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A new closed-form solution for optimal portfolio selection with liquidity risk

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Abstract. In the literature on optimal portfolio selection problems, it is rare that closed-form solutions are found. It is even more so when liquidity risk needs to be taken into consideration. In this paper, we present a closed-form solution for the optimal weights of a portfolio that consists of a risky and riskless asset under a new key assumption that the liquidity risk is directly proportional to the wealth of the portfolio invested in the risky asset. The solution found is for the Constant Relative Risk Aversion (CRRA) utility function, after successfully solving the associated HJB (Hamilton-Jacobi-Bellman) equation exactly. Due to the presence of liquidity risk, the research findings reveal that the optimal weights are consistently lower than those found by [19]. Finally, the quantitative impact of the proposed solution is discussed.

Keywords: portfolio choice problem; log-return assumption; liquidity cost; closed-form solution.

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1 Introduction

The optimal investment and consumption constitute a pivotal role within mathematical finance and economics. The basic idea behind this problem revolves around identifying optimal trading strategies, often referred to as portfolio weights, which maximize an investor expected utility. Notably, the foundational contribution by [19] ushered in the dynamic portfolio theory framework. This seminal work gained substantial attention from both researchers and practitioners in the realm of financial economics [1, 8]. In this paper, we refer to the [19] model as Merton's model. In scenarios where asset prices follow geometric returns, Merton's model shows that the optimal investment strategy of an agent exhibits direct proportionality to excess returns and inverse proportionality to asset variance. Despite its significance, Merton's model relies on a certain set of assumptions that, in actuality, deviate from behavior observed in real financial markets. These assumptions can be broadly classified into two categories. The first category depends on asset price dynamics, a domain extensively scrutinized by numerous researchers(for instance [13, 23]) resulting in an extensive body of literature. In cases where these assumptions are untenable, recent work by [18] and related references therein have presented closed-form solutions for optimal portfolio choice and consumption problems.

The second set of assumptions followed by Merton's model is based on market conditions and trading modalities, commonly known as trading frictions. These encompass factors such as transaction costs, liquidity costs, and execution costs. Within the existing body of literature, the introduction of trading frictions to the portfolio choice problem significantly increases the complexity of the task of finding a solution. By assuming proportional transaction costs, [6] demonstrated that the existence of such costs curtails continuous trading for agents. Consequently, their optimal investment strategy exhibits a non-traded region. Building upon this, [7] incorporated transaction costs and derived closed-form solutions for Hyperbolic Absolute Risk Aversion (HARA) utility functions. Furthermore, [21] extended the analysis to an infinite time horizon while accounting for transaction costs, drawing upon the framework of viscosity solutions of the Hamilton-Jacobi-Bellman (HJB) equation. The literature on transaction costs is vast, and many studies are available (for comprehensive exploration, readers can refer to [16] and references therein). However, it is noteworthy that the literature concerning liquidity risk remains notably limited.

Highlighting the existence of liquidity risks, [22] contributed insights by establishing illiquidity as a market-wide systemic risk rather than being confined to the realm of asset-specific risk. Furthermore, [20] meticulously detailed the impact of liquidity risk on stock returns. Drawing from a broader context, [3] engaged in a comprehensive examination of the illiquidity premium across stock markets spanning 45 countries. What they discovered was that this extra return, known as the "illiquidity premium", could be measured using various methods. These included analyzing the monthly returns of stocks that are less liquid comparatively or calculating a numerical measure of stock illiquidity obtained from statistical analyses like cross-section Fama-MacBeth regressions. Evidently, the existing literature collectively underscores that liquidity issues significantly pervade financial markets, albeit without a standardized approach for measurement. Distinct authors adopt diverse methodologies to quantify market liquidity risk, reflecting the multifaceted nature of this phenomenon.

The works [11, 15] have significantly enriched our understanding of liquidity risk by presenting diverse measures to quantify it. Although liquidity has multiple facets, a large number of economic researchers have focused on investigating the effects of liquidity risk on asset prices ([2, 4, 14]). There are also some researchers who explored how liquidity risk influences options, and they have come up with clear solutions for pricing options in these cases [5, 12]. In our paper, we are interested in finding optimal weights over time when the market faces liquidity risk.

The investigation into how liquidity risk affects the decisions about optimal weights has only recently begun to attract attention; the research in this area is still in its early stages, and there is no consensus reached on the definition of liquidity risk. There are few research papers that work on dynamic portfolio choice problems considering liquidity risk. [24] consider the simulation and regression approach to find the solution of dynamic portfolio theory with liquidity cost, they extend the classical least squares Monte Carlo algorithm to incorporate switching costs, corresponding to transaction costs and transient liquidity costs. The work of [10] considers non-linear price impact and finds asymptotic explicit formulas up to a structural constant that depends only on the curvature of the price impact function for the portfolio choice. He also considered a different form of liquidity cost which relies not only on the number of assets one buys (sells) but also on the asset price and wealth of the investor at any time t. The work of [16] provides closed-form solution to an optimal investment and consumption problem for a CARA agent, who faces execution costs when trading correlated risky assets with return predictability.

The highly tractable framework of [9] and [16] assumes arithmetic returns follow some random process with return-predicting factors. However, this approach sometimes results in negative asset prices, prompting further exploration. To address this issue, [17] considers geometric returns, but they only find the first-order approximate solutions of the problem using asymptotic expansion of the value function, while this paper derives an exact analytical solution. They assume liquidity cost is directly proportional to variance. They also assume liquidity cost depends only on the number of assets one buys (sells), regardless of the price of the asset at that time. This implies a uniform liquidity cost for all assets when the same quantity is bought or sold, regardless of individual asset prices.

This paper addresses the aforementioned issues by introducing novel contributions. Firstly, we focus on employing geometric returns (logarithmic returns) without incorporating a return-predicting factor. This effectively resolves the challenge of negative asset pricing. Additionally, we present an innovative approach to measuring liquidity risk, which effectively mitigates the concern of uniform liquidity cost across all assets.

Our proposed liquidity cost model is contingent upon the invested (or divested) amount in the risky asset, rather than mere quantity. Subsequently, we succeeded in deriving closed-form solutions for the power utility function (specifically, Constant Relative Risk Aversion - CRRA). Notably, our analytical optimal trading strategy, determined through closed-form calculations, exhibits an inverse correlation with the liquidity parameter Λ. In cases of pronounced market illiquidity ($\Lambda >> 1$), our model advocates for reduced allocation to illiquid assets - a prudent financial strategy.

However, the imposition of liquidity costs imposes constraints on extensive trading activities for agents, akin to Merton's model. An intriguing avenue for extension involves augmenting this model with geometric returns and return-predicting factors, potentially yielding further closed-form solutions. While this direction remains unexplored in our current work, it holds promise for future research.

Central to this paper is the aspiration to craft a model that echoes classical foundations and offers precise solutions to the portfolio choice problem with liquidity risk. In pursuit of this goal, we meticulously verify our newly derived solution through numerical examples. These examples effectively illustrate the economic ramifications of our assumptions and lend empirical support to our model's implications.

The organization of the rest of the paper is as follows. In Section 2, we propose a dynamic portfolio selection model with a new form of liquidity cost, under log-returns. In Section 3, we find the closed-form solution to the problem and the main results of the problem. In Section 4, we provide numerical results and economic explanations of our results. Section 5 concludes the paper.

2 Model

2.1 Asset dynamics and liquidity costs

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete filtered probability space, where $\mathcal{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ denotes the filtration generated by 1-dimensional Brownian motion $(W_t^S)_{t\geq 0}$. Consider a portfolio comprising a single risky asset and a risk-free asset. We only take into account the fact that the risky asset is experiencing liquidity challenges. The primary focus lies in the investor's endeavor to allocate their wealth strategically between these two assets, both of which are subject to continuous trading dynamics over the interval $[0, T]$. The dynamics of the risk-free asset $(B_t)_{t\geq 0}$ is given as,

$$\frac{dB_t}{B_t} = rdt,$$

where $r > 0$ denotes the risk-free interest rate. The risky asset $(S_t)_{t\geq 0}$ follows the log return and has the following dynamics,

$$\frac{dS_t}{S_t} = \mu dt + \sigma_S dW_t^S,$$

where μ denotes the return of the risky asset and $\sigma_S > 0$ denotes the volatility of the risky asset S_t. The symbol w_t denotes the proportion of investor wealth invested in the risky asset(S_t) at time $t \in [0, T]$.

Assumption 1. Inspired by the insights from [9], we adopt a trading framework wherein the investor's trading intensity (τ_t) at time t is governed by the following differential equation,

$$\frac{dw_t}{dt} = \tau_t. \tag{2.1}$$

The above equation implies the instantaneous change in the proportion $(w_t)_{t\geq 0}$ of the total wealth invested in the risky asset, where $(\tau_t)_{t\geq 0}$ is subsequently referred to as the **trading strategy** of the investor. It is important to note that there are no constraints on the sign of τ_t for $t > 0$. At any given point in time $t > 0$, if $\tau_t > 0$, the investor allocates $\tau_t X_t$ amount of money into the risky asset within the interval $[t, t + dt)$. Conversely, if $\tau_t < 0$, the investor withdraws (divests) $\tau_t X_t$ amount of money from their portfolio that is invested in the risky asset at time t. Here, X_t denotes the investor's total wealth at time t. Mathematically, Equation (2.1) signifies our consideration of a smooth or absolutely continuous portfolio. Next, we will specify the nature of liquidity cost and the dynamics of the wealth equation.

Remark 1. It should be remarked that we have assumed that an investor would continuously trade with a non-zero trading intensity of τ_t. Of course, this is in line with our overall assumption that there is no transaction cost taken into account in the current model, as otherwise, the investor may choose to "rest" for a period of time, albeit small, with no trading action at all. In other words, we have assumed that the investor has always been actively involved in trading, either buying or selling the risky asset, in order to maximize his/her utility. This assumption may deviate from the conventional ones, but it delivers the needed tractability for our model. It also makes financial sense for super-active traders.

Assumption 2. This assumption specifies the nature of liquidity cost. We assume the market is not liquid enough to buy (sell) any number of risky assets, and the liquidity cost is directly proportional to the amount invested in the risky asset. Following [17], we denote S_t^E to be the execution price when trading $\tau_t dt$ proportion of the wealth in the risky asset at time $t \geq 0$, the trade has a transient linear price impact on the asset price S_t as shown below,

$$S_t^E \triangleq S_t + \frac{1}{2}\Lambda \tau_t X_t, \tag{2.2}$$

where $\Lambda > 0$ is a parameter that measures the level of liquidity and is often referred to as Kyle's lambda. When the market experiences high illiquidity, Λ tends to be greater than or equal to 1. This implies that acquiring the asset comes at higher costs. In essence, a higher value of Λ corresponds to increased market illiquidity. In the context of this paper, our primary focus revolves around scenarios where assets encounter liquidity challenges. With reference to Equation (2.2), if the investor opts to purchase the risky asset ($\tau_t > 0$), this action would push the price of the risky asset upwards by $\frac{1}{2}\Lambda\tau_t X_t$. Conversely, if the investor withdraws funds from the risky asset (meaning they sell some of the risky assets), i.e., $\tau_t < 0$, this would exert a dampening effect on the price of the risky asset by $\frac{1}{2}\Lambda\tau_t X_t$. In terms of the liquidity price, the total cost of investing in a risky asset per time unit of trading with intensity τ_t is expressed as follows,

$$TC(\tau_t dt) \triangleq \tau_t X_t dt + \frac{1}{2}\Lambda \tau_t^2 X_t dt, \tag{2.3}$$

where TC denotes the total cost. The first term on the right side denotes the cost of trading $\tau_t dt$ proportion of the risky asset at the market price S_t, and the second term captures the liquidity cost of trading $\tau_t dt$ units. From Equation (2.3), the assumption of the linear transient price impact of trade size τ leads to quadratic illiquidity cost $\frac{1}{2}\Lambda\tau_t^2 X_t$, indicating the large trade size induces higher illiquidity costs. For the convenience of the reader, all mathematical symbols appearing in this paper are listed in Table 1 together with their meanings.

Table 1. Notation and their meanings used throughout the paper.

Notation	Meaning
X_t	Investor's total wealth at time t
w_t	Proportion of total wealth invested in the risky asset at time t
τ_t	Proportion of total wealth invested in the risky asset between any time t to t+dt
μ	Expected return of the stock
σ_S	Volatility of the stock price
Λ	Liquidity parameter
α	Risk-aversion parameter of CRRA utility
r	Risk-free interest rate

Remark 2. In the paper by [17], the liquidity cost is solely tied to the quantity of risky assets bought or sold, irrespective of the actual asset price. This implies that if we consider two assets, let's call them S_1 and S_2, with initial prices of 1000 and 10 respectively, and we purchase the same quantity of both assets, the resulting illiquidity cost would be identical in both scenarios. This, however, does not align with economic reality. Indeed, in real-world situations, higher asset prices tend to correlate with increased illiquidity concerns, as the potential pool of buyers and sellers is typically more limited. In our approach, we address this concern by anchoring the liquidity cost to the amount invested in risky assets, rather than merely the number of assets. This modification offers a more accurate representation and simultaneously rectifies the issue mentioned above.

We invest w_t proportion of wealth invested in the risky asset (S_t) and the remaining $(1 - w_t)$ proportion of wealth invested in the risk-free asset (B_t) at a time t. Effectively managing her wealth, the investor strives to allocate funds between the risky and risk-free assets while netting the liquidity costs, and applying Ito's lemma and the self-financing condition, the dynamics of the investor's wealth admit the following form,

$$\frac{dX_t}{X_t} = \big(w(t)(\mu - r) + r - 0.5\Lambda\tau_t^2\big)dt + w(t)\sigma_S dW_t^S.$$

Remark 3. This model is designed to address scenarios where the market experiences illiquidity challenges. Consequently, the parameter Λ is strictly positive $(\Lambda > 0)$. This assumption ensures that the trading strategy remains finite and feasible, avoiding the unrealistic outcome of an infinite trading strategy that would arise if Λ were zero. Therefore, we adopt $\Lambda > 0$ as a fundamental assumption to maintain the model's applicability in realistic market conditions. It should be noted that, in the above wealth dynamics, only the liquidity cost is subtracted explicitly, not the basic trading cost. This happened because the basic self-financing wealth equation already incorporates the standard cost of acquiring and holding the assets at the market price, but not the additional liquidity cost. Therefore, we subtract only this incremental liquidity term in the wealth dynamics, rather than the basic trading cost.

2.2 Utility maximization problem

We consider an expected utility maximization problem of the CRRA (power utility function) agent with liquidity costs. Specifically, the CRRA agent chooses a trading strategy to maximize his terminal wealth. The utility function we are using in this paper is defined below,

$$U(x) = x^\alpha/\alpha,$$

where $\alpha > 0$ is the risk aversion parameter.

Problem 1. An investor seeks to find an optimal trading strategy to maximize expected utility over terminal wealth,

$$\max_{\tau \in A} \mathbf{E}[U(X_T)|w(0) = w, X(0) = x],$$

where $\mathbf{E}$ is the expectation operator and the above maximization problem is subject to the following dynamics,

$$\begin{cases} dw_t = \tau_t dt, \\ \frac{dX_t}{X_t} = (w_t(\mu - r) + r - \frac{1}{2}\Lambda\tau_t^2)dt + w(t)\sigma_S dW_t^S, \end{cases}$$

and A is the set of admissible strategies satisfying,

$$\int_0^T \|\tau(s)\|\, ds \; < \infty.$$

So far, we have established the model, and in the next section, we will identify the associated HJB equation for the model and solve it in a clear and explicit manner. For notational convenience, we may write τ_t simply as τ, although the meaning remains unchanged and the control still implicitly depends on time.

3 Optimal trading strategies

In this section, we present a closed-form solution for Problem 1 using the dynamic programming method. To tackle Problem 1, we define a value function as

$$V(0, w, X) = \max_{\tau \in A} \mathbf{E}[U(X(T))|w(0) = w, X(0) = x].$$

Using the well-established concept of dynamic programming, the continuous value function $V(t, w, X)$ that ensures smooth outcomes for Problem 1 follows the HJB equation, which can be expressed as,

$$\max_{\tau \in \mathbf{R}}\{V_t + (w(\mu - r) + r - \frac{1}{2}\Lambda\tau^2)xV_x + \tau V_w + \frac{1}{2}w^2\sigma_S^2 x^2 V_{xx} = 0\}, \tag{3.1}$$

with the terminal condition $V(T, w, X) = \frac{X^\alpha}{\alpha}$.

3.1 A closed-form solution

We make the assumption that Problem 1 is well-defined, and the value function associated with it meets the requirements of the HJB equation described in Equation (3.1). By applying the first-order conditions concerning $\tau \in \mathbb{R}$, we deduce that for an optimal trading strategy τ^*, we obtain the following form as,

$$\tau^*(t, w, X) = \frac{1}{\Lambda x} \frac{\partial_w V(t, w, X)}{\partial_x V(t, w, X)},$$

for $\partial_x V(t, w, X) \neq 0$. Here, we denote τ^* to be the function of (t, w, X) to indicate that the investor's optimal trading strategy τ^* depends on the current value of portfolio position w and her wealth X. Now, Equation (3.1) changes to,

$$V_t + (w(\mu - r) + r)xV_x + \frac{1}{2}w^2\sigma_S^2x^2V_{xx} + \frac{1}{2\Lambda x}\frac{V_w^2}{V_x} = 0. \tag{3.2}$$

The above equation is a highly non-linear partial differential equation in V with three independent variables. Now, using the homogeneity condition of the utility function, we have,

$$V(x, t, w) = \frac{x^\alpha}{\alpha}h(t, w).$$

Employing the form of value function mentioned above, we can write the initial three-variable Equation (3.2) into a two-variable partial differential equation (PDE). The relevant partial derivatives are as follows,

$$V_t = \frac{x^\alpha}{\alpha}h_t;\ v_w = \frac{x^\alpha}{\alpha}h_w;\ V_x = x^{\alpha-1}h;\ V_{xx} = (\alpha-1)x^{\alpha-2}h,$$

Putting everything back into Equation (3.2), we obtain,

$$h_t + \alpha(w(\mu - r) + r)h + \frac{1}{2}\alpha(\alpha-1)w^2\sigma_S^2h + \frac{1}{2\Lambda\alpha}\frac{h_w^2}{h} = 0. \tag{3.3}$$

Now, consider a trial solution in the affine form, which is represented as,

$$h(t, w) = e^{A(t)+B(t)w+C(t)w^2},$$

with terminal condition $h(T, w) = 1$. Now, as a result of the simplification, Equation (3.3) transforms into an ODE which is given as,

$$\dot{A(t)} + \dot{B(t)}w + \dot{C(t)}w^2 + \alpha(w(\mu - r) + r) + \frac{1}{2}\alpha(\alpha-1)w^2\sigma_S^2 + \frac{1}{2\Lambda\alpha}[B(t) + C(t)w]^2 = 0,$$

where $\dot{A(t)}$ denotes the derivative of A with respect to time t. Similarly, $\dot{B(t)}, \dot{C(t)}$ represent their derivative with respect to t respectively. Following the work of [16], comparing the coefficients of $(\cdot)w$, $(\cdot)w^2$ and constant yields the following system of differential equations,

$$\begin{cases} \dot{A}(t) + \dfrac{1}{2\Lambda\alpha}B^2(t) + \alpha r = 0, \\ \dot{B}(t) + \dfrac{1}{\Lambda\alpha}B(t)C(t) + \alpha(\mu - r) = 0, \\ \dot{C}(t) + \dfrac{1}{2\Lambda\alpha}C^2(t) + \dfrac{1}{2}\alpha(\alpha-1)\sigma_S^2 = 0, \end{cases}$$

with terminal conditions $A(T) = 0, B(T) = 0, C(T) = 0$. Now we will solve for $C(t)$ and then for $B(t)$ and $A(t)$ respectively. Finally, our nonlinear PDE reduces to a system of ordinary differential equations, and our task is now to solve this system in order to obtain the final closed-form solution. Obviously, the $C(t)$ can be solved by the separable variable method, and the corresponding solution is given by,

$$C(t) = p\left[(e^{q(t-T)} - 1)/(e^{q(t-T)} + 1)\right],$$

where p and q are constants defined as,

$$p = \sigma_S\alpha\sqrt{(1-\alpha)\Lambda}\,; \quad q = \sigma_S\sqrt{\frac{1-\alpha}{\Lambda}}.$$

Now substituting the expression for $C(t)$ into the above system of equations to determine $B(t)$, we obtain,

$$\frac{dB(t)}{dt} + q\left[(e^{q(t-T)} - 1)/(e^{q(t-T)} + 1)\right]B(t) = -\alpha(\mu - r),$$

which is a linear differential equation in $B(t)$. Furthermore, to solve $B(t)$, the integrating factor (I.F.) will be,

$$I.F. = (e^{q(t-T)} - 1)^2/e^{q(t-T)},$$

and finally the corresponding solution for $B(t)$ is given as,

$$B(t) = \frac{\alpha(\mu - r)}{(e^{q(t-T)} - 1)^2}\left[\frac{1}{q} + 2e^{q(t-T)}(t - T) - \frac{e^{2q(t-T)}}{q}\right].$$

At last, $A(t)$ can be worked out by direct integration. However, our core focus lies in the values of $B(t)$ and $C(t)$. These values are of paramount importance as they enable us to make a meaningful comparison between our results and the performance of Merton's portfolio, a vital aspect of our analysis. Finally, the optimal trading strategy τ^* has the following solution,

$$\tau^*(t, w, X) = \frac{1}{\alpha\Lambda}[B(t) + 2C(t)w(t)].$$

Until now, we have successfully derived a closed-form solution for the optimal portfolio with geometric returns, when the underlying price needs to be adjusted due to liquidity risk. In the next section, the accuracy of the newly derived solutions will be verified through numerical experiments and the financial interpretation of the results as well.

Although we verify our results through numerical analysis, which will be shown in the next section, but to ensure mathematical rigorous of our results we also provide a verification theorem and show that our solution satisfies the corresponding HJB equation.

Theorem 1. *(Verification Theorem): Let take the solution of above value function which is defined as,*

$$V(t,x,w) = \frac{x^\alpha}{\alpha}\exp\bigl(A(t) + B(t)w + C(t)w^2\bigr),$$

where the functions $A(t)$, $B(t)$ and $C(t)$ are given explicitly and satisfy the system of ODEs which are mentioned above, together with the terminal conditions $A(T) = B(T) = C(T) = 0$. The solution found above can be represented as,

$$\tau^*(t,x,w) = \frac{1}{\Lambda x}\frac{V_w(t,x,w)}{V_x(t,x,w)}.$$

To ensure that a solution is indeed optimal and correctly verified, the following conditions must hold,

1. *The pair (V, τ^*) satisfies the original HJB equation with terminal condition $V(T,x,w) = X^\alpha/\alpha$.*
2. *For any admissible strategy τ, the following inequality holds*

$$V(t,x,w) \;\geq\; \mathbb{E}[U(X_T^\tau)].$$

3. *Under the strategy τ^*, we have*

$$V(t,x,w) \;=\; \mathbb{E}[U(X_T^{\tau^*})].$$

If the conditions stated above are satisfied by the candidate pair (V, τ^*), then we may conclude that V indeed represents the value function of the optimisation problem and that τ^* constitutes the optimal trading strategy.

Proof. To verify our solution, we proceed in several steps. Firstly, we identify τ^* as the maximiser of the Hamiltonian, which allows us to remove the maximum operator from the HJB equation. We then show that the HJB is satisfied exactly when we substitute both the candidate value function V and the control τ^* into it, and finally we check the value function inequalities.

Step 1: HJB equation and Hamiltonian.

The original HJB equation for V can be written as,

$$0 = \max_{\tau\in\mathbb{R}}\Bigl\{V_t + (w(\mu - r) + r)xV_x + \tfrac{1}{2}w^2\sigma_S^2x^2V_{xx} + \tau V_w - \tfrac{1}{2}\Lambda x\tau^2V_x\Bigr\}. \quad (3.4)$$

To organise the terms and isolate the effect of the control, we define the Hamiltonian as,

$$\mathcal{H}(t,x,w,\tau;V) := V_t + (w(\mu-r)+r)xV_x + \tfrac{1}{2}w^2\sigma_S^2x^2V_{xx} + \tau V_w - \tfrac{1}{2}\Lambda x\tau^2 V_x.$$

The HJB equation (3.4) simply states that, at each state (t,x,w), the value function must choose the control τ that maximises this Hamiltonian.

Step 2: First and second-order conditions for τ^*.

Basically, if the candidate solution is correct, it must satisfy both the first and second-order conditions. From the first-order condition, we have,

$$\frac{\partial \mathcal{H}}{\partial \tau} = V_w - \Lambda x\tau V_x = 0.$$

After substituting the derived τ^* into the first-order condition, it can be easily verified that the condition is indeed satisfied, as shown below,

$$V_w - \Lambda x\left(\frac{1}{\Lambda x}\frac{V_w}{V_x}\right)V_x = V_w - V_w = 0.$$

Hence, the first-order condition is satisfied by the newly derived solution. Next, we examine whether our solution also satisfies the second-order condition. To do so, the second partial derivative is given by,

$$\frac{\partial^2 \mathcal{H}}{\partial \tau^2} = -\Lambda x V_x.$$

Since $x > 0$, $\Lambda > 0$, and $V_x > 0$ under CRRA utility, the second derivative is strictly negative. The positivity of V_x follows from the fact that CRRA utility is strictly increasing in wealth, so the value function must also be strictly increasing, implying $V_x > 0$. Hence the Hamiltonian is strictly concave in τ, and the control τ^* obtained from the first-order condition is the unique maximiser. Because τ^* achieves the maximum in (3.4), the maximisation operator can be removed. Thus, the HJB equation can now be written as an equality by substituting $\tau = \tau^*$. Therefore the final PDE can be written as,

$$0 = V_t + (w(\mu-r)+r)xV_x + \tfrac{1}{2}w^2\sigma_S^2x^2V_{xx} + \tau^*V_w - \tfrac{1}{2}\Lambda x(\tau^*)^2V_x. \tag{3.5}$$

Step 3: Substituting V and τ^* into the HJB.

We now substitute the explicit form of V and the control τ^* into (3.5) to verify that our solution. The final value function is defined as,

$$V(t,x,w) = \frac{x^\alpha}{\alpha}\exp\big(A(t)+B(t)w+C(t)w^2\big),$$

for which we have,

$$\begin{aligned}
V_t &= \frac{x^\alpha}{\alpha}\exp\big(A(t)+B(t)w+C(t)w^2\big)\big(A'(t)+B'(t)w+C'(t)w^2\big),\\
V_x &= x^{\alpha-1}\exp\big(A(t)+B(t)w+C(t)w^2\big),\\
V_w &= \frac{x^\alpha}{\alpha}\exp\big(A(t)+B(t)w+C(t)w^2\big)\big(B(t)+2C(t)w\big),\\
V_{xx} &= (\alpha-1)x^{\alpha-2}\exp\big(A(t)+B(t)w+C(t)w^2\big).
\end{aligned}$$

Substituting these expressions and the value of τ^* into the right hand side of (3.5) gives,

$$\Longrightarrow \frac{x^\alpha}{\alpha} e^{A+Bw+Cw^2}\left(A' + B'w + C'w^2\right) + (w(\mu - r) + r)x\, x^{\alpha-1} e^{A+Bw+Cw^2} + \tfrac{1}{2} w^2 \sigma_S^2 x^2\, (\alpha-1) x^{\alpha-2} e^{A+Bw+Cw^2} + \frac{1}{2\Lambda x}\frac{V_w^2}{V_x}.$$

For the last term, we computed,

$$\frac{V_w^2}{V_x} = \frac{\left(\frac{x^\alpha}{\alpha} e^{A+Bw+Cw^2}(B+2Cw)\right)^2}{x^{\alpha-1} e^{A+Bw+Cw^2}} = \frac{x^{\alpha+1}}{\alpha^2} e^{A+Bw+Cw^2}(B+2Cw)^2,$$

so that,

$$\frac{1}{2\Lambda x}\frac{V_w^2}{V_x} = \frac{x^\alpha}{2\Lambda\alpha^2} e^{A+Bw+Cw^2}(B+2Cw)^2.$$

Hence the right hand side of (3.5) converted to,

$$\Longrightarrow x^\alpha e^{A+Bw+Cw^2}\left[\frac{1}{\alpha}\left(A' + B'w + C'w^2\right) + (w(\mu - r) + r) + \tfrac{1}{2}(\alpha-1)\sigma_S^2 w^2 + \frac{1}{2\Lambda\alpha^2}(B+2Cw)^2\right].$$

Now, after substituting the candidate value function V and the control τ^* into the rhs of the HJB equation and factoring out the positive term $x^\alpha e^{A+Bw+Cw^2}$, we define,

$$E(w) := \frac{1}{\alpha}\left(A'(t) + B'(t)w + C'(t)w^2\right) + \left(w(\mu - r) + r\right) + \tfrac{1}{2}(\alpha-1)\sigma_S^2 w^2 + \frac{1}{2\Lambda\alpha^2}\left(B(t) + 2C(t)w\right)^2. \tag{3.6}$$

We first expand the quadratic term,

$$\left(B + 2Cw\right)^2 = B^2 + 4BC\,w + 4C^2 w^2,$$

and hence

$$\frac{1}{2\Lambda\alpha^2}\left(B + 2Cw\right)^2 = \frac{B^2}{2\Lambda\alpha^2} + \frac{2BC}{\Lambda\alpha^2}w + \frac{2C^2}{\Lambda\alpha^2}w^2.$$

Substituting this into (3.6) gives,

$$E(w) = \frac{1}{\alpha}A' + \frac{1}{\alpha}B'w + \frac{1}{\alpha}C'w^2 + r + w(\mu - r) + \tfrac{1}{2}(\alpha-1)\sigma_S^2 w^2 + \frac{B^2}{2\Lambda\alpha^2} + \frac{2BC}{\Lambda\alpha^2}w + \frac{2C^2}{\Lambda\alpha^2}w^2.$$

Next, we collect constant, linear, and quadratic terms in w and factor $\frac{1}{\alpha}$:

$$E(w) = \frac{1}{\alpha}\left(A' + \alpha r + \frac{B^2}{2\Lambda\alpha}\right) + \frac{w}{\alpha}\left(B' + \alpha(\mu - r) + \frac{2BC}{\Lambda\alpha}\right) + \frac{w^2}{\alpha}\left(C' + \tfrac{1}{2}\alpha(\alpha-1)\sigma_S^2 + \frac{2C^2}{\Lambda\alpha}\right).$$

It can be observed that each of the bracketed expressions appearing in the decomposition of $E(w)$ coincides exactly with the left-hand sides of the ODE system derived earlier for $A(t)$, $B(t)$, and $C(t)$. Therefore, upon substituting these three identities into the expression for $E(w)$, every bracketed term vanishes identically, and hence,

$$E(w) = 0 \qquad \text{for all } w.$$

Finally, since the Hamiltonian reduces to,

$$\mathcal{H}(t, x, w, \tau^*; V) = x^\alpha e^{A(t)+B(t)w+C(t)w^2}\, E(w)$$

and the prefactor $x^\alpha e^{A(t)+B(t)w+C(t)w^2}$ is strictly positive, and the vanishing of $E(w)$ implies that,

$$\mathcal{H}(t, x, w, \tau^*; V) = 0.$$

Hence, the candidate value function V, together with the optimal control τ^*, *indeed satisfies the HJB equation.*

Step 5: Value function inequalities and optimality.

Finally, we show that V dominates the expected utility of any other strategy and that the maximum is attained under τ^*. Let τ be an arbitrary admissible control and let (X_t, w_t) be the associated state process. Applying Itô's formula to $V(t, X_t, w_t)$, we obtain,

$$dV(t, X_t, w_t) = \mathcal{H}(t, X_t, w_t, \tau_t; V)\, dt + dM_t,$$

where M_t is a local martingale. By the HJB equation and the optimality of τ^*, we have,

$$\mathcal{H}(t, x, w, \tau_t; V) \leq \mathcal{H}(t, x, w, \tau^*; V) = 0,$$

the drift of $V(t, X_t, w_t)$ is non-positive for any admissible strategy. Thus $V(t, X_t, w_t)$ is a supermartingale, and using the terminal condition $V(T, X_T, w_T) = U(X_T)$ we obtain,

$$\mathbb{E}[U(X_T^\tau)] \leq V(t, x, w),$$

which proves the desired inequality for any admissible strategy τ.

When $\tau = \tau^*$, the drift term vanishes and $V(t, X_t, w_t)$ becomes a true martingale. Therefore,

$$V(t, x, w) = \mathbb{E}\big[V(T, X_T, w_T)\big] = \mathbb{E}\big[U(X_T^{\tau^*})\big],$$

which shows that the upper bound is attained by τ^* and completes the proof. Combining these steps shows that V is the value function and that τ^* is the optimal trading strategy. $\square$

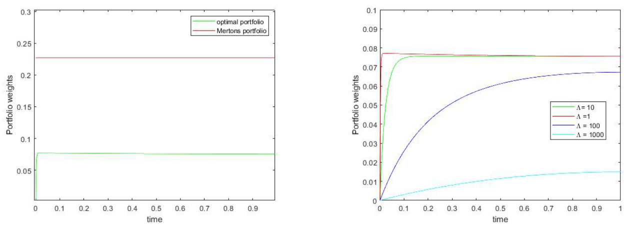
4 Results and discussions

To discuss the impact of liquidity on the optimal choice for a fundamental portfolio that involves a riskless and risky asset, the results calculated from the new closed-form solution are compared in this section with those presented in the classical paper of [19], which assumes that the risky asset is perfectly liquid. Let

$$w_M^* = \frac{(\mu - r)}{\sigma_S^2(1-\alpha)},$$

be the optimal weights found in [19] and assume a specific set of parameters as follows:

$$\mu = 0.2, r = 0.1, \Lambda = 1, \alpha = 0.1, \sigma_S = 0.7, w = 0.$$



(a) Comparison between Merton and optimal portfolio weights

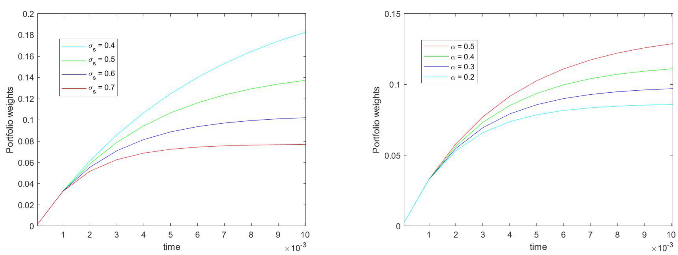
(b) The evolution of optimal portfolio with different levels of liquidity

Figure 1. Sample path of optimal portfolio weight and *Merton*.

Figure 1(a) displays the optimal allocations w^* and w_M^*, respectively, as a function of time for the given parameters. From this figure, the impact of liquidity can be clearly observed; the optimal weights with liquidity being taken into consideration are always less than those without. Financially, this makes perfect sense as the cost of liquidity is proportional to the amount one has allocated to the risky asset as assumed in Equation (2.2). Therefore, market friction in the form of liquidity risk should naturally discourage the investor from putting more money into the basket of his/her risky asset. When liquidity cost comes into the role, it can be seen that the optimal weights (w^*) increase gradually and then become steady. In other words, with the presence of liquidity risk, the investor's tolerance of the risk associated with the risky asset is damped when his/her tolerance of the total risk in the portfolio is maintained unchanged. It shows that the optimal portfolio is more conservative than the Merton portfolio in the presence of liquidity costs. Here, we assume that at time $t = 0$, the investor enters the market with all their funds allocated to a risk-free asset. From this initial position, the investor then actively trades with the objective of maximizing returns.

Figure 1(b) examines the influence of the liquidity parameter (Λ), keeping the other parameters unchanged. This figure reveals that in the scenario of extremely illiquid assets ($\Lambda = 1000$), the optimal weights are lower compared to a less illiquid asset ($\Lambda = 100$). So we allocate relatively less weight to the illiquid asset, compared to the case of a more liquid asset. Our findings save the investor from putting more money into an illiquid asset. This sounds logical, as the illiquidity penalty depends on the amount of money you invest in that asset at any time t, the more one invests the more is the penalty. A larger Λ means more liquidity challenges in the market. Our findings save the investor from putting more money into an illiquid asset.

In Figure 2(a), we are comparing optimal weights (w^*) for different levels of volatility, keeping the other parameters the same. From a financial standpoint, when the volatility of an asset rises, it generally leads to a decrease in our investment allocation for that asset. Our findings echo this financial insight. As evidenced in Figure 2(a), we discern that more volatility in the risky asset corresponds to a reduced allocation of funds to that asset, all else being constant. This visual representation corroborates that our optimal portfolio weights are indeed sensitive and responsive to changes in volatility.



(a) The evolution of optimal portfolio weights with different levels of volatility

(b) The evolution of optimal portfolio weights with different levels of risk aversion parameter

Figure 2. Optimal portfolio weight with respect σ_s and α.

To analyze the impact of the risk aversion parameter (α), we vary the α and keep the other parameters unchanged. A higher value of α implies a more risk-seeking investor. This phenomenon is illustrated in Figure 2(b), where we observe that an increased risk aversion parameter translates to a larger proportion of funds being invested in the risky asset. This alignment with our intuition is rooted in the investor's willingness to embrace more risk, subsequently leading to greater emphasis being placed on risky assets within our results.

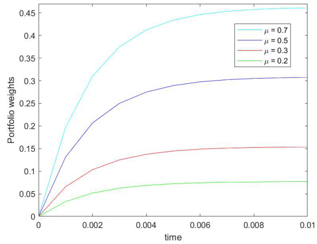


Figure 3. The evolution of optimal portfolio weights with different levels of mean.

Lastly, we delve into the impact of the mean (μ) on optimal portfolio weights, while keeping other parameters the same. Drawing from the classical insights presented in [19], it's established that portfolio weights are proportionate to the mean of the asset. In simpler terms, a higher asset mean correlates with larger investments in that asset. Remarkably, our findings align harmoniously with this rationale. Referring to Figure 3, we discern that elevated means of risky assets correspond to higher optimal portfolio weights allocated to those assets. This alignment not only resonates financially but is also logically grounded. Therefore, our outcomes effectively mirror the financial justifications exhibited in the Merton model, while also accommodating the presence of liquidity issues in the market.

5 Conclusions

This paper addresses the role of liquidity risk in dynamic portfolio theory for a CRRA utility function within a finite time frame. The approach involves considering geometric returns for the stock price and employing a quadratic form for liquidity cost. Notably, we succeed in solving a non-linear HJB equation by using the homogeneity condition of the power utility function that leads to a closed-form solution for the optimal weights.

Our closed-form optimal trading strategy sheds light on the dynamics of agent trading behavior. Specifically, we observe a pattern where trading actions initially increase over time and eventually stabilize, but always less than the classical solution found in [19] model. These findings are justifiable as the influence of liquidity cost curbs the agent's ability to engage in significant trades, aligning with the constraints observed in Merton's portfolio strategy.

Furthermore, our numerical analysis not only validates the findings but also demonstrates that our results align with the classical financial strategies as well. Our solution demonstrates a time-dependent impact on the agent's optimal portfolio, which later converges to a constant configuration similar to Merton's portfolio strategy.

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