

Variable viscosity effect on two layered flow of blood through time dependent stenosed artery with slip velocity

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Abstract. This study investigates the influence of variable viscosity and slip velocity on flow rate, resistance to flow, and wall shear stress within a specific range of parameters. A two-layered model is developed, where the core region of the blood is modeled as a Casson-type non-Newtonian fluid, and the plasma layer is assumed to behave as a Newtonian fluid. The results are presented graphically and discussed thoroughly. Additionally, the findings are compared with previously reported experimental results.

Keywords: angular frequency; Casson flow; hematocrit; plug velocity; pressure gradient; slip velocity; stenosis; variable viscosity; viscosity index; volumetric flow rate; wall shear stress; Womersley parameter; yield stress.

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1 Introduction

Due to the complex nature of blood, the exact characteristics of blood flow through a stenosed arterial tube remain unknown, despite its crucial role in any blood circulatory disorder. The accumulation of fatty substances in the lumen of the artery leads to stenosis, resulting in increased resistance to blood flow. Time-dependent arterial stenosis is one of the most common forms of arterial disease. Various cardiovascular diseases, like atherosclerosis, heart attacks, and brain hemorrhages, are worsened by arterial constriction.

The probability of stenosis formation rises in arteries with high curvature, as reported by Sharma et al. [26]. Their study also showed that blood generates greater entropy as the intensity of viscous heating and thermal radiation increases. Shabir et al. [24] demonstrated that the curvature of blood vessels significantly influences blood flow velocity. Hussain et al. [11, 12] conducted a numerical study of pulsatile flow of a nano-fluid (blood) through a curved

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artery with stenosis and post-stenotic dilatation to examine the effects of thermophoresis and Brownian motion. In another study, Mekheimer et al. [15] theoretically investigated blood flow through a catheterized curved artery containing a time-dependent overlapping stenosis.

The impact of various factors influencing arterial blood flow has been explored through mathematical modeling in numerous studies. Awad et al. [3] reported that higher values of wall elasticity parameters promote smoother blood flow through a stenotic section of the artery. Zaher et al. [30] confirmed that increasing parameters related to wall flexibility makes blood flow easier through the stenosed area. Bali et al. [4] examined how a magnetic field affects blood flow resistance when there is arterial constriction. Chakraborty et al. [7] built a model to evaluate how body acceleration influences unsteady blood flow with time-dependent arterial stenosis. The effect of time-dependent stenosis on blood flow has also been explored by Mishra et al. [16] and Khan et al. [18]. Shabir et al. [25] investigated the electro-osmotic effects in time-dependent blood flow through a W-shaped stenotic conduit driven by a pulsatile pressure gradient.

To study blood flow characteristics through a constricted artery, several researchers have proposed mathematical models assuming constant viscosity [1, 4, 7]. However, since viscosity depends on hematocrit, incorporating variable viscosity provides more accurate results for determining flow rate, flow resistance, and wall shear stress. Shit et al. [27] developed a model for blood flow through a stenosed artery that incorporates variable viscosity. Mekheimer et al. [14] conducted a theoretical study on the combined effects of variable viscosity and thermal conductivity on pulsatile blood flow in an asymmetric channel.

Blood flow behavior also depends on shear rate. At high shear rates, blood flow exhibits the characteristics of a Newtonian fluid in large-diameter arteries, whereas at low shear rates, it displays non-Newtonian properties, especially when the artery diameter is relatively small [10, 19]. Several studies modeled blood flow by considering blood as a Herschel-Bulkley-type non-Newtonian fluid. Ali et al. [1] presented a mathematical model to examine the non-Newtonian behavior of blood in the presence of overlapping arterial stenosis. However, blood more accurately represents a Casson-type non-Newtonian fluid, as it precisely captures the yield stress caused by red blood cell (RBC) aggregation, provides a closer fit to experimental rheological data, and reflects the micro-structural characteristics of blood, particularly at physiological shear rates. In [8], Charm and Kurland experimentally demonstrated that the Casson-type fluid model is a suitable representation of blood flow through a narrow arterial tube at low shear rates. Chaturani and Samy [9] considered Casson-type blood flow and developed a mathematical model for pulsatile flow through a stenosed artery. Blair and Spanner [6] observed that blood exhibits characteristics of a Casson-type non-Newtonian fluid at moderate shear rates, making it a suitable model for blood flow. Aroesty and Gross [2] also formulated a mathematical model for pulsatile blood flow through a narrow, uniform arterial tube, treating blood as a Casson-type non-Newtonian fluid.

Instead of assuming uniform blood flow behavior throughout the entire artery, many researchers (Ponalagusamy et al. [17], Sankar et al. [22]) pro-

posed a two-layered mathematical model for blood flow through a constricted artery to study flow velocity and resistance to flow. In these models, blood in the outer region of the artery is treated as a Newtonian fluid, while the inner core region is assumed to exhibit non-Newtonian behavior. Several recent studies investigated how variations in viscosity affect the two-fluid model of blood flowing through constricted blood vessels [3, 21, 23, 24, 25, 26, 30].

Furthermore, the slip velocity on the arterial wall plays a significant role in reducing hydrodynamic resistance, pressure drop, and wall shear stress within a stenosed artery, while also increasing the flow rate. Biswas et al. [5] demonstrated that slip velocity influences blood flow in a stenosed artery when body acceleration is present, using a two-layer pulsatile flow model. In [28], the effect of slip on pulsatile blood flow through a stenosed arterial segment subjected to periodic body acceleration was examined.

This study presents a mathematical model to investigate the effects of variable viscosity and slip velocity on the flow rate, resistance to flow, and wall shear stress of blood flowing through an artery with a time-dependent stenosis. The blood flow is modeled with two layers: a cell-free plasma layer treated as a Newtonian fluid, and a core layer composed of a suspension of red blood cells in plasma, modeled as a Casson-type non-Newtonian fluid.

2 Mathematical formulation

To develop the mathematical formulation, an axially symmetric, two dimensional, laminar, and pulsatile flow of blood through an inelastic arterial blood vessel is considered. Assuming the artery is sufficiently long, the effects of blood flow at the entrance and exit are negligible and can be ignored. The flow consists of two layers: a cell-free plasma layer, modeled as a Newtonian fluid, and a core layer comprising a suspension of red blood cells in plasma, which is modeled as a Casson-type non-Newtonian fluid (see Figure 1).

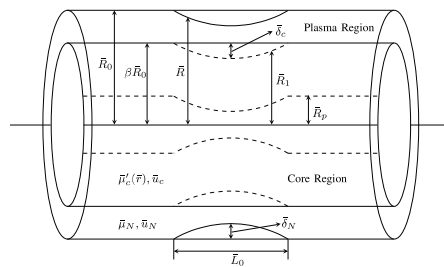


Figure 1. Constricted arterial tube for two fluid model.

To present the two-layered mathematical model, cylindrical coordinate $(\bar{r}, \bar{\phi}, \bar{z})$ is considered, where $\bar{r}$ is radial coordinates, $\bar{\phi}$ is the azimuthal angle, and $\bar{z}$ is the axial coordinate. In case of mild stenosis, $\frac{\delta}{R_0} \ll 1$ and therefore $\frac{\partial \bar{p}}{\partial \bar{r}} = 0$.

Thus, the governing equations are

$$\bar{\rho}_c \frac{\partial \bar{u}_c}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{\tau}_c), \quad 0 \leq \bar{r} \leq \bar{R}_1(\bar{t}, \bar{z}), \quad (2.1)$$

$$\bar{\rho}_N \frac{\partial \bar{u}_N}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{\tau}_N), \quad \bar{R}_1(\bar{t}, \bar{z}) \leq \bar{r} \leq \bar{R}(\bar{t}, \bar{z}), \quad (2.2)$$

where $\bar{\rho}_c$ and $\bar{\rho}_N$ are densities, $\bar{u}_c$ and $\bar{u}_N$ are velocities in core and plasma region respectively, $\bar{\tau}_c$ and $\bar{\tau}_N$ are shear stresses, $\bar{p}$ is the pressure and $\bar{t}$ is time. The correlation between shear stress and strain rate is expressed as

$$\begin{aligned} \bar{\tau}_c^{\frac{1}{2}} &= \bar{\tau}_0^{\frac{1}{2}} + \left[\bar{\mu}'_c(\bar{r}) \left(-\frac{\partial \bar{u}_c}{\partial \bar{r}} \right) \right]^{\frac{1}{2}}, \quad \bar{\tau}_c \geq \bar{\tau}_0, \bar{R}_p \leq \bar{r} \leq \bar{R}_1(\bar{t}, \bar{z}), \\ \frac{\partial \bar{u}_c}{\partial \bar{r}} &= 0, \quad \bar{\tau}_c \leq \bar{\tau}_0, 0 \leq \bar{r} \leq \bar{R}_p, \\ \bar{\tau}_N &= \bar{\mu}_N \left(-\frac{\partial \bar{u}_N}{\partial \bar{r}} \right), \quad \bar{R}_1(\bar{t}, \bar{z}) \leq \bar{r} \leq \bar{R}(\bar{t}, \bar{z}), \end{aligned} \quad (2.3)$$

where $\bar{\mu}'_c(\bar{r})$ represents the radial varying viscosity in core region and $\bar{R}_p$ is the plug core(flow) radius.

The variation in blood viscosity is influenced by several factors, including shear rate, temperature, protein concentration, and various pathological conditions like anemia and diabetes. It is observed from experimental results that the core region is not completely cell-free; therefore, the viscosity in the core region, $\bar{\mu}'_c(\bar{r})$, may be expressed in the form [23]

$$\bar{\mu}'_c(\bar{r}) = \bar{\mu}_c / \left[1 - k + k (\bar{r}/\bar{R}_0)^n \right],$$

where $k = 1 + \beta h_m$, $\bar{\mu}_c$ is the coefficient of viscosity, h_m is the maximum hematocrit having the value 2.5 (Mazumdar et. al. [13]), τ_0 is the yield stress, n is the viscosity index. Furthermore, the blood exhibits radially varying viscosity due to nonuniform RBC concentration, known as hematocrit, and this form successfully captures the viscosity behavior, including high viscosity at the center ($\bar{r} = 0$) and low viscosity near the wall ($\bar{r} = \bar{R}_0$).

The boundary conditions are given by

$$\begin{aligned} \bar{\tau}_c &\text{ is finite at } \bar{r} = 0, \\ \bar{\tau}_c &= \bar{\tau}_N, \bar{u}_c = \bar{u}_N \quad \text{at } \bar{r} = \bar{R}_1(\bar{t}, \bar{z}), \\ \bar{u}_N &= \bar{u}_s \quad \text{at } \bar{r} = \bar{R}(\bar{t}, \bar{z}), \end{aligned}$$

where $\bar{u}_s$ represents the slip velocity. In the core and peripheral region, the geometry of time-dependent constriction is given by [16, 21]

$$\bar{R}_1(\bar{t}, \bar{z}) = \begin{cases} \gamma \bar{R}_0 - \bar{\delta}_c (1 - e^{-\bar{t}\bar{\omega}}) \left[1 + \cos(\pi \bar{z}/2\bar{z}_o) \right], & \text{if } -2\bar{z}_o \leq \bar{z} \leq 2\bar{z}_o, \\ \gamma \bar{R}_0, & \text{otherwise,} \end{cases}$$

$$\bar{R}(\bar{t}, \bar{z}) = \begin{cases} \bar{R}_0 - \bar{\delta}_N (1 - e^{-\bar{t}\bar{\omega}}) \left[1 + \cos(\pi \bar{z}/2\bar{z}_0) \right], & \text{if } -2\bar{z}_0 \leq \bar{z} \leq 2\bar{z}_0, \\ \bar{R}_0, & \text{otherwise,} \end{cases}$$

where $\bar{L}_0$ is the length of the stenotic region and equal to $4\bar{z}_0$, $\bar{\delta}_c$ and $\bar{\delta}_N$ represent the maximum height of the stenosis in core and peripheral region respectively, $\bar{R}(\bar{t}, \bar{z})$, $\bar{R}_1(\bar{t}, \bar{z})$ are radii of the constricted artery with peripheral and core region respectively, $\bar{R}_0$ be the radius of the normal artery, $\gamma\bar{R}_0$ represent the radius of the core region for normal artery, γ is the fraction of the core radius to the radius of the normal artery.

Introducing the non-dimensional quantities

$$\begin{aligned} \tau_c &= \frac{\bar{\tau}_c \bar{R}_0}{U_0 \bar{\mu}_c}, \quad \tau_N = \frac{\bar{\tau}_N \bar{R}_0}{U_0 \bar{\mu}_N}, \quad L = \frac{\bar{L}}{\bar{R}_0}, \quad z_0 = \frac{\bar{z}_0}{\bar{R}_0}, \quad R_p = \frac{\bar{R}_p}{\bar{R}_0}, \quad p = \frac{\bar{p} \bar{R}_0}{U_0 \bar{\mu}_N}, \\ r &= \frac{\bar{r}}{\bar{R}_0}, \quad t = \bar{t} \bar{\omega}, \quad L_0 = \frac{\bar{L}_0}{\bar{R}_0}, \quad \alpha^2 = \frac{\bar{R}_0^2 \bar{\omega} \bar{\rho}_N}{\bar{\mu}_N}, \quad \rho_0 = \frac{\bar{\rho}_N}{\bar{\rho}_c}, \quad u_c = \frac{\bar{u}_c}{U_0}, \\ u_N &= \frac{\bar{u}_N}{U_0}, \quad u_s = \frac{\bar{u}_s}{U_0}, \quad \tau_0 = \frac{\bar{\tau}_0 \bar{R}_0}{U_0 \bar{\mu}_c}, \quad \delta_c = \frac{\bar{\delta}_c}{\bar{R}_0}, \quad \delta_N = \frac{\bar{\delta}_N}{\bar{R}_0}, \quad R(z) = \frac{\bar{R}(\bar{z})}{\bar{R}_0}, \\ R(t, z) &= \frac{\bar{R}(\bar{t}, \bar{z})}{\bar{R}_0}, \quad R_1(z) = \frac{\bar{R}_1(\bar{z})}{\bar{R}_0}, \quad R_1(t, z) = \frac{\bar{R}_1(\bar{t}, \bar{z})}{\bar{R}_0}, \end{aligned}$$

where α is the Womersley parameter and ω is the angular frequency of the blood flow. The pressure gradient is taken as

$$-\frac{\partial p}{\partial z} = q(z) f(t),$$

where the pressure gradient at steady state is represented by $q(z)$. $f(t)$ is defined as equal to $1 + A \sin(t)$, A represents the pressure gradient amplitude. Using non-dimensional variables as given above, Equations (2.1)–(2.3) take the form

$$\frac{\alpha^2}{\rho_0} \frac{\partial u_c}{\partial t} = q(z) f(t) - \frac{1}{r} \frac{\partial}{\partial r} (r \tau_c), \quad (2.4)$$

$$\alpha^2 \frac{\partial u_N}{\partial t} = q(z) f(t) - \frac{1}{r} \frac{\partial}{\partial r} (r \tau_N), \quad (2.5)$$

$$(\sqrt{\tau_c} - \sqrt{\tau_0})^2 (1 - k + k r^n) = - \frac{\partial u_c}{\partial r}, \quad \tau \geq \tau_0, \quad R_p \leq r \leq R_1(t, z),$$

$$\frac{\partial u_c}{\partial r} = 0, \quad \tau \leq \tau_0, \quad 0 \leq r \leq R_p,$$

$$\tau_N = - \frac{\partial u_N}{\partial r}, \quad R_1(t, z) \leq r \leq R(t, z),$$

In non-dimensional form, the boundary conditions are given by

$$\tau_c \text{ is finite at } r = 0,$$

$$\tau_c = \tau_N, \quad u_c = u_N \quad \text{at } r = R_1(t, z),$$

$$u_N = u_s \quad \text{at } r = R(t, z).$$

In core and peripheral regions, the geometry of the stenosis is given by

$$R_1(t, z) = \begin{cases} \gamma - \delta_c(1 - e^{-t}) \left[1 + \cos(\pi z/2z_o) \right], & \text{if } -2z_o \leq z \leq 2z_o, \\ \gamma, & \text{otherwise} \end{cases}$$

$$R(t, z) = \begin{cases} 1 - \delta_N(1 - e^{-t}) \left[1 + \cos(\pi z/2z_o) \right], & \text{if } -2z_o \leq z \leq 2z_o, \\ 1, & \text{otherwise.} \end{cases}$$

The volumetric flow rate is given by

$$Q = 8 \int_0^{R(t,z)} ru(z, r, t) dr.$$

3 Steady flow of blood

Under a steady flow of blood, the governing equations are

$$p_s = \begin{cases} \frac{1}{r} \frac{d}{dr}(r\tau_c), & 0 \leq r \leq R_1(z), \\ \frac{1}{r} \frac{d}{dr}(r\tau_N), & R_1(z) \leq r \leq R(z), \end{cases} \quad (3.1)$$

$$\begin{aligned} -\frac{\partial u_c}{\partial r} &= (\sqrt{\tau_c} - \sqrt{\tau_0})^2 (1 - k + kr^n), & R_p \leq r \leq R_1(z), \\ \frac{\partial u_c}{\partial r} &= 0, & \tau_c \leq \tau_0, 0 \leq r \leq R_p, \\ -\frac{\partial u_N}{\partial r} &= \tau_N, & R_1(z) \leq r \leq R(z) \end{aligned} \quad (3.2)$$

with the following boundary conditions

$$\begin{aligned} \tau_c &\text{ is finite at } r = 0, \\ \tau_c &= \tau_N, u_c = u_N \text{ at } r = R_1(z), \\ u_N &= u_s \text{ at } r = R(z), \end{aligned} \quad (3.3)$$

where $p_s = (-\frac{dp}{dz})$ is the non-dimensional steady state pressure gradient.

From (3.1) and using the boundary conditions (3.3)

$$\begin{aligned} \tau_c &= \frac{p_s r}{2}, & 0 \leq r \leq R_1(z), \\ \tau_N &= \frac{p_s r}{2}, & R_1(z) \leq r \leq R(z), \\ -\frac{\partial u_c}{\partial r} &= \left(\sqrt{p_s r/2} - \sqrt{\tau_0} \right)^2 (1 - k + kr^n). \end{aligned}$$

Integrating (3.2) and using the boundary condition $u_c = u_N$ at $r = R_1(z)$

$$\begin{aligned} u_N &= u_s + \frac{p_s(R^2 - r^2)}{4}, \\ u_c &= u_s + \frac{p_s(R^2 - R_1^2)}{4} - \frac{p_s(1-k)(r^2 - R_1^2)}{4} - \tau_0(1-k)(r - R_1) \\ &\quad + \frac{2\sqrt{2p_s\tau_0}}{3}(1-k)\left(r^{\frac{3}{2}} - R_1^{\frac{3}{2}}\right) - \frac{p_s k}{2(n+2)}\left(r^{n+2} - R_1^{n+2}\right) \\ &\quad - \frac{\tau_0 k}{(n+1)}\left(r^{n+1} - R_1^{n+1}\right) + \frac{2k\sqrt{2p_s\tau_0}}{(2n+3)}\left[r^{\frac{(2n+3)}{2}} - R_1^{\frac{(2n+3)}{2}}\right]. \end{aligned}$$

The plug velocity is obtained from core velocity u_c by putting $r = R_p$. Thus, the plug velocity is given by

$$\begin{aligned} u_p &= u_s + \frac{p_s(R^2 - R_1^2)}{4} - \frac{p_s(1-k)(R_p^2 - R_1^2)}{4} - \tau_0(1-k)(R_p - R_1) \\ &\quad + \frac{2\sqrt{2p_s\tau_0}}{3}(1-k)\left(R_p^{\frac{3}{2}} - R_1^{\frac{3}{2}}\right) - \frac{p_s k}{2(n+2)}\left(R_p^{n+2} - R_1^{n+2}\right) \\ &\quad - \frac{\tau_0 k}{(n+1)}\left(R_p^{n+1} - R_1^{n+1}\right) + \frac{2k\sqrt{2p_s\tau_0}}{(2n+3)}\left[R_p^{\frac{(2n+3)}{2}} - R_1^{\frac{(2n+3)}{2}}\right]. \end{aligned}$$

In core and peripheral regions, the geometry of the stenosis for steady state flow is given by

$$\begin{aligned} R_1(z) &= \begin{cases} \gamma - \delta_c \left[1 + \cos(\pi z/2z_o)\right], & \text{if } -2z_o \leq z \leq 2z_o, \\ \gamma, & \text{otherwise,} \end{cases} \\ R(z) &= \begin{cases} 1 - \delta_N \left[1 + \cos(\pi z/2z_o)\right], & \text{if } -2z_o \leq z \leq 2z_o, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

The volumetric flow rate Q_s is given by [29]

$$\begin{aligned} Q_s &= 8 \int_0^{R(z)} r u(z, r) dr \\ &= 8 u_p(z, R_p) \int_0^{R_p} r dr + 8 \int_{R_p}^{R_1(z)} r u_c(z, r) dr + 8 \int_{R_1(z)}^{R(z)} r u_N(z, r) dr. \end{aligned}$$

The resistance to flow λ_s is given by $\lambda_s = p_s L / Q_s$, where L represents the length of the blood vessel.

4 Solutions for pulsatile flow of blood

Since Womersley parameter α is assumed to be small, expressions for velocity distribution u_c and shear stress τ_c are expressed in the form [21]

$$\begin{aligned} u_c(z, r, t) &= u_{c0}(z, r, t) + \alpha^2 u_{c1}(z, r, t) + \dots, \\ \tau_c(z, r, t) &= \tau_{c0}(z, r, t) + \alpha^2 \tau_{c1}(z, r, t) + \dots, \\ u_N(z, r, t) &= u_{N0}(z, r, t) + \alpha^2 u_{N1}(z, r, t) + \dots, \\ \tau_N(z, r, t) &= \tau_{N0}(z, r, t) + \alpha^2 \tau_{N1}(z, r, t) + \dots \end{aligned} \quad (4.1)$$

The boundary conditions in non-dimensional form are given as

$$\begin{aligned} \tau_{c0} \text{ and } \tau_{c1} \text{ are both finite at } r = 0, \\ \tau_{c0} = \tau_{N0}, \tau_{c1} = \tau_{N1}, u_{c0} = u_{N0}, u_{c1} = u_{N1} \text{ at } r = R_1(t, z), \\ u_{N0} = u_{s0}, u_{N1} = u_{s1} \text{ at } r = R(t, z). \end{aligned} \quad (4.2)$$

Putting (4.1) in Equations (2.4), (2.5) and equating the coefficients of α^0 ,

$$\frac{\partial}{\partial r}(r\tau_{c0}) = rq(z)f(t), \quad (4.3)$$

$$(\sqrt{\tau_{c0}} - \sqrt{\tau_0})^2 (1 - k + kr^n) = -\frac{\partial u_{c0}}{\partial r}, \quad (4.4)$$

$$\frac{\partial}{\partial r}(r\tau_{N0}) = rq(z)f(t), \quad (4.5)$$

$$\frac{\partial}{\partial r}(u_{N0}) = -\tau_{N0}. \quad (4.6)$$

Integrating (4.3), (4.5) and using the boundary conditions (4.2),

$$\tau_{c0} = \frac{r}{2}q(z)f(t), \quad \tau_{N0} = \frac{r}{2}q(z)f(t).$$

Integrating (4.6) and using the boundary conditions (4.2),

$$u_{N0} = u_{s0} + \frac{q(z)f(t)}{4}(R^2 - r^2). \quad (4.7)$$

Again integrating (4.4) and then using the boundary conditions (4.2),

$$\begin{aligned} u_{c0} &= u_{s0} + \frac{q(z)f(t)}{4}(R^2 - R_1^2) \\ &\quad + (\sqrt{\tau_{c0}} - \sqrt{\tau_0})^2 \left[(1 - k)(R_1 - r) + \frac{k}{n+1}(R_1^{n+1} - r^{n+1}) \right]. \end{aligned} \quad (4.8)$$

The zeroth order plug velocity can be obtained from (4.8) as $u_{p0} = u_{c0}$ at $r = R_{p0}$.

Similarly after substituting (4.1) in Equations (2.4) and (2.5) and equating the coefficients of α^2 results

$$\frac{1}{\rho_0} \frac{\partial u_{c0}}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (r\tau_{c1}), \quad (4.9)$$

$$\frac{\tau_{c1}(\sqrt{\tau_{c0}} - \sqrt{\tau_0})}{\sqrt{\tau_{c0}}} (1 - k + kr^n) = -\frac{\partial u_{c1}}{\partial r},$$

$$\frac{\partial u_{N0}}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (r\tau_{N1}), \quad (4.10)$$

$$-\frac{\partial u_{N1}}{\partial r} = \tau_{N1}. \quad (4.11)$$

Substituting (4.8) in Equation (4.9) and then integrating we get after using boundary condition (4.2)

$$\begin{aligned} \tau_{c1} = & -\frac{q(z)f'(t)}{8\rho_0} (R^2 - R_1^2) r - \frac{(1-k)q(z)f'(t)}{2\rho_0} \left(R_1 \frac{r^2}{3} - \frac{r^3}{4} \right) \\ & + \frac{(1-k)f'(t)}{\rho_0} \sqrt{\frac{\tau_0 q(z)}{2f(t)}} \left(\frac{2}{5} R_1 r^{\frac{3}{2}} - \frac{2}{7} r^{\frac{5}{2}} \right) - \frac{q(z)f'(t)k}{2(n+1)\rho_0} \left(\frac{R_1^{n+1}r^2}{3} - \frac{r^{n+3}}{n+4} \right) \\ & + \frac{kf'(t)}{(n+1)\rho_0} \sqrt{\frac{\tau_0 q(z)}{2f(t)}} \left(\frac{2}{5} R_1^{n+1} r^{\frac{3}{2}} - \frac{2}{2n+7} r^{\frac{2n+5}{2}} \right). \end{aligned}$$

Integrating (4.9) and using the boundary conditions (4.2),

$$u_{c1} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11}, \quad (4.12)$$

where the terms I_1 to I_{11} are given in Appendix A.

Integrating (4.10) and using the boundary conditions the expression of τ_{N1} is obtained in the form as

$$\begin{aligned} \tau_{N1} = & -\frac{q(z)f'(t)}{4} \left(\frac{R^2 r}{2} - \frac{r^3}{4} \right) + \frac{q(z)f'(t)}{4r} \left(\frac{R^2 R_1^2}{2} - \frac{R_1^4}{4} \right) \\ & + R_1 \left[-\frac{q(z)f'(t)}{8r\rho_0} (R^2 - R_1^2) R_1 - \frac{(1-k)q(z)f'(t)}{2r\rho_0} \left(\frac{R_1^3}{12} \right) \right. \\ & + \frac{(1-k)f'(t)}{r\rho_0} \sqrt{\frac{\tau_0 q(z)}{2f(t)}} \left(\frac{4}{35} R_1^{\frac{5}{2}} \right) - \frac{q(z)f'(t)k}{2r(n+1)\rho_0} \left(\frac{R_1^{n+3}}{3} - \frac{R_1^{n+3}}{n+4} \right) \\ & \left. + \frac{kf'(t)}{r(n+1)\rho_0} \sqrt{\frac{\tau_0 q(z)}{2f(t)}} \left(\frac{2}{5} R_1^{\frac{2n+5}{2}} - \frac{2}{2n+7} R_1^{\frac{2n+5}{2}} \right) \right]. \end{aligned}$$

Similarly by integrating (4.11) and using the boundary conditions the expres-

sion of u_{N1} is obtained in the form as

$$u_{N1} = \frac{q(z)f'(t)}{4} \left(\frac{R^2 r^2}{4} - \frac{r^4}{16} \right) + \log \left(\frac{r}{R} \right) f'(t) \left[\frac{q(z)(2R^2 - R_1^2)R_1^2}{16} \right. \\ \left. - \frac{q(z)(R^2 - R_1^2)R_1^2}{8\rho_0} - (1-k)\frac{q(z)R_1^4}{24\rho_0} + (1-k)\frac{4\sqrt{\tau_0 q(z)}R_1^{\frac{7}{2}}}{7\rho_0\sqrt{2f}} \right. \\ \left. - \frac{q(z)kR_1^{n+4}}{6\rho_0(n+4)} + k\frac{4\sqrt{\tau_0 q(z)}R_1^{\frac{2n+7}{2}}}{5\rho_0(2n+7)\sqrt{2f(t)}} \right] + u_{s1} - \frac{3q(z)f'(t)R^4}{64}. \quad (4.13)$$

The wall shear stress τ_w and the shear stress at the core layer τ_c can be obtained as

$$\tau_w = \left[\tau_{N0} + \alpha^2 \tau_{N1} \right]_{r=R}, \quad \tau_c = \left[\tau_{c0} + \alpha^2 \tau_{c1} \right]_{r=R_p}.$$

The plug velocity, u_p , for pulsatile flow is obtained as

$$u_p = u_{p0} + \alpha^2 u_{p1},$$

where u_{p0} and u_{p1} are obtained by r with R_{p0} and R_{p1} in (4.8) and (4.12), respectively.

The velocity u_c and u_N in the core and plasma layers are respectively given by

$$u_c = u_{c0} + \alpha^2 u_{c1}, \quad u_N = u_{N0} + \alpha^2 u_{N1},$$

where u_{c0} , u_{c1} , u_{N0} and u_{N1} are given in (4.8), (4.12), (4.7) and (4.13) respectively.

The volumetric flow rate for pulsatile flow, Q is thus becomes

$$Q = 8u_p \int_0^{R_p} r \, dr + \int_{R_p}^{R_1} 8r \left[(u_p|_{R_p=0}) - \tau_0(1-k)r - \frac{p_s}{4}(1-k)r^2 \right. \\ \left. + \sqrt{2p_s\tau_0}(1-k)\frac{2}{3}r^{1.5} - \tau_0\frac{k}{(n+1)}r^{n+1} - \frac{p_sk}{2(n+2)}r^{n+2} \right. \\ \left. + \frac{2k}{(2n+3)}\sqrt{2p_s\tau_0}r^{\frac{2n+3}{2}} \right] dr + \int_{R_1}^R 8 \left[\frac{p_s}{4}(R^2r - r^3) + u_sr \right] dr.$$

Resistance to flow is given by $\lambda = \frac{P_0 - P_L}{Q}$, where P_0 and P_L are pressures at $z = 0$ and $z = L$ respectively.

5 Numerical results and discussions

5.1 Analysis for steady flow

Figures 2 and 3 show the variation of the core velocity for different values of viscosity index (n) and slip velocity (u_s). It is observed that core velocity (u_c) decreases as the radial distance (r) increases with constant slip velocity. It

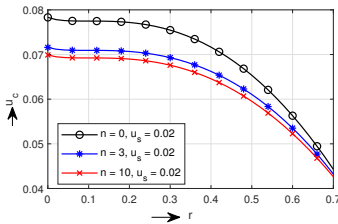


Figure 2. Core velocity profile vs radial distance for different n .

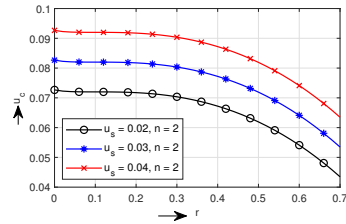


Figure 3. Core velocity profile vs radial distance for different u_s .

is also observed that core velocity (u_c) decreases as n increases, whereas the opposite effect occurs when slip velocity (u_s) increases.

Figures 4 and 5 represent the variation of plug velocity (u_p) for different viscosity indices and slip velocities. It is found that as the yield stress (τ_0) increases, the plug velocity (u_p) decreases. It is clear from the figures that plug velocity (u_p) decreases for increasing n , but the opposite phenomenon occurs when slip velocity u_s increases.

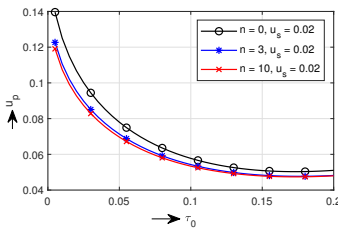


Figure 4. Plug velocity vs yield stress for different n .

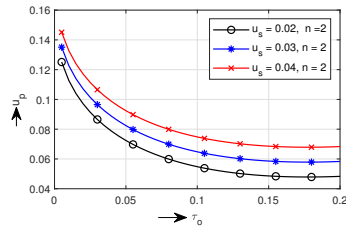


Figure 5. Plug velocity vs yield stress for different u_s .

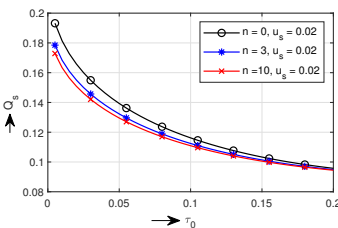


Figure 6. Volumetric flow rate vs yield stress for different n .

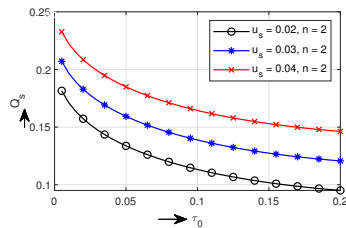


Figure 7. Volumetric flow rate vs yield stress for different u_s .

Figures 6–13 illustrate the variation of volumetric flow rate (Q_s) with changes in yield stress (τ_0) and height of the stenosis (δ_N), viscosity parameter (k) for different values of viscosity index (n), slip velocity (u_s), and pressure gradient (p_s). It is evident from the figures that the volumetric flow rate decreases as

the yield stress, stenosis height, and viscosity parameters increase; however, the pressure gradient exhibits the opposite trend. Additionally, the volumetric flow rate decreases with an increase in (n) and increases as the slip velocity increases.

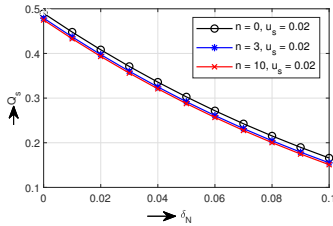


Figure 8. Volumetric flow rate vs δ_N for different n .

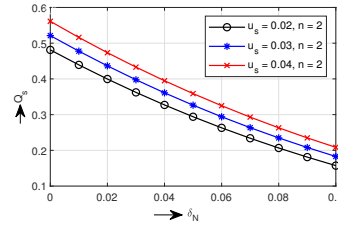


Figure 9. Volumetric flow rate vs δ_N for different u_s .

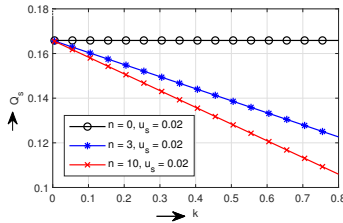


Figure 10. Volumetric flow rate vs k for different n .

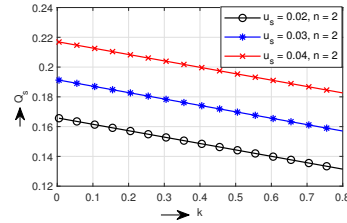


Figure 11. Volumetric flow rate vs k for different u_s .

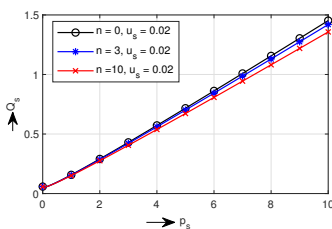


Figure 12. Volumetric flow rate vs p_s for different n .

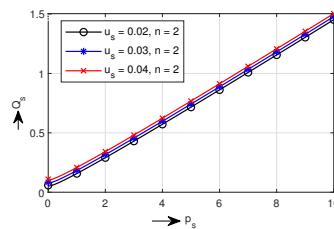


Figure 13. Volumetric flow rate vs p_s for different u_s .

Figures 14–19 depict the behavior of flow resistance with respect to variations in stenosis height (δ_N), viscosity parameter (k), and yield stress (τ_0). It has been observed that, in all instances, the flow resistance (λ_s) increases with the increased values of the above parameters. Furthermore, it is noted that the flow resistance (λ_s) increases as n increases, whereas it decreases with an increase in slip velocity (u_s).

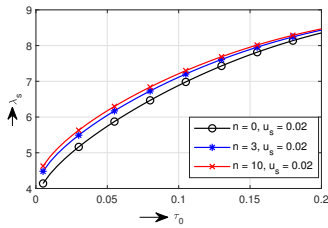


Figure 14. Resistance to flow vs τ_0 for different n .

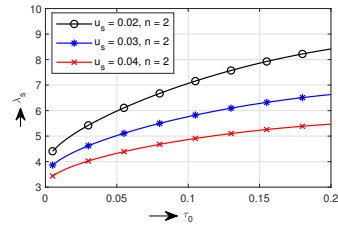


Figure 15. Resistance to flow vs τ_0 for different u_s .

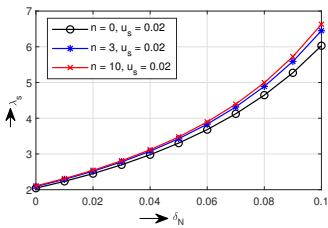


Figure 16. Resistance to flow vs δ_N for different n .

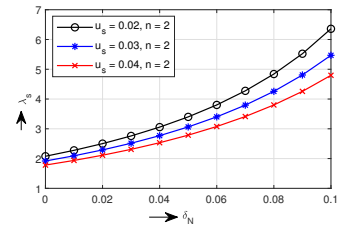


Figure 17. Resistance to flow vs δ_N for different u_s .

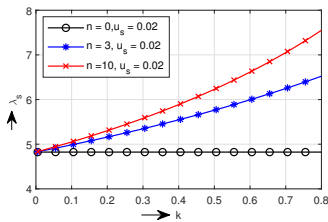


Figure 18. Resistance to flow vs k for different n .

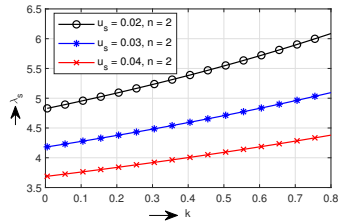


Figure 19. Resistance to flow vs k for different u_s .

5.2 Analysis for pulsatile flow

Figures 20 and 21 illustrate the variation of core velocity (u_c) in the case of pulsatile flow. It increases as the yield stress (τ_0) and the slip velocity (u_s) increase. Similar phenomena are observed regarding the variation of n and u_s , analogous to the behavior in steady flow conditions.

Figures 22–25 present the variation of plug velocity for pulsatile flow (u_p) as a function of yield stress and time for various values of n and u_s . It is observed that plug velocity increases with an increase in yield stress. However, in the case of time variation, it is found that u_p initially increases until $t = 8$, after which it gradually decreases for all permissible parameter values.

The variation of wall shear stress for pulsatile flow (τ_w) as a function of time for different values of the viscosity index (n) and the amplitude of the

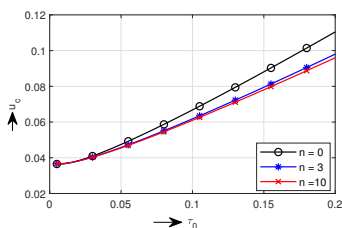


Figure 20. Core velocity profile vs τ_0 for different n .

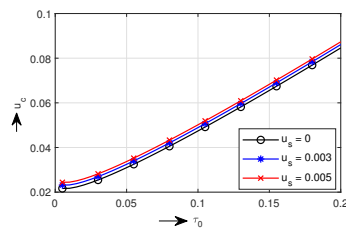


Figure 21. Core velocity profile vs τ_0 for different u_s .

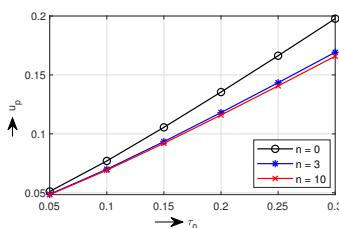


Figure 22. Plug velocity vs τ_0 for different n .

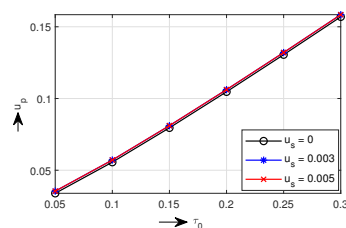


Figure 23. Plug velocity vs τ_0 for different u_s .

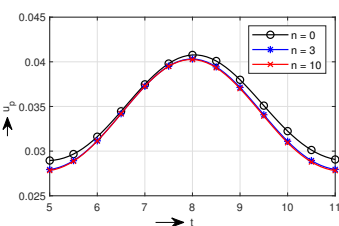


Figure 24. Plug velocity vs time for different n .

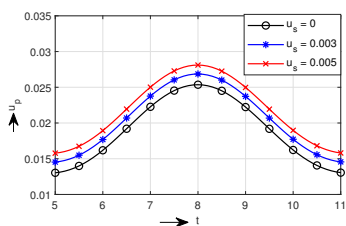


Figure 25. Plug velocity vs time for different u_s .

pressure gradient (A) is depicted in Figures 26–27. It is observed that the wall shear stress (τ_w) follows a sinusoidal variation.

In the case of pulsatile flow, the volumetric flow rate (Q) exhibits an increase with the yield stress (τ_0), as demonstrated in Figure 28. However, over time, it sustains an increase until approximately 8 seconds, after which it diminishes, as evidenced in Figures 29–31 for various values of n, k, A respectively.

Figures 32–35 illustrate the variations in flow resistance for pulsatile flow. These figures clearly demonstrate that resistance to flow decreases as yield stress increases, whereas it increases with a higher viscosity index. As time progresses, resistance to flow decreases up to a specific fixed point in time, after which it increases again for constant values of other parameters. It is observed that the flow resistance exhibits a sinusoidal pattern over time.

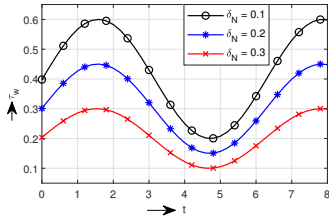


Figure 26. Wall shear stress vs time for **Figure 27.** Wall shear stress vs time for different δ_N .

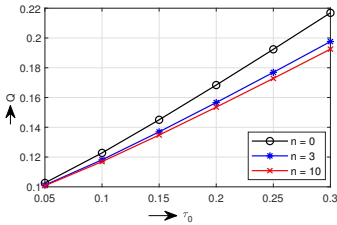
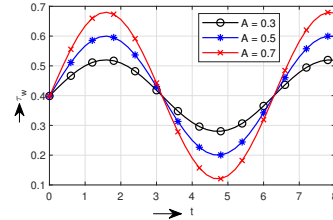


Figure 28. Volumetric flow rate Q vs τ_0 **Figure 29.** Volumetric flow rate Q vs time for different n .

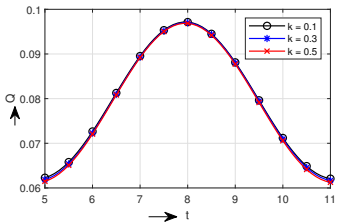
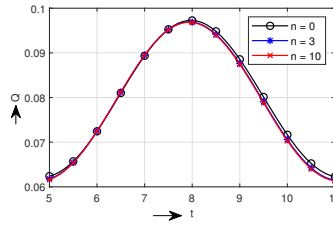


Figure 30. Volumetric flow rate Q vs τ_0 **Figure 31.** Volumetric flow rate Q vs time for different A .

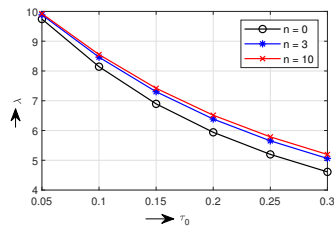
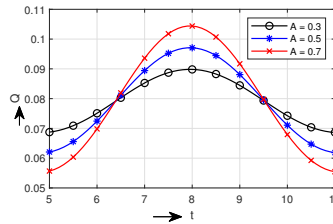


Figure 32. Resistance to flow rate λ vs τ_0 for different n .

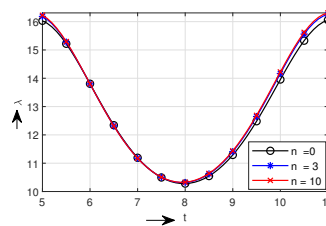


Figure 33. Resistance to flow rate λ vs time for different n .

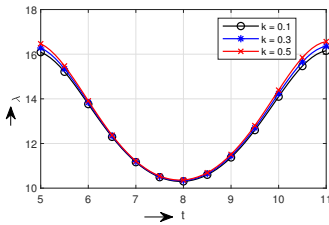


Figure 34. Resistance to flow rate λ vs time for different k .

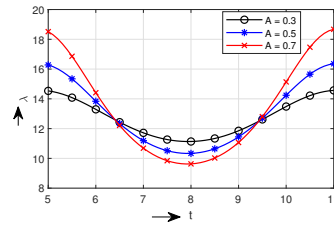


Figure 35. Resistance to flow rate λ vs time for different A .

6 Validation of results

The validation of the presented theoretical analysis has been performed by comparing the results obtained with those reported by Biswas and Chakraborty [5], who studied two-layer pulsatile blood flow in a stenosed artery under the influence of body acceleration and wall slip. The results shown in this paper closely align with the findings of Biswas et al., as illustrated in Figures 36–38. This close agreement confirms the accuracy and reliability of the proposed model. In [20], the variation of volumetric flow rate in the Rubinow and Keller model is validated experimentally. Awad et al. compared the results of their derived model with those obtained from the Rubinow and Keller model [3]. The variation of volumetric flow rate Q for different radii of blood vessels in the proposed model is compared in Table 1 with results from both the Rubinow and Keller model [20] and the Awad et al. model [3].

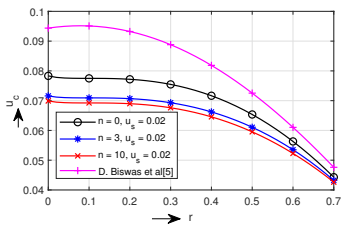


Figure 36. Validation of results for core velocity profile vs radial distance.

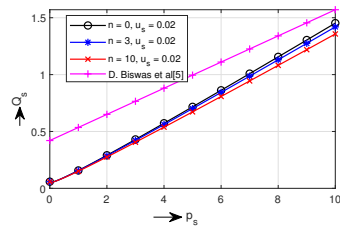


Figure 37. Validation of results for volumetric flow rate Q_s vs pressure gradient.

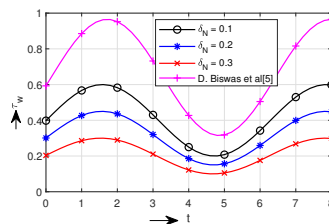


Figure 38. Validation of results for wall shear stress vs time.

Table 1. Comparison of volumetric flow rate for different radius with Rubinow and Keller model [20] and Awad et al model [3] ($n = 3$, $\tau_0 = 0.2$).

$R(Z)$	Q in Rubinow and Keller model [20]	Q in Awad et al model [3]	Q in Proposed model
0.1	0.000012	0.00000013	0.000014
0.2	0.00033	0.000013	0.00021
0.3	0.0039	0.00033	0.0010
0.4	0.028	0.0039	0.0033

7 Conclusions

The variation in blood viscosity plays a crucial role in various cardiovascular diseases because it influences blood flow in constricted arteries. However, there is currently no method available to measure blood viscosity for treating many arterial disorders. Slip velocity also impacts blood flow in a stenosed artery by significantly reducing hydrodynamic resistance, pressure drop, and wall shear stress, while simultaneously increasing the flow rate. This paper presents a mathematical model that accounts for both variable viscosity and slip velocity to examine their effects on flow rate, resistance to flow, and wall shear stress of blood flowing through an artery with a time-dependent stenosis. A two-layer blood flow model is considered, comprising a Newtonian fluid for the outer, cell-free plasma layer and a Casson-type non-Newtonian fluid for the inner core layer containing red blood cells, as this model more accurately characterizes blood properties within a stenosed artery.

It has been observed that improving blood flow requires a reduction in blood viscosity. Additionally, blood flow can be enhanced (under any arterial occlusion) by increasing slip velocity. The results of the proposed model are compared with those reported in the literature earlier, ensuring the validity of the proposed model. Therefore, the present work may be helpful for better treatment of any arterial diseases.

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Appendix A

The terms of (4.12) are expressed as

$$\begin{aligned}
 I_1 &= q(z)f'(t) \frac{(R^2 - R_1^2)}{8\rho_0} \left[\frac{(1-k)r^2}{2} + \frac{kr^{n+2}}{n+2} \right. \\
 &\quad \left. - \frac{2}{3}(1-k) \frac{\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} r^{\frac{3}{2}} - 2 \frac{k\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \frac{r^{n+1.5}}{2n+3} \right], \\
 I_2 &= \frac{q(z)f'(t)(1-k)}{2\rho_0} \left[\frac{(1-k)R_1r^3}{9} + \frac{kR_1r^{n+3}}{3n+9} - \frac{(1-k)r^4}{16} - \frac{kr^{n+4}}{4n+16} \right. \\
 &\quad \left. - \frac{(1-k)\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{2R_1r^{2.5}}{15} - \frac{r^{3.5}}{14} \right) - \frac{k\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{2R_1r^{n+2.5}}{6n+15} - \frac{r^{n+3.5}}{4n+14} \right) \right], \\
 I_3 &= \frac{(k-1)f'(t)\sqrt{\tau_0 q(z)}}{\rho_0\sqrt{2f(t)}} \left[(1-k) \left(\frac{4R_1r^{2.5}}{25} - \frac{4r^{3.5}}{49} \right) \right. \\
 &\quad + k \left(\frac{4R_1r^{n+2.5}}{10n+25} - \frac{4r^{n+3.5}}{14n+49} \right) - \frac{(1-k)\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{R_1r^2}{5} - \frac{2r^3}{21} \right) \\
 &\quad \left. - \frac{k\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{2R_1r^{n+2}}{5n+10} - \frac{2r^{n+3}}{7n+21} \right) \right], \\
 I_4 &= \frac{q(z)f'(t)k}{\rho_0(2n+2)} \left[(1-k) \left(\frac{R_1^{n+1}r^3}{9} - \frac{r^{n+4}}{(n+4)^2} \right) \right. \\
 &\quad + k \left(\frac{R_1^{n+1}r^{n+3}}{3n+9} - \frac{r^{2n+4}}{(n+4)(2n+4)} \right) \\
 &\quad - (1-k) \frac{\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{2R_1^{n+1}r^{2.5}}{15} - \frac{2r^{n+3.5}}{(n+4)(2n+7)} \right) \\
 &\quad \left. - \frac{k\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{2R_1^{n+1}r^{n+2.5}}{6n+15} - \frac{2r^{2n+3.5}}{(n+4)(4n+7)} \right) \right], \\
 I_5 &= -\frac{kf'(t)\sqrt{q(z)\tau_0}}{(n+1)\rho_0\sqrt{2f(t)}} \left[(1-k) \left(\frac{4R_1^{n+1}r^{2.5}}{25} - \frac{4r^{n+3.5}}{(2n+7)^2} \right) \right. \\
 &\quad + k \left(\frac{4R_1^{n+1}r^{n+2.5}}{10n+25} - \frac{4r^{2n+3.5}}{(2n+7)(4n+7)} \right) \\
 &\quad - (1-k) \frac{\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{R_1^{n+1}r^2}{5} - \frac{2r^{n+3}}{(2n+7)(n+3)} \right) \\
 &\quad \left. - \frac{k\sqrt{2\tau_0}}{\sqrt{q(z)f(t)}} \left(\frac{2R_1^{n+1}r^{n+2}}{5n+10} - \frac{2r^{2n+3}}{(2n+7)(2n+3)} \right) \right],
 \end{aligned}$$

$$\begin{aligned}
I_6 &= \log \left(\frac{R_1}{R} \right) q(z) f'(t) \left[\frac{(2R^2 - R_1^2) R_1^2}{16} - \frac{(R^2 - R_1^2) R_1^2}{8\rho_0} - (1-k) \frac{R_1^4}{24\rho_0} \right. \\
&\quad \left. + (1-k) \frac{4\sqrt{\tau_0} R_1^{3.5}}{7\rho_0 \sqrt{2f(t)q(z)}} - \frac{k R_1^{n+4}}{6\rho_0(n+4)} + k \frac{4\sqrt{\tau_0} R_1^{n+3.5}}{5\rho_0(2n+7) \sqrt{2f(t)q(z)}} \right], \\
I_7 &= \frac{q(z) f'(t)}{4} \left(\frac{R^2 R_1^2}{4} - \frac{R_1^4}{16} - \frac{3R^4}{16} \right) + u_{s1} - q(z) f'(t) \left[\frac{(1-k) R_1^2}{2} \right. \\
&\quad \left. + \frac{k R_1^{n+2}}{n+2} - (1-k) \frac{\sqrt{2\tau_0} 2R_1^{1.5}}{3\sqrt{q(z)f(t)}} - k \frac{\sqrt{2\tau_0} 2R_1^{n+1.5}}{(2n+3)\sqrt{q(z)f(t)}} \right] \frac{(R^2 - R_1^2)}{8\rho_0}, \\
I_8 &= \frac{q(z) f'(t)(k-1)}{2\rho_0} \left[7(1-k) \frac{R_1^4}{144} + \frac{k(n+7) R_1^{n+4}}{12(n+3)(n+4)} \right. \\
&\quad \left. - (1-k) \frac{\sqrt{2\tau_0} 13R_1^{3.5}}{210\sqrt{q(z)f(t)}} - k \frac{\sqrt{2\tau_0} (2n+13) R_1^{n+3.5}}{6(2n+5)(2n+7)\sqrt{q(z)f(t)}} \right], \\
I_9 &= \frac{(1-k) f'(t) \sqrt{\tau_0 q(z)}}{\rho_0 \sqrt{2f(t)}} \left[(1-k) \frac{96R_1^{3.5}}{1225} + k \frac{16(n+6) R_1^{n+3.5}}{35(2n+5)(2n+7)} \right. \\
&\quad \left. - (1-k) \frac{\sqrt{2\tau_0} 11R_1^3}{105\sqrt{q(z)f(t)}} - k \frac{\sqrt{2\tau_0} 2(2n+11) R_1^{n+3}}{35(n+2)(n+3)\sqrt{q(z)f(t)}} \right], \\
I_{10} &= -\frac{q(z) f'(t) k}{2\rho_0(n+4)} \left[(1-k) \frac{(n+7) R_1^{n+4}}{9(n+4)} + k \frac{(2n+7) R_1^{2n+4}}{6(n+2)(n+3)} \right. \\
&\quad \left. - (1-k) \frac{\sqrt{2\tau_0} 2(2n+13) R_1^{n+3.5}}{15(2n+7)\sqrt{q(z)f(t)}} - k \frac{\sqrt{2\tau_0} 2(4n+13) R_1^{2n+3.5}}{3(2n+5)(4n+7)\sqrt{q(z)f(t)}} \right], \\
I_{11} &= \frac{k f'(t) \sqrt{\tau_0 q(z)}}{(2n+7)\rho_0 \sqrt{2f(t)}} \left[(1-k) \frac{16(n+6) R_1^{n+3.5}}{25(2n+7)} + k \frac{32(n+3) R_1^{2n+3.5}}{5(2n+5)(4n+7)} \right. \\
&\quad \left. - (1-k) \frac{\sqrt{2\tau_0} (2n+11) R_1^{n+3}}{5(n+3)\sqrt{q(z)f(t)}} - k \frac{\sqrt{2\tau_0} 2(4n+11) R_1^{2n+3}}{5(n+2)(2n+3)\sqrt{q(z)f(t)}} \right].
\end{aligned}$$