



Uniform numerical method for singularly perturbed parabolic time delay reaction-diffusion problems arising in control theory

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Abstract. This paper introduces a uniform numerical scheme to find approximate solutions for singularly perturbed parabolic time delay reaction-diffusion problems. The scheme utilizes the Crank-Nicolson method for approximating time derivatives, combined with a novel finite difference method for spatial discretization. The stability and uniform convergence of the proposed scheme are investigated. The primary objective of this work is to demonstrate that the proposed scheme achieves a parameter-free error bound of order $O(k^2 + N^{-2})$. To validate the theoretical results, various numerical experiments have been conducted, showing that the proposed scheme yields superior results compared to some existing methods in the literature.

Keywords: singular perturbation problem; reaction-diffusion equation; Crank-Nicolson method; time delay.

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1 Introduction

In this study, we examine singularly perturbed time delay reaction-diffusion problem of the form:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \mu(x)u(x,t) + \zeta(x,t)u(x,t - \tau) = s(x,t), & (x,t) \in \mathfrak{S}, \\ u(0,t) = \lambda_l(t), & t \in [0, 2], \\ u(1,t) = \lambda_r(t), & t \in [0, 2], \\ u(x,t) = \lambda_b(x,t), & (x,t) \in [0, 1] \times [-\tau, 0], \end{cases} \quad (1.1)$$

where $\mathfrak{S} = (0, 1) \times [0, 2]$, $\varepsilon(0 < \varepsilon \ll 1)$ is the perturbation parameter, and τ is the delay parameter. The functions $\mu, \zeta, \lambda_l, \lambda_b, \lambda_r$ and s are assumed to be

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sufficiently smooth and bounded that satisfy

$$\mu(x) \geq \mu^* > 0, \zeta(x, t) \geq \zeta^* > 0, (x, t) \in [0, 1] \times [0, 2].$$

As $\varepsilon \rightarrow 0$ the solutions of Equation (1.1) exhibit dual boundary layers. The characteristics of the reduced problem of (1.1) (after setting $\varepsilon = 0$) are the vertical lines $x = \text{constant}$, which implies that any boundary layers arising in the solution are of parabolic type.

Singularly perturbed delay differential equations (SPDDEs) arise frequently in science and engineering. For example, chemical kinetics [5], chemical processes [1], control theory [20], semiconductor drift-diffusion model [12], problems with water quality in river networks [19]. Various applications of partial differential problems with time lag are given in [22]. The control system model for the furnace used in metal sheet production is an example of SPDDEs, represented by the equation:

$$\frac{\partial u(x, t)}{\partial t} - \varepsilon \frac{\partial^2 u(x, t)}{\partial x^2} = vg(u(x, t - \tau)) + c [f(u(x, t - \tau)) - u(x, t)],$$

where u is the temperature distribution in a metal sheet moving at an instantaneous material strip velocity v and heated by a distributed temperature source given by the function f ; both v and f are dynamically changing by a controller monitoring the current temperature distribution. The controller's finite speed causes a fixed delay of length τ [21].

Nowadays, the study of unsteady problems such as singularly perturbed problems [2, 6], inverse boundary problems [7, 8, 14], and Black-scholes problems [18] have gained great interest from the scientific community.

The development of solution methodology for singularly perturbed delay differential problems (SPDDPs) has received remarkable attention from researchers, mainly due to the importance of accuracy independent of the perturbation parameter. Boundary layers and interior layers that arise from ε ($0 < \varepsilon \ll 1$) and delay terms, respectively, present additional complexity to solve SPDDPs. To tackle these challenges, techniques such as the fitted operator method [4] and fitted mesh method [15] have been established. The development of robust numerical algorithms for singularly perturbed parabolic differential difference convection-dominated problems with time delay is exhaustively designed. For instance, parameter-uniform numerical schemes have been developed for singularly perturbed parabolic differential difference convection-dominated problems with time delay by [3] and the references therein.

Several researchers suggested an ε -uniformly convergent computational methods for singularly perturbed parabolic differential-difference reaction-dominated problems with time delay. For instance, the scholars in [13] have presented a uniformly convergent numerical method consisting of the Crank-Nicolson method for the temporal discretization and an exponentially fitted tension spline method for the spatial discretization for solving singularly perturbed parabolic differential-difference reaction-dominated problems with time delay. The researcher in [16] have constructed an exponentially fitted cubic B-spline method in space with the implicit-Euler method in time discretization for sin-

gularly perturbed parabolic reaction-diffusion problems with general time delay. The authors in [17] fitting the Numerov method for singularly perturbed parabolic partial differential equations with a small negative shift in the temporal variable.

Kumar and Kumar [11] solved singularly perturbed parabolic differential-difference reaction-dominated problems with time delay using the hybrid scheme on a generalized Shishkin mesh in spatial direction and the implicit Euler scheme on a uniform mesh in time direction. Kumar and Ravi Kanth [10] proposed a numerical method using the Crank-Nicolson method for temporal discretization and the tension spline scheme on a non-uniform Shishkin mesh for spatial discretization. Nevertheless, most of the published papers have ordered one. Thus, there is a substantial interest in the numerical community to develop higher-order numerical algorithms for SPDDPs. The primary purpose of this work is to formulate and analyze high-order robust numerical schemes for singularly perturbed time delay reaction-dominated problems. In this work, we employ the Crank-Nicolson method to temporal semi-discretization and the novel finite difference method to spatial discretization.

The rest of this article is organized as follows: Section 2 is discussed about the properties of the continuous problem. The derivation of the numerical scheme is discussed in Section 3. Convergence analysis of the proposed method is presented in Section 4. In Section 5, numerical examples and results are given to confirm the theoretical investigations. Finally, the paper ends with Section 6 with the conclusion.

Notation: The norm $\|\cdot\|$ is denoted for the maximum norm which is defined as $\|u\| = \max_{\mathfrak{S}} |u(x, t)|$. C denote generic positive constants independent of ε , mesh points and mesh sizes.

2 Properties of continuous problem

To ensure a unique solution for Equation (1.1), it is assumed that the functions $\mu(x)$, $b(x, t)$, and $s(x, t)$ are Holder's continuous, and the compatibility conditions at the corner points $(0, 0)$, $(1, 0)$, $(0, -\tau)$, and $(1, -\tau)$ hold:

$$\begin{aligned} \lambda_b(0, 0) &= \lambda_l(0), \quad \lambda_b(1, 0) = \lambda_r(0) \\ \begin{cases} \frac{\partial \lambda_l(0)}{\partial t} - \varepsilon \frac{\partial^2 \lambda_b(0,0)}{\partial x^2} + \mu(0)\lambda_b(0, 0) = -\zeta(0, 0)\lambda_b(0, -\tau) + s(0, 0), \\ \frac{\partial \lambda_r(0)}{\partial t} - \varepsilon \frac{\partial^2 \lambda_b(0,0)}{\partial x^2} + \mu(1)\lambda_b(1, 0) = -\zeta(1, 0)\lambda_b(1, -\tau) + s(1, 0). \end{cases} \end{aligned}$$

Under the above suitable conditions Equation (1.1) has unique solution, which exhibits twin boundary layers of width $O(\sqrt{\varepsilon})$ at $x = 0$ and $x = 1$.

The differential operator $\mathcal{L}_\varepsilon = \frac{\partial}{\partial t} - \varepsilon \frac{\partial^2}{\partial x^2} + \mu(x) + \zeta(x, t)$ in Equation (1.1) satisfies the following maximum principle.

Lemma 1. (Continuous maximum principle) Assume that $\mu, \zeta \in C^0(\mathfrak{S})$ and let $u \in C^2(\mathfrak{S}) \cap C^0(\bar{\mathfrak{S}})$. Suppose that $u \geq 0$ on $\partial\mathfrak{S} = \bar{\mathfrak{S}} - \mathfrak{S}$. Then, $\mathcal{L}_\varepsilon u \geq 0$ in \mathfrak{S} implies that $u \geq 0$ in $\bar{\mathfrak{S}}$.

Proof. See reference [9]. \square

Lemma 2. Let u be the solution of Equation (1.1), then the following bound holds:

$$\|u\| \leq (1 + \mu^*T) \max \{ \|\mathcal{L}_\varepsilon u\|, \|u\|_{\partial\mathfrak{S}} \},$$

Proof. See reference [13]. \square

Lemma 3. Derivatives of the solution of Equation (1.1) satisfy the following bound:

$$\left| \frac{\partial^{a+b}u}{\partial x^a \partial t^b} \right| \leq C \left[1 + \varepsilon^{-a/2} \left(\exp \left(-x\sqrt{\mu^*/\varepsilon} \right) + \exp \left(-(1-x)\sqrt{\mu^*/\varepsilon} \right) \right) \right],$$

for $a = 0, 1, 2, 3, 4$ and $b = 0, 1, 2$.

Proof. See reference [16]. \square

3 Construction of the numerical scheme

3.1 Temporal discretization

Let M be the number of mesh points in the discretization of the interval $[0, T]$, and m is the number of mesh points in the discretization of the interval $[-\tau, 0]$. We assume the delay τ is an integer multiple of the time step k (i.e $\tau = rk$, for some positive integer r) to simplify implementation and analysis. The time domain $[0, T]$ is discretized using a uniform mesh with time step k as

$$\begin{cases} \mathfrak{S}_t^M = \{0 = t_0 < t_1 < t_2 < \dots < t_j = \tau < \dots < t_{M-1} < t_M = T, k = T/M\}, \\ \mathfrak{S}_t^m = \{-\tau = t_{-j} < t_{-j+1} < \dots < t_{-1} < t_0 < 0, k = \tau/m\}. \end{cases}$$

Using the Crank-Nicolson method, the problem in Equation (1.1) is discretized as follows:

$$\begin{cases} \mathcal{L}_\varepsilon^k U^{j+1}(x) = H^{j+1}(x), \\ U^{j+1}(0) = \lambda_l^{j+1}, U^{j+1}(1) = \lambda_r^{j+1}, \end{cases} \tag{3.1}$$

where

$$\mathcal{L}_\varepsilon^k U^{j+1}(x) = -\varepsilon \frac{\partial^2 U^{j+1}(x)}{\partial x^2} + (2/k + \mu(x)) U^{j+1}(x)$$

and

$$H^{j+1}(x) = \begin{cases} \varepsilon \frac{\partial^2 U^j(x)}{\partial x^2} + (2/k - \mu(x)) U^j(x) - (\zeta^{j+1} + \zeta^j) \lambda_b^{j+1}(x) + f^{j+1}(x) + f^j(x), \text{ for } j = 0(1)m, \\ \varepsilon \frac{\partial^2 U^j(x)}{\partial x^2} + (2/k - \mu(x)) U^j(x) - (\zeta^{j+1} + \zeta^j) U^{j-m}(x) + f^{j+1}(x) + f^j(x), \text{ for } j = m + 1(1)M - 1. \end{cases}$$

Lemma 4. Suppose that $\left| \frac{\partial^n u(x,t)}{\partial t^n} \right| \leq C, (x, t) \in [0, 1] \times [0, T], 0 \leq n \leq 2$. Then, the local error estimate given by

$$\|e_{j+1}\| \leq C_1 (k)^3, \quad j = 1(1)M - 1.$$

Proof. Using Taylor’s series approximation for $u(x, t_j)$ and $u(x, t_{j+1/2})$ centering at $t_{j+1/2}$, we have

$$u(x, t_{j+1}) = u(x, t_{j+1/2}) + \frac{k}{2}u_t(x, t_{j+1/2}) + \frac{k^2}{8}u_{tt}(x, t_{j+1/2}) + O(k^3), \tag{3.2}$$

$$u(x, t_j) = u(x, t_{j+1/2}) - \frac{k}{2}u_t(x, t_{j+1/2}) + \frac{k^2}{8}u_{tt}(x, t_{j+1/2}) + O(k^3). \tag{3.3}$$

From Equations (3.2) and (3.3), we have

$$\frac{u(x, t_{j+1}) - u(x, t_j)}{k} = u_t(x, t_{j+1/2}) + O(k^2). \tag{3.4}$$

Substituting Equation (3.1) into Equation (3.4), we obtain

$$\frac{u(x, t_{j+1}) - u(x, t_j)}{k} = \varepsilon \frac{\partial^2 u(x, t_{j+1})}{\partial x^2} - (2/k + \mu(x)) u(x, t_{j+1}) + H^{j+1}(x) + O(k^2).$$

Since the error $e_{j+1} = u(x, t_{j+1}) - U^{j+1}(x)$ satisfies the semi discrete difference scheme

$$\mathcal{L}_\varepsilon^k e_{j+1} = O(k^3), \quad e_{j+1}(0) = 0 = e_{j+1}(1).$$

Hence, by applying the maximum principle, we get

$$\|e_{j+1}\| \leq C_1(k)^3, \quad j = 1(1)M.$$

□

Lemma 5. *The global error in temporal discretization at t_j time step is given by*

$$\|E_j\| \leq C_2(k)^2, \quad j = 1(1)M.$$

Proof. From Lemma 4 it follows that

$$\begin{aligned} \|E_j\| &= \left\| \sum_{g=1}^j \right\| \leq \|e_1\| + \|e_2\| + \dots + \|e_j\| \\ &\leq C_1 j(k)^3, \quad \text{using Lemma 4} \\ &\leq C_1(jk)(k^2) \leq C_1 T(k)^2 \text{ as } jk \leq T, \\ \|E_j\| &\leq C_2(k)^2, \quad C_2 = C_1 T. \end{aligned}$$

□

Lemma 6. *The solution $U^{j+1}(x)$ of semi-discretized scheme (3.1) and its derivatives satisfies*

$$\left| \frac{\partial^a U^{j+1}(x)}{\partial x^a} \right| \leq C \left[1 + \varepsilon^{-a/2} \left(\exp\left(-x\sqrt{\mu^*/\varepsilon}\right) + \exp\left(-(1-x)\sqrt{\mu^*/\varepsilon}\right) \right) \right],$$

for $a = 0, 1, 2, 3, 4$.

Proof. See reference [16]. □

3.2 Spatial discretization

In this section, we use the finite difference method for the spatial discretization of Equation (3.1) with $h = \frac{1}{N}$, where N is the number of nodal points in the spatial direction. Let $U^{j+1}(x)$ be a smooth function in the interval $[0, 1]$. By applying Taylor's series expansion, we can express $U_{i\pm h}^{j+1} \approx U_{i\pm 1}^{j+1}$ as:

$$U_{i+h}^{j+1} \approx U_{i+1}^{j+1} = U_i^{j+1} + h \frac{dU_i^{j+1}}{dx} + \frac{h^2}{2!} \frac{d^2U_i^{j+1}}{dx^2} + \frac{h^3}{3!} \frac{d^3U_i^{j+1}}{dx^3} + \frac{h^4}{4!} \frac{d^4U_i^{j+1}}{dx^4} + \frac{h^5}{5!} \frac{d^5U_i^{j+1}}{dx^5} + \frac{h^6}{6!} \frac{d^6U_i^{j+1}}{dx^6} + \frac{h^7}{7!} \frac{d^7U_i^{j+1}}{dx^7} + \frac{h^8}{8!} \frac{d^8U_i^{j+1}}{dx^8} + O(h^9) \quad (3.5)$$

and

$$U_{i-h}^{j+1} \approx U_{i-1}^{j+1} = U_i^{j+1} - h \frac{dU_i^{j+1}}{dx} + \frac{h^2}{2!} \frac{d^2U_i^{j+1}}{dx^2} - \frac{h^3}{3!} \frac{d^3U_i^{j+1}}{dx^3} + \frac{h^4}{4!} \frac{d^4U_i^{j+1}}{dx^4} - \frac{h^5}{5!} \frac{d^5U_i^{j+1}}{dx^5} + \frac{h^6}{6!} \frac{d^6U_i^{j+1}}{dx^6} - \frac{h^7}{7!} \frac{d^7U_i^{j+1}}{dx^7} + \frac{h^8}{8!} \frac{d^8U_i^{j+1}}{dx^8} + O(h^9). \quad (3.6)$$

Adding Equations (3.5) and (3.6), we get

$$U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1} = \frac{2h^2}{2!} \frac{d^2U_i^{j+1}}{dx^2} + \frac{2h^4}{4!} \frac{d^4U_i^{j+1}}{dx^4} + \frac{2h^6}{6!} \frac{d^6U_i^{j+1}}{dx^6} + \frac{2h^8}{8!} \frac{d^8U_i^{j+1}}{dx^8} + O(h^{10}). \quad (3.7)$$

Operating differential operator $\frac{d^2}{dx^2}$ on both sides of Equation (3.7), we get

$$\frac{d^2U_{i-1}^{j+1}}{dx^2} - 2\frac{d^2U_i^{j+1}}{dx^2} + \frac{d^2U_{i+1}^{j+1}}{dx^2} = \frac{2h^2}{2!} \frac{d^4U_i^{j+1}}{dx^4} + \frac{2h^4}{4!} \frac{d^6U_i^{j+1}}{dx^6} + \frac{2h^6}{6!} \frac{d^8U_i^{j+1}}{dx^8} + \frac{2h^8}{8!} \frac{d^{10}U_i^{j+1}}{dx^{10}} + O(h^{12}). \quad (3.8)$$

By solving Equation (3.8) for $\frac{h^4}{12} \frac{d^6U_i^{j+1}}{dx^6}$ gives:

$$\frac{h^4}{12} \frac{d^6U_i^{j+1}}{dx^6} = \frac{d^2U_{i-1}^{j+1}}{dx^2} - 2\frac{d^2U_i^{j+1}}{dx^2} + \frac{d^2U_{i+1}^{j+1}}{dx^2} - \frac{2h^2}{2!} \frac{d^4U_i^{j+1}}{dx^4} - \frac{2h^6}{6!} \frac{d^8U_i^{j+1}}{dx^8} - \frac{2h^8}{8!} \frac{d^{10}U_i^{j+1}}{dx^{10}} + O(h^{12}). \quad (3.9)$$

Substituting Equation (3.9) into Equation (3.7), we obtain

$$U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1} = \frac{h^2}{30} \left(\frac{d^2U_{i-1}^{j+1}}{dx^2} + 28\frac{d^2U_i^{j+1}}{dx^2} + \frac{d^2U_{i+1}^{j+1}}{dx^2} \right) + R, \quad (3.10)$$

where $R = \frac{h^4}{20} \frac{d^4 U_i^{j+1}}{dx^4} - \frac{13h^8}{302400} \frac{d^8 U_i^{j+1}}{dx^8} + O(h^{10})$.

Equation (3.1) can be discretized and rewritten as

$$\begin{aligned}
 & -\varepsilon \left(\frac{\partial^2 U^{j+1}(x_\alpha)}{\partial x^2} + \frac{\partial^2 U^j(x_\alpha)}{\partial x^2} \right) \\
 & = \begin{cases} - (2/k + \mu(x_\alpha)) U^{j+1}(x_\alpha) + (2/k - \mu(x_\alpha)) U^j(x_\alpha) - \\ (\zeta^{j+1} + \zeta^j) \lambda_b^{j+1} + f^{j+1}(x_\alpha) + f^j(x_\alpha), \text{ for } j = 0(1)m, \\ - (2/k + \mu(x_\alpha)) U^{j+1}(x_\alpha) + (2/k - \mu(x_\alpha)) U^j(x_\alpha) - \\ (\zeta^{j+1} + \zeta^j) U^{j-m}(x_\alpha) + f^{j+1}(x_\alpha) + f^j(x_\alpha), \\ \text{for } j = m + 1(1)M - 1, \end{cases} \tag{3.11}
 \end{aligned}$$

where $\alpha = i, i \pm 1$.

To handle the effect of the perturbation parameter ε on the solution profiles of Equation (3.11), we use the exponential fitting factor [13]:

$$\sigma = \frac{h^2/4\varepsilon\mu_i}{\left(\sinh(h/2)\sqrt{\mu_i/\varepsilon} \right)^2}. \tag{3.12}$$

Then

$$\begin{aligned}
 & -\sigma \left(\frac{\partial^2 U^{j+1}(x_\alpha)}{\partial x^2} + \frac{\partial^2 U^j(x_\alpha)}{\partial x^2} \right) \\
 & = \begin{cases} - (2/k + \mu(x_\alpha)) U^{j+1}(x_\alpha) + (2/k - \mu(x_\alpha)) U^j(x_\alpha) - \\ (\zeta^{j+1} + \zeta^j) \lambda_b^{j+1} + f^{j+1}(x_\alpha) + f^j(x_\alpha), \text{ for } j = 0(1)m \\ - (2/k + \mu(x_\alpha)) U^{j+1}(x_\alpha) + (2/k - \mu(x_\alpha)) U^j(x_\alpha) - \\ (\zeta^{j+1} + \zeta^j) U^{j-m}(x_\alpha) + f^{j+1}(x_\alpha) + f^j(x_\alpha), \\ \text{for } j = m + 1(1)M - 1. \end{cases}
 \end{aligned}$$

Using Equation (3.10) into Equation (3.12), we obtain

$$A_i^- u_{i-1}^{j+1} + A_i^c u_i^{j+1} + A_i^+ u_{i+1}^{j+1} = B_i^- u_{i-1}^j + B_i^c u_i^j + B_i^+ u_{i+1}^j + F^j, \tag{3.13}$$

where

$$\begin{aligned}
 A_i^- &= \frac{-\sigma}{h^2} + \frac{1}{30} (2/k + \mu_{i-1}), & B_i^- &= \frac{\sigma}{h^2} + \frac{1}{30} (2/k - \mu_{i-1}), \\
 A_i^c &= \frac{2\sigma}{h^2} + \frac{28}{30} (2/k + \mu_i), & B_i^c &= \frac{-2\sigma}{h^2} + \frac{28}{30} (2/k - \mu_i), \\
 A_i^+ &= \frac{-\sigma}{h^2} + \frac{1}{30} (2/k + \mu_{i+1}), & B_i^+ &= \frac{\sigma}{h^2} + \frac{1}{30} (2/k - \mu_{i+1}),
 \end{aligned}$$

and

$$F^j = \frac{1}{30} \left(s_{i-1}^{j+1} + s_{i-1}^j + 28(s_i^{j+1} + s_i^j) + s_{i+1}^{j+1} + s_{i+1}^j \right) - \begin{cases} \frac{1}{30} \left(\zeta_{i-1}^{j+1} + \zeta_{i-1}^j \right) \lambda_b^{j+1}(x_{i-1}) + \frac{28}{30} \left(\zeta_i^{j+1} + \zeta_i^j \right) \lambda_b^{j+1}(x_i) + \\ \frac{1}{30} \left(\zeta_{i+1}^{j+1} + \zeta_{i+1}^j \right) \lambda_b^{j+1}(x_{i+1}), \text{ for } j = 0(1)m, \\ \frac{1}{30} \left(\zeta_{i-1}^{j+1} + \zeta_{i-1}^j \right) u_{i-1}^{j-m} + \frac{28}{30} \left(\zeta_i^{j+1} + \zeta_i^j \right) u_i^{j-m} + \\ \frac{1}{30} \left(\zeta_{i+1}^{j+1} + \zeta_{i+1}^j \right) u_{i+1}^{j-m}, \text{ for } j = m + 1(1)M - 1. \end{cases}$$

4 Convergence analysis

Lemma 7. For a fixed number of mesh N and for positive integer r as $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} \max_{01 \leq i \leq N-1} \frac{\exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0, \quad \lim_{\varepsilon \rightarrow 0} \max_{01 \leq i \leq N-1} \frac{\exp(C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0.$$

Proof. See reference [13]. □

Lemma 8. If ϑ_i^{j+1} be any mesh function such that $\vartheta_0^{j+1} = 0 = \vartheta_N^{j+1}$. Then,

$$\left| \vartheta_i^{j+1} \right| \leq \frac{1}{\mu^*} \max_{1 \leq a \leq N-1} \left| \mathcal{L}_\varepsilon^{k,h} \vartheta_a^{j+1} \right|.$$

Proof. See reference [13]. □

Lemma 9. The discrete solution satisfies the following error bound:

$$\left\| U^{j+1}(x) - U_i^{j+1} \right\| \leq CN^{-2}.$$

Proof. The truncation error of the scheme (3.13) is given by

$$\left| \mathcal{L}^k U^{j+1}(x) - \mathcal{L}^k U_i^{j+1} \right| = -\varepsilon U_{xx}^{j+1}(x) + \eta(x) U^{j+1}(x) - \left[-\frac{\varepsilon\sigma}{h^2} \left(U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1} \right) \right] - \left[\eta U_{i-1}^{j+1} + 28\eta U_i^{j+1} + \eta U_{i+1}^{j+1} \right],$$

where $\eta = 1/30(2/k + \mu_i)$. Using Taylor’s series expansion of the terms $U_{i\pm 1}^{j+1}$, we have

$$U_{i\pm 1}^{j+1} = U^{j+1}(x_i) \pm hU_x^{j+1}(x_i) + \frac{h^2}{2} U_{xx}^{j+1}(x_i) \pm \frac{h^3}{6} U_{xxx}^{j+1}(x_i) + \frac{h^4}{24} U_{xxxx}^{j+1}(x_i) \pm \frac{h^5}{120} U_{xxxxx}^{j+1}(x_i) + \dots \tag{4.1}$$

Substituting Equation (4.1) into Equation (4.1), we obtain

$$\left| \mathcal{L}^k U^{j+1}(x) - \mathcal{L}^k U_i^{j+1} \right| = -\varepsilon U_{xx}^{j+1}(x_i) + \eta(x_i) U^{j+1}(x) - \left[-\frac{\varepsilon\sigma}{h^2} \left(h^2 U_{xx}^{j+1}(x_i) + \frac{h^4}{12} U_{xxxx}^{j+1}(x_i) \right) \right] - \left[\eta h^2 U_{xx}^{j+1}(x_i) + \eta \frac{h^4}{12} U_{xxxx}^{j+1}(x_i) \right]. \tag{4.2}$$

Using Taylor's series expansion on fitting factor (σ), we have

$$\sigma = 1 - \frac{\gamma_i^2 h^2}{12} + \frac{\gamma_i^4 h^4}{240} + O(h^5). \quad (4.3)$$

Taking Equation (4.3) into Equation (4.2), we get

$$\left| \mathcal{L}^k U^{j+1}(x) - \mathcal{L}^k U_i^{j+1} \right| = \left(\frac{\varepsilon h^2}{12} (U_{xx}^{j+1}(x_i) \gamma_i^2 + U_{xxxx}^{j+1}(x_i)) + \eta U_{xx}^{j+1}(x_i) \right) + h^4 \left(\varepsilon \frac{\gamma_i^4}{240} U_{xx}^{j+1}(x_i) - \frac{\gamma_i^4}{144} U_{xx}^{j+1}(x_i) + \frac{\eta}{12} U_{xx}^{j+1}(x_i) \right) + h^6 \left(\frac{\varepsilon \gamma_i^4}{2880} U_{xxxx}^{j+1}(x_i) \right).$$

Using Lemma 6 together with Lemma 7 and the relation $N^{-2} > N^{-4} > N^{-6} > \dots$ the discrete scheme satisfies the bound

$$\left| \mathcal{L}^k U^{j+1}(x) - \mathcal{L}^k U_i^{j+1} \right| \leq CN^{-2}.$$

Using the bound in Lemma 8, we obtain

$$\left\| U^{j+1}(x) - U_i^{j+1} \right\| \leq CN^{-2}.$$

□

Theorem 1. *Let u and U be the solutions of (1.1) and (3.13), respectively. Then, the following uniform error bound holds*

$$\|u - U\| \leq C \left(k^2 + N^{-2} \right).$$

Proof. The proof follows from the error estimate for the temporal and spatial discretization. □

5 Numerical results and discussion

We considered several examples for validating the theoretical analysis. Since there is no exact solution for the case considered, we computed the maximum absolute errors using the double mesh principle [3].

$$e_\varepsilon^{N,M} = \max_{(x_i, t_{j+1}) \in \mathfrak{S}^{N,M}} \left| \left(U^{N,M}(x_i, t_{j+1}) - U^{2N,2M}(x_i, t_{j+1}) \right) \right|,$$

where $U^{N,M}(x_i, t_{j+1})$ and $U^{2N,2M}(x_i, t_{j+1})$ denote the numerical solution obtained in $\mathfrak{S}^{N,M}$ and $\mathfrak{S}^{2N,2M}$ with N and M mesh intervals in the spatial and the temporal directions, respectively. The corresponding rate of convergence is computed by

$$r_\varepsilon^{N,M} = \log_2 \left(e_\varepsilon^{N,M} / e_\varepsilon^{2N,2M} \right).$$

The parameter uniform maximum absolute error and uniform order of convergence are calculated by

$$e^{N,M} = \max_\varepsilon \left(e_\varepsilon^{N,M} \right), \quad r^{N,M} = \log_2 \left(e^{N,M} / e^{2N,2M} \right).$$

Example 1. Consider the problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (\frac{1+x^2}{2})u(x,t) + u(x,t - \tau) = t^3, (x,t) \in (0,1) \times [0,2], \\ u(0,t) = 0, u(1,t) = 0, t \in [0,2], \\ u(x,t) = 0, (x,t) \in [0,1] \times [-\tau,0]. \end{cases}$$

Table 1. Maximum absolute error and the rate of convergence for Example 1.

$\varepsilon \downarrow$	$N = 2^5$ M=20	$N = 2^6$ M=40	$N = 2^7$ M=80	$N = 2^8$ M=160	$N = 2^9$ M=320
10^{-0}	1.2971e-03 1.9319	3.3996e-04 1.9833	8.5977e-05 1.9959	2.1556e-05 1.9989	5.3930e-06
10^{-2}	1.9505e-02 2.0259	4.7894e-03 2.0072	1.1914e-03 2.0018	2.9748e-04 2.0005	7.4345e-05
10^{-8}	2.0244e-02 2.0346	4.9410e-03 2.0094	1.2272e-03 2.0024	3.0630e-04 2.0006	7.6542e-05
10^{-12}	2.0244e-02 2.0346	4.9410e-03 2.0094	1.2272e-03 2.0024	3.0630e-04 2.0006	7.6542e-05
10^{-16}	2.0244e-02 2.0346	4.9410e-03 2.0094	1.2272e-03 2.0024	3.0630e-04 2.0006	7.6542e-05
10^{-20}	2.0244e-02 2.0346	4.9410e-03 2.0094	1.2272e-03 2.0024	3.0630e-04 2.0006	7.6542e-05
$e^{N,M}$	2.0244e-02	4.9410e-03	1.2272e-03	3.0630e-04	7.6542e-05
$r^{N,M}$	2.0346	2.0094	2.0024	2.0006	

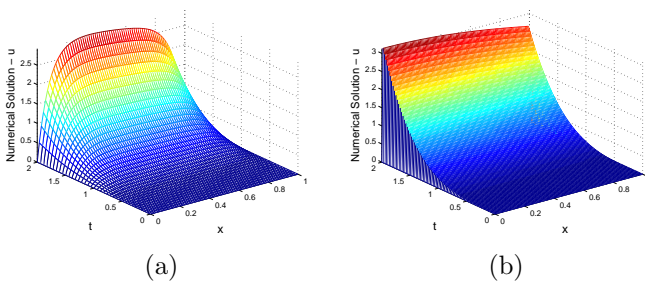


Figure 1. 3D view of numerical solution for Example 1 on (a) $\varepsilon = 2^{-6}$, on (b) $\varepsilon = 2^{-20}$.

Example 2. Consider the problem

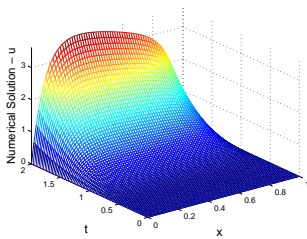
$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (\frac{1.1+x^2}{2})u(x,t) + u(x,t - \tau) = t^3, (x,t) \in (0,1) \times [0,2], \\ u(0,t) = 0, u(1,t) = 0, t \in [0,2], \\ u(x,t) = 0, (x,t) \in [0,1] \times [-\tau,0]. \end{cases}$$

Example 3. Consider the problem

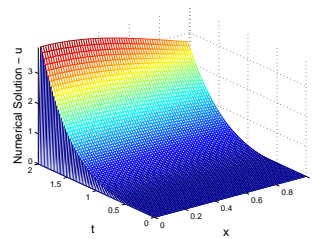
$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + x^2 u(x,t) + u(x, t - \tau) = t^3, (x, t) \in (0, 1) \times [0, 2], \\ u(0, t) = 0, u(1, t) = 0, t \in [0, 2], \\ u(x, t) = 0, (x, t) \in [0, 1] \times [-\tau, 0]. \end{cases}$$

Table 2. Maximum absolute error and the rate of convergence for Example 2.

$\varepsilon \downarrow$	$N = 2^5$ M=20	$N = 2^6$ M=40	$N = 2^7$ M=80	$N = 2^8$ M=160	$N = 2^9$ M=320
10^{-0}	1.2861e-03 1.9319	3.3707e-04 1.9834	8.5245e-05 1.9959	2.1372e-05 1.9989	5.3470e-06
10^{-2}	1.8992e-02 2.0225	4.6744e-03 2.0063	1.1635e-03 2.0016	2.9055e-04 2.0004	7.2617e-05
10^{-4}	1.9662e-02 2.0297	4.8155e-03 1.8246	1.3595e-03 1.9700	3.4701e-04 1.9991	8.6805e-05
10^{-8}	1.9703e-02 2.0309	4.8213e-03 2.0086	1.1982e-03 2.0021	2.9911e-04 2.0005	7.4749e-05
10^{-12}	1.9703e-02 2.0309	4.8213e-03 2.0086	1.1982e-03 2.0021	2.9911e-04 2.0005	7.4749e-05
10^{-16}	1.9703e-02 2.0309	4.8213e-03 2.0086	1.1982e-03 2.0021	2.9911e-04 2.0005	7.4749e-05
10^{-20}	1.9703e-02 2.0309	4.8213e-03 2.0086	1.1982e-03 2.0021	2.9911e-04 2.0005	7.4749e-05
$e^{N,M}$	1.9703e-02	4.8213e-03	1.3595e-03	3.4701e-04	8.6805e-05
$r^{N,M}$	2.0309	1.8263	1.9700	1.9991	



(a)



(b)

Figure 2. 3D view of numerical solution for Example 3 on (a) $\varepsilon = 2^{-6}$, on (b) $\varepsilon = 2^{-20}$.

The computed $e_{\varepsilon}^{N,M}$, $e^{N,M}$, $r_{\varepsilon}^{N,M}$, and $r^{N,M}$ for Examples 1–3 are tabulated in Tables 1–4. The results in Tables 1–4 depict that the proposed numerical method is accurate in order $O(k^2 + N^{-2})$ which aligns with the theoretical predictions. The numerical solutions obtained by the numerical scheme presented

Table 3. Maximum absolute error and the rate of convergence for Example 3.

$\varepsilon \downarrow$	$N = 2^5$ M=20	$N = 2^6$ M=40	$N = 2^7$ M=80	$N = 2^8$ M=160	$N = 2^9$ M=320
10^{-0}	1.3807e-03 1.9316	3.6193e-04 1.9833	9.1539e-05 1.9958	2.2952e-05 1.9989	5.7422e-06
10^{-2}	2.5050e-02 2.0590	6.0116e-03 2.0156	1.4867e-03 2.0039	3.7068e-04 2.0010	9.2606e-05
10^{-4}	2.6775e-02 2.0756	6.3522e-03 2.0262	1.5595e-03 1.9995	3.9001e-04 2.0005	9.7472e-05
10^{-8}	2.6860e-02 2.0769	6.3664e-03 2.0199	1.5698e-03 2.0050	3.9108e-04 2.0013	9.7685e-05
10^{-12}	2.6860e-02 2.0769	6.3664e-03 2.0199	1.5698e-03 2.0050	3.9108e-04 2.0013	9.7685e-05
10^{-16}	2.6860e-02 2.0769	6.3664e-03 2.0199	1.5698e-03 2.0050	3.9108e-04 2.0013	9.7685e-05
10^{-20}	2.6860e-02 2.0769	6.3664e-03 2.0199	1.5698e-03 2.0050	3.9108e-04 2.0013	9.7685e-05
$e^{N,M}$	2.6860e-02	6.3664e-03	1.5698e-03	3.9108e-04	9.7685e-05
$r^{N,M}$	2.0769	2.0199	2.0050	2.0013	

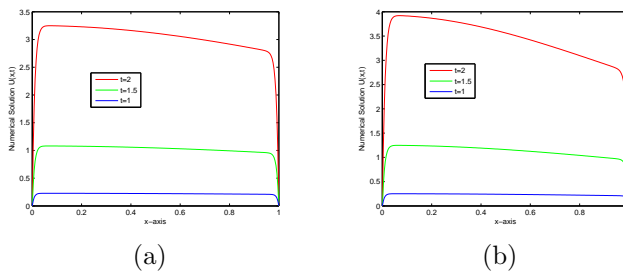


Figure 3. Numerical solution on (a) Example 1, on (b) Example 3.

in Examples 1 and 3 are shown in Figures 1 and 2, respectively, showing the boundary layer formation on the solution is given as $\varepsilon \rightarrow 0$.

From Figures 3 (a) and (b), one can conclude that as $\varepsilon \rightarrow 0$ dual boundary layer is created near $x = 0$ and $x = 1$. In general, it is observed from Figures 2–3 that when goes small, a dual boundary layer is created at the two endpoints of the underlying interval. The comparison in Table 4 reveals that the proposed method is more accurate than the methods in [13, 17] and [10].

Table 4. Comparison of uniform errors $e^{N,M}$ and uniform order of convergence $r^{N,M}$ for Example 1.

$\varepsilon \downarrow$	$N = 2^6$ M=20	$N = 2^7$ M=40	$N = 2^8$ M=80	Method↓	$N = 2^7$ M=8	$N = 2^8$ M=32
PM				PM		
10^{-8}	1.937e-2 2.0268	4.752e-3 2.0074	1.182e-3		4.983e-2 2.0539	1.200e-2
10^{-10}	1.959e-2 2.0294	4.798e-3 2.0081	1.193e-3		5.060e-2 2.0608	1.213e-2
10^{-12}	1.966e-2 2.0303	4.813e-3 2.0084	1.196e-3		5.086e-2 2.0633	1.217e-2
10^{-14}	1.969e-2 2.0306	4.818e-3 2.0089	1.1972e-3		5.095e-2 2.0642	1.218e-2
10^{-16}	1.970e-2 2.0309	4.820e-3 2.0078	1.199e-3		5.098e-2 2.0645	1.219e-2
10^{-18}	1.970e-2 2.0310	4.821e-3 2.0191	1.189e-3		5.099e-2 2.0646	1.219e-2
10^{-20}	1.971e-2 2.0312	4.821e-3 2.0086	1.198e-3		1.219e-2 2.0646	5.099e-2
10^{-22}	1.971e-2 2.0312	4.821e-3 2.0086	1.198e-3		5.099e-2 2.0646	1.219e-2
$e^{N,M}$	1.971e-2	4.821e-3	1.198e-3		5.099e-2	1.219e-2
$r^{N,M}$	2.0312	2.0086			2.0646	
Results in [13]				Results in [10]		
$e^{N,M}$	8.929e-2	2.422e-2	4.951e-3		2.39e-1	1.28e-1
$r^{N,M}$	1.8824	2.2903			0.9008	0.9556
Results in [17]						
$e^{N,M}$	5.3429e-02	2.7108e-02	1.3653e-02			
$r^{N,M}$	0.97890	0.98950				

6 Conclusions

Uniform numerical algorithm is presented to solve the singularly perturbed parabolic time delay reaction-diffusion problem. It utilizes the Crank-Nicolson method for time derivative discretization and the novel finite difference approach for spatial derivatives. The stability and convergence analysis of the scheme are discussed and proved. Several test examples are presented to illustrate the efficiency of the proposed method. Numerical results suggest second-order uniform convergence in both space and time directions. The findings depict that the proposed scheme achieves greater accuracy and a higher order of convergence compared to existing methods in the literature.

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References

- [1] G. Adomian and R. Rach. Nonlinear stochastic differential delay equations. *J. Math. Anal. Appl.*, **91**(1):94–101, 1983. [https://doi.org/10.1016/0022-247X\(83\)90094-X](https://doi.org/10.1016/0022-247X(83)90094-X).
- [2] C. Clavero and J.C. Jorge. Efficient numerical methods for semilinear one dimensional parabolic singularly perturbed convection-diffusion systems. *Appl. Numer. Math.*, **198**:461–473, 2024. <https://doi.org/10.1016/j.apnum.2024.02.006>.
- [3] I.T. Daba, N.E. Kebede and G.G. Gonfa. Extended B-spline collocation method for singularly perturbed time delay parabolic differential-difference problems. *Int. J. Comput. Methods*, **22**(01):2450046, 2025. <https://doi.org/10.1142/S0219876224500464>.
- [4] E.P. Doolan, J.J.H. Miller and W.H.A. Schilders. *Uniform numerical methods for problems with initial and boundary layers*. Boole Press, Dublin, 1980.
- [5] I.R. Epstein. Delay effects and differential delay equations in chemical kinetics. *Int. Rev. Phys. Chem.*, **11**(1):135–160, 1992. <https://doi.org/10.1080/01442359209353268>.
- [6] F.W. Gelu, I.T. Daba, W.G. Melesse and G.D. Kebede. Fitted mesh numerical method for two-parameter singularly perturbed partial differential equations with large time lag. *Partial Differ. Equ. Appl. Math.*, **11**:100844, 2024. <https://doi.org/10.1016/j.padiff.2024.100844>.
- [7] M.J. Huntul. Recovering a source term in the higher-order pseudo-parabolic equation via cubic spline functions. *Phys. Scr.*, **97**(3):035004, 2022. <https://doi.org/10.1088/1402-4896/ac54d0>.
- [8] M.J. Huntul, A.T. Ramazanov and Y.T. Mehraliyev. On the solvability of an inverse boundary problem for the equation of transverse oscillation of the rod. *Math. Methods Appl. Sci.*, **47**(18):14629–14638, 2024. <https://doi.org/10.1002/mma.10294>.
- [9] K. Khari and V. Kumar. Finite element analysis of the singularly perturbed parabolic reaction-diffusion problems with retarded argument. *Numer. Methods Partial Differential Equations*, **38**(4):997–1014, 2022. <https://doi.org/10.1002/num.22785>.
- [10] P.M.M. Kumar and A.S.V. Ravi Kanth. Computational study for a class of time-dependent singularly perturbed parabolic partial differential equation through tension spline. *Comput. Appl. Math.*, **39**:1–19, 2020. <https://doi.org/10.1007/s40314-020-01278-5>.
- [11] S. Kumar and M. Kumar. High order parameter-uniform discretization for singularly perturbed parabolic partial differential equations with time delay. *Comput. Math. Appl.*, **68**(10):1355–1367, 2014. <https://doi.org/10.1016/j.camwa.2014.09.004>.
- [12] B.J. McCartin. Discretization of the semiconductor device equations. In *New Problems and New Solutions for Device and Process Modelling*, pp. 72–82, Dublin, 1985. Boole Press.

- [13] E.A. Megiso, M.M. Woldaregay and T.G. Dinka. Fitted tension spline method for singularly perturbed time delay reaction diffusion problems. *Math. Probl. Eng.*, **2022**(1):8669718, 2022. <https://doi.org/10.1155/2022/8669718>.
- [14] Y.T. Mehraliyev, M.J. Huntul and E.I. Azizbayov. Simultaneous identification of the right-hand side and time-dependent coefficients in a two-dimensional parabolic equation. *Math. Model. Anal.*, **29**(1):90–108, 2024. <https://doi.org/10.3846/mma.2024.17974>.
- [15] J.J.H. Miller, E. O’Riordan and G.I. Shishkin. *Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions*. World Scientific, Singapore, 1996.
- [16] N.T. Negero. A robust fitted numerical scheme for singularly perturbed parabolic reaction–diffusion problems with a general time delay. *Results Phys.*, **51**:106724, 2023. <https://doi.org/10.1016/j.rinp.2023.106724>.
- [17] R.N. Rao and P.P. Chakravarthy. A fitted Numerov method for singularly perturbed parabolic partial differential equation with a small negative shift arising in control theory. *Numer. Math. Theory Methods Appl.*, **7**(1):23–40, 2014. <https://doi.org/10.4208/nmtma.2014.1316nm>.
- [18] S. Sultan and Z. Zhang. A stable time-dependent mesh method for generalized credit rating migration problem. *J. Nonlinear Math. Phys.*, **30**(4):1774–1803, 2023. <https://doi.org/10.1007/s44198-023-00157-x>.
- [19] A.N. Tikhonov and A.A. Samarskii. *Equations of Mathematical Physics*. Courier Corporation, North Chelmsford, MA, 2013. ISBN 9780486641140. Reprint of the 1963 Pergamon Press edition
- [20] A. Van Harten and J.M. Schumacher. On a class of partial functional differential equations arising in feed-back control theory. In *North-Holland Mathematics Studies*, volume 31, pp. 161–179. Elsevier, 1978. [https://doi.org/10.1016/S0304-0208\(08\)70556-5](https://doi.org/10.1016/S0304-0208(08)70556-5).
- [21] P.K.C. Wang. Asymptotic stability of a time-delayed diffusion system. *J. Appl. Mech.*, **30**(4):500–504, 1963. <https://doi.org/10.1115/1.3636609>.
- [22] J. Wu. *Theory and applications of partial functional differential equations*, volume 119. Springer Science & Business Media, New York, 2012.