3), referred to as the E-conditions, which will lead to s additional nonlinear equations. Therefore, we need to solve $(3s^2 + s - 2)/2$ nonlinear equations for some of the $2s^2 - 1$ unknown entries of the coefficient matrices $\mathbf{B}$, $\overline{\mathbf{A}}$, $\mathbf{R}$, and $\overline{\mathbf{R}}$.

It can be seen that following the solutions to equations (5.2) and (5.3), several method coefficients remain undetermined as free parameters. Here, these parameters are used to construct methods with a large stability region or even stability interval. Additionally, to evaluate and compare the stability regions of the proposed methods, we consider TSPMs with RKS properties of orders 1 to 4, as detailed in [2]. The authors in [2] developed a practical variable stepsize/order (VSVO) code leveraging this class of methods. It should be noted that the stability regions of these methods coincide with those of explicit RK methods of orders $p = s \leq 4$.

5.1 Explicit STSPMs of order one

In this subsection, we investigate deriving one-stage explicit STSPMs of order $p = 1$ with the abscissa vector $c = 1$. For these methods, we have $\mathbf{B} = 1$, $\overline{\mathbf{A}} = \overline{a}$, $\mathbf{R} = \overline{\mathbf{R}} = 0$. According to the formula (5.1), this results in $\mathbf{A} = 1$. One can see that for the variable stepsizes, the value of $\mathbf{A}$, calculated using formula (4.3), is also equal to 1. Therefore, the coefficient matrices of these methods are independent of stepsize ratios.

The stability polynomial of these methods has the form

$$R(z) = 1 + z + \overline{a}z^2.$$

The maximum length of the stability interval for these methods is achieved for the error constant $E_2 \approx 0.356$, which is $\approx (-6.95, 0)$. Therefore, from (5.3), we have $\overline{a} = \frac{737}{5120}$. Alternatively, setting $\overline{a} = \frac{1}{4}$ results in a smaller error constant $E_2 = \frac{1}{4}$, along with a considerable stability interval $\approx (-4, 0)$. In Figure 1, the absolute stability regions of the explicit STSPMs (with large and small error constants) alongside explicit TSPM method of order one have been plotted. Notably, it is clear that the stability regions for the explicit STSPMs with different error constants are larger than that for the TSPM of the same order.

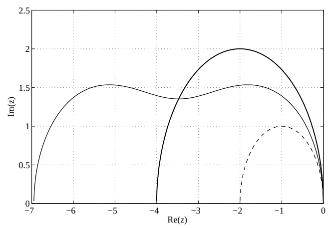


Figure 1. Regions of absolute stability for STSPM of order $p = 1$ for $\overline{a} = \frac{737}{5120}$ (thin solid line), for $\overline{a} = \frac{1}{4}$ (thick solid line) and TSPM of order $p = 1$ (dashed line).

5.2 Explicit STSPMs of order two

Here, we investigate the construction of two-stage explicit STSPMs of order $p = 2$ with the abscissa vector $c = [0 \;\; 1]^T$ equipped with RKS property.

The stability polynomial corresponding to these methods takes the form (4.4). Thus, according to E-condition (5.3) with (5.4), we have E_3 and E_4 as the constants of the methods. By setting $b_1 = \frac{1}{4}$, applying the order conditions (5.1) to compute the matrix $\mathbf{A}$, and solving the system of nonlinear equations arising from the RKS conditions (5.2) and E-condition (5.3) for the error constants $E_3 = -E_4 = -\frac{1}{100}$ results in a method with a favorable balance between the accuracy and stability. The coefficient matrices of the derived method are

$$\mathbf{B} = e \cdot \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \; \overline{\mathbf{A}} = \begin{bmatrix} \frac{9}{64} & \frac{3}{64} \\ -\frac{137}{4800} & -\frac{137}{14400} \end{bmatrix}, \; \mathbf{R} = \begin{bmatrix} 0 & 0 \\ \frac{16}{15} & 0 \end{bmatrix}, \; \overline{\mathbf{R}} = \begin{bmatrix} 0 & 0 \\ \frac{152}{225} & 0 \end{bmatrix},$$

and interval of absolute stability is $\approx (-3.63, 0)$. The matrix $\mathbf{A}$ computed by formula (4.3) for the method in a variable stepsize environment is given by

$$\mathbf{A} = \begin{bmatrix} \frac{3\delta_n^2+6}{16\delta_n} & -\frac{3\delta_n^2-6}{16\delta_n} \\ \frac{11\delta_n^2+30}{80\delta_n} & -\frac{33\delta_n^2+16\delta_n-90}{240\delta_n} \end{bmatrix}.$$

The region of absolute stability of the constructed explicit STSPMs has been plotted in Figure 2, along with that of the explicit TSPM of order two. Moreover, we have added the absolute stability region of the explicit STSPMs in which the free parameters, except $b_1 = \frac{1}{4}$, are obtained in such a way that the method has a maximum area of the stability region which leads to $E_3 \approx \frac{386}{4641}$ and $E_4 \approx \frac{69}{1856}$ and the stability interval $\approx (-8.88, 0)$. Clearly, the stability regions for the explicit STSPMs are larger than those for the TSPM method of the same order.

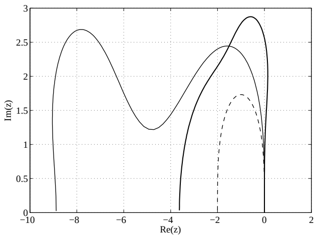


Figure 2. Regions of absolute stability for STSPM of order $p = 2$ with large error constant (thin solid line), with small error constant (thick solid line) and TSPM of order $p = 2$ (dashed line).

5.3 Explicit STSPMs of order three

In this subsection, we construct three-stage explicit STSPMs of order $p = 3$ with the abscissa vector $c = [0 \;\; \frac{1}{2} \;\; 1]^T$ equipped with RKS property.

The stability polynomial corresponding to these methods takes the form (4.4). Thus, according to E-condition (5.3) with (5.4), we have E_4, E_5, and E_6 as the constants of the methods. It should be noted that for deriving the coefficient matrices of the methods of orders greater than or equal to three, solving the nonlinear equations arising from (5.2) and (5.3) symbolically is not possible. Here, this system of nonlinear equations is numerically solved through the `fsolve.m` subroutine in MATLAB, employing the 'levenberg-marquardt' algorithm. Then, applying the order conditions (5.1) to compute the matrix $\mathbf{A}$, and numerically solving the system of nonlinear equations arising from (5.2) and (5.3) for the values $E_4 = 10^{-3}$, $E_5 = 0.27 \times 10^{-2}$, and $E_6 = 0.11 \times 10^{-2}$, we derive a method that strikingly balances the accuracy and stability. The coefficient matrices of the constructed method are given as

$$\mathbf{B} = e \cdot \begin{bmatrix} -0.08348102307442 & 0.414486043118231 & 0.668994979956186 \end{bmatrix},$$

$$\overline{\mathbf{A}} = \begin{bmatrix} 0.083871481282502 & -0.047835100013298 & 0.016760184563685 \\ 0.106634214262270 & -0.086047346176656 & 0.048804581818 \\ 0.100161763102066 & -0.106266604919018 & 0.073569348928976 \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0.422013981685835 & 0 & 0 \\ 0.171812092400260 & 0.699392761176122 & 0 \end{bmatrix},$$

$$\overline{\mathbf{R}} = \begin{bmatrix} 0 & 0 & 0 \\ 0.179135800997617 & 0 & 0 \\ 0.088625822919000 & 0.100721777496547 & 0 \end{bmatrix}.$$

The interval of absolute stability for this method is $\approx (-7.37, 0)$. Figure 3 presents the region of absolute stability for the constructed explicit STSPMs of order three, alongside that of the explicit TSPM of the same order. Clearly, the stability region for the explicit STSPM is significantly larger than that for the TSPM of the same order.

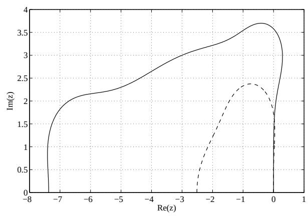


Figure 3. Regions of absolute stability for STSPM of order $p = 3$ (solid line) and TSPM of order $p = 3$ (dashed line).

The matrix **A** for the method in a variable stepsize environment is computed by the formula (4.3). This leads to a matrix with the entries of lengthy expressions in terms of δ_n, which is not reported here and also will be not reported for the methods of orders four and five in the next subsections.

5.4 Explicit STSPMs of order four

Here, we are going to construct four-stage explicit STSPMs of order $p = 4$ with the abscissa vector $c = [0\ \frac{1}{3}\ \frac{2}{3}\ 1]^T$ equipped with RKS property.

The stability polynomial corresponding to these methods takes the form (4.4). Thus, according to E-condition (5.3) with (5.4), we have E_{4+i}, $i = 1, 2, 3, 4$, as the constants of the methods. Again, applying the order conditions (5.1) to compute the matrix **A** and numerically solving the system of nonlinear equations arising from (5.2) and (5.3) for the values $E_5 = 10^{-4}$, $E_6 = 0.28 \times 10^{-3}$, $E_7 = 0.11 \times 10^{-3}$, and $E_8 = 0.02 \times 10^{-3}$, we derive a method that maintains a balance between the accuracy and stability. The entries of the coefficient matrices of the derived method are

$b_1 = -2.13364983823225,\ \ b_2 = 3.48787969569445,\ \ b_3 = -0.963051124082518,$
$b_4 = 0.608821266620,\ \ \overline{a}_{11} = 0.251312480029256,\ \ \overline{a}_{12} = 0.220719436749542,$
$\overline{a}_{13} = -0.304085051254224,\ \ \overline{a}_{14} = -0.096749206480088,\ \ \overline{a}_{21} = 0.005302091474058,$
$\overline{a}_{22} = -0.112355410642796,\ \ \overline{a}_{23} = 0.062868461821452,\ \ \overline{a}_{24} = -0.019412257064537,$
$\overline{a}_{31} = -0.269902891996826,\ \ \overline{a}_{32} = 0.056556764925979,\ \ \overline{a}_{33} = -0.027755018013074,$
$\overline{a}_{34} = 0.048111621772497,\ \ \overline{a}_{41} = -0.678650538505617,\ \ \overline{a}_{42} = -0.246677279176850,$
$\overline{a}_{43} = 0.394117991310805,\ \ \overline{a}_{44} = 0.191890649847907,\ \ r_{21} = -0.192019876450987,$
$r_{31} = -1.232666430414977,\ \ r_{32} = 0.418772173658379,\ \ r_{41} = -0.984769574547910,$

$r_{42} = -0.520902729218407$, $r_{43} = 0.738370811443188$, $\overline{r}_{21} = -0.017224290350414$,
$\overline{r}_{31} = -0.086518568370296$, $\overline{r}_{32} = 0.027389668154099$, $\overline{r}_{41} = -0.119550782154535$,
$\overline{r}_{42} = -0.036241064274140$, $\overline{r}_{43} = 0.056896671139028$.

The achieved interval of absolute stability for this method is $\approx (-10.07, 0)$. Figure 4 represents the region of absolute stability for the derived explicit STSPM of order four, alongside that of the explicit TSPM of the same order. One can see that the stability region for the explicit STSPM is considerably more extensive than that of the TSPM method of the same order.

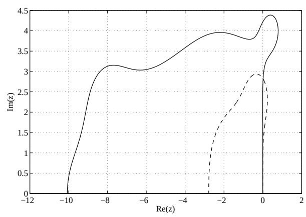


Figure 4. Regions of absolute stability for STSPM of order $p = 4$ (solid line) and TSPM of order $p = 4$ (dashed line).

5.5 Explicit STSPMs of order five

In this subsection, we investigate five-stage explicit STSPMs of order $p = 5$ with the abscissa vector $c = [0\ \frac{1}{4}\ \frac{1}{2}\ \frac{3}{4}\ 1]^T$ equipped with RKS property.

The stability polynomial corresponding to these methods takes the form (4.4). Thus, according to E-condition (5.3) with (5.4), we have E_{5+i}, $i = 1, \ldots, 5$, as the constants of the methods. For the methods of orders greater than or equal to five, it is not possible to produce ϕ_{ij} of the stability polynomial $\phi(w, z)$ using symbolic manipulation tools such as Mathematica or Maple. Therefore, we need to utilize a different alternative approach. To address this, we will utilize the Fourier series approach, which has previously been discussed in [5] (see also [15,25]). Now, setting $E_6 = 10^{-5}$, $E_7 \approx 0.16 \times 10^{-4}$, $E_8 \approx 0.07 \times 10^{-4}$, $E_9 \approx 0.02 \times 10^{-4}$, and $E_{10} \approx 0.02 \times 10^{-5}$, applying the order conditions (5.1) to compute matrix $\mathbf{A}$, and numerically solving the system of nonlinear equations arising from (5.2) and (5.3) using the Fourier series approach, we derive a method with a nice balance between the accuracy and stability. The entries of the coefficient matrices of the derived method are

$b_1 = -3.31058370546993$, $b_2 = 4.65079833480428$, $b_3 = 1.41074731122409$,
$b_4 = -1.77845963544530$, $b_5 = 0.027497694886857$, $\overline{a}_{11} = -0.242967508966694$,
$\overline{a}_{12} = 0.282302004623456$, $\overline{a}_{13} = 0.739434955152069$, $\overline{a}_{14} = -0.002620522681910$,
$\overline{a}_{15} = 0.000463220106798$, $\overline{a}_{21} = -0.376209189974481$, $\overline{a}_{22} = -0.047987954774102$,
$\overline{a}_{23} = 0.981654630373294$, $\overline{a}_{24} = -0.122295429260655$, $\overline{a}_{25} = 0.002051053762716$,
$\overline{a}_{31} = -0.483062138169932$, $\overline{a}_{32} = -0.293123160957164$, $\overline{a}_{33} = 1.168508710011867$,
$\overline{a}_{34} = -0.211694546044386$, $\overline{a}_{35} = 0.003227436683576$, $\overline{a}_{41} = -0.690340815870138$,

$\overline{a}_{42} = -0.811480081386055$, $\overline{a}_{43} = 1.542927259266087$, $\overline{a}_{44} = -0.394203681643298$, $\overline{a}_{45} = 0.005633625796402$, $\overline{a}_{51} = -2.812375744930116$, $\overline{a}_{52} = -5.323659116952385$, $\overline{a}_{53} = 4.795877833119971$, $\overline{a}_{54} = -2.568794473543036$, $\overline{a}_{55} = 0.036983673661145$, $r_{21} = 0.608927934594683$, $r_{31} = 1.000036638795209$, $r_{32} = 0.152832523980261$, $r_{41} = 1.108353674429744$, $r_{42} = 1.770699336020147$, $r_{43} = -0.959694175697170$, $r_{51} = -1.775617238588581$, $r_{52} = 1.733397491990266$, $r_{53} = -1.396104451843886$, $r_{54} = 3.239799700664664$, $\overline{r}_{21} = 0.026401059553080$, $\overline{r}_{31} = 0.014069627272872$, $\overline{r}_{32} = 0.054735530274526$, $\overline{r}_{41} = -0.029245142983725$, $\overline{r}_{42} = 0.147862304904127$, $\overline{r}_{43} = 0.182855334040056$, $\overline{r}_{51} = 3.958934774781318$, $\overline{r}_{52} = -2.281989507297899$, $\overline{r}_{53} = -1.430926300347974$, $\overline{r}_{54} = 0.306972014632235$.

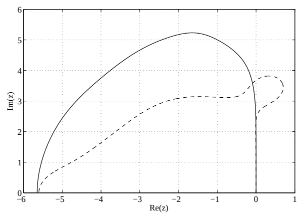


Figure 5. Regions of absolute stability for STSPM of order $p = 5$ (solid line) and RK method of order $p = 5$ (dashed line).

The interval of absolute stability for this method is $\approx (-5.65, 0)$. The regions of absolute stability for the constructed explicit STSPM of order five and for the 6-stage explicit Runge–Kutta method of order five, have been plotted in Figure 5. It is clear that the stability region of the proposed explicit STSPM is larger than that of the Runge–Kutta method of the same order.

6 Numerical experiments

This section is devoted to demonstrating the accuracy and efficiency of the proposed STSPMs and verifying the theoretical results on the order of convergence of these methods. We will test the methods on a selection of well-known non-stiff or mildly stiff problems, examining their performance in both fixed and variable stepsize environments. Furthermore, the results of TSPMs with orders up to four are reported for comparison. Also, we give the number of function evaluations, `nfe`, in terms of the global error of these methods and compare them with those of TSPMs with $p = s$.

To initiate the integration process using the proposed methods, we must first determine the starting vector $Y_i^{[0]}$, $i = 1, 2. \dots, s$. Since, for all the proposed methods, the first component of the abscissa vector is zero, it follows that $Y_1^{[0]} = y_0$. To derive approximations for $Y_i^{[0]}$, $i = 2, 3, \dots, s$, we utilize the `ode45` command from the MATLAB ODE suite. Moreover, we report the global error at the endpoint of integration as the difference between the solutions computed by the proposed methods and the code ode45 with tolerances $Atol = Rtol = 2.22045 \times 10^{-14}$.

The proposed methods are evaluated through computational experiments conducted on the following problems:

Example 1. The nonlinear system of ODEs [16]

$$\begin{cases} y_1'(x) = -\left(4+\epsilon^{-1}\right) y_1(x) + \epsilon^{-1} y_2(x)^4, & y_1(0)=1, \\ y_2'(x) = y_1(x) - y_2(x)\left(1+y_2(x)^3\right), & y_2(0)=1, \end{cases}$$

with $\epsilon = 10^{-1}$, the integration interval $x \in [0,2]$, and the exact solution $\left[y_1(x), y_2(x)\right]^T = [\exp(-4x), \exp(-x)]^T$.

Example 2. The famous nonlinear Van der Pol system [14]

$$\begin{cases} y_1'(x) = y_2(x), & y_1(0)=2, \\ y_2'(x) = (1-y_1(x)^2)y_2(x) - y_1(x), & y_2(0)=0, \end{cases}$$

with the integration interval $x \in [0,20]$.

Example 3. The Rigid Body system [14]

$$\begin{cases} y_1'(x) = y_2(x)y_3(x), & y_1(0)=0, \\ y_2'(x) = -y_1(x)y_3(x), & y_2(0)=1, \\ y_3'(x) = -0.51 y_1(x)y_2(x), & y_3(0)=1, \end{cases}$$

with the integration interval $x \in [0,10]$.

Example 4. The reaction–diffusion equation (Brusselator with diffusion) [14]

$$\begin{cases} \dfrac{\partial u}{\partial t} = A + u^2 v - (B+1)u + \alpha \dfrac{\partial^2 u}{\partial x^2}, \\ \dfrac{\partial v}{\partial t} = Bu - u^2 v + \alpha \dfrac{\partial^2 v}{\partial x^2}, \end{cases}$$

with $0 \le x \le 1$ which using the method of lines for the diffusion terms, the solutions $u(x_i,t)$ and $v(x_i,t)$, for $x_i = i/(\overline{N}+1)$, $i = 1,2,\ldots,\overline{N}$ and a given integer $\overline{N}$, can be approximated as the solutions $u_i(t)$ and $v_i(t)$ of system of ODEs

$$\begin{cases} u_i' = A + u_i^2 v_i - (B+1)u_i + \dfrac{\alpha}{(\Delta x)^2}(u_{i-1} - 2u_i + u_{i+1}), \\ v_i' = Bu_i - u_i^2 v_i + \dfrac{\alpha}{(\Delta x)^2}(v_{i-1} - 2v_i + v_{i+1}), \end{cases}$$

for $i = 1,2,\ldots,\overline{N}$. We consider $\overline{N} = 25$ leading to a mildly stiff problem in a higher dimension $2 \cdot N = 50$. Following [14], we take the parameters of the problem as $A = 1$, $B = 3$, $\alpha = 1/50$, $\Delta x = 1/(\overline{N}+1)$, the initial values as $u_i(0) = 1 + \sin(2\pi x_i)$, and $v_i(0) = 3$, for $i = 1,2,\ldots,\overline{N}$, and Dirichlet boundary conditions as $u_0 = u_{\overline{N}+1} = 1$ and $v_0 = v_{\overline{N}+1} = 3$, with $t_{out} = 10$.

6.1 Fixed stepsize experiments

In this subsection, we present the numerical results of the methods in a fixed stepsize environment to demonstrate the accuracy of the proposed STSPMs. Also, this serves to confirm the order of these methods when applied to the integration of non-stiff or mildly stiff problems. To this end, we apply the methods to Example 1, implementing them with various fixed stepsizes h. The results of our numerical experiments are presented in Figure 6, where the logarithm of the global error for each method is plotted versus $\log(h)$ for different values of h. These results clearly confirm both the accuracy and the expected theoretical order of convergence of the proposed methods. Moreover, as illustrated in this Figure, the results obtained with the developed STSPMs demonstrate greater accuracy compared to the TSPMs of equivalent order outlined in [2].

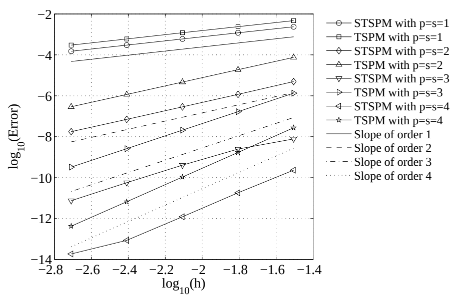


Figure 6. The numerical results of the STSPMs and TSPMs for Example 1.

6.2 Variable stepsize experiments

In this subsection, we discuss the results of numerical experiments for the proposed STSPMs of the orders $2 \le p \le 5$ in a variable stepsize (VS) environment using the rapidly changing stepsize pattern as [15]

$$h_{n+1} = \rho^{(-1)^n \sin(4\pi n/(X-x_0))} h_n,$$

where the initial stepsize h_0 is calculated as $(X - x_0)/N$, for a given integer N as the number of steps. We uniformly rescale the resulting grid points x_n for $n = 0, 1, \dots, N$ so that $x_N = X$. In our numerical experiments, we consider two different values for ρ as $\rho = 2$ and $\rho = 4$. In each case, we provide a numerical estimation of the convergence order p, computed by the formula $\mathrm{O}_N := \log(\mathtt{ge}_1/\mathtt{ge}_2)/\log(N_2/N_1)$. Here, $\mathtt{ge}_1$ and $\mathtt{ge}_2$ represent the global errors associated with the methods implemented with N_1 and N_2 steps, respectively.

The results of numerical experiments for Examples 1–4 are given in Tables 1–7. These results vividly demonstrate the high accuracy of the methods and provide a numerical estimation of their convergence order, aligning perfectly with theoretical expectations. Once more, when these methods are applied in a variable stepsize mode, the tables demonstrate that the developed STPMs yield more accurate results compared to the TSPMs of the same order examined in [2].

Moreover, in Figures 7 and 8, the global errors of the STSPMs and TSPMs of orders $p = 2, 3, 4$ have been plotted in terms of the required number of function evaluations, `nfe`, for the case $\rho = 2$. In these figures, `nfe` indicates the number of function f evaluations for TSPMs, and the total evaluations of functions f and g for STSPMs. These results show that the developed methods are more cost-effective than TSPMs. It should be mentioned that these are simple comparisons for the case that the second derivative function g is explicitly known and is not computed by the formula $f_y f$. Generally, a fair comparison can be carried out through a practical implementation of the methods in a variable stepsize environment, where, in order to reduce computational effort, the evaluation of the Jacobian matrix at each stage and step is avoided. In fact, the same approximation for the Jacobian matrix may be used not just over a single step, but over many steps. This will be addressed in future work.

Table 1. The numerical results of the STSPMs and TSPMs with $\rho = 2$ for Example 1.

	N	500	1000	2000	4000	8000
STSPM with $p = s = 2$	ge	7.42e-8	1.84e-8	4.57e-9	1.14e-9	2.85e-10
	O_N		2.01	2.00	2.00	2.00
TSPM with $p = s = 2$	ge	1.22e-6	3.04e-7	7.60e-8	1.90e-8	4.75e-9
	O_N		2.00	2.00	2.00	2.00
	N	100	200	400	800	1600
STSPM with $p = s = 3$	ge	4.42e-9	8.05e-10	1.16e-10	1.54e-11	1.99e-12
	O_N		2.46	2.80	2.91	2.95
TSPM with $p = s = 3$	ge	3.61e-7	4.44e-8	5.50e-9	6.84e-10	8.54e-11
	O_N		3.03	3.01	3.01	3.00
	N	100	150	200	250	300
STSPM with $p = s = 4$	ge	4.80e-11	1.00e-11	3.24e-12	1.35e-12	6.55e-13
	O_N		3.87	3.92	3.94	3.95
TSPM with $p = s = 4$	ge	4.77e-9	9.27e-10	2.91e-10	1.19e-10	5.70e-11
	O_N		4.04	4.03	4.02	4.02
	N	100	120	140	160	180
STSPM with $p = s = 5$	ge	9.85e-13	3.17e-13	1.22e-13	5.55e-14	2.58e-14
	O_N		6.22	6.19	5.90	3.98

7 Concluding remarks and future work

In this paper, we introduced a new class of numerical methods for solving ODEs, referred to as STSPMs, as an extension of two-step peer methods. We thoroughly derived the order conditions and analyzed the stability behavior of the proposed methods. Moreover, we constructed some examples of the explicit STSPMs up to order five possessing the RKS property. The proposed explicit STSPMs have more desirable stability properties than explicit TSPMs of the same order. The results of some numerical experiments conducted in both fixed

Table 2. The numerical results of the STSPMs and TSPMs with $\rho = 2$ for Example 2.

	N	1000	2000	4000	8000	16000
STSPM with $p = s = 2$	ge	2.63e-4	7.22e-5	1.88e-5	4.80e-6	1.21e-6
	O_N		1.87	1.94	1.97	1.99
TSPM with $p = s = 2$	ge	3.53e-3	8.99e-4	2.27e-4	5.69e-5	1.42e-5
	O_N		1.97	1.99	1.99	2.00
	N	1000	2000	4000	8000	16000
STSPM with $p = s = 3$	ge	5.80e-6	7.11e-7	8.79e-8	1.09e-8	1.36e-9
	O_N		3.03	3.02	3.01	3.00
TSPM with $p = s = 3$	ge	3.62e-5	4.39e-6	5.40e-7	6.69e-8	8.33e-9
	O_N		3.04	3.02	3.01	3.01
	N	3000	3500	4000	4500	5000
STSPM with $p = s = 4$	ge	1.60e-10	9.63e-11	6.07e-11	3.92e-11	2.66e-11
	O_N		3.31	3.46	3.71	3.68
TSPM with $p = s = 4$	ge	4.94e-9	2.68e-9	1.58e-9	9.86e-10	6.48e-10
	O_N		3.31	3.46	3.71	3.68
	N	2000	2250	2500	2750	3000
STSPM with $p = s = 5$	ge	7.32e-11	3.74e-11	2.18e-11	1.40e-11	9.69e-12
	O_N		5.70	5.15	4.66	4.19

Table 3. The numerical results of the STSPMs and TSPMs with $\rho = 4$ for Example 2.

	N	1000	2000	4000	8000	16000
STSPM with $p = s = 2$	ge	4.60e-4	1.34e-4	3.56e-5	9.18e-6	2.33e-6
	O_N		1.78	1.91	1.96	1.98
TSPM with $p = s = 2$	ge	3.77e-3	9.58e-4	2.41e-4	6.05e-5	1.52e-5
	O_N		1.98	1.99	2.00	2.00
	N	1000	2000	4000	8000	16000
STSPM with $p = s = 3$	ge	2.11e-5	2.57e-6	3.17e-7	3.94e-8	4.91e-9
	O_N		3.04	3.02	3.01	3.00
TSPM with $p = s = 3$	ge	4.33e-5	5.22e-6	6.40e-7	7.93e-8	9.86e-9
	O_N		3.05	3.03	3.01	3.01
	N	3000	4000	5000	6000	7000
STSPM with $p = s = 4$	ge	9.38e-10	3.29e-10	1.42e-10	7.03e-11	3.80e-11
	O_N		3.64	3.76	3.86	3.99
TSPM with $p = s = 4$	ge	7.48e-9	2.41e-9	1.00e-9	4.86e-10	2.64e-10
	O_N		3.93	3.95	3.96	3.95
	N	2000	2250	2500	2750	3000
STSPM with $p = s = 5$	ge	2.32e-10	1.21e-10	6.89e-11	4.23e-11	2.78e-11
	O_N		5.53	5.34	5.11	4.84

Table 4. The numerical results of the STSPMs and TSPMs with $\rho = 2$ for Example 3.

	N	500	1000	2000	4000	8000
STSPM with $p = s = 2$	ge	9.52e-6	2.03e-6	4.73e-7	1.20e-7	3.02e-8
	O_N		2.23	2.10	1.98	1.99
TSPM with $p = s = 2$	ge	8.85e-5	2.26e-5	5.71e-6	1.43e-6	3.59e-7
	O_N		1.97	1.99	1.99	2.00
	N	500	1000	2000	4000	8000
STSPM with $p = s = 3$	ge	3.78e-7	4.98e-8	6.38e-9	8.09e-10	1.03e-10
	O_N		2.92	2.96	2.98	2.97
TSPM with $p = s = 3$	ge	2.67e-6	3.39e-7	4.26e-8	5.35e-9	6.72e-10
	O_N		2.98	2.99	2.99	2.99
	N	125	250	500	1000	2000
STSPM with $p = s = 4$	ge	2.63e-7	1.84e-8	1.29e-9	8.44e-11	4.60e-12
	O_N		3.84	3.84	3.93	4.20
TSPM with $p = s = 4$	ge	6.87e-6	4.47e-7	2.83e-8	1.78e-9	1.12e-10
	O_N		3.94	3.98	3.99	3.99
	N	200	300	400	500	600
STSPM with $p = s = 5$	ge	1.87e-9	2.42e-10	5.71e-11	1.92e-11	8.36e-12
	O_N		5.04	5.02	4.89	4.56

Table 5. The numerical results of the STSPMs and TSPMs with $\rho = 4$ for Example 3.

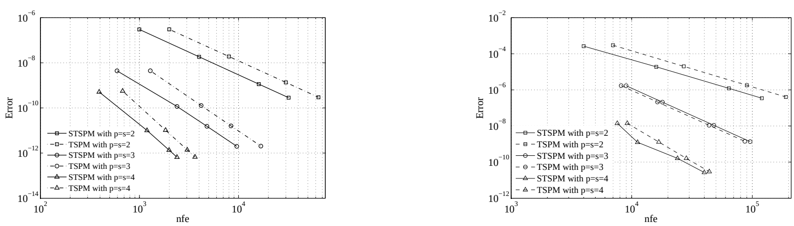
	N	500	1000	2000	4000	8000
STSPM with $p = s = 2$	ge	2.15e-5	4.40e-6	9.75e-7	2.40e-7	6.08e-8
	O_N		2.29	2.17	2.02	1.98
TSPM with $p = s = 2$	ge	1.07e-4	2.73e-5	6.89e-6	1.73e-6	4.34e-7
	O_N		1.97	1.98	1.99	2.00
	N	500	1000	2000	4000	8000
STSPM with $p = s = 3$	ge	1.59e-6	2.00e-7	2.51e-8	3.14e-9	3.94e-10
	O_N		2.99	3.00	3.00	2.99
TSPM with $p = s = 3$	ge	3.87e-6	4.90e-7	6.16e-8	7.72e-9	9.68e-10
	O_N		2.98	2.99	3.00	3.00
	N	125	250	500	1000	2000
STSPM with $p = s = 4$	ge	8.37e-7	8.62e-8	6.47e-9	4.38e-10	2.75e-11
	O_N		3.28	3.74	3.88	3.99
TSPM with $p = s = 4$	ge	1.31e-5	8.58e-7	5.47e-8	3.45e-9	2.17e-10
	O_N		3.94	3.97	3.99	3.99
	N	200	300	400	500	600
STSPM with $p = s = 5$	ge	1.19e-8	1.60e-9	3.71e-10	1.20e-10	4.82e-11
	O_N		4.96	5.07	5.07	4.99

Table 6. The numerical results of the STSPMs and TSPMs with $\rho = 2$ for Example 4.

	N	1000	2000	4000	8000	16000
STSPM with $p = s = 2$	ge	1.00e-5	2.44e-6	6.01e-7	1.49e-7	3.72e-8
	O_N		2.04	2.02	2.01	2.00
TSPM with $p = s = 2$	ge	1.11e-4	2.80e-5	7.01e-6	1.75e-6	4.39e-7
	O_N		1.99	2.00	2.00	2.00
	N	1000	2000	4000	8000	16000
STSPM with $p = s = 3$	ge	9.53e-8	1.06e-8	1.24e-9	1.50e-10	1.98e-11
	O_N		3.17	3.09	3.04	2.98
TSPM with $p = s = 3$	ge	5.53e-7	6.49e-8	7.85e-9	9.65e-10	1.22e-10
	O_N		3.17	3.09	3.05	3.02
	N	1000	1250	1500	1750	2000
STSPM with $p = s = 4$	ge	3.00e-10	1.32e-10	6.71e-11	3.76e-11	2.28e-11
	O_N		3.67	3.73	3.75	3.74
TSPM with $p = s = 4$	ge	8.58e-9	3.51e-9	1.69e-9	9.11e-10	5.34e-10
	O_N		4.01	4.01	4.01	4.01

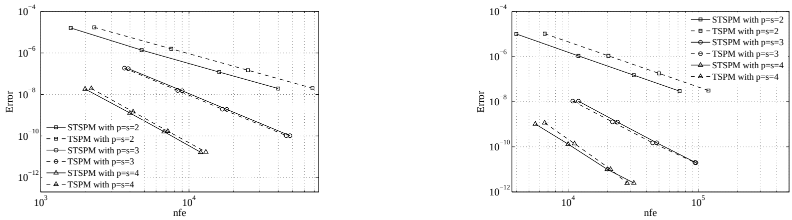
Table 7. The numerical results of the STSPMs and TSPMs with $\rho = 4$ for Example 4.

	N	1000	2000	4000	8000	16000
STSPM with $p = s = 2$	ge	2.02e-5	4.92e-6	1.21e-6	3.02e-7	7.53e-8
	O_N		2.04	2.02	2.01	2.00
TSPM with $p = s = 2$	ge	1.35e-4	3.38e-5	8.47e-6	2.12e-6	5.30e-7
	O_N		1.99	2.00	2.00	2.00
	N	1000	2000	4000	8000	16000
STSPM with $p = s = 3$	ge	3.06e-7	3.69e-8	4.53e-9	5.61e-10	7.12e-11
	O_N		3.05	3.03	3.01	2.98
TSPM with $p = s = 3$	ge	7.94e-7	9.35e-8	1.13e-8	1.39e-9	1.74e-10
	O_N		3.09	3.05	3.02	3.00
	N	1500	1750	2000	2250	2500
STSPM with $p = s = 4$	ge	2.88e-10	1.67e-10	1.03e-10	6.69e-11	4.55e-11
	O_N		3.54	3.61	3.65	3.67
TSPM with $p = s = 4$	ge	3.18e-9	1.72e-9	1.01e-9	6.32e-10	4.15e-10
	O_N		3.98	3.98	3.98	3.99



a) b)

Figure 7. The global error versus the number of function evaluations of the methods with $\rho = 2$ for Example 1 a) and Example 2 b).



a) b)

Figure 8. The global error versus the number of function evaluations of the methods with $\rho = 2$ for Example 3 a) and Example 4 b).

and variable stepsize environments, demonstrated the efficiency and accuracy of the proposed methods and verified the theoretical order of convergence.

Future work will address the construction of new methods within the proposed class of the methods, based on their specific applications (for non-stiff or stiff ODEs) as well as their computational architectures (parallel or sequential). Furthermore, the practical implementation of these methods in variable stepsize environments will be explored.

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References

[1] A. Abdi and D. Conte. Implementation of second derivative general linear methods. *Calcolo*, **57**(20):1–29, 2020. https://doi.org/10.1007/s10092-020-00370-w.

[2] A. Abdi, G. Hojjati, Z. Jackiewicz, H. Podhaisky and M. Sharifi. On the implementation of explicit two–step peer methods with Runge–Kutta stability. *Appl. Numer. Math.*, **186**:213–227, 2023. https://doi.org/10.1016/j.apnum.2023.01.015.

[3] A. Abdi, G. Hojjati and L. Taheri Koltape. Extended SDBDF–type methods based on linear barycentric rational interpolants for ODEs. *Bull. Iran. Math. Soc.*, **48**:3255–3268, 2022. https://doi.org/10.1007/s41980-022-00694-z.

[4] H. Brunner. *Collocation methods for Volterra integral and related functional equations.* Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511543234.

[5] J.C. Butcher and Z. Jackiewicz. Construction of high order diagonally implicit multistage integration methods for ordinary differential equations. *Appl. Numer. Math.*, **27**:1–12, 1998. https://doi.org/10.1016/S0168-9274(97)00109-8.

[6] J.R. Cash. Second derivative extended backward differentiation formula for the numerical integration of stiff systems. *SIAM J. Numer. Anal.*, **18**:21–36, 1981. https://doi.org/10.1137/0718003.

[7] P.C. Chakravarti and M.S. Kamel. Stiffly stable second derivative multistep methods with higher order and improved stability regions. *BIT*, **23**:75–83, 1983. https://doi.org/10.1007/BF01937327.

[8] R.P.K. Chan and A.Y.J. Tsai. On explicit two–derivative Runge–Kutta methods. *Numer. Algorithms*, **53**:171–194, 2010. https://doi.org/10.1007/s11075-009-9349-1.

[9] D. Conte, G. Pagano and B. Paternoster. Two–step peer methods with equation–dependent coefficients. *Comput. Appl. Math.*, **41**(140):1–21, 2022. https://doi.org/10.1007/s40314-022-01844-z.

[10] R. D'Ambrosio, M. Ferro, Z. Jackiewicz and B. Paternoster. Two–step almost collocation methods for ordinary differential equations. *Numer. Algorithms*, **53**:195–217, 2010. https://doi.org/10.1007/s11075-009-9280-5.

[11] W.H. Enright. Second derivative multistep methods for stiff ordinary differential equations. *SIAM J. Numer. Anal.*, **11**:321–331, 1974. https://doi.org/10.1137/0711029.

[12] N. Guglielmi and M. Zennaro. On the zero–stability of variable stepsize multistep methods: the spectral radius approach. *Numer. Math.*, **88**:445–458, 2001. https://doi.org/10.1007/s211-001-8010-0.

[13] G.K. Gupta. Implementing second–derivative multistep methods using the Nordsieck polynomial representation. *Math. Comput.*, **32**:13–18, 1978. https://doi.org/10.1090/S0025-5718-1978-0478630-7.

[14] E. Hairer, S.P. Nørsett and G. Wanner. *Solving ordinary differential equations I. Nonstiff problems.* Springer-Verlag, Berlin, 1993. https://doi.org/10.1007/978-3-540-78862-1.

[15] Z. Jackiewicz. *General linear methods for ordinary differential equations.* John Wiley & Sons, Hoboken, NJ, 2009. https://doi.org/10.1002/9780470522165.

[16] P. Kaps. Rosenbrock–type methods. In G. Dahlquist and R. Jeltsch(Eds.), *Numerical Methods for Solving Stiff Initial Value Problems, Proceeding, Oberwolfach, 1981.* Institut für Geometrie und Praktische Mathematik, RWTH Aachen, 1981.

[17] M. Klinge, R. Weiner and H. Podhaisky. Optimally zero stable explicit peer methods with variable nodes. *BIT*, **58**:331–345, 2018. https://doi.org/10.1007/s10543-017-0691-8.

[18] I. Lie and S.P. Nørsett. Superconvergence for multistep collocation. *Math. Comput.*, **52**(185):65–79, 1989. https://doi.org/10.1090/S0025-5718-1989-0971403-5.

[19] J.I. Montijano, H. Podhaisky, L. Rández and M. Calvo. A family of L–stable singly implicit peer methods for solving stiff IVPs. *BIT*, **59**:483–502, 2019. https://doi.org/10.1007/s10543-018-0734-9.

[20] H. Podhaisky, B.A. Schmitt and R. Weiner. Design, analysis and testing of some parallel two-step W-methods for stiff systems. *Appl. Numer. Math.*, **42**:381–395, 2002. https://doi.org/10.1016/S0168-9274(01)00162-3.

[21] H. Podhaisky, R. Weiner and B.A. Schmitt. Rosenbrock–type 'Peer' two–step methods. *Appl. Numer. Math.*, **53**:409–420, 2005. https://doi.org/10.1016/j.apnum.2004.08.021.

[22] B.A. Schmitt and R. Weiner. Parallel two-step W-methods with peer variables. *SIAM J. Numer. Anal.*, **42**:265–282, 2004. https://doi.org/10.1137/S0036142902411057.

[23] B.A. Schmitt, R. Weiner and K. Erdmann. Implicit parallel peer methods for stiff initial value problems. *Appl. Numer. Math.*, **53**:457–470, 2005. https://doi.org/10.1016/j.apnum.2004.08.019.

[24] B.A. Schmitt, R. Weiner and H. Podhaisky. Multi-implicit peer two-step W-methods for parallel time integration. *BIT*, **45**:197–217, 2005. https://doi.org/10.1007/s10543-005-2635-y.

[25] M. Sharifi, A. Abdi, M. Braś and G. Hojjati. A class of explicit second derivative general linear methods for non-stiff ODEs. *Math. Model. Anal.*, **29**(4):621–640, 2024. https://doi.org/10.3846/mma.2024.19325.

[26] M. Sharifi, A. Abdi, M. Braś and G. Hojjati. On implicit second derivative two-step peer methods with RK stability for ODEs. *Appl. Numer. Math.*, **220**:329–345, 2026. https://doi.org/10.1016/j.apnum.2025.10.019.

[27] A.Y.J. Tsai, R.P.K. Chan and S. Wang. Two–derivative Runge–Kutta methods for PDEs using a novel discretization approach. *Numer. Algorithms*, **65**:687–703, 2014. https://doi.org/10.1007/s11075-014-9823-2.

[28] R. Weiner, K. Biermann, B.A. Schmitt and H. Podhaisky. Explicit two–step peer methods. *Comput. Math. Appl.*, **55**:609–619, 2008. https://doi.org/10.1016/j.camwa.2007.04.026.

[29] R. Weiner, B.A. Schmitt and H. Podhaisky. Parallel Peer two-step W-methods and their application to MOL-systems. *Appl. Numer. Math.*, **48**:425–439, 2004. https://doi.org/10.1016/j.apnum.2003.10.005.

[30] R. Weiner, B.A. Schmitt, H. Podhaisky and S. Jebens. Superconvergent explicit two–step peer methods. *J. Comput. Appl. Math.*, **223**:753–764, 2009. https://doi.org/10.1016/j.cam.2008.02.014.