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On an inverse coefficient problem for a drug war reaction-diffusion system via an optimization approach

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Abstract. In this paper, we study a coefficients inversion problem of a coupled system controlled by three reaction-diffusion equations describing a simple dynamic model of a drug epidemic in an idealized community from the final measurement data. Firstly, the optimization theory is used to transform the given problem into an optimal control problem, and the existence of minimizer is established. Then the stability estimates of the Lipschitz type for the three spatially varying coefficients are proved, where the upper bounds are given by some Lebesgue norms of the final measure.

Keywords: coefficient inverse problem; coupled reaction-diffusion system; optimal control problem; stability.

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1 Introduction

Nowadays, inverse problems based on PDE models have become one of the fastest growing areas in applied mathematics. This is due to the pressing need for solutions to inverse problems in other disciplines and in engineering and technology. For example, in geological exploration [4], we may use data from seismic waves to infer underground structures; in medical imaging [18], we use X-ray or MRI (magnetic resonance imaging) to diagnose the inside of the human body; in environmental science [12], we use the distribution of pollutant concentrations to determine the location and intensity of the source of pollution, and so on.

Inverse problems are challenging because they are often ill-posed. This means that even if solutions exist, they may not be unique and are extremely sensitive to small perturbations in the data. This leads to the need for special mathematical techniques and algorithms to solve inverse problems, such as regularization methods [2, 5, 9], Bayesian estimation [3], and various optimization techniques [1,11,22,29] to ensure the stability and reliability of the solution.

In this paper, we are interested in the inverse problem of determining the spatial coefficients $a_{ii}(i = 1, 2, 3)$ in a reaction-diffusion system with the following initial and boundary value conditions:

$$\begin{cases} u_t - u_{xx} + a_{11}(x)u + a_{12}(x)v + a_{13}(x)w = 0, & (x,t) \in Q = I \times (0,T], \\ v_t - v_{xx} + a_{21}(x)u + a_{22}(x)v + a_{23}(x)w = 0, & (x,t) \in Q, \\ w_t - w_{xx} + a_{31}(x)u + a_{32}(x)v + a_{33}(x)w = 0, & (x,t) \in Q, \\ u(x,0) = u_0, v(x,0) = v_0, w(x,0) = w_0, & x \in I, \\ u = v = w = 0, & (x,t) \in \partial I \times (0,T], \end{cases} \tag{1.1}$$

where the interval $I = (0, 1)$ and $T > 0$ is a constant.

System (1.1) is a drug war weakly coupled reaction-diffusion system, a simple dynamic model of a drug epidemic in an idealized community if all the conditions for a definite solution are known. In this model, the population is divided into three subgroups $u(x, t)$, $v(x, t)$ and $w(x, t)$, which denote the susceptible, infective and law enforcement levels at (x, t), respectively. Here we assume three reaction coefficients $a_{11}(x)$, $a_{22}(x)$ and $a_{33}(x)$ are unknown, so the above-mentioned system is under-determined. However, it is assumed that it is possible to provide three kinds of population levels at terminal time. So our inverse problem is to determine reaction coefficients (a_{11}, a_{22}, a_{33}) in problem (1.1) from the additional final data $(u(x, T), v(x, T), w(x, T))$ given by

$$u(x,T) = g_1(x), \; v(x,T) = g_2(x), \; w(x,T) = g_3(x), \; x \in I, \tag{1.2}$$

where the given functions $g_1(x)$, $g_2(x)$ and $g_3(x)$ are three known field data.

The aim of this paper is to obtain stability estimates of an inverse problem for simultaneous determination of three coefficients $a_{11}(x)$, $a_{22}(x)$ and $a_{33}(x)$ in reaction-diffusion system (1.1) by final value measured data (1.2).

In recent years, the inverse problems of coupled systems controlled by multiple equations have aroused many researchers' interest, and some researchers have discussed such problems. For instance, Cristofol et al. in [7] studied the inverse problem of simultaneous identification of two discontinuous diffusion coefficients for a one-dimensional coupled parabolic system under a single component condition. In [23, 24], Srati and Oulmelk et al. investigated theoretically and numerically the inverse identification of two unknown spatial coefficients under additional conditions for a system controlled by two coupled fractional diffusion equations by the terminal data. In [19,20], Li and Liu et al. respectively studied the inverse problems of the coefficient of the interaction term in the Lotka-Volterra model and some coefficients in a coupled nonlinear parabolic system by using the boundary observations of non-negative solutions. In [13,14], Ait Ben Hassi et al. studied several inverse problems of determining four space-dependent coefficients and simultaneous inversion of radiative potentials and initial temperatures in parabolic equations with dynamic boundary conditions by employing the Carleman estimate.

In addition, some scholars have studied the stability of inverse problems in systems controlled by multiple equations. e.g., Wu et al. in [27] discussed the inverse problem of determining the coefficient of spatial change in a strongly coupled reaction-diffusion system from internal observation data on arbitrary subdomain, and used a new Carleman estimate to derive the Hölder stability of this inverse problem. Chorfi et al. in [6] demonstrated conditional stability estimates for linear and semilinear systems in general backward semilinear strongly coupled parabolic systems. In [26], Wu et al. studied the inverse problem of determining the source term of spatial change in a thermoelastic medium with memory effect, proved the Carleman estimate for a general strongly coupled hyperbolic system, and obtained the Carleman estimate for a hyperbolic thermoelastic system. Based on this estimate, the stability of the inverse source problem is finally established only by making displacement measurements over a given subdomain over a sufficiently large period of time.

In this paper, the inverse spatial coefficient problem of the reaction-diffusion system is discussed, and the stability estimation is obtained by using the optimal control framework. Our main contributions are reflected in the following two points:

1. Our model consists of three coupled reaction-diffusion equations;
2. Lipschitz stability of coefficient inversion for nonlinear inverse problems based on terminal measurements.

It should be emphasized that though, for simplicity, we have studied the stability estimate for a one dimensional diffusion system, it can be extended to higher dimensional systems. For instance, the dimension $d = 2, 3$. Here we need to assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz's smooth boundary, and to make higher assumptions about the regularity of the unknown coefficients and other definite solution data. On the other hand, as possible generalizations, it seems interesting to study the same problem with the observation taken at different moments T_k for the component u_k. Or in a strongly coupled system, it may be possible to uniquely reconstruct unknown coefficients by measuring a single component at different times (e.g., such as $u(x,T_1), u(x,T_2), u(x,T_3)$). However, such issues require further investigations and here we restrict ourselves to the issue within the existing framework.

The main goal of our work can be expressed as follows. Let $(\tilde{u},\tilde{v},\tilde{w})$ be the solution of the following system:

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} + \tilde{a}_{11}(x)\tilde{u} + a_{12}(x)\tilde{v} + a_{13}(x)\tilde{w} = 0, & (x,t) \in Q, \\ \tilde{v}_t - \tilde{v}_{xx} + a_{21}(x)\tilde{u} + \tilde{a}_{22}(x)\tilde{v} + a_{23}(x)\tilde{w} = 0, & (x,t) \in Q, \\ \tilde{w}_t - \tilde{w}_{xx} + a_{31}(x)\tilde{u} + a_{32}(x)\tilde{v} + \tilde{a}_{33}(x)\tilde{w} = 0, & (x,t) \in Q, \\ \tilde{u}(x,0) = u_0, \tilde{v}(x,0) = v_0, \tilde{w}(x,0) = w_0, & x \in I, \\ \tilde{u} = \tilde{v} = \tilde{w} = 0, & (x,t) \in \partial I \times (0,T]. \end{cases} \tag{1.3}$$

Set $U = u - \tilde{u},\ V = v - \tilde{v},\ W = w - \tilde{w}$, by subtracting (1.3) from (1.1) we get a sensitive problem of (1.1) as follows:

$$\begin{cases} U_t - U_{xx} + a_{11}U + a_{12}V + a_{13}W = -\mathcal{A}_1\tilde{u}, & (x,t) \in Q, \\ V_t - V_{xx} + a_{21}U + a_{22}V + a_{23}W = -\mathcal{A}_2\tilde{v}, & (x,t) \in Q, \\ W_t - W_{xx} + a_{31}U + a_{32}V + a_{33}W = -\mathcal{A}_3\tilde{w}, & (x,t) \in Q, \\ U(x,0) = 0, V(x,0) = 0, W(x,0) = 0, & x \in I, \\ U = V = W = 0, & (x,t) \in \partial I \times (0,T], \end{cases} \tag{1.4}$$

where $\mathcal{A}_1 = a_{11} - \tilde{a}_{11},\ \mathcal{A}_2 = a_{22} - \tilde{a}_{22}$ and $\mathcal{A}_3 = a_{33} - \tilde{a}_{33}$.

Theorem 1. *Let (u, v, w) and $(\tilde{u}, \tilde{v}, \tilde{w})$ be the solutions of the systems (1.1) and (1.3), respectively. Assume the coefficients a_{ii}, $\tilde{a}_{ii}$ $(i = 1, 2, 3)$ satisfy (2.1)–(2.2) defined in Section 2, and there exists a point $x_0 \in I$ such that $a_{ii}(x_0) = \tilde{a}_{ii}(x_0)$. Then for small T, there exists a constant $C > 0$, which depends only on T and I, satisfying*

$$\begin{aligned}
&\max_{x\in\bar{I}} |a_{11} - \tilde{a}_{11}|^2 + \max_{x\in\bar{I}} |a_{22} - \tilde{a}_{22}|^2 + \max_{x\in\bar{I}} |a_{33} - \tilde{a}_{33}|^2 \\
&\quad \le C \int_I \big(|g_1 - \tilde{g}_1|^2 + |g_2 - \tilde{g}_2|^2 + |g_3 - \tilde{g}_3|^2\big)dx,
\end{aligned}$$

where g_1, g_2, g_3, $\tilde{g}_1$, $\tilde{g}_2$ and $\tilde{g}_3$ are the values of the solutions of the systems (1.1) and (1.3) at final time $t = T$.

Remark 1. In above theorem, we assume that T is small. In fact, for cases where T is not small, we can also obtain the similar result, as long as the initial value of the system is small or the coefficient of the non-diagonal potential term is small, that is, there are small τ defined in (3.16) and small C_1 defined in Lemma 1. On the other hand, under the condition $\mathcal{A}_i(x_0) = 0$, it means that the unknown coefficients a_{ii} are essentially detectable at some specific points x_0, and there is no need to reconstruct them using inverse problem methods. This is also in line with the actual situation. Furthermore, it can be found from the following proof that the condition is merely for obtaining the conditional stability estimation in the sense of L^∞-norm in the space of continuous functions. If it is merely an estimation of the stability of the L^2-norm, it can be deleted. Or we can obtain a relatively rough stability estimate of L^∞-norm through the Sobolev embedding and Poincaré's inequality in the absence of condition $\mathcal{A}_i(x_0) = 0$.

Remark 2. In this article, we only discussed the problem of coefficient inversion in the case where the initial values are completely known. However, for cases where the initial values of the system are unknown or only some of the initial values are unknown, the method used in this paper is not applicable. This is also a type of problem with significant research value and challenge, because in reality the initial state of a system is often unknown. This will be one of the issues we need to study in the future, namely the coefficient inversion problem with unknown initial states.

This paper is organized as follows: In Section 2, we transform the inverse problem into the optimal control problem, and prove the existence of the minimum, and obtain some necessary optimality conditions. Using these conditions and estimates, we complete the stability estimates in Section 3. In Section 4, we end the paper with a brief concluding remark.

2 Optimal control problem

For $0 < \alpha < 1$, suppose that coefficients a_{ij} $(i, j = 1, 2, 3)$ and initial data u_0, v_0 and w_0 satisfy

$$a_{ij}(x) \in C^\alpha(\bar{I}) \ (i, j = 1, 2, 3), \ \text{and} \ u_0(x), \ v_0(x), \ w_0(x) \in C^{2,\alpha}(\bar{I}), \tag{2.1}$$

and the additional measured data $g_1(x),\ g_2(x),\ g_3(x) \in L^2(I)$ and also satisfy the homogeneous Dirichlet boundary condition. Now we define an admissible set

$$\mathcal{M}_{ad} = \left\{ a_{ii}(x) : 0 < \underline{a}_i \le a_{ii}(x) \le \bar{a}_i,\ \nabla a_{ii} \in L^2(I), i = 1,2,3 \right\}, \tag{2.2}$$

where the constants $\underline{a}_i,\ \bar{a}_i,\ i = 1,2,3$ are given. The optimal control problem is stated as follows: Find $\big(a_{11}^*(x), a_{22}^*(x), a_{33}^*(x)\big) \in \mathcal{M}_{ad}$ satisfying

$$(a_{11}^*, a_{22}^*, a_{33}^*) = \underset{a_{11},a_{22},a_{33}\in\mathcal{M}_{ad}}{\arg\min}\ \mathcal{J}(a_{11}, a_{22}, a_{33}), \tag{2.3}$$

where

$$\begin{aligned} \mathcal{J}(a_{11}, a_{22}, a_{33}) &= \frac{1}{2}\int_I \Big(|u(x,T; a_{11}, a_{22}, a_{33}) - g_1(x)|^2 \\ &+ |v(x,T; a_{11}, a_{22}, a_{33}) - g_2(x)|^2 + |w(x,T; a_{11}, a_{22}, a_{33}) - g_3(x)|^2 \Big) dx \\ &+ \frac{\mu}{2}\int_I \Big(|\nabla a_{11}|^2 + |\nabla a_{22}|^2 + |\nabla a_{33}|^2 \Big)\, dx. \end{aligned} \tag{2.4}$$

Here, (u, v, w) is the solution of the system (1.1) associated with (a_{11}, a_{22}, a_{33}), and μ is the regularization parameter.

The following existence result can be proved by means of the well-known Schauder theory of parabolic equations and the maximum principle [10,17,28].

Theorem 2. *Let $0 < \alpha < 1$ and the coefficients $a_{ij}(x),\ u_0, v_0, w_0$ satisfy (2.1) $(i, j = 1,2,3)$. Then there exists a unique solution $u(x,t),\ v(x,t),\ w(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q})$ to system (1.1).*

Theorem 3. *Let $0 < \alpha < 1/2,\ g_i \in L^2(I)$ and the coefficients $a_{ij}(x),\ u_0, v_0, w_0$ satisfy (2.1)–(2.2) $(i, j = 1,2,3)$. Then there is a minimizer $(\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}) \in \mathcal{M}_{ad}$, such that*

$$\mathcal{J}(\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}) = \min_{a_{11},a_{22},a_{33}\in\mathcal{M}_{ad}} \mathcal{J}(a_{11}, a_{22}, a_{33}).$$

Proof. According to the definition of $\mathcal{J}$, note that this functional is non-negative. Moreover, since the admissible set $\mathcal{M}_{ad}$ is non-empty, $\mathcal{J}$ admits a greatest lower bound.

Let $(a_{11}^n, a_{22}^n, a_{33}^n)$ be a minimizing sequence, then we get

$$\begin{aligned} \inf_{a_{11},a_{22},a_{33}\in\mathcal{M}_{ad}} \mathcal{J}(a_{11}, a_{22}, a_{33}) &\le \mathcal{J}(a_{11}^n, a_{22}^n, a_{33}^n) \\ &\le \inf_{a_{11},a_{22},a_{33}\in\mathcal{M}_{ad}} \mathcal{J}(a_{11}, a_{22}, a_{33}) + \frac{1}{n},\ \forall n \in \mathbb{Z}^+. \end{aligned}$$

From $\mathcal{J}(a_{11}^n, a_{22}^n, a_{33}^n) \le C$ and the definition of admissible sets, we can easily infer that

$$\|\nabla a_{11}^n\|_{L^2(I)} + \|\nabla a_{22}^n\|_{L^2(I)} + \|\nabla a_{33}^n\|_{L^2(I)} \le C,$$

where the constant C is independent of n. Then for $0 < \alpha \leq \frac{1}{2}$, $H^1(I)$ can de embedded in $C^\alpha(I)$ continuously. Thus we obtain

$$verta_{11}^n\|_{C^{\frac{1}{2}}(I)} + \|a_{22}^n\|_{C^{\frac{1}{2}}(I)} + \|a_{33}^n\|_{C^{\frac{1}{2}}(I)} \leq C.$$

Therefore, according to Theorem 2, we derive

$$\|u_n\|_{C^{\frac{1}{2},\frac{1}{4}}(\bar{Q})} + \|v_n\|_{C^{\frac{1}{2},\frac{1}{4}}(\bar{Q})} + \|w_n\|_{C^{\frac{1}{2},\frac{1}{4}}(\bar{Q})} \leq C,$$

where (u_n, v_n, w_n) is the solution of (1.1) corresponding to the coefficients $(a_{11}^n, a_{22}^n, a_{33}^n)$. And for any $Q_0 \Subset Q$, we also get

$$\|u_n\|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(Q_0)} + \|v_n\|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(Q_0)} + \|w_n\|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(Q_0)} \leq C.$$

Thus, there exists a subsequence of $(u_n, v_n, w_n, a_{11}^n, a_{22}^n, a_{33}^n)$, which is also expressed by $(u_n, v_n, w_n, a_{11}^n, a_{22}^n, a_{33}^n)$, such that

$$\begin{aligned} &(a_{11}^n, a_{22}^n, a_{33}^n) \longrightarrow (\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}) \in C^{\frac{1}{2}}(I) \quad \text{uniformly on} \quad C^\alpha(\bar{I}), \\ &(u_n, v_n, w_n) \longrightarrow (\bar{u}, \bar{v}, \bar{w}) \quad \text{uniformly on} \quad C^{\alpha,\frac{\alpha}{2}}(\bar{Q}) \cap C_{loc}^{2+\alpha,1+\frac{\alpha}{2}}(Q). \end{aligned}$$

Therefore, we replace $(u, v, w, a_{11}, a_{22}, a_{33})$ in (1.1) by $(u_n, v_n, w_n, a_{11}^n, a_{22}^n, a_{33}^n)$ and pass to the limit, we can see that $(\bar{u}, \bar{v}, \bar{w}, \bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33})$ satisfies the system (1.1). In addition, by means of the Lebesgue control convergence theorem and the weak lower semi-continuity of the L^2-norm, we have

$$\mathcal{J}(\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}) \leq \liminf_{n\to\infty} \mathcal{J}(a_{11}^n, a_{22}^n, a_{33}^n) = \min_{a_{11},a_{22},a_{33}\in\mathcal{M}_{ad}} \mathcal{J}(a_{11}, a_{22}, a_{33}).$$

Then,

$$\mathcal{J}(\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}) = \min_{a_{11},a_{22},a_{33}\in\mathcal{M}_{ad}} \mathcal{J}(a_{11}, a_{22}, a_{33}).$$

Therefore, $(\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}) := (a_{11}, a_{22}, a_{33})$ is one of the optimal solutions to the optimal control problem (2.2)–(2.4). This concludes the proof of this theorem. □

In that follows, we will investigate the necessary conditions for the optimal control problem (2.3). Firstly we give an adjoint problem of (1.1).

Suppose (p, q, r) is a solution to the following adjoint system of (1.1):

$$\begin{cases} -p_t - p_{xx} + a_{11}p + a_{21}q + a_{31}r = 0, & (x,t) \in Q, \\ -q_t - q_{xx} + a_{12}p + a_{22}q + a_{32}r = 0, & (x,t) \in Q, \\ -r_t - r_{xx} + a_{13}p + a_{23}q + a_{33}r = 0, & (x,t) \in Q, \\ p(x,T) = u(x,T; a_{11}, a_{22}, a_{33}) - g_1(x), & x \in I, \\ q(x,T) = v(x,T; a_{11}, a_{22}, a_{33}) - g_2(x), & x \in I, \\ r(x,T) = w(x,T; a_{11}, a_{22}, a_{33}) - g_3(x), & x \in I, \\ p = q = r = 0, & (x,t) \in \partial I \times (0,T], \end{cases} \tag{2.5}$$

where $i = 1, 2, 3$, and $g_1,\ g_2,\ g_3$ are the measured data of system (1.1) at the final time $t = T$.

Theorem 4. *Assume that* (a_{11}, a_{22}, a_{33}) *is a solution of the optimal control problem (2.3). Then there exists a set of functions* $(u, v, w, p, q, r; a_{11}, a_{22}, a_{33})$ *which satisfy*

$$\begin{aligned}&\int_Q pu(a_{11}-h)dtdx + \int_Q qv(a_{22}-k)dtdx + \int_Q rw(a_{33}-s)dtdx \\ &+ \mu \int_I \Big[\nabla a_{11}\cdot\nabla(h-a_{11}) + \nabla a_{22}\cdot\nabla(k-a_{22}) + \nabla a_{33}\cdot\nabla(s-a_{33})\Big]dx \geq 0,\end{aligned} \tag{2.6}$$

for any $h,\ k,\ s \in \mathcal{M}_{ad}$.

Proof. For any $h,\ k,\ s \in \mathcal{M}_{ad}$ and $0 \leq \delta \leq 1$, we consider

$$\begin{aligned}&a_{11}^\delta = (1-\delta)a_{11} + \delta h \in \mathcal{M}_{ad},\ a_{22}^\delta = (1-\delta)a_{22} + \delta k \in \mathcal{M}_{ad},\\ &a_{33}^\delta = (1-\delta)a_{33} + \delta s \in \mathcal{M}_{ad}.\end{aligned}$$

Let $(u_\delta, v_\delta, w_\delta)$ be the solution of the system (1.1) corresponding to the perturbed coefficients $(a_{11}^\delta, a_{22}^\delta, a_{33}^\delta)$, then we have

$$\begin{aligned}\mathcal{J}(a_{11}^\delta, a_{22}^\delta, a_{33}^\delta) =& \frac{1}{2}\int_I \Big(|u_\delta(\cdot,T)-g_1|^2 + |v_\delta(\cdot,T)-g_2|^2 + |w_\delta(\cdot,T)-g_3|^2\Big)\,dx \\ &+ \frac{\mu}{2}\int_I \Big(|\nabla a_{11}^\delta|^2 + |\nabla a_{22}^\delta|^2 + |\nabla a_{33}^\delta|^2\Big)\,dx.\end{aligned}$$

Now take the Fréchet derivative for $\mathcal{J}$, we obtain

$$\begin{aligned}\frac{d\mathcal{J}}{d\delta}\Big|_{\delta=0} =& \int_I \Bigg(\big[u_\delta(\cdot,T)-g_1\big]\left(\frac{\partial u_\delta}{\partial\delta}\right)\Bigg|_{\delta=0} + \big[v_\delta(\cdot,T)-g_2\big]\left(\frac{\partial v_\delta}{\partial\delta}\right)\Bigg|_{\delta=0} \\ &+ \big[w_\delta(\cdot,T)-g_3\big]\left(\frac{\partial w_\delta}{\partial\delta}\right)\Bigg|_{\delta=0}\Bigg)dx \\ &+ \mu\int_I \big[\nabla a_{11}\cdot\nabla(h-a_{11}) + \nabla a_{22}\cdot\nabla(k-a_{22}) + \nabla a_{33}\cdot\nabla(s-a_{33})\big]\,dx.\end{aligned}$$

Furthermore, as (a_{11}, a_{22}, a_{33}) is an optimal solution, so

$$\frac{d\mathcal{J}}{d\delta}\Big|_{\delta=0} \geq 0. \tag{2.7}$$

If we set $(\bar{u}_\delta, \bar{v}_\delta, \bar{w}_\delta) = \left(\frac{\partial u_\delta}{\partial\delta}, \frac{\partial v_\delta}{\partial\delta}, \frac{\partial w_\delta}{\partial\delta}\right)$, then $(\bar{u}_\delta, \bar{v}_\delta, \bar{w}_\delta)$ satisfies the system with coefficients $(a_{11}^\delta, a_{12}, a_{13}, a_{21}, a_{22}^\delta, a_{23}, a_{31}, a_{32}, a_{33}^\delta)$ as follows:

$$\begin{cases}(\bar{u}_\delta)_t - (\bar{u}_\delta)_{xx} + a_{11}^\delta\bar{u}_\delta + (h-a_{11})u_\delta + a_{12}\bar{v}_\delta + a_{13}\bar{w}_\delta = 0, \quad (x,t)\in Q,\\ (\bar{v}_\delta)_t - (\bar{v}_\delta)_{xx} + a_{21}\bar{u}_\delta + a_{22}^\delta\bar{v}_\delta + (k-a_{22})v_\delta + a_{23}\bar{w}_\delta = 0, \quad (x,t)\in Q,\\ (\bar{w}_\delta)_t - (\bar{w}_\delta)_{xx} + a_{31}\bar{u}_\delta + a_{32}\bar{v}_\delta + a_{33}^\delta\bar{w}_\delta + (s-a_{33})w_\delta = 0, \quad (x,t)\in Q,\\ \bar{u}_\delta(x,0)=0, \bar{v}_\delta(x,0)=0, \bar{w}_\delta(x,0)=0, \quad x\in I,\\ \bar{u}_\delta = \bar{v}_\delta = \bar{w}_\delta = 0, \qquad (x,t)\in\partial I\times(0,T].\end{cases}$$

Taking $\xi = \bar{u}_\delta\big|_{\delta=0}$, $\eta = \bar{v}_\delta\big|_{\delta=0}$, $\varphi = \bar{w}_\delta\big|_{\delta=0}$, we obtain that (ξ, η, φ) satisfies the following system:

$$\begin{cases} \xi_t - \xi_{xx} + a_{11}\xi + a_{12}\eta + a_{13}\varphi = (a_{11} - h)u, & (x,t) \in Q = I \times (0,T], \\ \eta_t - \eta_{xx} + a_{21}\xi + a_{22}\eta + a_{23}\varphi = (a_{22} - k)v, & (x,t) \in Q, \\ \varphi_t - \varphi_{xx} + a_{31}\xi + a_{32}\eta + a_{33}\varphi = (a_{33} - s)w, & (x,t) \in Q, \\ \xi(x,0) = 0, \eta(x,0) = 0, \varphi(x,0) = 0, & x \in I, \\ \xi = \eta = \varphi = 0, & (x,t) \in \partial I \times (0,T], \end{cases} \tag{2.8}$$

where $u = u_\delta\big|_{\delta=0}$, $v = v_\delta\big|_{\delta=0}$, and $w = w_\delta\big|_{\delta=0}$. From (2.7), we have

$$\int_I \Big(\big[u(\cdot,T) - g_1\big]\,\xi(\cdot,T) + \big[v(\cdot,T) - g_2\big]\,\eta(\cdot,T) + \big[w(\cdot,T) - g_3\big]\,\varphi(\cdot,T)\Big) dx$$
$$+\mu \int_I \big[\nabla a_{11} \cdot \nabla(h - a_{11}) + \nabla a_{22} \cdot \nabla(k - a_{22}) + \nabla a_{33} \cdot \nabla(s - a_{33})\big]\, dx \geq 0.$$

From (2.5), the above equation can be rewritten as

$$\begin{aligned} &\int_I \Big(p(\cdot,T)\xi(\cdot,T) + q(\cdot,T)\eta(\cdot,T) + r(\cdot,T)\varphi(\cdot,T)\Big) dx \\ &+ \mu \int_I \Big[\nabla a_{11} \cdot \nabla(h - a_{11}) + \nabla a_{22} \cdot \nabla(k - a_{22}) + \nabla a_{33} \cdot \nabla(s - a_{33})\Big] dx \geq 0. \end{aligned} \tag{2.9}$$

Suppose that (p, q, r) is a solution for the system (2.5). Multiplying the first equation of (2.5) (expressed as $(2.5)_1$) by ξ and applying Green's theorem, we get

$$\begin{aligned} 0 &= \int_Q \xi(-p_t - p_{xx} + a_{11}p + a_{21}q + a_{31}r)\,dtdx \\ &= -\int_I \left(\xi p\Big|_0^T - \int_0^T \xi_t p dt \right) dx - \int_0^T \left(\xi p_x\Big|_0^1 - \xi_x p\Big|_0^1 + \int_0^1 \xi_{xx} p dx \right) dt \\ &+ \int_Q \Big(a_{11}p\xi + a_{21}q\xi + a_{31}r\xi\Big) dtdx = -\int_I \xi(\cdot,T)p(\cdot,T)dx + \int_Q \xi_t p dtdx \\ &- \int_Q \xi_{xx} p dtdx + \int_Q \Big(a_{11}p\xi + a_{21}q\xi + a_{31}r\xi\Big) dtdx \\ &= -\int_I \xi(\cdot,T)p(\cdot,T)dx + \int_Q p\Big(\xi_t - \xi_{xx} + a_{11}\xi + a_{12}\eta + a_{13}\varphi\Big) dtdx \\ &+ \int_Q \Big(a_{21}q\xi - a_{12}p\eta\Big) + \Big(a_{31}r\xi - a_{13}p\varphi\Big) dtdx. \end{aligned}$$

From system (2.8), one can get

$$\begin{aligned} \int_I \xi(\cdot,T)p(\cdot,T)dx &= \int_Q pu(a_{11} - h)dtdx \\ &+ \int_Q \Big(a_{21}q\xi - a_{12}p\eta\Big) + \Big(a_{31}r\xi - a_{13}p\varphi\Big) dtdx. \end{aligned} \tag{2.10}$$

In the same way, from $(2.5)_i$ $(i=2,3)$ and (2.8) we obtain

$$\int_I \eta(\cdot,T)q(\cdot,T)dx$$
$$=\int_Q qv(a_{22}-k)dtdx+\int_Q \Big(a_{12}p\eta-a_{21}q\xi\Big)+\Big(a_{32}r\eta-a_{23}q\varphi\Big)dtdx,$$

and

$$\begin{aligned}&\int_I \varphi(\cdot,T)r(\cdot,T)dx\\ =&\int_Q rw(a_{33}-s)dtdx+\int_Q \Big(a_{13}p\varphi-a_{31}r\xi\Big)+\Big(a_{23}q\varphi-a_{32}r\eta\Big)dtdx.\end{aligned} \tag{2.11}$$

By substituting (2.10)–(2.11) into (2.9) to obtain

$$\int_Q pu(a_{11}-h)dtdx+\int_Q qv(a_{22}-k)dtdx+\int_Q rw(a_{33}-s)dtdx$$
$$+\mu\int_I\Big[\nabla a_{11}\cdot\nabla(h-a_{11})+\nabla a_{22}\cdot\nabla(k-a_{22})+\nabla a_{33}\cdot\nabla(s-a_{33})\Big]dx\geq 0.$$

One can easily complete the proof of Theorem 4. □

3 Main results

This section provides stability estimates of the inverse problem of finding three smooth coefficients $a_{11}(x)$, $a_{22}(x)$ and $a_{33}(x)$ in a given parabolic system. These results (Lemmas 1–4) can be obtained by the standard L^2 energy estimation [8], but we still present the proof process here for the completeness of the paper and to obtaining a more precise right-hand side control constant. We firstly give an estimate of a solution to the sensitive problem (1.4).

Lemma 1. *Suppose that (U,V,W) is a solution of the system (1.4). Then, we have the following estimate:*

$$\begin{aligned}&\max_{0\leq t\leq T}\int_I\big(|U|^2+|V|^2+|W|^2\big)dx\\ \leq&\exp(C_1T)\Big[\max_{x\in\bar I}|\mathcal{A}_1|^2\int_Q|\tilde u|^2dtdx+\max_{x\in\bar I}|\mathcal{A}_2|^2\int_Q|\tilde v|^2dtdx\\ &+\max_{x\in\bar I}|\mathcal{A}_3|^2\int_Q|\tilde w|^2dtdx\Big],\end{aligned} \tag{3.1}$$

where $C_1=3+\sum_{i,j=1,i\neq j}^3\max_{x\in\bar I}|a_{ij}(x)|^2$.

Proof. Multiply the first equation of (1.4) (denoted as $(1.4)_1$) by U and integrate over I, we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2_{L^2(I)}+\int_I|U_x|^2dx+\int_I a_{11}|U|^2dx=-\int_I a_{12}UVdx-\int_I a_{13}UWdx$$
$$-\int_I\mathcal{A}_1U\tilde u dx.$$

Using the assumption of the coefficient a_{11} and Cauchy's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U\|^2_{L^2(I)} + \int_I |U_x|^2 dx + \int_I \underline{a}_1 |U|^2 dx \\
\leq &\frac{1}{2}\int_I \left(|U|^2 + |a_{12}V|^2\right) dx + \frac{1}{2}\int_I \left(|U|^2 + |a_{13}W|^2\right) dx \\
&+ \frac{1}{2}\int_I \left(|U|^2 + |\mathcal{A}_1 \tilde{u}|^2\right) dx \\
\leq &\frac{3}{2}\int_I |U|^2 dx + \frac{1}{2}\max_{x\in\bar{I}} |a_{12}|^2 \int_I |V|^2 dx + \frac{1}{2}\max_{x\in\bar{I}} |a_{13}|^2 \int_I |W|^2 dx \\
&+ \frac{1}{2}\max_{x\in\bar{I}} |\mathcal{A}_1|^2 \int_I |\tilde{u}|^2 dx.
\end{aligned} \tag{3.2}$$

Similarly according to $(1.4)_i$ and the assumption of $a_{ii}, i = 2, 3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V\|^2_{L^2(I)} + \int_I |V_x|^2 dx + \int_I \underline{a}_2 |V|^2 dx \\
\leq &\frac{3}{2}\int_I |V|^2 dx + \frac{1}{2}\max_{x\in\bar{I}} |a_{21}|^2 \int_I |U|^2 dx + \frac{1}{2}\max_{x\in\bar{I}} |a_{23}|^2 \int_I |W|^2 dx \\
&+ \frac{1}{2}\max_{x\in\bar{I}} |\mathcal{A}_2|^2 \int_I |\tilde{v}|^2 dx,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|W\|^2_{L^2(I)} + \int_I |W_x|^2 dx + \int_I \underline{a}_3 |W|^2 dx \\
\leq &\frac{3}{2}\int_I |W|^2 dx + \frac{1}{2}\max_{x\in\bar{I}} |a_{31}|^2 \int_I |U|^2 dx + \frac{1}{2}\max_{x\in\bar{I}} |a_{32}|^2 \int_I |V|^2 dx \\
&+ \frac{1}{2}\max_{x\in\bar{I}} |\mathcal{A}_3|^2 \int_I |\tilde{w}|^2 dx.
\end{aligned} \tag{3.3}$$

Now, combining (3.2)–(3.3), we get

$$\begin{aligned}
&\frac{d}{dt}\left[\|U\|^2_{L^2(I)} + \|V\|^2_{L^2(I)} + \|W\|^2_{L^2(I)}\right] \\
\leq &C_1\left(\|U\|^2_{L^2(I)} + \|V\|^2_{L^2(I)} + \|W\|^2_{L^2(I)}\right) + \max_{x\in\bar{I}} |\mathcal{A}_1|^2 \int_I |\tilde{u}|^2 dx \\
&+ \max_{x\in\bar{I}} |\mathcal{A}_2|^2 \int_I |\tilde{v}|^2 dx + \max_{x\in\bar{I}} |\mathcal{A}_3|^2 \int_I |\tilde{w}|^2 dx,
\end{aligned}$$

where $C_1 = 3 + \sum_{i,j=1,i\neq j}^{3} \max_{x\in\bar{I}} |a_{ij}(x)|^2$.

Multiply both sides by $\exp(-C_1 t)$ and shift the terms, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\exp(-C_1 t)\left(\|U\|^2_{L^2(I)} + \|V\|^2_{L^2(I)} + \|W\|^2_{L^2(I)}\right)\right] \\
\leq &\exp(-C_1 t)\Big(\max_{x\in\bar{I}} |\mathcal{A}_1|^2 \int_I |\tilde{u}|^2 dx + \max_{x\in\bar{I}} |\mathcal{A}_2|^2 \int_I |\tilde{v}|^2 dx \\
&+ \max_{x\in\bar{I}} |\mathcal{A}_3|^2 \int_I |\tilde{w}|^2 dx\Big).
\end{aligned}$$

Therefore, by integrating from 0 to t, we obtain

$$\begin{aligned}
&\|U\|^2_{L^2(I)} + \|V\|^2_{L^2(I)} + \|W\|^2_{L^2(I)} \\
\leq &\exp(C_1 t)\Big(\max_{x\in \bar{I}} |\mathcal{A}_1|^2 \int_I \int_0^t \exp(-C_1 s)|\tilde{u}|^2 ds dx \\
&+ \max_{x\in \bar{I}} |\mathcal{A}_2|^2 \int_I \int_0^t \exp(-C_1 s)|\tilde{v}|^2 ds dx \\
&+ \max_{x\in \bar{I}} |\mathcal{A}_3|^2 \int_I \int_0^t \exp(-C_1 s)|\tilde{w}|^2 ds dx \Big) \\
\leq &\exp(C_1 T)\Big(\max_{x\in \bar{I}} |\mathcal{A}_1|^2 \int_Q |\tilde{u}|^2 dt dx + \max_{x\in \bar{I}} |\mathcal{A}_2|^2 \int_Q |\tilde{v}|^2 dt dx \\
&+ \max_{x\in \bar{I}} |\mathcal{A}_3|^2 \int_Q |\tilde{w}|^2 dt dx \Big).
\end{aligned}$$

This is the end of the proof of Lemma 1. □

Let (a_{11}, a_{22}, a_{33}) and $(\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_{33})$ be two minimizers of the optimal control problem (2.3) corresponding to (g_1, g_2, g_3) and $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$, respectively. $(\tilde{u}, \tilde{v}, \tilde{w})$ is solution of the state system (1.1) associated with $(\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_{33})$, and $(\tilde{p}, \tilde{q}, \tilde{r})$ is solution of the adjoint system (2.5) associated with $(\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_{33})$. Now let $\mathcal{P} = p - \tilde{p}$, $\mathcal{Q} = q - \tilde{q}$ and $\mathcal{R} = r - \tilde{r}$, thus adjoint system (2.5) becomes

$$\begin{cases}
-\mathcal{P}_t - \mathcal{P}_{xx} + a_{11}\mathcal{P} + a_{21}\mathcal{Q} + a_{31}\mathcal{R} = -\mathcal{A}_1\tilde{p}, & (x,t) \in Q, \\
-\mathcal{Q}_t - \mathcal{Q}_{xx} + a_{12}\mathcal{P} + a_{22}\mathcal{Q} + a_{32}\mathcal{R} = -\mathcal{A}_2\tilde{q}, & (x,t) \in Q, \\
-\mathcal{R}_t - \mathcal{R}_{xx} + a_{13}\mathcal{P} + a_{23}\mathcal{Q} + a_{33}\mathcal{R} = -\mathcal{A}_3\tilde{r}, & (x,t) \in Q, \\
\mathcal{P}(x,T) = U(x,T) - (g_1 - \tilde{g}_1), & x \in I, \\
\mathcal{Q}(x,T) = V(x,T) - (g_2 - \tilde{g}_2), & x \in I, \\
\mathcal{R}(x,T) = W(x,T) - (g_3 - \tilde{g}_3)), & x \in I, \\
\mathcal{P} = \mathcal{Q} = \mathcal{R} = 0, & (x,t) \in \partial I \times (0,T].
\end{cases} \tag{3.4}$$

Now, we give an estimate of a solution to the adjoint problem (2.5) as follows.

Lemma 2. *Let $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ be a solution of system (3.4). Then we have the following estimate:*

$$\begin{aligned}
&\max_{0\leq t\leq T} \int_I \left(|\mathcal{P}|^2 + |\mathcal{Q}|^2 + |\mathcal{R}|^2\right) dx \\
&\leq 2\exp(2C_1 T)\Bigg(\max_{x\in \bar{I}} |\mathcal{A}_1|^2 \int_Q \left(|\tilde{p}|^2 + |\tilde{u}|^2\right) dt dx \\
&+ \max_{x\in \bar{I}} |\mathcal{A}_2|^2 \int_Q \left(|\tilde{q}|^2 + |\tilde{v}|^2\right) dt dx + \max_{x\in \bar{I}} |\mathcal{A}_3|^2 \int_Q \left(|\tilde{r}|^2 + |\tilde{w}|^2\right) dt dx \\
&+ \int_I \left(|g_1 - \tilde{g}_1|^2 + |g_2 - \tilde{g}_2|^2 + |g_3 - \tilde{g}_3|^2\right) dx \Bigg),
\end{aligned}$$

where C_1 is the constant defined in Lemma 1.

Proof. Multiply the first equation of (3.4) (denoted as $(3.4)_1$) by $\mathcal{P}$ and integrate over I to get

$$\int_I \left(-\mathcal{P}\mathcal{P}_t - \mathcal{P}\mathcal{P}_{xx} + a_{11}\mathcal{P}^2 + a_{21}\mathcal{P}\mathcal{Q} + a_{31}\mathcal{P}\mathcal{R}\right) dx = \int_I -\mathcal{A}_1\mathcal{P}\tilde{p}dx,$$

using the assumption of the coefficient a_{11} and Cauchy's inequality, we have

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|\mathcal{P}\|^2_{L^2(I)} + \int_I |\mathcal{P}_x|^2 dx + \int_I \underline{a}_1|\mathcal{P}|^2 dx \le \frac{1}{2}\int_I \left(|\mathcal{P}|^2 + |a_{21}\mathcal{Q}|^2\right) dx \\
&\quad + \frac{1}{2}\int_I \left(|\mathcal{P}|^2 + |a_{31}\mathcal{R}|^2\right) dx + \frac{1}{2}\int_I \left(|\mathcal{P}|^2 + |\mathcal{A}_1\tilde{p}|^2\right) dx \\
&\quad \le \frac{3}{2}\int_I |\mathcal{P}|^2 dx + \frac{1}{2}\max_{x\in\bar{I}}|a_{21}|^2 \int_I |\mathcal{Q}|^2 dx + \frac{1}{2}\max_{x\in\bar{I}}|a_{31}|^2 \int_I |\mathcal{R}|^2 dx \\
&\quad\quad + \frac{1}{2}\max_{x\in\bar{I}}|\mathcal{A}_1|^2 \int_I |\tilde{p}|^2 dx.
\end{aligned} \tag{3.5}$$

Similarly according to $(3.4)_i$ and the assumption of $a_{ii}, i = 2, 3$, we have

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}\|^2_{L^2(I)} + \int_I |\mathcal{Q}_x|^2 dx + \int_I \underline{a}_2|\mathcal{Q}|^2 dx \\
&\le \frac{3}{2}\int_I |\mathcal{Q}|^2 dx + \frac{1}{2}\max_{x\in\bar{I}}|a_{12}|^2 \int_I |\mathcal{P}|^2 dx + \frac{1}{2}\max_{x\in\bar{I}}|a_{32}|^2 \int_I |\mathcal{R}|^2 dx \\
&\quad + \frac{1}{2}\max_{x\in\bar{I}}|\mathcal{A}_2|^2 \int_I |\tilde{q}|^2 dx,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|\mathcal{R}\|^2_{L^2(I)} + \int_I |\mathcal{R}_x|^2 dx + \int_I \underline{a}_3|\mathcal{R}|^2 dx \\
&\le \frac{3}{2}\int_I |\mathcal{R}|^2 dx + \frac{1}{2}\max_{x\in\bar{I}}|a_{13}|^2 \int_I |\mathcal{P}|^2 dx + \frac{1}{2}\max_{x\in\bar{I}}|a_{23}|^2 \int_I |\mathcal{Q}|^2 dx \\
&\quad + \frac{1}{2}\max_{x\in\bar{I}}|\mathcal{A}_3|^2 \int_I |\tilde{r}|^2 dx.
\end{aligned} \tag{3.6}$$

Now, by combining (3.5)–(3.6), we obtain

$$\begin{aligned}
&-\frac{d}{dt}\left[\|\mathcal{P}\|^2_{L^2(I)} + \|\mathcal{Q}\|^2_{L^2(I)} + \|\mathcal{R}\|^2_{L^2(I)}\right] \\
&\le C_1\left(\|\mathcal{P}\|^2_{L^2(I)} + \|\mathcal{Q}\|^2_{L^2(I)} + \|\mathcal{R}\|^2_{L^2(I)}\right) \\
&\quad + \max_{x\in\bar{I}}|\mathcal{A}_1|^2 \int_I |\tilde{p}|^2 dx + \max_{x\in\bar{I}}|\mathcal{A}_2|^2 \int_I |\tilde{q}|^2 dx + \max_{x\in\bar{I}}|\mathcal{A}_3|^2 \int_I |\tilde{r}|^2 dx,
\end{aligned}$$

where $C_1 = 3 + \sum_{i,j=1,i\neq j}^{3} \max_{x\in\bar{I}} |a_{ij}(x)|^2$. Multiply both sides by $\exp(C_1 t)$ and shift the terms, we get

$$\begin{aligned}
&-\frac{d}{dt}\left[\exp(C_1 t)\left(\|\mathcal{P}\|^2_{L^2(I)} + \|\mathcal{Q}\|^2_{L^2(I)} + \|\mathcal{R}\|^2_{L^2(I)}\right)\right] \\
&\le \exp(C_1 t)\left(\max_{x\in\bar{I}}|\mathcal{A}_1|^2 \int_I |\tilde{p}|^2 dx + \max_{x\in\bar{I}}|\mathcal{A}_2|^2 \int_I |\tilde{q}|^2 dx + \max_{x\in\bar{I}}|\mathcal{A}_3|^2 \int_I |\tilde{r}|^2 dx\right).
\end{aligned}$$

Therefore, by integrating from t to T, we obtain

$$\begin{aligned}
&\exp(C_1 t)\left(\|\mathcal{P}(\cdot,t)\|^2_{L^2(I)}+\|\mathcal{Q}(\cdot,t)\|^2_{L^2(I)}+\|\mathcal{R}(\cdot,t)\|^2_{L^2(I)}\right)\\
\le&\max_{x\in\bar I}|\mathcal{A}_1|^2\int_I\int_t^T\exp(C_1 s)|\tilde p|^2dsdx+\max_{x\in\bar I}|\mathcal{A}_2|^2\int_I\int_t^T\exp(C_1 s)|\tilde q|^2dsdx\\
&+\max_{x\in\bar I}|\mathcal{A}_3|^2\int_I\int_t^T\exp(C_1 s)|\tilde r|^2dsdx\\
&+\exp(C_1 T)\left(\|\mathcal{P}(\cdot,T)\|^2_{L^2(I)}+\|\mathcal{Q}(\cdot,T)\|^2_{L^2(I)}+\|\mathcal{R}(\cdot,T)\|^2_{L^2(I)}\right).
\end{aligned}$$

Setting

$$\begin{aligned}
I_1:=&\max_{x\in\bar I}|\mathcal{A}_1|^2\int_I\int_t^T\exp(C_1 s)|\tilde p|^2dsdx+\max_{x\in\bar I}|\mathcal{A}_2|^2\int_I\int_t^T\exp(C_1 s)|\tilde q|^2dsdx\\
&+\max_{x\in\bar I}|\mathcal{A}_3|^2\int_I\int_t^T\exp(C_1 s)|\tilde r|^2dsdx,
\end{aligned}$$

according to $(3.4)_4$–$(3.4)_6$ we get

$$\begin{aligned}
&\exp(C_1 t)\left(\|\mathcal{P}\|^2_{L^2(I)}+\|\mathcal{Q}\|^2_{L^2(I)}+\|\mathcal{R}\|^2_{L^2(I)}\right)\\
&\le I_1+\exp(C_1 T)\bigg(\int_I\big|U(x,T)-(g_1-\tilde g_1)\big|^2dx+\int_I\big|V(x,T)-(g_2-\tilde g_2)\big|^2dx\\
&+\int_I\big|W(x,T)-(g_3-\tilde g_3)\big|^2dx\bigg).
\end{aligned}$$

Thus, by (3.1), one can get

$$\begin{aligned}
&\|\mathcal{P}\|^2_{L^2(I)}+\|\mathcal{Q}\|^2_{L^2(I)}+\|\mathcal{R}\|^2_{L^2(I)}\le\exp(-C_1 t)I_1+2\exp\big(C_1(T-t)\big)\\
&\times\bigg(\int_I\big|U(x,T)\big|^2dx+\int_I\big|V(x,T)\big|^2dx+\int_I\big|W(x,T)\big|^2dx\bigg)\\
&+2\exp\big(C_1(T-t)\big)\bigg(\int_I\big|g_1-\tilde g_1\big|^2dx+\int_I\big|g_2-\tilde g_2\big|^2dx\\
&+\int_I\big|g_3-\tilde g_3\big|^2dx\bigg)\le 2\exp(2C_1T)\bigg(\max_{x\in\bar I}|\mathcal{A}_1|^2\int_Q(|\tilde p|^2+|\tilde u|^2)dtdx\\
&+\max_{x\in\bar I}|\mathcal{A}_2|^2\int_Q(|\tilde q|^2+|\tilde v|^2)dtdx+\max_{x\in\bar I}|\mathcal{A}_3|^2\int_Q(|\tilde r|^2+|\tilde w|^2)dtdx\\
&+\int_I\left(\big|g_1-\tilde g_1\big|^2+\big|g_2-\tilde g_2\big|^2+\big|g_3-\tilde g_3\big|^2\right)dx\bigg).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3. *Let (u,v,w) be a solution for system (1.1). Then we have the following estimate:*

$$\max_{0\le t\le T}\int_I\left(|u|^2+|v|^2+|w|^2\right)dx\le\exp(C_1T)\left(\|u_0\|^2_{L^2(I)}+\|v_0\|^2_{L^2(I)}+\|w_0\|^2_{L^2(I)}\right), \tag{3.7}$$

where C_1 is the constant defined in Lemma 1.

Proof. Multiplying $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$ by u, v and w respectively, and integrating over I, we obtain the following inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)}+\int_I|u_x|^2dx+\int_I a_{11}|u|^2dx\\ \le&\int_I|u|^2dx+\frac{1}{2}\max_{x\in\bar{I}}|a_{12}|^2\int_I|v|^2dx+\frac{1}{2}\max_{x\in\bar{I}}|a_{13}|^2\int_I|w|^2dx,\\ &\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(I)}+\int_I|v_x|^2dx+\int_I a_{22}|v|^2dx\\ \le&\int_I|v|^2dx+\frac{1}{2}\max_{x\in\bar{I}}|a_{21}|^2\int_I|u|^2dx+\frac{1}{2}\max_{x\in\bar{I}}|a_{23}|^2\int_I|w|^2dx\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(I)}+\int_I|w_x|^2dx+\int_I a_{33}|w|^2dx\\ \le&\int_I|w|^2dx+\frac{1}{2}\max_{x\in\bar{I}}|a_{31}|^2\int_I|u|^2dx+\frac{1}{2}\max_{x\in\bar{I}}|a_{32}|^2\int_I|v|^2dx.\end{aligned}\tag{3.9}$$

By coupling (3.8)–(3.9), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|u\|^2_{L^2(I)}+\|v\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\right]\\ &+\int_I\left(|u_x|^2+|v_x|^2+|w_x|^2\right)dx+\int_I\left(a_{11}u^2+a_{22}v^2+a_{33}w^2\right)dx\\ \le&\Bigg(1+\frac{1}{2}\max_{x\in\bar{I}}|a_{12}|^2+\frac{1}{2}\max_{x\in\bar{I}}|a_{13}|^2+\frac{1}{2}\max_{x\in\bar{I}}|a_{21}|^2+\frac{1}{2}\max_{x\in\bar{I}}|a_{23}|^2\\ &+\frac{1}{2}\max_{x\in\bar{I}}|a_{31}|^2+\frac{1}{2}\max_{x\in\bar{I}}|a_{32}|^2\Bigg)\left(\|u\|^2_{L^2(I)}+\|v\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\right)\\ \le&\frac{C_1}{2}\left(\|u\|^2_{L^2(I)}+\|v\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\right).\end{aligned}$$

Under the assumptions of a_{11}, a_{22} and a_{33}, we have

$$\frac{1}{2}\frac{d}{dt}\left[\|u\|^2_{L^2(I)}+\|v\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\right]\le\frac{C_1}{2}\left(\|u\|^2_{L^2(I)}+\|v\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\right).$$

Multiply both sides by $\exp(-C_1t)$ and shift the terms, we get

$$\frac{d}{dt}\left[\exp(-C_1t)\left(\|u\|^2_{L^2(I)}+\|v\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\right)\right]\le 0,$$

thus, by integrating from 0 to t, we obtain

$$\begin{aligned}&\|u\|^2_{L^2(I)}+\|v\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\\ \le&\exp(C_1t)\left(\|u(\cdot,0)\|^2_{L^2(I)}+\|v(\cdot,0)\|^2_{L^2(I)}+\|w(\cdot,0)\|^2_{L^2(I)}\right)\\ \le&\exp(C_1T)\left(\|u_0\|^2_{L^2(I)}+\|v_0\|^2_{L^2(I)}+\|w_0\|^2_{L^2(I)}\right).\end{aligned}$$

The proof can be completed. □

Lemma 4. *Let (p,q,r) be a solution of system (2.5). Then we have the following estimate:*

$$\max_{0\le t\le T}\int_I \left(|p|^2+|q|^2+|r|^2\right)dx \le 2\exp(2C_1T)\Big(\|u_0\|^2_{L^2(I)}+\|v_0\|^2_{L^2(I)} + \|w_0\|^2_{L^2(I)}+\int_I\left(|g_1|^2+|g_2|^2+|g_3|^2\right)dx\Big),$$

where C_1 is the constant defined in Lemma 1.

Proof. Multiplying $(2.5)_1$, $(2.5)_2$ and $(2.5)_3$ by p, q and r respectively, and integrating over I, we obtain

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|p\|^2_{L^2(I)}+\int_I|p_x|^2dx+\int_I a_{11}|p|^2dx\\ &\le\int_I|p|^2dx+\frac12\max_{x\in\bar I}|a_{21}|^2\int_I|q|^2dx+\frac12\max_{x\in\bar I}|a_{31}|^2\int_I|r|^2dx,\\ &-\frac{1}{2}\frac{d}{dt}\|q\|^2_{L^2(I)}+\int_I|q_x|^2dx+\int_I a_{22}|q|^2dx\\ &\le\int_I|q|^2dx+\frac12\max_{x\in\bar I}|a_{12}|^2\int_I|p|^2dx+\frac12\max_{x\in\bar I}|a_{32}|^2\int_I|r|^2dx\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|r\|^2_{L^2(I)}+\int_I|r_x|^2dx+\int_I a_{33}|r|^2dx\\ &\le\int_I|r|^2dx+\frac12\max_{x\in\bar I}|a_{13}|^2\int_I|p|^2dx+\frac12\max_{x\in\bar I}|a_{23}|^2\int_I|q|^2dx.\end{aligned}\tag{3.11}$$

By coupling (3.10)–(3.11), we get

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\left[\|p\|^2_{L^2(I)}+\|q\|^2_{L^2(I)}+\|r\|^2_{L^2(I)}\right]\\ &+\int_I\left(|p_x|^2+|q_x|^2+|r_x|^2\right)dx+\int_I\left(a_{11}p^2+a_{22}q^2+a_{33}r^2\right)dx\\ &\le\Big(1+\frac12\max_{x\in\bar I}|a_{12}|^2+\frac12\max_{x\in\bar I}|a_{13}|^2+\frac12\max_{x\in\bar I}|a_{21}|^2+\frac12\max_{x\in\bar I}|a_{23}|^2\\ &+\frac12\max_{x\in\bar I}|a_{31}|^2+\frac12\max_{x\in\bar I}|a_{32}|^2\Big)\left(\|p\|^2_{L^2(I)}+\|q\|^2_{L^2(I)}+\|r\|^2_{L^2(I)}\right)\\ &\le\frac{C_1}{2}\left(\|p\|^2_{L^2(I)}+\|q\|^2_{L^2(I)}+\|r\|^2_{L^2(I)}\right).\end{aligned}$$

According to the hypotheses on a_{11}, a_{22} and a_{33}, we have the following result

$$-\frac{1}{2}\frac{d}{dt}\left[\|p\|^2_{L^2(I)}+\|q\|^2_{L^2(I)}+\|r\|^2_{L^2(I)}\right]\le\frac{C_1}{2}\left(\|p\|^2_{L^2(I)}+\|q\|^2_{L^2(I)}+\|r\|^2_{L^2(I)}\right).$$

Multiply both sides by $\exp(C_1t)$ and shift the terms, we get

$$-\frac{1}{2}\frac{d}{dt}\left[\exp(C_1t)\left(\|p\|^2_{L^2(I)}+\|q\|^2_{L^2(I)}+\|r\|^2_{L^2(I)}\right)\right]\le 0.$$

Therefore, by integration from t to T, we have

$$\|p(\cdot,t)\|^2_{L^2(I)} + \|q(\cdot,t)\|^2_{L^2(I)} + \|r(\cdot,t)\|^2_{L^2(I)} \\ \leq \exp(C_1(T-t))\left(\|p(\cdot,T)\|^2_{L^2(I)} + \|q(\cdot,T)\|^2_{L^2(I)} + \|r(\cdot,T)\|^2_{L^2(I)}\right).$$

According to $(2.5)_4$–$(2.5)_6$ we get

$$\|p\|^2_{L^2(I)} + \|q\|^2_{L^2(I)} + \|r\|^2_{L^2(I)} \leq 2\exp(C_1(T-t) \\ \times\Bigg(\|u(\cdot,T)\|^2_{L^2(I)} + \|v(\cdot,T)\|^2_{L^2(I)} + \|w(\cdot,T)\|^2_{L^2(I)} \\ + \int_I \left(|g_1|^2 + |g_2|^2 + |g_3|^2\right) dx\Bigg) \leq 2\exp(2C_1T) \\ \times\left(\|u_0\|^2_{L^2(I)} + \|v_0\|^2_{L^2(I)} + \|w_0\|^2_{L^2(I)} + \int_I \left(|g_1|^2 + |g_2|^2 + |g_3|^2\right) dx\right)$$

This completes the proof of the Lemma 4. □

Now, with the help of the lemmas we established above, we are ready to prove the main result of this paper. For the proof of the local stability estimates, we follow certain ideas used to reconstruct the source terms of the phase field system [15] and the local fluctuations of the Black-Scholes equations [16].

Now we are in a position to prove Theorem 1.1.

Proof. We give **Proof of Theorem 1.1**

Let us begin the proof by taking $h = \tilde{a}_{11},\ k = \tilde{a}_{22},\ s = \tilde{a}_{33}$ in (2.6), we have

$$\int_Q pu(a_{11} - \tilde{a}_{11})dtdx + \int_Q qv(a_{22} - \tilde{a}_{22})dtdx + \int_Q rw(a_{33} - \tilde{a}_{33})dtdx \quad (3.12) \\ + \mu\int_I \Big[\nabla a_{11}\cdot\nabla(\tilde{a}_{11} - a_{11}) + \nabla a_{22}\cdot\nabla(\tilde{a}_{22} - a_{22}) + \nabla a_{33}\cdot\nabla(\tilde{a}_{33} - a_{33})\Big]dx \geq 0,$$

and by taking $a_{11}=\tilde{a}_{11}$, $a_{22}=\tilde{a}_{22}$, $a_{33}=\tilde{a}_{33}$ when $h=a_{11}$, $k = a_{22}$, $s = a_{33}$ we get

$$\int_Q \tilde{p}\tilde{u}(\tilde{a}_{11} - a_{11})dtdx + \int_Q \tilde{q}\tilde{v}(\tilde{a}_{22} - a_{22})dtdx + \int_Q \tilde{r}\tilde{w}(\tilde{a}_{33} - a_{33})dtdx \quad (3.13) \\ + \mu\int_I \Big[\nabla \tilde{a}_{11}\cdot\nabla(a_{11} - \tilde{a}_{11}) + \nabla \tilde{a}_{22}\cdot\nabla(a_{22} - \tilde{a}_{22}) + \nabla \tilde{a}_{33}\cdot\nabla(a_{33} - \tilde{a}_{33})\Big]dx \geq 0,$$

where (u,v,w) and $(\tilde{u},\tilde{v},\tilde{w})$ are solutions of the system (1.1) with the coefficients a_{ij} $(i,j = 1,2,3)$ and $(\tilde{a}_{11}, a_{12}, a_{13}, a_{21}, \tilde{a}_{22}, a_{23}, a_{31}, a_{32}, \tilde{a}_{33})$, respectively. (p,q,r) and $(\tilde{p},\tilde{q},\tilde{r})$ are solutions of the corresponding adjoint systems (2.5). Now combining the inequality (3.12) and (3.13), and using Cauchy's inequality, we obtain

$$\begin{aligned}
&\mu\Big(\int_I |\nabla(a_{11}-\tilde{a}_{11})|^2dx+\int_I |\nabla(a_{22}-\tilde{a}_{22})|^2dx+\int_I |\nabla(a_{33}-\tilde{a}_{33})|^2dx\Big)\\
&\le \int_Q \mathcal{A}_1\Big(pU+\mathcal{P}\tilde{u}\Big)dtdx+\int_Q \mathcal{A}_2\Big(qV+\mathcal{Q}\tilde{v}\Big)dtdx+\int_Q \mathcal{A}_3\Big(rW+\mathcal{R}\tilde{w}\Big)dtdx\\
&\le\frac{1}{2}\int_Q\Big(|\mathcal{A}_1 p|^2+|U|^2+|\mathcal{A}_1\tilde{u}|^2+|\mathcal{P}|^2\Big)dtdx\\
&+\frac{1}{2}\int_Q\Big(|\mathcal{A}_2 q|^2+|V|^2+|\mathcal{A}_2\tilde{v}|^2+|\mathcal{Q}|^2\Big)dtdx\\
&+\frac{1}{2}\int_Q\Big(|\mathcal{A}_3 r|^2+|W|^2+|\mathcal{A}_3\tilde{w}|^2+|\mathcal{R}|^2\Big)dtdx\\
&\le\frac{1}{2}\max_{x\in\bar{I}}|\mathcal{A}_1|^2\int_Q(|p|^2+|\tilde{u}|^2)dtdx\\
&+\frac{1}{2}\max_{x\in\bar{I}}|\mathcal{A}_2|^2\int_Q(|q|^2+|\tilde{v}|^2)dtdx+\frac{1}{2}\max_{x\in\bar{I}}|\mathcal{A}_3|^2\int_Q(|r|^2+|\tilde{w}|^2)dtdx\\
&+\frac{1}{2}\int_Q\Big(|U|^2+|V|^2+|W|^2+|\mathcal{P}|^2+|\mathcal{Q}|^2+|\mathcal{R}|^2\Big)dtdx. \qquad (3.14)
\end{aligned}$$

According to Lemmas 1–2, we have

$$\begin{aligned}
&\int_Q\Big(|U|^2+|V|^2+|W|^2+|\mathcal{P}|^2+|\mathcal{Q}|^2+|\mathcal{R}|^2\Big)dtdx\\
&\le 4T\exp(2C_1T)\Bigg(\max_{x\in\bar{I}}|\mathcal{A}_1|^2\int_Q\Big(|\tilde{p}|^2+|\tilde{u}|^2\Big)dtdx\\
&+\max_{x\in\bar{I}}|\mathcal{A}_2|^2\int_Q\Big(|\tilde{q}|^2+|\tilde{v}|^2\Big)dtdx+\max_{x\in\bar{I}}|\mathcal{A}_3|^2\int_Q|\Big(|\tilde{r}|^2+|\tilde{w}|^2\Big)dtdx\\
&+\int_I\Big(|g_1-\tilde{g}_1|^2+|g_2-\tilde{g}_2|^2+|g_3-\tilde{g}_3|^2\Big)dx\Bigg). \qquad (3.15)
\end{aligned}$$

Furthermore, from the analogues of Lemma 3 and Lemma 4, there exists a constant

$$\tau=\|u_0\|^2_{L^2(I)}+\|v_0\|^2_{L^2(I)}+\|w_0\|^2_{L^2(I)}+\sum_{i=1}^{3}\|g_i\|^2_{L^2(I)} \qquad (3.16)$$

such that

$$\int_Q\Big(|\tilde{u}|^2+|\tilde{v}|^2+|\tilde{w}|^2\Big)dtdx\le T\exp(C_1T)\tau,$$

$$\int_Q\Big(|\tilde{p}|^2+|\tilde{q}|^2+|\tilde{r}|^2\Big)dtdx\le 2T\exp(2C_1T)\tau.$$

Besides, considering $\mathcal{A}_i(x_0)=0$ and using the Hölder's inequality, we have

$$\max_{x\in\bar{I}}|\mathcal{A}_i(x)|=\max_{x\in\bar{I}}\left|\int_{x_0}^{x}\mathcal{A}_i'(y)dy\right|\le\left|\int_I|\nabla\mathcal{A}|^2dy\right|^{\frac{1}{2}}. \qquad (3.17)$$

Substitute (3.15)–(3.17) above into (3.14), we obtain

$$
\begin{aligned}
&\max_{x\in\bar I}|\mathcal{A}_1|^2+\max_{x\in\bar I}|\mathcal{A}_2|^2+\max_{x\in\bar I}|\mathcal{A}_3|^2\\
&\le \frac{1}{2\mu}\max_{x\in\bar I}|\mathcal{A}_1|^2\Big(T\exp(C_1T)\tau+2T\exp(2C_1T)\tau\Big)\\
&+\frac{1}{2\mu}\max_{x\in\bar I}|\mathcal{A}_2|^2\Big(T\exp(C_1T)\tau+2T\exp(2C_1T)\tau\Big)\\
&+\frac{1}{2\mu}\max_{x\in\bar I}|\mathcal{A}_3|^2\Big(T\exp(C_1T)\tau+2T\exp(2C_1T)\tau\Big)\\
&+\frac{1}{2\mu}4T\exp(2C_1T)\Bigg(\max_{x\in\bar I}|\mathcal{A}_1|^2\Big(T\exp(C_1T)\tau+2T\exp(2C_1T)\tau\Big)\\
&+\max_{x\in\bar I}|\mathcal{A}_2|^2\Big(T\exp(C_1T)\tau+2T\exp(2C_1T)\tau\Big)\\
&+\max_{x\in\bar I}|\mathcal{A}_3|^2\Big(T\exp(C_1T)\tau+2T\exp(2C_1T)\tau\Big)\\
&+\int_I\Big(|g_1-\tilde g_1|^2+|g_2-\tilde g_2|^2+|g_3-\tilde g_3|^2\Big)\,dx\Bigg)\\
&\le \frac{2}{\mu}T\exp(2C_1T)\tau\left(\max_{x\in\bar I}|\mathcal{A}_1|^2+\max_{x\in\bar I}|\mathcal{A}_2|^2+\max_{x\in\bar I}|\mathcal{A}_3|^2\right)\\
&+\frac{8}{\mu}T^2\exp(4C_1T)\tau\left(\max_{x\in\bar I}|\mathcal{A}_1|^2+\max_{x\in\bar I}|\mathcal{A}_2|^2+\max_{x\in\bar I}|\mathcal{A}_3|^2\right)\\
&+\frac{2}{\mu}T\exp(2C_1T)\left(\int_I\Big(|g_1-\tilde g_1|^2+|g_2-\tilde g_2|^2+|g_3-\tilde g_3|^2\Big)\,dx\right)\\
&=C_T\left(\max_{x\in\bar I}|\mathcal{A}_1|^2+\max_{x\in\bar I}|\mathcal{A}_2|^2+\max_{x\in\bar I}|\mathcal{A}_3|^2\right)\\
&+\frac{1}{\mu}T\exp(2C_1T)\left(\int_I\Big(|g_1-\tilde g_1|^2+|g_2-\tilde g_2|^2+|g_3-\tilde g_3|^2\Big)\,dx\right),
\end{aligned}
\tag{3.18}
$$

where $C_T=\frac{2T}{\mu}\exp(2C_1T)\tau(1+4T\exp(2C_1T))$. Now choosing $T>0$ such that $C_T<1$, one can complete the proof. □

Remark 3. From Theorem 1, we see that if the final measurements of the systems (1.1) and (1.3) are equal, that is

$$u(x,T)=\tilde u(x,T),\ v(x,T)=\tilde v(x,T)\text{ and }w(x,T)=\tilde w(x,T),$$

then the data $a_{11},\ a_{22},\ a_{33}$ can be uniquely determined, i.e., for some small $T_0>0$ there is $a_{11}=\tilde a_{11},\ a_{22}=\tilde a_{22}$ and $a_{33}=\tilde a_{33}$ in I. In fact, from (3.18), it follows that

$$\max_{x\in\bar I}|\mathcal{A}_1|^2+\max_{x\in\bar I}|\mathcal{A}_2|^2+\max_{x\in\bar I}|\mathcal{A}_3|^2\le C_T\left(\max_{x\in\bar I}|\mathcal{A}_1|^2+\max_{x\in\bar I}|\mathcal{A}_2|^2+\max_{x\in\bar I}|\mathcal{A}_3|^2\right).$$

Similarly, choosing $T_0>0$ such that $C_{T_0}<1$ leads to the conclusion that

$$\max_{x\in\bar I}|\mathcal{A}_1|^2+\max_{x\in\bar I}|\mathcal{A}_2|^2+\max_{x\in\bar I}|\mathcal{A}_3|^2\le 0.$$

So we deduce that $a_{ii}(x)\equiv\tilde a_{ii}(x), i=1,2,3$ for all $x\in\bar I$.

Remark 4. For the coefficient reconstruction problem in the non-diagonal case, the method proposed in this paper is only applicable to some cases, that is, when the unknown coefficients are located in different rows and columns, and is not applicable to all other cases. For instance, when reconstructing the three coefficients (a_{11}, a_{12}, a_{13}) (i.e. in the same row), we cannot obtain a result similar to that of Lemma 1. We can only obtain the estimation of the first equation U with respect to the unknown coefficient of the disturbance. Such an estimation is insufficient in the proof of conditional stability. Similarly, when reconstructing coefficients like (a_{11}, a_{21}, a_{31}) (in the same column), results similar to those of Lemma 2 cannot be obtained either. In addition, Lemmas 3-4 also need to be adjusted accordingly. The reason is that when the diagonal coefficient a_{ii} is perturbated, the estimates of the above two lemmas remain unchanged. However, when perturbating non-diagonal coefficients, the estimates in the above lemma need to be adjusted accordingly.

4 Concluding remarks

This paper mainly focuses on the inverse problem of determining three spatial coefficients in a reaction-diffusion system based on final measurement data. Firstly, the given problem is transformed into an optimal control problem using optimization theory, and the existence of a minimizer is proven. Subsequently, stability estimates for the three spatially varying coefficients are derived, with upper bounds provided by the Lebesgue norm of the final measurement. In the following work, we will continue to deepen the study of the application of optimization theory to inverse problems, such as inverse non-diagonal potential terms problem or inverse coefficient problem in anomalous diffusion system (for instance, our previous situation regarding a single equation [21, 25]). Try to introduce more advanced optimization algorithms and techniques to improve the accuracy and efficiency of parameter identification. At the same time, it is considered to extend the method to more complex strongly coupled reaction-diffusion systems.

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