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Extension of a Bohr-Jessen type theorem for the Epstein zeta-function in short intervals

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Abstract. Let Q be a positive definite $n \times n$ matrix, $n \in 2\mathbb{N}$, $n \geqslant 4$. The Epstein zeta-function $\zeta(s;Q)$, defined for $\operatorname{Re} s > \frac{n}{2}$, is given by $\zeta(s;Q) = \sum_{\underline{x}\in\mathbb{Z}^n\setminus\{\underline{0}\}}(\underline{x}^T Q \underline{x})^{-s}$, and has a meromorphic continuation to the whole complex plane. Let $T^{27/82} \leqslant H \leqslant T^{1/2}$. In this paper, we prove a limit theorem on weak convergence for $\frac{1}{H}\text{meas}\left\{t \in [T, T+H] : \zeta(\sigma+it;Q) \in A\right\}$, $A \in \mathcal{B}(\mathbb{C})$, as $T \to \infty$, where $\mathcal{B}(\mathbb{C})$ is the Borel σ-algebra on $\mathbb{C}$. The limit measure is explicitly given. The result extends a known theorem obtained for the interval $[0, T]$.

Keywords: Epstein zeta-function; limit theorem; Haar probability measure; weak convergence.

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1 Introduction

It is well known that the value distribution of many functions, including the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \quad \sigma > 1,$$

is quite chaotic, and it is difficult to give precise statements about their concrete values. In turn, H. Bohr proposed the following statistical approach: one fixes a set $A \subset \mathbb{C}$ and considers the density of values of $\zeta(s)$ belonging to A. In [3], H. Bohr jointly with B. Jessen proved that, for $\sigma > 1$ and for every rectangle R with edges parallel to axis, the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{J}\left\{t \in [0,T] : \log\zeta(\sigma+it) \in R\right\},$$

exists, where $\mathrm{J}A$ denotes the Jordan measure of $A \subset \mathbb{C}$. In [4], the latter result has been extended to the region $\sigma > \frac{1}{2}$.

In the middle of the 20th century, the theory of weak convergence of probability measures was created, and it became convenient to state Bohr-Jessen type theorems as limit theorems on weakly convergent probability measures.

We begin by recalling the essential definitions. Let $\mathbb{X}$ be a topological space, and $\mathcal{B}(\mathbb{X})$ the Borel σ-field on $\mathbb{X}$. Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We say that P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$) if, for every real bounded continuous function g on $\mathbb{X}$,

$$\lim_{n\to\infty} \int_{\mathbb{X}} g \,\mathrm{d}P_n = \int_{\mathbb{X}} g \,\mathrm{d}P.$$

Using the above definition, the Bohr-Jessen type theorem for $\zeta(s)$ can be stated in a modern form: suppose that $\mathrm{meas}A$ denotes the Lebesgue measure of $A \subset \mathbb{R}$, and $\sigma > \frac{1}{2}$ is fixed; then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P such that

$$P_T(A) = \frac{1}{T}\mathrm{meas}\left\{t \in [0,T] : \zeta(\sigma + it) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P as $T \to \infty$ (see, for example, [17]).

Owing to B. Bagchi's contribution, the concept of weak convergence of probability measures has become widely used, especially in theorems established in the space of analytic functions, since the proofs of universality theorems on the approximation of analytic functions by shifts of zeta-functions are based on such results. The contributions of Bohr and Jessen, as well as a probabilistic approach to value-distribution of various zeta-functions, have influenced subsequent developments. For example, in a series of papers, this probabilistic technique was extended to the analysis of the value-distribution for modified Mellin transforms of powers of the Riemann zeta-function, see [19, 25]. We observe that, in this case, the limit distribution is degenerated at the point $s = 0$.

Among the various analytic functions studied in number theory, the Epstein zeta-function stands out as a particularly interesting object. The value distribution of $\zeta(s; Q)$, as with other zeta-functions, is chaotic; therefore, it motivates the application of a probabilistic approach to its characterization.

Let Q be a positive definite $n \times n$ matrix, $n \in \mathbb{N}$. For $\underline{x} \in \mathbb{Z}^n$, denote $Q[\underline{x}] = \underline{x}^T Q \underline{x}$, where $\underline{x}^T$ is the transpose of $\underline{x}$. The Epstein zeta-function $\zeta(s; Q)$, for $\sigma > \frac{n}{2}$, is defined by the series

$$\zeta(s; Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s}.$$

Moreover, $\zeta(s; Q)$ admits an analytic continuation to the whole complex plane, except for a simple pole at $s = \frac{n}{2}$, and

$$\operatorname*{Res}_{s=n/2} \zeta(s; Q) = \pi^{\frac{n}{2}} \left(\Gamma\left(n/2\right)\sqrt{\det Q}\right)^{-1},$$

where, $\Gamma(s)$ is the Euler gamma-function. The function $\zeta(s;Q)$ was introduced by P. Epstein in [9]. He also obtained the functional equation of Riemann type for $\zeta(s;Q)$. Let Q^{-1} stand for the inverse of Q. Then, in [9], it was given that

$$\pi^{-s}\Gamma(s)\zeta(s;Q)=\sqrt{\det Q}\pi^{s-\frac{n}{2}}\Gamma\left(n/2-s\right)\zeta\left(n/2-s;Q^{-1}\right),\quad s\in\mathbb{C}.$$

Properties of the function $\zeta(s;Q)$, of course, depend on the matrix Q, and are quite different from those of $\zeta(s)$. For example, there are matrices Q such that $\zeta(s;Q)$ has zeros in the region of absolute convergence $\sigma>\frac{n}{2}$, while $\zeta(s)\neq 0$ for $\sigma>1$. Moreover, for certain matrices, the Riemann hypothesis for $\zeta(s;Q)$ is not true: there are zeros off the critical line $\sigma=\frac{n}{4}$. Also, it is proved that zeros of $\zeta(s;Q)$, in general, are not symmetric with respect to the line $\sigma=\frac{n}{4}$ [28]. On the other hand, it has been established [27] that imaginary parts of the zeros of $\zeta(s;Q)$, as of $\zeta(s)$, are uniformly distributed modulo 1. Recent investigations concerning the values of Epstein zeta-functions include a formula for a sum of values of $\zeta(s;Q)$ over the nontrivial zeros of $\zeta(s)$ presented in [10].

The Epstein zeta-function has wide-ranging applications in both pure and applied sciences. For example, in mathematics, $\zeta(s;Q)$ as an automorphic form for the unimodular group, plays a significant role in algebraic number theory. Beyond mathematics, it is an important tool in the study of crystallography [13], quantum field theory [7], temperature and energy related problems [5,8].

The first attempt was made successful in [20] to prove a limit theorem in the sense of weak convergence of probability measures for the Epstein zeta-function $\zeta(s;Q)$. To state the result, we need to give a description of the situation in question. Suppose that $Q[\underline{x}]\in\mathbb{Z}$ for all $\underline{x}\in\mathbb{Z}^n\setminus\{\underline{0}\}$. Fomenko proved that then the representation

$$\zeta(s;Q)=\zeta(s;E_Q)+\zeta(s;F_Q)$$

holds, where $\zeta(s;E_Q)$ is the zeta-function of a certain Eisenstein series, and $\zeta(s;F_Q)$ the zeta-function of a modular form of weight $\frac{n}{2}$. Moreover, suppose that $n\in 2\mathbb{N}$ and $n\geqslant 4$. Then, it is known (see [14, 15]) that the Eisenstein series $E_Q(s)$ is a modular form of level q, where $q\in\mathbb{N}$ is such that $q(2Q)^{-1}$ is an integral matrix, and, for $\sigma>\frac{n-1}{2}$,

$$\zeta(s;Q)=\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{a_{kl}}{k^s l^s}L(s,\chi_k)L\left(s-\frac{n}{2}+1,\psi_l\right)+\sum_{m=1}^{\infty}\frac{b_Q(m)}{m^s}. \tag{1.1}$$

Here k and l are positive divisors of q, χ_k and ψ_l are Dirichlet characters modulo q/k and q/l, respectively, $L(s,\chi_k)$ and $L(s,\psi_l)$ are the corresponding Dirichlet L-functions, and $a_{kl}\in\mathbb{C}$ are certain complex coefficients. Furthermore, the series with coefficients $b_Q(m)$ is absolutely convergent for $\sigma>\frac{n-1}{2}$. The representation (1.1) is fundamental to the study of the function $\zeta(s;Q)$.

Now, we define the limit measure in the main theorem of [20]. Let

$$\varOmega=\prod_{p\in\mathbb{P}}\{s\in\mathbb{C}:|s|=1\},$$

where $\mathbb{P}$ is the set of all prime numbers. The set Ω consists of all functions from $\mathbb{P}$ into the unit circle. With the product topology and pointwise multiplication, Ω is a compact topological group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. This leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega = (\omega(p) : p \in \mathbb{P})$ elements of Ω, and extend the functions $\omega(p)$, $p \in \mathbb{P}$, to the set $\mathbb{N}$ by using the formula

$$\omega(m) = \prod_{p^l \| m} \omega^l(p), \quad m \in \mathbb{N},$$

where $p^l \parallel m$ means that $p^l \mid m$, but $p^{l+1} \nmid m$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the complex-valued random element

$$\begin{aligned}\zeta(\sigma, \omega; Q) &= \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}\omega(k)\omega(l)}{k^\sigma l^\sigma} L(\sigma, \omega, \chi_k) L\left(\sigma - \frac{n}{2} + 1, \omega, \psi_l\right) \\ &+ \sum_{m=1}^{\infty} \frac{b_Q(m)\omega(m)}{m^\sigma}, \quad \sigma > \tfrac{n-1}{2},\end{aligned}$$

where, for a Dirichlet character χ, and almost all ω,

$$L(s, \omega, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)\omega(m)}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)\omega(p)}{p^s}\right)^{-1}, \quad \sigma > \frac{1}{2}.$$

Let $P_\zeta = P_{\zeta,\sigma,Q}$ be the distribution of the random element $\zeta(\sigma, \omega; Q)$, that is,

$$P_\zeta(A) = m_H \left\{\omega \in \Omega : \zeta(\sigma, \omega; Q) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

In [20], the weak convergence of the probability measure

$$P_{T,\zeta}(A) = P_{T,\zeta,\sigma,Q}(A) = \frac{1}{T}\text{meas}\left\{t \in [0, T] : \zeta(\sigma + it; Q) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

where meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, was analysed. The main result of the mentioned paper is the following statement.

Theorem 1. *Suppose that $\sigma > \frac{n-1}{2}$ is fixed. Then $P_{T,\zeta}$ converges weakly to the measure P_ζ as $T \to \infty$.*

Theorem 1 has been further developed in subsequent works [21, 22, 23, 24] in several directions, including the use of discrete shifts, certain generalized shifts, and the study of joint distributions. Furthermore, the papers [11] and [12] are devoted to two-dimensional limit theorems for the Epstein and Hurwitz zeta-functions. In these works, the measures are defined by a density in the interval of length T. In this study, we aim to decrease the length of T, i.e., to prove that Theorem 1 remains valid in short intervals.

Several limit theorems for zeta-functions in the space of analytic functions have been obtained in the papers on universality in short intervals. The first result of this type was given in [18] for the Riemann zeta-function. Subsequently, such limit theorems have been established for zeta-functions of certain

cusp forms, as well as for the Lerch and Hurwitz zeta-functions.
We now proceed to formulate the main result of the paper. For $A \in \mathcal{B}(\mathbb{C})$, set

$$P_{T,H,\zeta}(A) = P_{T,H,\zeta,\sigma,Q}(A) = \frac{1}{H}\text{meas}\left\{t \in [T, T+H] : \zeta(\sigma+it;Q) \in A\right\}.$$

Theorem 2. *Suppose that* $\sigma > \frac{n-1}{2}$ *is fixed, and* $T^{27/82} \leqslant H \le T^{1/2}$. *Then,* $P_{T,H,\zeta}$ *converges weakly to the measure* P_ζ *as* $T \to \infty$.

Thus, Theorem 2 extends Theorem 1 to the interval of length at least $T^{27/82}$. The proof of Theorem 2 is based on the representation (1.1), and on the mean square estimate for the Hurwitz zeta-functions obtained in [26].

2 Case of absolute convergence

We start with a limit lemma for the probability measures on $(\Omega, \mathcal{B}(\Omega))$ in short intervals. For $A \in \mathcal{B}(\Omega)$, define

$$Q_{T,H}(A) = \frac{1}{H}\text{meas}\Big\{t \in [T, T+H] : \Big(p^{-it} : p \in \mathbb{P}\Big) \in A\Big\}.$$

Lemma 1. *Suppose that* $H \to \infty$ *as* $T \to \infty$. *Then* $Q_{T,H}$ *converges weakly to the Haar measure* m_H *as* $T \to \infty$.

Proof. Consider the Fourier transform $f_{T,H}(\underline{k})$, $\underline{k} = \big(k_p : k_p \in \mathbb{Z}, p \in \mathbb{P}\big)$, of $Q_{T,H}$ given by

$$f_{T,H}(\underline{k}) = \int_\Omega \Big(\prod_{p\in\mathbb{P}}{}^{*} \omega^{k_p}(p)\Big) \mathrm{d}Q_{T,H},$$

where the star "*" shows that only a finite number of integers k_p are not equal to zero. Thus, by the definition of $Q_{T,H}$,

$$f_{T,H}(\underline{k}) = \frac{1}{H}\int_T^{T+H} \prod_{p\in\mathbb{P}}{}^{*} p^{-ik_p t}\mathrm{d}t = \frac{1}{H}\int_T^{T+H} \exp\Big\{-it\sum_{p\in\mathbb{P}}{}^{*} k_p \log p\Big\}\mathrm{d}t. \quad (2.1)$$

Obviously,

$$f_{T,H}(\underline{0}) = 1. \quad (2.2)$$

The set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers. Therefore, for $\underline{k} \neq \underline{0}$,

$$A(\underline{k}) \overset{\text{def}}{=} \sum_{p\in\mathbb{P}}{}^{*} k_p \log p \neq 0.$$

In view of (2.1), we have

$$f_{T,H}(\underline{k}) = \frac{\exp\{-iTA(\underline{k})\} - \exp\{-i(T+H)A(\underline{k})\}}{iHA(\underline{k})}.$$

This and (2.2) show that

$$\lim_{T\to\infty} f_{T,H}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k}=\underline{0}, \\ 0 & \text{if } \underline{k}\neq\underline{0}. \end{cases}$$

Since the right-hand side of that equality is the Fourier transform of the measure m_H, and Ω is a compact group, the lemma follows. □

Fix a number $\theta > \frac{1}{2}$, and, for $m, N \in \mathbb{N}$, set

$$w_N(m) = \exp\left\{-\left(m/N\right)^{\theta}\right\},$$

where $\exp\{a\} = e^a$. Return to representation (1.1), and define

$$L_N\left(s-\frac{n}{2}+1, \psi_l\right) = \sum_{m=1}^{\infty} \frac{\psi_l(m) w_N(m)}{m^s},$$

and

$$\zeta_N(s;Q) = \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s,\chi_k) L_N\left(s-\frac{n}{2}+1,\psi_l\right) + \sum_{m=1}^{\infty}\frac{b_Q(m)}{m^s}.$$

Since $w_N(m)$ decreases exponentially with respect to m, and $|\psi_l(m)| \leqslant 1$, the series for $L_N(s-\frac{n}{2}+1,\psi_l)$ is absolutely convergent in any half-plane $\sigma > \sigma_0$. Moreover, the series for $L(s,\chi_k)$ is absolutely convergent for $\sigma > \frac{n-1}{2}$, assuming that $n \geqslant 4$. These remarks show that the function $\zeta_N(s;Q)$ is a combination of absolutely convergent Dirichlet series for $\sigma > \frac{n-1}{2}$.

Next, we consider the weak convergence of the probability measure

$$\begin{aligned} P_{T,H,N,\zeta}(A) &= P_{T,H,N,\zeta,\sigma}(A) \\ &= \frac{1}{H}\text{meas}\left\{t\in[T,T+H] : \zeta_N(\sigma+it;Q)\in A\right\}, \quad A\in\mathcal{B}(\mathbb{C}). \end{aligned}$$

Lemma 2. *Suppose that $\sigma > \frac{n-1}{2}$ is fixed, and $H\to\infty$ as $T\to\infty$. Then, on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_{N,\sigma,\zeta}$ such that $P_{T,H,N,\zeta}$ converges weakly to $P_{N,\sigma,\zeta}$ as $T\to\infty$.*

Proof. For $\omega\in\Omega$, define

$$\begin{aligned} \zeta_N(s,\omega;Q) = &\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{a_{kl}\omega(k)\omega(l)}{k^s l^s} L(s,\omega,\chi_k) L_N\left(s-\frac{n}{2}+1,\omega,\psi_l\right) \\ &+\sum_{m=1}^{\infty}\frac{b_Q(m)\omega(m)}{m^s}, \end{aligned}$$

where

$$L_N\left(s-\frac{n}{2}+1,\omega,\psi_l\right) = \sum_{m=1}^{\infty}\frac{\psi_l(m)\omega(m)w_N(m)}{m^s}.$$

Consider the function $v_{N,\sigma} : \Omega \to \mathbb{C}$ given by the formula

$$v_{N,\sigma}(\omega) = \zeta_N(\sigma, \omega; Q), \quad \omega \in \Omega.$$

Since $\zeta_N(\sigma, \omega; Q)$ is a combination of absolutely convergent Dirichlet series, thus, uniformly in ω, the function $v_{N,\sigma}$ is continuous with respect to the topology of Ω. Moreover, by the definition of $v_{N,\sigma}$,

$$v_{N,\sigma}\left((p^{-it} : p \in \mathbb{P})\right) = \zeta_N(\sigma + it; Q). \tag{2.3}$$

Consequently, we have a situation which allows to apply a method on preservation of weak convergence under continuous mappings, see Section 5.1 of [2]. Actually, in view of (2.3), for all $A \in \mathcal{B}(\mathbb{C})$,

$$P_{T,H,N,\zeta}(A)=\frac{1}{H}\text{meas}\left\{t\in[T,T+H] : \left(p^{-it} : p\in\mathbb{P}\right) \in v_{N,\sigma}^{-1}A\right\}=Q_{T,H}(v_{N,\sigma}^{-1}A),$$

where $Q_{T,H}$ is the measure from Lemma 1. Define

$$P_{N,\sigma,\zeta,Q} = Q_{T,H}v_{N,\sigma}^{-1},$$

where

$$Q_{T,H}v_{N,\sigma}^{-1}(A) = Q_{T,H}(v_{N,\sigma}^{-1}A).$$

Then, Theorem 5.1 of [2] together with Lemma 1 imply that $P_{N,\sigma,\zeta,Q}$ converges weakly to $m_H v_{N,\sigma}^{-1}$ as $T \to \infty$. Thus, $P_{N,\sigma,\zeta} = m_H v_{N,\sigma}^{-1}$. □

3 Approximation results

To derive a limit theorem for $\zeta(s; Q)$ from Lemma 2, we need the approximation result for $\zeta(s; Q)$ by $\zeta_N(s; Q)$ in short intervals. The definition of $\zeta_N(s; Q)$ and (1.1) show that it suffices to do this for $L\left(s - \frac{n}{2} + 1, \psi_l\right)$ by $L_N\left(s - \frac{n}{2} + 1, \psi_l\right)$ in short intervals. The result is based on the mean square estimate for Dirichlet L-functions in short intervals. Suppose that χ is an arbitrary Dirichlet character modulo q and $L(s, \chi)$ is the corresponding Dirichlet L-function.

Lemma 3. *Suppose that* $\frac{1}{2} < \sigma \leqslant \frac{7}{12}$ *is fixed, and* $T^{27/82} \leqslant H \leqslant T^{\sigma}$. *Then, the estimate*

$$\int_{T-H}^{T+H} |L(\sigma + it, \chi)|^2 \mathrm{d}t \ll_{\sigma,q} H \tag{3.1}$$

holds.

Here, the notation $a \ll b$, $a \in \mathbb{C}$, $b > 0$, means that there exists a constant $c = c(\theta) > 0$ such that $|a| \leqslant cb$.

Proof. We will use a mean square estimate in short intervals for the Hurwitz zeta-function. Let $0 < \alpha \leqslant 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$, for $\sigma > 1$, is defined by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and, as the Riemann zeta-function $\zeta(s) = \zeta(s, 1)$, has analytic continuation to the whole complex plane, except for the point $s = 1$ which is a simple pole with residue 1. In [26], it was obtained that under hypotheses of the lemma, the estimate

$$\int_{T-H}^{T+H} |\zeta(\sigma + it, \alpha)|^2 \mathrm{d}t \ll_{\sigma,\alpha} H \tag{3.2}$$

for the Hurwitz zeta function is valid. We note that Hurwitz zeta- and Dirichlet L-functions are connected by the following relation [6]

$$L(s, \chi) = \frac{1}{q^s} \sum_{m=1}^{q} \chi(m) \zeta\left(s, m/q\right).$$

From this, we have that

$$|L(\sigma + it, \chi)|^2 \ll_q \left| \sum_{m=1}^{q} \zeta\left(\sigma + it, \frac{m}{q}\right) \right|^2 \ll_q \max_{1 \leqslant m \leqslant q} \left| \zeta\left(\sigma + it, \frac{m}{q}\right) \right|^2. \tag{3.3}$$

Thus, (3.3) and (3.2) gives (3.1). □

It is important to emphasize that one of the key aspects in this lemma is to minimize the exponent of T in the lower bound for H. For example, in the case of the Riemann zeta-function, a universality theorem in short intervals for which $T^{1/3}(\log T)^{26/15} \le H \le T$ was established in [18], and the lower bound for H was later improved to 1273/4053 in [1].

Now, we recall that

$$L_N(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m) w_N(m)}{m^s}.$$

Lemma 4. *Suppose that $\theta > \frac{1}{2}$ is fixed, and $\kappa_N(s) = \frac{N^s}{\theta}\Gamma\left(\frac{s}{\theta}\right)$ with $N \in \mathbb{N}$. Then, for $\sigma > \frac{1}{2}$, the representation*

$$L_N(s, \chi) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} L(s + z, \chi) \kappa_N(z) \mathrm{d}z$$

is valid.

Proof. The representation stated in the lemma follows from the classical Mellin formula

$$\frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} \Gamma(z) b^{-z} \mathrm{d}z = \mathrm{e}^{-b}, \quad a, b > 0,$$

the definitions of $L_N(s, \chi)$, $\kappa_N(s)$, and inequalities $\theta > \frac{1}{2}$, $\sigma > \frac{1}{2}$. □

We turn to considering the approximation of $L(s, \chi)$ by $L_N(s, \chi)$ in mean.

Lemma 5. *Suppose that* $\frac{1}{2} < \sigma \leqslant \frac{7}{12}$ *is fixed, and* $T^{27/82} \leqslant H \leqslant T^{1/2}$. *Then,*

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \left| L\left(\sigma+it,\chi\right) - L_N\left(\sigma+it,\chi\right) \right| \mathrm{d}t = 0.$$

Proof. Take $\theta = \frac{1}{2}+\varepsilon$, and $\frac{1}{2}+2\varepsilon \leqslant \sigma \leqslant \frac{7}{12}$, $\varepsilon > 0$, and $\theta_1 = \frac{1}{2}+\varepsilon-\sigma$. Then $\theta_1 < 0$ and $\theta_1 \geqslant \varepsilon - \frac{1}{12}$. Hence, the function $L(s+z,\chi)\kappa_N(z)$ has in the strip $\theta_1 \leqslant \mathrm{Re}s \leqslant \theta$ a simple pole at the point $z=0$ (the pole of $\Gamma(z)$), and a simple pole at the point $z = 1-s$ if χ is the principal character (the pole of $L(s+z,\chi)$). Therefore, the integral representation of Lemma 5 and the residue theorem give that

$$L_N(s,\chi) - L(s,\chi) = \frac{1}{2\pi i} \int_{\theta_1-i\infty}^{\theta_1+i\infty} L(s+z,\chi)\kappa_N(z)\mathrm{d}z + r_n(s),$$

with

$$r_n(s) = \operatorname*{Res}_{z=1-s} L(s+z,\chi)\kappa_N(z) = \begin{cases} \kappa_N(1-s) & \text{if } \chi \text{ is the principal character,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have

$$\begin{aligned} L(\sigma+it,\chi) - L_N(\sigma+it,\chi) &\ll \int_{-\infty}^{\infty} \left| L\left(1/2+\varepsilon+it+iu,\chi\right) \right| \\ &\quad \times \left| \kappa_N\left(1/2+\varepsilon-\sigma+iu\right) \right| \mathrm{d}u + |\kappa_N(1-\sigma-it)| \\ &= \left(\int_{-\infty}^{-\log^2 T} + \int_{-\log^2 T}^{\log^2 T} + \int_{\log^2 T}^{\infty} \right) \left| L\left(1/2+\varepsilon+it+iu,\chi\right) \right| \\ &\quad \times \left| \kappa_N\left(1/2+\varepsilon-\sigma+iu\right) \right| \mathrm{d}u + |\kappa_N(1-\sigma-it)|. \end{aligned} \tag{3.4}$$

Recall that uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, for fixed $\sigma_1 < \sigma_2$, the estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{3.5}$$

is valid. This and the definition of $\kappa_N(z)$ imply

$$\kappa_N\left(1/2+\varepsilon-\sigma\right) \ll N^{1/2+\varepsilon-\sigma} \exp\left\{-\frac{c}{\theta}|u|\right\} \ll_\varepsilon N^{-\varepsilon} \exp\{-c_1|u|\}, \ c_1 > 0. \tag{3.6}$$

Thus, we obtain

$$\begin{aligned} &\left(\int_{-\infty}^{-\log^2 T} + \int_{\log^2 T}^{\infty} \right) \left| L\left(1/2+\varepsilon+it+iu,\chi\right) \right| \left| \kappa_N\left(1/2+\varepsilon-\sigma+iu\right) \right| \mathrm{d}u \\ &\ll_{q,\varepsilon} N^{-\varepsilon} \left(\int_{-\infty}^{-\log^2 T} + \int_{\log^2 T}^{\infty} \right) \left(|t|^{\frac{1}{2}} + |u|^{\frac{1}{2}} \right) \exp\{-c_1|u|\}\mathrm{d}u \\ &\ll_{q,\varepsilon} N^{-\varepsilon} \left(1 + |t|^{\frac{1}{2}} \right) \exp\{-c_2 \log^2 T\}, \quad c_2 > 0, \end{aligned}$$

as in view of (3.3) and the estimate (see [26])

$$\zeta(\sigma+it,\alpha) \ll_{\sigma,\alpha} |t|^{\frac{1}{2}}, \quad \sigma \geqslant \frac{1}{2},$$

it follows that

$$L\left(1/2+\varepsilon+it+iu,\chi\right) \ll_{q,\varepsilon} \left(|t|^{\frac{1}{2}}+|u|^{\frac{1}{2}}\right).$$

From this and (3.4), we find

$$\begin{aligned} I \overset{\text{def}}{=} \frac{1}{H}\int_T^{T+H} \left|L(\sigma+it,\chi)-L_N(\sigma+it,\chi)\right| \mathrm{d}t \ll_{q,\varepsilon} \int_{-\log^2 T}^{\log^2 T} \\ \times \left(\frac{1}{H}\int_T^{T+H} \left|L\left(1/2+\varepsilon+it+iu,\chi\right)\right| \mathrm{d}t\right) \left|\kappa_N\left(1/2+\varepsilon-\sigma+iu\right)\right| \mathrm{d}u \\ + \frac{1}{H}\int_T^{T+H} |\kappa_N(1-\sigma-it)|\mathrm{d}t + \frac{1}{H}N^{-\varepsilon}\exp\{-c_2\log^2 T\}\int_T^{T+H}\left(1+|t|^{\frac{1}{2}}\right)\mathrm{d}t \\ \overset{\text{def}}{=} J_1+J_2+J_3. \end{aligned}$$

The Cauchy-Schwarz inequality gives

$$\frac{1}{H}\int\limits_T^{T+H} \left|L\left(1/2+\varepsilon+it+iu,\chi\right)\right| \mathrm{d}t \ll \left(\frac{1}{H}\int\limits_{T-H-|u|}^{T+H+|u|} \left|L\left(1/2+\varepsilon+it,\chi\right)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}. \tag{3.7}$$

For $T^{27/82} \leqslant H \leqslant T^{1/2}$ and $|u| \leqslant \log^2 T$, it follows that

$$T^{\frac{27}{82}} \leqslant H+|u| \leqslant T^{\frac{1}{2}}+\log^2 T \leqslant T^{\frac{1}{2}+\varepsilon}, \quad T \to \infty.$$

Therefore, for the mean square in (3.7), Lemma 3 is applicable, and we find that

$$\frac{1}{H}\int_T^{T+H} \left|L\left(\frac{1}{2}+\varepsilon+it+iu,\chi\right)\right| \mathrm{d}t \ll_{q,\varepsilon} \left(\frac{1}{H}\left(H+|u|\right)\right)^{\frac{1}{2}} \ll_{q,\varepsilon} \left(1+|u|\right)^{\frac{1}{2}}.$$

From this and (3.6), we have

$$J_1 \ll_{q,\varepsilon} N^{-\varepsilon}\int_{-\log^2 T}^{\log^2 T} (1+|u|)^{\frac{1}{2}} \exp\{-c_1|u|\}\mathrm{d}u \ll_{q,\varepsilon} N^{-\varepsilon}. \tag{3.8}$$

Applying (3.5) once more, we get

$$\kappa_N\left(1-\sigma-it\right) \ll_{\varepsilon} N^{1-\sigma}\exp\left\{-\frac{c}{\theta}|t|\right\} \ll_{\varepsilon} N^{1/2-2\varepsilon}\exp\{-c_2|t|\}, \quad c_2>0.$$

Therefore, the estimate

$$\begin{aligned} J_2 \ll_{\varepsilon} N^{\frac{1}{2}-2\varepsilon}\frac{1}{H}\int_T^{T+H}\exp\{-c_2 t\}\,\mathrm{d}t \ll N^{\frac{1}{2}-2\varepsilon}\frac{1}{H} \\ \times \left(\exp\left\{-\frac{c_2}{2}T\right\}\int_T^{T+H}\exp\left\{-\frac{c_2}{2}t\right\}\mathrm{d}t\right) \ll_{\varepsilon} N^{\frac{1}{2}-2\varepsilon}\exp\left\{-\frac{c_2}{2}T\right\}. \end{aligned} \tag{3.9}$$

Moreover,

$$J_3 \ll N^{-\varepsilon} \exp\{-c_2 \log^2 T\}\sqrt{T}.$$

This, and (3.7)–(3.9) yield

$$I \ll_{\varepsilon,q} N^{-\varepsilon} + N^{\frac{1}{2}-2\varepsilon} \exp\left\{-\frac{c_2}{2}T\right\} + N^{-\varepsilon}\sqrt{T}\exp\{-c_2 \log^2 T\}.$$

Now, letting $T \to \infty$ and then $N \to \infty$, we obtain the equality of the lemma. □

Lemma 6. *Suppose that* $\sigma > \frac{n-1}{2}$ *is fixed, and* $T^{27/82} \leqslant H \leqslant T^{1/2}$. *Then,*

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \left|\zeta\left(\sigma+it;Q\right) - \zeta_N\left(\sigma+it;Q\right)\right| \mathrm{d}t = 0.$$

Proof. By (1.1) and the definition of $\zeta_N(s;Q)$,

$$\begin{aligned}\zeta(\sigma+it;Q) - \zeta_N(\sigma+it;Q) &= \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}}{k^{\sigma+it}l^{\sigma+it}} L(\sigma+it,\chi_k)\\ &\times \Big(L\left(\sigma+it-n/2+1,\psi_l\right) - L_N\left(\sigma+it-n/2+1,\psi_l\right)\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\zeta(\sigma+it;Q) - \zeta_N(\sigma+it;Q) &\ll \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{|a_{kl}|}{k^{\sigma}l^{\sigma}} \left|L(\sigma+it,\chi_k)\right|\\ &\times \Big|L\left(\sigma+it-n/2+1,\psi_l\right) - L_N\left(\sigma+it-n/2+1,\psi_l\right)\Big|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \left|\zeta\left(\sigma+it;Q\right) - \zeta_N\left(\sigma+it;Q\right)\right| \mathrm{d}t\\ &\ll \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{|a_{kl}|}{k^{\sigma}l^{\sigma}} \limsup_{N\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \Big| L\left(\sigma+it-n/2+1,\psi_l\right)\\ &- L_N\left(\sigma+it-n/2+1,\psi_l\right)\Big| \mathrm{d}t = 0\end{aligned}$$

in virtue of Lemma 5, and this proves the lemma. □

4 Proof of Theorem 2

We recall that $P_{N,\zeta,\sigma}$ is the limit measure in Lemma 2. The measure $P_{N,\zeta,\sigma}$ is independent of H. Therefore, we may use the results obtained in [20].

Lemma 7. [20, Lemma 8]. *The sequence* $\{P_{N,\zeta,\sigma} : N \in \mathbb{N}\}$ *is relatively compact, i.e., each sequence of* $\{P_{N,\zeta,\sigma}\}$ *contains a subsequence weakly convergent to a certain probability measure on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

Additionally, we state some well known assertions on convergence in distribution and weak convergence. Let X_N, $N \in \mathbb{N}$, and X be $\mathbb{X}$-valued random elements, and Q_N and Q their distributions. We say that X_N converges to X in distribution $\left(X_N \xrightarrow[N\to\infty]{\mathcal{D}} X\right)$ if $Q_N \xrightarrow[N\to\infty]{\mathrm{w}} Q$.

Lemma 8. [2, Theorem 4.2]. *Let $Y_N, X_{1N}, X_{2N}, \ldots$ be $\mathbb{X}$-valued random elements defined on the same probability space with measure ν, and the space $(\mathbb{X}, d)$ is separable. Suppose that, for each k,*

$$X_{kN} \xrightarrow[N\to\infty]{\mathcal{D}} X_k \quad \text{and} \quad X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

and let, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{N\to\infty} \nu\left\{d\left(X_{kN}, Y_N\right) \geqslant \varepsilon\right\} = 0.$$

Then, $Y_N \xrightarrow[N\to\infty]{\mathcal{D}} X$.

Proof of Theorem 2. In view of Lemma 7, there exists a subsequence $\{P_{N_r,\zeta,\sigma}\} \subset \{P_{N,\zeta,\sigma}\}$ and a probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that

$$P_{N_r,\zeta,\sigma} \xrightarrow[r\to\infty]{\mathrm{w}} P_\sigma. \tag{4.1}$$

Let $\xi_{T,H}$ be a random variable defined on the probability space with measure ν, and uniformly distributed in the interval $[T, T+H]$. For $\sigma > \frac{n-1}{2}$, define the $\mathbb{C}$-valued random element

$$X_{T,H,N} = X_{T,H,N}(\sigma) = \zeta_N(\sigma + i\xi_{T,H}; Q).$$

Introduce one more $\mathbb{C}$-valued random element

$$X_{T,H} = X_{T,H}(\sigma) = \zeta(\sigma + i\xi_{T,H}; Q).$$

Now, an application of Lemma 6 and the above definitions, for $\sigma > \frac{n-1}{2}$ and $\varepsilon > 0$, gives

$$\begin{aligned}
&\lim_{N\to\infty} \limsup_{T\to\infty} \nu\left\{\left|X_{T,H}(\sigma) - X_{T,H,N}(\sigma)\right| \geqslant \varepsilon\right\} \\
&\quad = \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{H} \mathrm{meas}\left\{t \in [T, T+H] : \left|\zeta(\sigma+it; Q) - \zeta_N(\sigma+it; Q)\right| \geqslant \varepsilon\right\} \\
&\quad \leqslant \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_T^{T+H} \left|\zeta(\sigma+it; Q) - \zeta_N(\sigma+it; Q)\right| \mathrm{d}t = 0.
\end{aligned} \tag{4.2}$$

Moreover, by Lemma 2,

$$X_{T,H,N} \xrightarrow[T\to\infty]{\mathcal{D}} X_N,$$

where $X_N = X(\sigma)$ is a $\mathbb{C}$-valued random element having the distribution $P_{N,\zeta,\sigma}$. So, the relation (4.1) can be rewritten in the form

$$X_{N_r} \xrightarrow[r\to\infty]{\mathcal{D}} P_\sigma.$$

Together with (4.2), this allows us to apply Lemma 8. Thus, we have

$$X_{T,H} \xrightarrow[T\to\infty]{\mathcal{D}} P_\sigma, \tag{4.3}$$

or, equivalently, $P_{T,H,\zeta} \xrightarrow[T\to\infty]{\mathrm{w}} P_\sigma$.

Relation (4.3) shows that the measure P_σ is independent of the sequence $P_{N_r,\zeta,\sigma}$. Therefore, the relation

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P_\sigma \tag{4.4}$$

holds.

It remains to identify the measure P_σ. For this, we apply the proof of Theorem 1. Let ξ_T be a random variable uniformly distributed in $[0,T]$,

$$\hat{X}_{T,N} = \hat{X}_{T,N}(\sigma) = \zeta_N(\sigma + i\xi_T; Q)$$

and

$$\hat{X}_T = \hat{X}_T(\sigma) = \zeta(\sigma + i\xi_T; Q).$$

In [20], it is obtained that

$$\hat{X}_{T,N} \xrightarrow[T\to\infty]{\mathcal{D}} X_N, \tag{4.5}$$

and, for $\varepsilon > 0$,

$$\lim_{N\to\infty} \limsup_{T\to\infty} \nu\left\{\left|\hat{X}_T(\sigma) - \hat{X}_{T,N}(\sigma)\right| \geqslant \varepsilon\right\} = 0.$$

From this, (4.4) and (4.5), it follows the relation

$$\hat{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P_\sigma.$$

Moreover, using the Birkhoff-Khintchine ergodic theorem [16], it is proved that

$$P_\sigma = P_{\zeta,\sigma}.$$

Thus, $P_{T,H,\zeta} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\zeta,\sigma}$, and the theorem is proved. □

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