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Analysis of the Darcy-Brinkman flow with viscous dissipation and non-homogeneous thermal boundary condition

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Abstract. This study investigates the steady-state Darcy-Brinkman flow within a thin, saturated porous domain, focusing on the effects of viscous dissipation and non-homogeneous boundary condition for the temperature. Employing asymptotic techniques with respect to the domain's thickness, we rigorously derive the simplified coupled model describing the fluid flow. The mathematical analysis is based on deriving the sharp a priori estimates and proving the compactness results of the rescaled functions. The resulting limit model incorporates contributions of viscous dissipation and thermal boundary conditions and thus could prove useful in the engineering applications involving porous media.

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1 Introduction

The Darcy-Brinkman model [4] serves as a main framework for analyzing fluid flow through porous media, particularly when accounting for both Darcy resistance and viscous shear effects. This model extends Darcy's law [7] by incorporating a viscous term, thereby enabling the study of flows in media with moderate to high permeability where shear effects are non-negligible. The usage of the Brinkman's extension of the Darcy law has been justified in numerous works, see e.g., [2, 15, 18, 22].

In thin-domain geometries where one dimension is significantly smaller than the others, the flow behavior exhibits unique characteristics due to the pronounced influence of boundary layers and the confinement of the flow. We

refer the reader to monograph [19] and the references therein. Such configu-
rations naturally appear in engineering applications like microfluidic devices
and lubrication systems (see e.g. [23, 24]). The analysis of non-isothermal flow
in these domains necessitates careful consideration of the interplay between
viscous forces and thermal effects, see e.g. [3, 17, 21].

Viscous dissipation, the process by which mechanical energy is converted
into thermal energy due to viscous forces, frequently plays an important role
in the thermal analysis of fluid flows. In porous media, this effect can lead
to significant temperature elevations, especially under conditions of high flow
velocities or low thermal conductivity. Many practical applications exhibit this
phenomenon, starting from geological processes (petroleum reserves, geother-
mal reservoirs) to industrial applications (catalytic reactors, porous journal
bearings). Studies have shown that viscous dissipation can substantially alter
temperature distributions, and we refer to [16] for the overview of the obtained
results.

The aim of the present paper is to analyze the Darcy-Brinkman system
given by

$$\begin{cases} 2\mu_{\mathrm{eff}}\mathrm{div}(\mathbb{D}[\mathbf{u}]) - \dfrac{\mu}{K}\mathbf{u} = \nabla p - \mathbf{f}, \\ \qquad\qquad\qquad \mathrm{div}(\mathbf{u}) = 0, \end{cases} \tag{1.1}$$

in a three-dimensional thin domain

$$\Omega^{\varepsilon} = \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, \ 0 < x_3 < \varepsilon h\left(x'\right)\}, \quad 0 < \varepsilon \ll 1.$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the filter velocity, p is the pressure, $\mathbf{f} = (f_1, f_2, f_3)$ is the
momentum source term, μ is the dynamic viscosity coefficient, μ_{eff} denotes the
effective viscosity of the Brinkman term, while K stands for the permeability
of the porous medium.

To obtain the thermodynamic closure of the Darcy-Brinkman model, we
couple (1.1) with the heat equation

$$-k\Delta T = \Phi(\mathbf{u}, \mu, \mu_{\mathrm{eff}}, K).$$

Here T is the temperature, k is the thermal conductivity, whereas the viscous
dissipation function Φ is defined by

$$\Phi(\mathbf{u}, \mu, \mu_{\mathrm{eff}}, K) = \frac{\mu}{K}|\mathbf{u}|^2 + 2\mu_{\mathrm{eff}}|\mathbb{D}[\mathbf{u}]|^2. \tag{1.2}$$

The formula (1.2), proposed by Al-Hadhrami et al. [1], ensures the correct
asymptotic behavior for a wide range of permeability values K and, thus, has
been widely accepted when the Brinkman second-order term appears in $(1.1)_1$.
The first term in (1.2) results from the internal heating needed to extrude the
fluid through the porous medium (Darcy dissipation), while the second term
comes from the frictional heating due to dissipation.

The analytical investigations of the viscous dissipation effects in porous
media flow are rather sparse throughout the literature. One can mostly find
the analysis for 2D channel flows with only Darcy dissipation appearing in (1.2)
($K \to 0$), see e.g., [12, 13, 14]. A rigorous derivation of an asymptotic model for

a 3D thin domain involving viscous dissipation term (1.2) has been provided in [20]. The approach is based on the multiscale expansion technique and the abstract result from [6] derived for the purpose of the thin-film flow through a domain with no porous structure inside.

In the above mentioned works, a simple zero boundary condition for the temperature has been imposed on the domain boundary. However, from the point of view of the applications, it is natural to allow the variations of the heat flux along the boundary. This leads to the presence of a non-homogeneous boundary condition for the temperature further complicating the analysis. In fact, such condition force us to change the modeling technique in order to capture the resulting thermal gradients and their impact on the overall flow. Following the approach from [5], the key idea is to the derive the sharp a priori estimates using the decomposition of the pressure (see Section 3) and prove the compactness results for the rescaled functions (see Section 4). Consequently, we are in position to pass to the limit in the non-linear term on the right-hand side of the temperature equation, previously ensuring the strong convergence of the velocity. As a main result formulated in Theorem 1, we obtain the homogenized model maintaining at the limit both the effects of viscous dissipation and thermal boundary condition and that represents our main contribution. By rigorously deriving a mathematical model that captures these complexities, we provide insights into the thermal and flow behaviors in such systems, hopefully contributing to the known engineering practice involving porous media.

2   Preliminaries and setting of the problem

As indicated in the Introduction, we consider a thin domain defined by

$$\Omega^{\varepsilon} = \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, \ 0 < x_3 < h_{\varepsilon}(x')\},$$

where the bottom of the fluid domain $\omega \subset \mathbb{R}^2$ has a Lipschitz boundary. The small parameter of the problem is ε and $h_{\varepsilon}(x') = \varepsilon h\left(x'\right)$ represents the real gap between the two surfaces. h is a $W^{1,\infty}$ function such that $0 < h_{\min} \leq h(x') < h_{\max}$ for all $(x', 0) \in \omega$.

The bottom, top and lateral boundaries of Ω^{ε} are respectively given by

$$\Gamma_0 = \left\{x \in \mathbb{R}^3 : x' \in \omega, \ x_3 = 0\right\}, \quad \Gamma_1^{\varepsilon} = \left\{x \in \mathbb{R}^3 : x' \in \omega, \ x_3 = h_{\varepsilon}(x')\right\},$$
$$\Gamma_{\ell}^{\varepsilon} = \partial \Omega^{\varepsilon} \setminus (\Gamma_0 \cup \Gamma_1^{\varepsilon}).$$

We define the rescaled sets, after a dilatation of the vertical variable, as

$$\Omega = \{z \in \mathbb{R}^3 : z' \in \omega, \ 0 < z_3 < h(z')\}, \ \Gamma_1 = \left\{z \in \mathbb{R}^3 : z' \in \omega, \ z_3 = h(z')\right\},$$
$$\Gamma_{\ell} = \partial \Omega \setminus (\Gamma_0 \cup \Gamma_1).$$

We denote by C a generic constant which can change from line to line.
In the sequel, we introduce the following notation. Let us consider a vector function $\mathbf{v} = (\mathbf{v}', v_3)$ with $\mathbf{v}' = (v_1, v_2)$ defined in Ω^{ε}. We have denoted by

$\mathbb{D} : \mathbb{R}^3 \to \mathbb{R}^3_{\text{sym}}$ the symmetric part of the velocity gradient, that is

$$
\mathbb{D}[\mathbf{v}] = \begin{pmatrix} \partial_{x_1} v_1 & \frac{1}{2}(\partial_{x_1} v_2 + \partial_{x_2} v_1) & \frac{1}{2}(\partial_{x_3} v_1 + \partial_{x_1} v_3) \\ \frac{1}{2}(\partial_{x_1} v_2 + \partial_{x_2} v_1) & \partial_{x_2} v_2 & \frac{1}{2}(\partial_{x_3} v_2 + \partial_{x_2} v_3) \\ \frac{1}{2}(\partial_{x_3} v_1 + \partial_{x_1} v_3) & \frac{1}{2}(\partial_{x_3} v_2 + \partial_{x_2} v_3) & \partial_{x_3} v_3 \end{pmatrix}.
$$

Moreover, for $\widetilde{\mathbf{v}} = (\widetilde{\mathbf{v}}', \widetilde{v}_3)$ a vector function and $\widetilde{\varphi}$ a scalar function, both defined in Ω, obtained from $\mathbf{v}$ and φ after a dilatation in the vertical variable, respectively, we will use the following operators

$$
\Delta_\varepsilon \widetilde{\mathbf{v}} = \Delta_{x'} \widetilde{\mathbf{v}} + \varepsilon^{-2} \partial^2_{z_3} \widetilde{\mathbf{v}}, \quad \Delta_\varepsilon \widetilde{\varphi} = \Delta_{x'} \widetilde{\varphi} + \varepsilon^{-2} \partial^2_{z_3} \widetilde{\varphi},
$$
$$
(D_\varepsilon \widetilde{\mathbf{v}})_{ij} = \partial_{x_j} \widetilde{v}_i, \quad i = 1,2,3, \ j = 1,2, \quad (D_\varepsilon \widetilde{\mathbf{v}})_{i3} = \varepsilon^{-1} \partial_{z_3} \widetilde{v}_i, \quad i = 1,2,3,
$$
$$
\nabla_\varepsilon \widetilde{\varphi} = (\nabla_{x'} \widetilde{\varphi}, \varepsilon^{-1} \partial_{z_3} \widetilde{\varphi})^t, \quad \text{div}_\varepsilon(\widetilde{\mathbf{v}}) = \text{div}_{x'}(\widetilde{\mathbf{v}}') + \varepsilon^{-1} \partial_{z_3} \widetilde{v}_3.
$$

Moreover, we define $\mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}] = \mathbb{D}_{x'}[\widetilde{\mathbf{v}}] + \varepsilon^{-1} \partial_{z_3}[\widetilde{\mathbf{v}}]$ as follows

$$
\mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}] = \begin{pmatrix} \partial_{x_1} \widetilde{v}_1 & \frac{1}{2}(\partial_{x_1} \widetilde{v}_2 + \partial_{x_2} \widetilde{v}_1) & \frac{1}{2}(\partial_{x_1} \widetilde{v}_3 + \varepsilon^{-1} \partial_{z_3} \widetilde{v}_1) \\ \frac{1}{2}(\partial_{x_1} \widetilde{v}_2 + \partial_{x_2} \widetilde{v}_1) & \partial_{x_2} \widetilde{v}_2 & \frac{1}{2}(\partial_{x_2} \widetilde{v}_3 + \varepsilon^{-1} \partial_{z_3} \widetilde{v}_2) \\ \frac{1}{2}(\partial_{x_1} \widetilde{v}_3 + \varepsilon^{-1} \partial_{z_3} \widetilde{v}_1) & \frac{1}{2}(\partial_{x_2} \widetilde{v}_3 + \varepsilon^{-1} \partial_{z_3} \widetilde{v}_2) & \varepsilon^{-1} \partial_{z_3} \widetilde{v}_3 \end{pmatrix},
$$

where $\mathbb{D}_{x'}[\widetilde{\mathbf{v}}]$ and $\partial_{z_3}[\widetilde{\mathbf{v}}]$ are respectively defined by

$$
\mathbb{D}_{x'}[\mathbf{v}] = \begin{pmatrix} \partial_{x_1} v_1 & \frac{1}{2}(\partial_{x_1} v_2 + \partial_{x_2} v_1) & \frac{1}{2}\partial_{x_1} v_3 \\ \frac{1}{2}(\partial_{x_1} v_2 + \partial_{x_2} v_1) & \partial_{x_2} v_2 & \frac{1}{2}\partial_{x_2} v_3 \\ \frac{1}{2}\partial_{x_1} v_3 & \frac{1}{2}\partial_{x_2} v_3 & 0 \end{pmatrix},
$$
$$
\partial_{z_3}[\mathbf{v}] = \begin{pmatrix} 0 & 0 & \frac{1}{2}\partial_{z_3} v_1 \\ 0 & 0 & \frac{1}{2}\partial_{z_3} v_2 \\ \frac{1}{2}\partial_{z_3} v_1 & \frac{1}{2}\partial_{z_3} v_2 & \partial_{z_3} v_3 \end{pmatrix}. \tag{2.1}
$$

We also define the following operators applied to $\mathbf{v}'$:

$$
\mathbb{D}_{x'}[\mathbf{v}'] = \begin{pmatrix} \partial_{x_1} v_1 & \frac{1}{2}(\partial_{x_1} v_2 + \partial_{x_2} v_1) & 0 \\ \frac{1}{2}(\partial_{x_1} v_2 + \partial_{x_2} v_1) & \partial_{x_2} v_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$
$$
\partial_{z_3}[\mathbf{v}'] = \begin{pmatrix} 0 & 0 & \frac{1}{2}\partial_{z_3} v_1 \\ 0 & 0 & \frac{1}{2}\partial_{z_3} v_2 \\ \frac{1}{2}\partial_{z_3} v_1 & \frac{1}{2}\partial_{z_3} v_2 & 0 \end{pmatrix}. \tag{2.2}
$$

2.1 Setting of the problem

As explained in the Introduction, we assume that the fluid flow is governed by the Darcy-Brinkman system coupled with the heat equation containing the

viscous dissipation term, namely:

$$\begin{cases} -2\mu_{\text{eff}}\,\text{div}(\mathbb{D}[\mathbf{u}^\varepsilon]) + \dfrac{\mu}{K_\varepsilon}\mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}^\varepsilon & \text{in } \varOmega^\varepsilon, \\[2mm] \text{div}(\mathbf{u}^\varepsilon) = 0 & \text{in } \varOmega^\varepsilon, \\[2mm] -k\varDelta T^\varepsilon = \dfrac{\mu}{K_\varepsilon}|\mathbf{u}^\varepsilon|^2 + 2\mu_{\text{eff}}|\mathbb{D}[\mathbf{u}^\varepsilon]|^2 & \text{in } \varOmega^\varepsilon. \end{cases} \qquad (2.3)$$

Here, the superscript ε is added to stress the dependence of the solution on the small parameter. We impose the standard no-slip boundary condition for the velocity, and allow the heat flux across the bottom of the fluid domain. In view of that, the system (2.3) is endowed with the following boundary conditions:

$$\mathbf{u}^\varepsilon = 0, \quad T^\varepsilon = 0 \quad \text{on } \varGamma_1^\varepsilon \cup \varGamma_\ell^\varepsilon, \qquad (2.4)$$

$$\mathbf{u}^\varepsilon = 0, \quad k\frac{\partial T^\varepsilon}{\partial n} = b^\varepsilon \quad \text{on } \varGamma_0. \qquad (2.5)$$

In addition, we make the following assumptions on the given data:

– We assume the following scaling of the parameter K^ε with respect to the small parameter ε (see [20]):

$$K^\varepsilon = \varepsilon^2 K, \quad \text{with} \quad K \in \mathbb{R}.$$

– We assume that the external source functions are independent of the variable x_3 and take the following scaling (see [3, 21]):

$$\mathbf{f}^\varepsilon = \varepsilon^{-2}(\mathbf{f}'(x'), 0), \quad \text{with} \quad f' \in L^2(\omega)^2.$$

– We assume the following scaling of the function b^ε with respect to the small parameter ε (see [3]):

$$b^\varepsilon = \varepsilon^{-1}b \quad \text{with} \quad b = \mathcal{O}(1). \qquad (2.6)$$

Remark 1. Imposing non-homogeneous Neumann boundary condition $(2.5)_2$ is physically relevant as it is directly motivated by the practical applications, where the heat flux across the boundary naturally appears. Engineering applications cover a large number of devices such as heat exchangers, chemical reactors, particle separators in the mineral processing industry (see e.g., [11]), etc. Based on the results from [3], one should expect that the magnitude of the right-hand side function b^ε plays an important with regards to the effective behavior of the flow. Choosing the scaling for b^ε as in (2.6) preserves the effects of the heat exchange in the limit model, as explained in Section 4 (Remark 3). Finally, it should be emphasized that the presented framework could be extended to a setting where the thermal boundary condition $(2.5)_2$ is described by the Robin boundary condition coming from the Newton cooling law (see e.g., [17]).

We consider the following functional framework on Ω^ε, where $1 < r < +\infty$ with $1/r + 1/r^* = 1$:

$$W^{1,r}_{\Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon}(\Omega^\varepsilon) = \{\psi \in W^{1,r}(\Omega^\varepsilon) : \psi = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon\},$$

$$L_0^r(\Omega^\varepsilon) = \Big\{\psi \in L^{r'}(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} \psi\, dx = 0\Big\}.$$

The weak formulation associated to (2.3)–(2.5) is obtained by multiplying $(2.3)_1$ by $\mathbf{v} \in H_0^1(\Omega^\varepsilon)^3$, $(2.3)_2$ by $\psi \in H_0^1(\Omega^\varepsilon)$ and $(2.3)_3$ by $\zeta \in W^{1,q^*}_{\Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon}$, $q^* = q/(q-1)$ with $q \in (1, 3/2)$, respectively, and formally integrating these identities over Ω^ε to get the following variational form:
Find $\mathbf{u}^\varepsilon \in H_0^1(\Omega^\varepsilon)^3$, $p^\varepsilon \in L_0^2(\Omega^\varepsilon)$ and $T^\varepsilon \in W^{1,q}_{\Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon}(\Omega^\varepsilon)$, with $q \in (1, 3/2)$, such that

$$\begin{cases} 2\mu_{\text{eff}} \displaystyle\int_{\Omega^\varepsilon} \mathbb{D}[\mathbf{u}^\varepsilon] : \mathbb{D}[\mathbf{v}]\, dx + \frac{\mu}{K}\varepsilon^{-2} \int_{\Omega^\varepsilon} \mathbf{u}^\varepsilon \cdot \mathbf{v}\, dx - \int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div}(\mathbf{v})\, dx \\ \qquad = \varepsilon^{-2} \displaystyle\int_{\Omega^\varepsilon} \mathbf{f}' \cdot \mathbf{v}'\, dx, \\ \displaystyle\int_{\Omega^\varepsilon} \mathbf{u}^\varepsilon \cdot \nabla\psi\, dx = 0, \\ k \displaystyle\int_{\Omega^\varepsilon} \nabla T^\varepsilon \cdot \nabla\zeta\, dx = \frac{\mu}{K}\varepsilon^{-2} \int_{\Omega^\varepsilon} |\mathbf{u}^\varepsilon|^2 \zeta\, dx + 2\mu_{\text{eff}} \int_{\Omega^\varepsilon} |\mathbb{D}[\mathbf{u}^\varepsilon]|^2 \zeta\, dx \\ \qquad + \displaystyle\int_{\Gamma_0} \varepsilon^{-1} b\, \zeta\, d\sigma, \end{cases} \tag{2.7}$$

for every $\mathbf{v} \in H_0^1(\Omega^\varepsilon)^3$, $\psi \in H_0^1(\Omega^\varepsilon)$ and $\zeta \in W^{1,q^*}_{\Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon}(\Omega^\varepsilon)$.

We observe that $q^* = 1/(q-1) > 3$, following [10], by Sobolev inequalities, $\zeta \in L^\infty(\Omega^\varepsilon)$ and the right-hand side of $(2.7)_3$ make sense.

The well-posedness of the described problem (2.3)–(2.5) can be established by adapting the proof from [3, Theorem 1] and [8, Chapitre 2, Théorème 2.2] (see also [6, Theorem 2.4]). Thus, based on these references, the system (2.3)–(2.5) admits at least one solution $(\mathbf{u}^\varepsilon, p^\varepsilon, T^\varepsilon) \in H_0^1(\Omega^\varepsilon)^3 \times L_0^2(\Omega^\varepsilon) \times W^{1,q}_{\Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon}(\Omega^\varepsilon)$ with $q \in (1, 3/2)$. We remark that the proof of well-posedness of (2.3)–(2.5) follows from [3, Theorem 1] (and also in [8, Chapitre 2, Théorème 2.2]) by changing the Tresca boundary condition on ω for no-slip condition, which is now simpler. Concerning [6, Theorem 2.4], the adaptation is to change the non-homogeneous Dirichlet boundary condition on the bottom for the no-slip condition.

In view of that, our goal in the present paper is to rigorously derive the effective model describing the asymptotic behavior of the process governed by (2.3)–(2.6). To accomplish that, we use the dilatation in the variable x_3 given by

$$z_3 = x_3/\varepsilon\,, \tag{2.8}$$

in order to have the functions defined in the open set independent of ε, denoted by Ω, and on the rescaled boundaries Γ_1 and Γ_ℓ. Consequently, the system (2.3)

becomes:

$$
\begin{cases}
-2\mu_{\text{eff}}\text{div}_\varepsilon(\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]) + \dfrac{\mu}{K}\varepsilon^{-2}\widetilde{\mathbf{u}}^\varepsilon + \nabla_\varepsilon\widetilde{p}^\varepsilon = \mathbf{f}^\varepsilon & \text{in } \Omega, \\
\hspace{6.5cm} \text{div}_\varepsilon(\widetilde{\mathbf{u}}^\varepsilon) = 0 & \text{in } \Omega, \\
-k\varDelta_\varepsilon\widetilde{T}^\varepsilon = \dfrac{\mu}{K}\varepsilon^{-2}|\widetilde{\mathbf{u}}^\varepsilon|^2 + 2\mu_{\text{eff}}|\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2 & \text{in } \Omega,
\end{cases}
\tag{2.9}
$$

and the following boundary conditions:

$$
\begin{aligned}
\widetilde{\mathbf{u}}^\varepsilon &= 0, \quad \widetilde{T}^\varepsilon = 0 \quad \text{on } \varGamma_1 \cup \varGamma_\ell, \\
\widetilde{\mathbf{u}}^\varepsilon &= 0, \quad k\nabla_\varepsilon\widetilde{T}^\varepsilon \cdot n = \varepsilon^{-1}b \quad \text{on } \varGamma_0.
\end{aligned}
\tag{2.10}
$$

The unknown functions in the above system are given by $\widetilde{\mathbf{u}}^\varepsilon(x', z_3) = \mathbf{u}^\varepsilon(x', \varepsilon z_3)$, $\widetilde{p}^\varepsilon(x', z_3) = p^\varepsilon(x', \varepsilon z_3)$ and $\widetilde{T}^\varepsilon(x', z_3) = T^\varepsilon(x', \varepsilon z_3)$ for a.e. $(x', z_3) \in \Omega$. The weak variational formulation of (2.9)–(2.10) now reads:

Find $\widetilde{\mathbf{u}}^\varepsilon \in H_0^1(\Omega)^3$, $\widetilde{p}^\varepsilon \in L_0^2(\Omega)$ and $\widetilde{T}^\varepsilon \in W_{\varGamma_1}^{1,q}(\Omega)$ with $q \in (1, 3/2)$, such that

$$
\begin{cases}
2\mu_{\text{eff}} \displaystyle\int_\Omega \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon] : \mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}]\, dx'dz_3 + \dfrac{\mu}{K}\varepsilon^{-2} \int_\Omega \widetilde{\mathbf{u}}^\varepsilon \cdot \widetilde{\mathbf{v}}\, dx'z_3 \\
\quad - \int_\Omega \widetilde{p}^\varepsilon \, \text{div}_\varepsilon(\widetilde{\mathbf{v}})\, dx'dz_3 = \displaystyle\int_\Omega \varepsilon^{-2}\mathbf{f}' \cdot \widetilde{\mathbf{v}}'\, dx'dz_3, \\
\displaystyle\int_\Omega \widetilde{\mathbf{u}}^\varepsilon \cdot \nabla_\varepsilon\widetilde{\psi}\, dx'dz_3 = 0, \\
\varepsilon^2 \displaystyle\int_\Omega k\nabla_\varepsilon\widetilde{T}^\varepsilon \cdot \nabla_\varepsilon\widetilde{\zeta}\, dx'dz_3 = \dfrac{\mu}{K}\int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2\widetilde{\zeta}\, dx'dz_3 \\
\quad + 2\mu_{\text{eff}}\varepsilon^2 \int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2\widetilde{\zeta}\, dx'dz_3 + \int_\omega b\,\widetilde{\zeta}\, dx',
\end{cases}
\tag{2.11}
$$

where $\widetilde{\mathbf{v}}, \widetilde{\psi}$ and $\widetilde{\zeta}$ are obtained respectively from $\mathbf{v}, \psi$ and ζ given in (2.7) by using the change of variable (2.8). Here, the spaces are the following

$$
\begin{aligned}
W_{\varGamma_1 \cup \varGamma_\ell}^{1,r}(\Omega) &= \{\widetilde{\psi} \in W^{1,r}(\Omega) \,:\, \psi = 0 \text{ on } \varGamma_1 \cup \varGamma_\ell\}, \\
L_0^r(\Omega) &= \Big\{\psi \in L^{r'}(\Omega) \,:\, \int_\Omega \widetilde{\psi}\, dx'dz_3 = 0\Big\}.
\end{aligned}
$$

Now, we aim to describe the asymptotic behavior of this new sequences $\widetilde{\mathbf{u}}^\varepsilon$, $\widetilde{p}^\varepsilon$ and $\widetilde{T}^\varepsilon$, as ε tends to zero.

3   A priori estimates

3.1   Estimates for velocity and temperature

To derive the desired estimates, let us recall a well-known technical result (see, e.g., [3]).

Lemma 1 [Poincaré's and Korn's inequalities]. *For all $\mathbf{v} \in W_0^{1,r}(\Omega^\varepsilon)^3$ and $\psi \in W_{\varGamma_1^\varepsilon \cup \varGamma_\ell^\varepsilon}^{1,r}(\Omega^\varepsilon)$, $1 < r < +\infty$, there hold the following inequalities*

$$
\|\mathbf{v}\|_{L^r(\Omega^\varepsilon)^3} \le C_1\varepsilon \|D\mathbf{v}\|_{L^r(\Omega^\varepsilon)^{3\times3}}, \quad \|D\mathbf{v}\|_{L^r(\Omega^\varepsilon)^{3\times3}} \le C_2 \left\|\mathbb{D}[\mathbf{v}]\right\|_{L^r(\Omega^\varepsilon)^{3\times3}},
$$

$$
\|\psi\|_{L^r(\Omega^\varepsilon)} \le C_1\varepsilon \|\nabla\psi\|_{L^r(\Omega^\varepsilon)^3},
$$

where C_1 and C_2 do not depend on ε. Moreover, for all $\widetilde{\mathbf{v}} \in W_0^{1,r}(\Omega)^3$ and $\widetilde{\psi} \in W_{\Gamma_1 \cup \Gamma_\ell}^{1,r}(\Omega)$, $1 < r < +\infty$, obtained from $\mathbf{v}$ and ψ from the change of variable (2.8), there hold the following inequalities

$$\|\widetilde{\mathbf{v}}\|_{L^r(\Omega)^3} \leq C_1 \varepsilon \|D_\varepsilon \widetilde{\mathbf{v}}\|_{L^r(\Omega)^{3\times3}}, \|D_\varepsilon \widetilde{\mathbf{v}}\|_{L^r(\Omega)^{3\times3}} \leq C_2 \left\|\mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}]\right\|_{L^r(\Omega)^{3\times3}}, \quad (3.1)$$

$$\left\|\widetilde{\psi}\right\|_{L^r(\Omega)} \leq C_1 \varepsilon \left\|\nabla_\varepsilon \widetilde{\psi}\right\|_{L^r(\Omega)^3}. \qquad (3.2)$$

Next, we give the *a priori* estimates for velocity and temperature.

Proposition 1 [Estimates for velocity and temperature]. *Assume $q \in (1, 3/2)$ and let $(\widetilde{\mathbf{u}}^\varepsilon, \widetilde{T}^\varepsilon)$ be a solution of the dilated problem (2.9)–(2.10). Then, there hold the following estimates*

$$\|\widetilde{\mathbf{u}}^\varepsilon\|_{L^2(\Omega)^3} \leq C, \quad \|D_\varepsilon \widetilde{\mathbf{u}}^\varepsilon\|_{L^2(\Omega)^{3\times3}} \leq C\varepsilon^{-1}, \quad \left\|\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}_\varepsilon]\right\|_{L^2(\Omega)^{3\times3}} \leq C\varepsilon^{-1}, (3.3)$$

$$\left\|\widetilde{T}^\varepsilon\right\|_{L^q(\Omega)} \leq C, \quad \left\|\nabla_\varepsilon \widetilde{T}^\varepsilon\right\|_{L^q(\Omega)^3} \leq C\varepsilon^{-1}. \qquad (3.4)$$

Remark 2. From the estimates given in Proposition 1, we also have the following estimates for $\mathbf{u}^\varepsilon$

$$\|\mathbf{u}^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{1}{2}}, \ \|D\mathbf{u}^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}} \leq C\varepsilon^{-\frac{1}{2}}, \ \left\|\mathbb{D}[\mathbf{u}_\varepsilon]\right\|_{L^2(\Omega_\varepsilon)^{3\times3}} \leq C\varepsilon^{-\frac{1}{2}},$$

which will be useful to derive estimates for pressure.

Proof. [Proof of Proposition 1] We divide the proof in two steps. In the first step we deduce estimates for velocity (3.3) and in the second step we derive estimates for temperature (3.4).

Step 1. Velocity estimates. Taking $\widetilde{\mathbf{v}} = \widetilde{\mathbf{u}}^\varepsilon$ as test function in $(2.11)_1$, we obtain

$$2\mu_{\text{eff}} \int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}_\varepsilon]|^2 dx'dz_3 + \frac{\mu}{K}\varepsilon^{-2} \int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2 dx'dz_3 = \int_\Omega \varepsilon^{-2}\mathbf{f}' \cdot \widetilde{\mathbf{u}}_\varepsilon' \, dx'dz_3,$$

where we have used that $\int_\Omega \widetilde{p}_\varepsilon \, \text{div}_\varepsilon(\widetilde{\mathbf{u}}_\varepsilon) = 0$, because $\text{div}_\varepsilon(\widetilde{\mathbf{u}}_\varepsilon) = 0$ in $\widetilde{\Omega}^\varepsilon$. Using the Cauchy-Schwarz inequality, $\mathbf{f}' \in L^2(\omega)^2$ and the Poincaré and Korn inequalities (3.1), we get

$$\left|\int_\Omega \varepsilon^{-2}\mathbf{f}' \cdot \widetilde{\mathbf{u}}_\varepsilon \, dx\right| \leq h_{\max}^{\frac{1}{2}}\varepsilon^{-2} \left\|f'\right\|_{L^2(\omega)^2} \|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)^3}$$

$$\leq h_{\max}^{\frac{1}{2}} C_1 \varepsilon^{-1} \|D_\varepsilon \widetilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)^{3\times3}} \leq h_{\max}^{\frac{1}{2}} C_1 C_2 \varepsilon^{-1} \left\|\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}_\varepsilon]\right\|_{L^2(\Omega)^{3\times3}},$$

which leads to

$$\mu_{\text{eff}} \left\|\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}_\varepsilon]\right\|_{L^2(\Omega)^{3\times3}}^2 + \frac{\mu}{K}\varepsilon^{-2} \|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)^3}^2 \leq h_{\max}^{\frac{1}{2}} C_1 C_2 \varepsilon^{-1} \left\|\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}_\varepsilon]\right\|_{L^2(\Omega)^{3\times3}}. \qquad (3.5)$$

On the one hand, this implies that

$$\left\|\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}_\varepsilon]\right\|_{L^2(\Omega)^{3\times3}} \leq C\varepsilon^{-1}, \qquad (3.6)$$

and so, again from the Poincaré and Korn inequalities (3.1), we deduce

$$\|D_\varepsilon \widetilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \leq C\varepsilon^{-1}, \quad \|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)^3} \leq C.$$

On the other hand, using (3.6) into (3.5), we also obtain

$$\frac{\mu}{K}\varepsilon^{-2}\|\widetilde{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)^3} \leq h^{\frac{1}{2}}_{\max}C_1 C_2 \varepsilon^{-2},$$

which also gives $\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)^3} \leq C$. This completes the velocity estimates.

Step 2. Temperature estimates. We follow [8, Lemme 2.4.2] in the case $r = 2$ and [3, Theorem 2] in the case $r = 2$ and $\hat{b} \equiv 0$, with some modifications. We define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \xi\,\mathrm{sign}(t)\int_0^{|t|}\frac{d\tau}{(1+\tau)^{\xi+1}}d\tau = \mathrm{sign}(t)\left[1 - \frac{1}{(1+|t|)^\xi}\right],$$

with $\xi > 0$. Then, it holds that $\varphi'(t) = \frac{\xi}{(1+|t|)^{\xi+1}}$.

We take $\widetilde{\zeta} = \varphi(\varepsilon\widetilde{T}^\varepsilon)$ as test function in $(2.11)_3$. Then, taking into account that

$$\int_\Omega k\nabla_\varepsilon(\varepsilon^2\widetilde{T}^\varepsilon)\cdot\nabla_\varepsilon\widetilde{\zeta}\,dx'dz_3 = \int_\Omega \xi k\frac{|\nabla_\varepsilon(\varepsilon\widetilde{T}^\varepsilon)|^2}{(1+|\varepsilon\widetilde{T}^\varepsilon|)^{\xi+1}}\,dx'dz_3,$$

we get that $(2.11)_3$ is rewritten as follows

$$\int_\Omega \frac{|\nabla_\varepsilon(\varepsilon\widetilde{T}^\varepsilon)|^2}{(1+|\varepsilon\widetilde{T}^\varepsilon|)^{\xi+1}}\,dx'dz_3 \leq \frac{\mu}{Kk\xi}\int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2\,dx'dz_3 + \frac{2\mu_{\mathrm{eff}}}{k\xi}\varepsilon^2\int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2\,dx'dz_3$$
$$+ \frac{1}{k\xi}\int_\omega |b|\,dx'.$$

By using estimates (3.3), we get

$$\left|\frac{\mu}{Kk\xi}\int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2\,dx'dz_3\right| \leq C\|\widetilde{\mathbf{u}}^\varepsilon\|^2_{L^2(\Omega)^3} \leq C,$$

$$\left|\frac{2\mu_{\mathrm{eff}}}{k\xi}\varepsilon^2\int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2\,dx'dz_3\right| \leq C\varepsilon^2\left\|\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]\right\|^2_{L^2(\Omega)^3} \leq C, \quad \left|\frac{1}{k\xi}\int_\omega b\,dx'\right| \leq C,$$

and then, we deduce

$$\int_\Omega \frac{|\nabla_\varepsilon(\varepsilon\widetilde{T}^\varepsilon)|^2}{(1+|\varepsilon\widetilde{T}^\varepsilon|)^{\xi+1}}\,dx'dz_3 \leq C. \tag{3.7}$$

Using Hölder's inequality with the exponents $2/q$ and $2/(2-q)$, for $q \in (1, 3/2)$, we obtain

$$\int_\Omega |\nabla_\varepsilon(\varepsilon\widetilde{T}^\varepsilon)|^q dx'dz_3 = \int_\Omega |\nabla_\varepsilon(\varepsilon\widetilde{T}^\varepsilon)|^q\frac{(1+|\varepsilon\widetilde{T}^\varepsilon|)^{(\xi+1)\frac{q}{2}}}{(1+|\varepsilon\widetilde{T}^\varepsilon|)^{(\xi+1)\frac{q}{2}}}dx'dz_3$$
$$\leq \left(\int_\Omega \frac{|\nabla_\varepsilon(\varepsilon\widetilde{T}^\varepsilon)|^2}{(1+|\varepsilon\widetilde{T}^\varepsilon|)^{\xi+1}}\,dx'dz_3\right)^{\frac{q}{2}}\left(\int_\Omega (1+|\varepsilon\widetilde{T}^\varepsilon|)^{(\xi+1)\frac{q}{2-q}}dx'dz_3\right)^{\frac{2-q}{2}}.$$

Now, we choose ξ such that $(\xi + 1)q/(2 - q) \leq q^\star = 3q/(3 - q)$, that is $\xi \leq (3 - 2q)/(3 - q)$. Using (3.7), we get

$$
\begin{aligned}
\int_\Omega |\nabla_\varepsilon(\varepsilon \widetilde{T}^\varepsilon)|^q dx' dz &\leq C \left(\int_\Omega (1 + |\varepsilon \widetilde{T}^\varepsilon|)^{q^\star} dx' dz_3 \right)^{\frac{2-q}{2}} \\
&\leq C 2^{\frac{(q^\star - 1)(2 - q)}{2}} \left(|\Omega|^{\frac{2-q}{2}} + \left(\int_\Omega |\varepsilon \widetilde{T}^\varepsilon|^{q^\star} dx' dz_3 \right)^{\frac{2-q}{2}} \right).
\end{aligned}
\tag{3.8}
$$

To obtain the estimate for $\nabla_\varepsilon(\varepsilon \widetilde{T}^\varepsilon)$ in $L^q(\Omega)^3$, it remains to estimate $\varepsilon \widetilde{T}^\varepsilon$ in $L^{q^\star}(\Omega)$. Thus, using the Poincaré-Sobolev inequality and (3.8), there exists a constant $C_\star > 0$ independent of ε such that

$$
\begin{aligned}
\left(\int_\Omega |\varepsilon \widetilde{T}^\varepsilon|^{q^\star} dx' dz_3 \right)^{1/q^\star} &\leq C_\star \left\| \nabla_{x', z_3}(\varepsilon \widetilde{T}^\varepsilon) \right\|_{L^q(\Omega)^3} \leq C_\star \left\| \nabla_\varepsilon(\varepsilon \widetilde{T}^\varepsilon) \right\|_{L^q(\Omega)^3} \\
&\leq C_\star 2^{\frac{(q^\star - 1)(2 - q)}{2q}} \left(|\Omega|^{\frac{2-q}{2q}} + \left(\int_\Omega |\varepsilon \widetilde{T}^\varepsilon|^{q^\star} dx' dz_3 \right)^{\frac{2-q}{2q}} \right).
\end{aligned}
\tag{3.9}
$$

On the other hand, for all $a > 0$, $c_1 > 0$, $c_2 > 0$ and $0 < s < t$, the following implication holds

$$
\text{If} \quad a^t \leq c_1 + c_2 a^s \quad \text{then} \quad a \leq \max\{1, (c_1 + c_2)^{1/(t-s)}\}.
$$

Hence, taking in (3.9) the following choices for a, c_1, c_2, s, t:

$$
a = \int_{\widetilde{\Omega}^\varepsilon} |\varepsilon \widetilde{T}^\varepsilon|^{q^\star} dx' dz_3, \quad c_1 = C_\star 2^{\frac{(q^\star - 1)(2 - q)}{2q}} |\Omega|^{\frac{2-q}{2q}}, \quad c_2 = C_\star 2^{\frac{(q^\star - 1)(2 - q)}{2q}},
$$

$$
t = \frac{1}{q^\star}, \quad s = \frac{2 - q}{2q}, \quad \frac{1}{t - s} = 6,
$$

we deduce that the integral term on the right-hand side of (3.9) satisfies

$$
\int_\Omega |\varepsilon \widetilde{T}^\varepsilon|^{q^\star} dx' dz_3 \leq A_\varepsilon = \max\{1, \beta\}, \quad \beta = \left(C_\star 2^{\frac{(q^\star - 1)(2 - q)}{2q}} \left(|\Omega|^{\frac{2-q}{2q}} + 1 \right) \right)^6.
$$

Then, it holds $\left(\int_\Omega |\varepsilon \widetilde{T}^\varepsilon|^{q^\star} dx' dz_3 \right)^{\frac{2-q}{2}} \leq \max\{1, \beta^{\frac{2-q}{2}}\}$, so, from (3.8), we have

$$
\int_\Omega |\nabla_\varepsilon(\varepsilon \widetilde{T}^\varepsilon)|^q dx' dz_3 \leq C,
$$

which gives (3.4)$_2$. By the Poincaré inequality (3.2), we deduce (3.4)$_1$ and so, the proof is finished. $\square$

3.2 Estimates for pressure

Let us first give a more accurate estimate for pressure p^ε. For this, we need to recall a version of the decomposition result for p^ε whose proof can be found in [5].

Proposition 2. *The following decomposition for $p^\varepsilon \in L^2_0(\Omega^\varepsilon)$ holds*

$$p^\varepsilon = p^\varepsilon_0 + p^\varepsilon_1, \tag{3.10}$$

where $p^\varepsilon_0 \in H^1(\mathcal{T}^2)$, which is independent of x_3, and $p^\varepsilon_1 \in L^2(\Omega^\varepsilon)$. Moreover, the following estimates hold

$$\varepsilon^{\frac{3}{2}} \|p^\varepsilon_0\|_{H^1(\omega)} + \|p^\varepsilon_1\|_{L^2(\Omega^\varepsilon)} \le C \|\nabla p^\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3},$$

that is

$$\|p^\varepsilon_0\|_{H^1(\omega)} \le C\varepsilon^{-\frac{3}{2}} \|\nabla p_\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3}, \quad \|p^\varepsilon_1\|_{L^2(\Omega^\varepsilon)} \le C \|\nabla p^\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3}. \tag{3.11}$$

Proof. This result follows from [5, Corollary 3.4] by assuming

$$\Omega^\varepsilon = \omega \times (\varepsilon\vartheta), \quad \forall \varepsilon > 0,$$

with $\omega \subset \mathbb{R}^2, \vartheta \subset \mathbb{R}$ connected Lipschitz open sets and exponent $q = 2$, see [5, Remark 3.1]. In the present case, $\vartheta = (0, h(x'))$ with $h \in W^{1,\infty}$ as we set in Section 2. Thus, Ω^ε satisfies the abstract conditions (3.3)–(3.5) given in [5], and so, [5, Corollary 3.4] can be applied. $\square$

We denote by $\widetilde{p}^\varepsilon_1$ the rescaled function associated with p^1_ε defined by $\widetilde{p}^\varepsilon_1(x', z_3) = p^\varepsilon_1(x', \varepsilon z_3)$ for a.e. $(x', z_3) \in \Omega$. As a consequence, we have the following result:

Corollary 1. The pressures p^ε_0, p^ε_1 and $\widetilde{p}^\varepsilon_1$ satisfy the following estimates

$$\|p^\varepsilon_0\|_{H^1(\omega)} \le C\varepsilon^{-2}, \quad \|p^\varepsilon_1\|_{L^2(\Omega^\varepsilon)} \le C\varepsilon^{-\frac{1}{2}}, \quad \|\widetilde{p}^\varepsilon_1\|_{L^2(\Omega)} \le C\varepsilon^{-1}. \tag{3.12}$$

Proof. In view of (3.11), to derive (3.12) we just need to obtain the estimate for ∇p^ε given by

$$\|\nabla p^\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3} \le C\varepsilon^{-\frac{1}{2}}. \tag{3.13}$$

To do this, we consider $\mathbf{v} \in H^1_0(\Omega^\varepsilon)^3$, and taking into account the variational formulation $(2.7)_1$, we get

$$\langle \nabla p^\varepsilon, \mathbf{v} \rangle = -2\mu_{\text{eff}} \int_{\Omega^\varepsilon} \mathbb{D}[\mathbf{u}_\varepsilon] : \mathbb{D}[\mathbf{v}]\, dx - \frac{\mu}{K}\varepsilon^{-2} \int_{\Omega^\varepsilon} \mathbf{u}^\varepsilon \cdot \mathbf{v}\, dx + \int_{\Omega^\varepsilon} \varepsilon^{-2}\mathbf{f}' \cdot \mathbf{v}'\, dx. \tag{3.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $H^{-1}(\Omega^\varepsilon)^3$ and $H^1_0(\Omega^\varepsilon)^3$. Estimating the terms on the right-hand side of (3.14) using Lemma 1 and Remark 2, we get

$$\left| 2\mu_{\text{eff}} \int_{\Omega^\varepsilon} \mathbb{D}[\mathbf{u}_\varepsilon] : \mathbb{D}[\mathbf{v}]\, dx \right| \le C \|\mathbb{D}[\mathbf{u}^\varepsilon]\|_{L^2(\Omega^\varepsilon)^{3\times3}} \|D\mathbf{v}\|_{L^2(\Omega^\varepsilon)^{3\times3}}$$

$$\le C\varepsilon^{-\frac{1}{2}} \|\mathbf{v}\|_{H^1_0(\Omega^\varepsilon)^3},$$

$$\left| \frac{\mu}{K}\varepsilon^{-2} \int_{\Omega^\varepsilon} \mathbf{u}^\varepsilon \cdot \mathbf{v}\, dx \right| \le C\varepsilon^{-2} \|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \|\mathbf{v}\|_{L^2(\Omega_\varepsilon)^3}$$

$$\le C \|D\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}} \|D\mathbf{v}\|_{L^2(\Omega_\varepsilon)^{3\times3}}$$

$$\le C\varepsilon^{-\frac{1}{2}} \|\mathbf{v}\|_{H^1_0(\Omega^\varepsilon)^3},$$

$$\left| \int_{\Omega^\varepsilon} \varepsilon^{-2}\mathbf{f}' \cdot \mathbf{v}'\, dx \right| \le C\varepsilon^{-\frac{1}{2}} \|D\mathbf{v}\|_{L^2(\Omega_\varepsilon)^{3\times3}} \le C\varepsilon^{-\frac{1}{2}} \|\mathbf{v}\|_{H^1_0(\Omega^\varepsilon)^3},$$

which together with (3.14) gives

$$\left|\langle\nabla p^\varepsilon, \mathbf{v}\rangle\right| \leq C\varepsilon^{-\frac{1}{2}}\|\mathbf{v}\|_{H_0^1(\Omega^\varepsilon)^3}, \quad \forall \mathbf{v} \in H_0^1(\Omega^\varepsilon)^3.$$

This gives the desired estimate (3.13), completing the proof. □

4 Convergence results and limit problem

For $1 < r < +\infty$, let us introduce the following sets:

$$V_{z_3}^r = \left\{v \in L^r(\Omega) \ : \ \partial_{z_3} v_i \in L^r(\Omega)\right\}, \ V_{z_3,\Theta}^r = \left\{v \in V_{z_3}^r(\Omega) \ : \ \mathbf{v} = 0 \text{ on } \Theta\right\}.$$

Proposition 3 [Compactness results for rescaled functions]. *For a subsequence of ε still denoted by ε, we have the following convergence results:*

- *There exists $\widetilde{\mathbf{u}}^\star = (\widetilde{u}_1^\star, \widetilde{u}_2^\star) \in (V_{z_3,\Gamma_0\cup\Gamma_1}^2)^2$, such that*

$$\widetilde{u}_i^\varepsilon \rightharpoonup \widetilde{u}_i^\star \text{ in } V_{z_3,\Gamma_0\cup\Gamma_1}^2, \quad i = 1, 2, \tag{4.1}$$

$$\widetilde{u}_3^\varepsilon \rightharpoonup 0 \text{ in } L^2(\Omega), \tag{4.2}$$

$$\varepsilon\partial_{x_j}\widetilde{u}_i^\varepsilon \rightharpoonup 0 \text{ in } L^2(\Omega), \quad i, j = 1, 2, \tag{4.3}$$

$$\operatorname{div}_{x'}\left(\int_0^{h(x')} \widetilde{\mathbf{u}}^\star(z)\,dz_3\right) = 0 \quad \text{in } \omega, \tag{4.4}$$

$$\left(\int_0^{h(x')} \widetilde{\mathbf{u}}^\star(z)\,dz_3\right)\cdot n = 0 \quad \text{on } \partial\omega. \tag{4.5}$$

- *There exists a function $\widetilde{p} \in L_0^2(\omega) \cap H^1(\omega)$, independent of z_3, such that*

$$\varepsilon^2 p_0^\varepsilon \rightharpoonup \widetilde{p}^\star \text{ in } H^1(\omega). \tag{4.6}$$

- *There exists $\widetilde{T}^\star \in V_{z_3,\Gamma_1}^q$ such that*

$$\widetilde{T}^\varepsilon \rightharpoonup \widetilde{T}^\star \text{ in } V_{z_3,\Gamma_1}^q. \tag{4.7}$$

$$\varepsilon\partial_{x_i}\widetilde{T}^\varepsilon \rightharpoonup 0 \text{ in } L^q(\Omega), \quad i = 1, 2. \tag{4.8}$$

Proof. We start by proving the convergence for the velocity $\widetilde{\mathbf{u}}^\varepsilon$, which is classical. From estimates (3.3), we deduce that there exists $\widetilde{\mathbf{u}} = (\widetilde{\mathbf{u}}^*, \widetilde{u}_3^\star) \in (V_{z_3,\Gamma_0\cup\Gamma_1}^2)^3$, with $\widetilde{\mathbf{u}} = 0$ on $\Gamma_0 \cup \Gamma_1$, such that

$$\widetilde{\mathbf{u}}^\varepsilon \rightharpoonup \widetilde{\mathbf{u}} \quad \text{in } (V_{z_3,\Gamma_0\cup\Gamma_1}^2)^3. \tag{4.9}$$

From (4.9), we also have that

$$\operatorname{div}_{x'}(\widetilde{\mathbf{u}}^\varepsilon)' \rightharpoonup \operatorname{div}_{x'}(\widetilde{\mathbf{u}}^\star) \quad \text{in } H^1(0, 1; H^{-1}(\omega)),$$

which implies (4.3). Then, by using $\operatorname{div}_\varepsilon(\widetilde{\mathbf{u}}^\varepsilon) = 0$ in Ω, we deduce that $\varepsilon^{-1}\partial_{z_3}\widetilde{u}_3^\varepsilon$ is bounded in $L^2(0, T; H^{-1}(\omega))$. Using then that $\widetilde{u}_3^\varepsilon = 0$ on Γ_1, we deduce that

$\varepsilon^{-1}\widetilde{u}_3^\varepsilon$ is bounded in $H^1(0, 1; H^{-1}(\omega))$, and therefore, together with (4.9), we deduce that $\widetilde{u}_3^\varepsilon$ tends to $\widetilde{u}_3^\star \equiv 0$. This completes the proof of (4.1) and (4.2).

Next, by taking $\widetilde{\psi} \in \mathcal{D}(\omega)$ in $(2.11)_2$, we deduce

$$\int_\Omega (\widetilde{\mathbf{u}}^\varepsilon)' \nabla_{x'} \widetilde{\psi}(x') \, dx' dz_3 = 0,$$

and passing to the limit by using (4.1), we deduce (4.4)–(4.5).

Convergences and free-divergence condition given in this proposition are obtained directly from the estimates given in Proposition 1 for velocity and temperature, see for instance [3, Theorem 3].

Concerning the pressure, from estimates of p_0^ε given in $(3.12)_1$, we get the convergence (4.6). Since $\widetilde{p}^\varepsilon$ has mean value zero, from the decomposition of the pressure, we have

$$0 = \int_\Omega \widetilde{p}^\varepsilon \, dx' dz_3 = h(x') \int_\omega p_0^\varepsilon(x') \, dx' + \int_\Omega \widetilde{p}_1^\varepsilon \, dx' dz_3.$$

Taking into account the convergence of $\varepsilon^2 p_0^\varepsilon$ to $\widetilde{p}^\star$ given in (4.6) and that

$$\left| \int_\Omega \varepsilon^2 \widetilde{p}_1^\varepsilon \, dx' dz_3 \right| \leq C\varepsilon \to 0, \tag{4.10}$$

we get $h(x') \int_\omega \widetilde{p}^\star \, dx' = 0$, and so, by the assumptions of $h(x')$, $\widetilde{p}$ has null mean value in ω.

Finally, convergences (4.7)–(4.8) are obtained from estimates (3.4) with similar arguments of the proof as for the velocity convergences given above. $\square$

Next, by using previous convergences, we derive the limit coupled model.

Theorem 1 [Limit model]. *The limit functions $\widetilde{\mathbf{u}}^\star$, $\widetilde{p}^\star$, $\widetilde{T}^\star$ given in Proposition 3 satisfy*

$$\begin{cases} -\mu_{\text{eff}} \partial_{z_3}^2 \widetilde{\mathbf{u}}^\star + \frac{\mu}{K} \hat{\mathbf{u}}^\star = \mathbf{f}'(x') - \nabla_{x'} \widetilde{p}^\star(x') & in \; \Omega, \\ \text{div}_{x'} \left(\int_0^{h(x')} \hat{\mathbf{u}}^\star \, dz_3 \right) = 0 & in \; \Omega, \\ -k \partial_{z_3}^2 \hat{T}^\star = \frac{\mu}{K} |\widetilde{\mathbf{u}}^\star|^2 + \mu_{\text{eff}} |\partial_{z_3} \widetilde{\mathbf{u}}^\star|^2 & in \; \Omega, \\ \widetilde{\mathbf{u}}^\star = 0 & on \; \Gamma_0 \cup \Gamma_1, \\ \widetilde{T}^\star = 0 & on \; \Gamma_1, \\ -k \partial_{z_3} \widetilde{T}^\star = b & on \; \Gamma_0. \end{cases} \tag{4.11}$$

Proof. We divide the proof in four steps. In the first step, we pass to the limit in the equation of velocity and, in the second and third ones, we obtain strong convergence of velocity. Finally, in the fourth step, we pass to the limit in the equation of temperature.

Step 1. To prove $(4.11)_1$, according to Proposition 3, we consider $(2.11)_1$ with $\widetilde{\mathbf{v}}$ replaced by $\widehat{\mathbf{v}_\varepsilon} = (\varepsilon^2 \widetilde{\mathbf{v}}', 0) \in \mathcal{D}(\Omega)^3$. This gives the following variational

formulation:

$$2\mu_{\text{eff}} \int_\Omega \varepsilon^2 \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon] : \mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}'] \, dx' dz_3 + \frac{\mu}{K} \int_\Omega \widetilde{\mathbf{u}}^\varepsilon \cdot \widetilde{\mathbf{v}}' \, dx' z_3$$

$$- \int_\Omega \varepsilon^2 \widetilde{p}^\varepsilon \, \text{div}_{x'}(\widetilde{\mathbf{v}}') \, dx' dz_3 = \int_\Omega \mathbf{f}' \cdot \widetilde{\mathbf{v}}' \, dx' dz_3. \tag{4.12}$$

Below, let us pass to the limit when ε tends to zero in each term of (4.12):

- First term on the left-hand side of (4.12). From the convergence (4.3), taking into account definitions (2.1)–(2.2), and since $\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon] : \mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}'] = \mathbb{D}_\varepsilon[(\widetilde{\mathbf{u}}^\varepsilon)'] : \mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}']$, we get

$$2\mu_{\text{eff}} \int_\Omega \varepsilon^2 \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon] : \mathbb{D}_\varepsilon[\widetilde{\mathbf{v}}'] \, dx' dz_3$$

$$= 2\mu_{\text{eff}} \int_\Omega \varepsilon^2 \mathbb{D}_{x'}[(\widetilde{\mathbf{u}}^\varepsilon)'] : \mathbb{D}_{x'}[\widetilde{\mathbf{v}}'] \, dx' dz_3 + 2\mu_{\text{eff}} \int_\Omega \partial_{z_3}[(\widetilde{\mathbf{u}}^\varepsilon)'] : \partial_{z_3}[\widetilde{\mathbf{v}}'] \, dx' dz_3$$

$$= 2\mu_{\text{eff}} \int_\Omega \partial_{z_3}[\widetilde{\mathbf{u}}^\star] : \partial_{z_3}[\widetilde{\mathbf{v}}'] \, dx' dz_3 + O_\varepsilon.$$

- Second term on the left-hand side of (4.12). From the convergence (4.1), we get

$$\frac{\mu}{K} \int_\Omega \widetilde{\mathbf{u}}^\varepsilon \cdot \widetilde{\mathbf{v}}' \, dx' z_3 = \frac{\mu}{K} \int_\Omega \widetilde{\mathbf{u}}^\star \cdot \widetilde{\mathbf{v}}' \, dx' z_3 + O_\varepsilon.$$

- Third term on the left-hand side of (4.12). From the decomposition of the pressure (3.10), convergence (4.6) and (4.10), we have

$$\int_\Omega \varepsilon^2 \widetilde{p}^\varepsilon \, \text{div}_{x'}(\widetilde{\mathbf{v}}') \, dx' dz_3$$

$$= \int_\Omega \varepsilon^2 p_0^\varepsilon(x') \, \text{div}_{x'}(\widetilde{\mathbf{v}}') \, dx' dz_3 + \int_\Omega \varepsilon^2 \widetilde{p}_1^\varepsilon \, \text{div}_{x'}(\widetilde{\mathbf{v}}') \, dx' dz_3$$

$$= \int_\Omega \widetilde{p}^\star(x') \, \text{div}_{x'}(\widetilde{\mathbf{v}}') \, dx' dz_3 + O_\varepsilon.$$

Therefore, passing to the limit in (4.12) as $\varepsilon \to 0$, by previous convergences, we deduce the following limit variational formulation

$$2\mu_{\text{eff}} \int_\Omega \partial_{z_3}[\widetilde{\mathbf{u}}^\star] : \partial_{z_3}[\widetilde{\mathbf{v}}'] \, dx' dz_3 + \frac{\mu}{K} \int_\Omega \widetilde{\mathbf{u}}^\star \cdot \widetilde{\mathbf{v}}' \, dx' z_3$$

$$- \int_\Omega \widetilde{p}^\star(x') \, \text{div}_{x'}(\widetilde{\mathbf{v}}') \, dx' dz_3 = \int_\Omega \mathbf{f}' \cdot \widetilde{\mathbf{v}}' \, dx' dz_3, \tag{4.13}$$

which, by density, holds for every $\widetilde{\mathbf{v}}' \in H^1_{\Gamma_0 \cup \Gamma_1}$. Taking into account that

$$\partial_{z_3}[\widetilde{\mathbf{u}}^\star] : \partial_{z_3}[\mathbf{v}'] = \frac{1}{2} \partial_{z_3} \widetilde{\mathbf{u}}^\star \cdot \partial_{z_3} \hat{\mathbf{v}}', \tag{4.14}$$

then (4.13) is equivalent to $(4.11)_{1,4}$.

Step 2. We prove the following property

$$\lim_{\varepsilon \to 0} \left(\frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2 \, dx' dz_3 + 2\mu_{\text{eff}} \varepsilon^2 \int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2 dx' dz_3 \right)$$
$$= \frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\star|^2 \, dx' dz_3 + 2\mu_{\text{eff}} \int_\Omega |\partial_{z_3}[\widetilde{\mathbf{u}}^\star]|^2 dx' dz_3. \tag{4.15}$$

To prove this, we take $\widetilde{\mathbf{u}}^\varepsilon$ as test function in $(2.11)_1$. Taking into account that the pressure term vanish because $\text{div}_\varepsilon(\widetilde{\mathbf{u}}^\varepsilon) = 0$ in Ω, passing to the limit in the right-hand side of $(2.11)_1$, we deduce

$$\lim_{\varepsilon \to 0} \left(\frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2 \, dx' dz_3 + 2\mu_{\text{eff}} \varepsilon^2 \int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2 dx' dz_3 \right) = \int_\Omega \mathbf{f}' \cdot \widetilde{u}^\star \, dx' dz_3. \tag{4.16}$$

Now, we take $\widetilde{\mathbf{u}}^\star$ as test function in (4.13). Taking into account that $\widetilde{p}^\star$ does not depend on z_3 and the divergence condition $\text{div}_{x'}(\int_0^{h(x')} \widetilde{\mathbf{u}}^\star dz_3) = 0$ in ω, we have

$$2\mu_{\text{eff}} \int_\Omega |\partial_{z_3}[\widetilde{\mathbf{u}}^\star]|^2 \, dx' dz_3 + \frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\star|^2 \, dx' z_3 = \int_\Omega \mathbf{f}' \cdot \widetilde{\mathbf{u}}^\star \, dx' dz_3.$$

This together with (4.16), implies (4.15).

Step 3. We prove strong convergences

$$\widetilde{\mathbf{u}}^\varepsilon \to (\widetilde{\mathbf{u}}^\star, 0) \quad \text{in } L^2(\Omega)^3, \quad \varepsilon \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon] \to \partial_{z_3}[\widetilde{\mathbf{u}}^\star] \quad \text{in } L^2(\Omega)^{3\times 3}. \tag{4.17}$$

To prove (4.17), it is enough to prove

$$E_\varepsilon := 2\mu_{\text{eff}} \int_\Omega |\varepsilon \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon] - \partial_{z_3}[\widetilde{\mathbf{u}}^\star]|^2 dx' dz_3 + \frac{\mu}{k} \int_\Omega |\widetilde{\mathbf{u}}^\varepsilon - (\widetilde{\mathbf{u}}^\star, 0)|^2 \, dx' dz_3$$

tends to zero. Developing the expression of E_ε, we have

$$E_\varepsilon = 2\mu_{\text{eff}} \int_\Omega |\varepsilon \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2 dx' dz_3 + 2\mu_{\text{eff}} \int_\Omega |\partial_{z_3}[\widetilde{\mathbf{u}}^\star]|^2 dx' dz_3$$
$$- 4\mu_{\text{eff}} \int_\Omega \varepsilon \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon] : \partial_{z_3}[\widetilde{\mathbf{u}}^\star] dx' dz_3$$
$$+ \frac{\mu}{k} \int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2 dx' dz_3 + \frac{\mu}{k} \int_\Omega |\widetilde{\mathbf{u}}^\star|^2 \, dx' dz_3 - 2\frac{\mu}{k} \int_\Omega (\widetilde{\mathbf{u}}^\varepsilon)' \cdot \widetilde{\mathbf{u}}^\star \, dx' dz_3.$$

By using property (4.15) and convergences (4.1)–(4.3), we easily deduce $E_\varepsilon \to 0$.

Step 4. To prove $(4.11)_3$, we consider $(2.11)_3$ with $\widetilde{\zeta} = \widetilde{\psi}(x')\widetilde{\phi}(z_3)$ with $\widetilde{\psi} \in \mathcal{D}(\omega)$ and $\widetilde{\phi} \in C^\infty([0,1])$ with $\widetilde{\phi} = 0$ on Γ_1. We pass to the limit in every term:

$$\varepsilon^2 \int_\Omega k \nabla_\varepsilon \widetilde{T}^\varepsilon \cdot \nabla_\varepsilon \widetilde{\zeta} \, dx' dz_3 = \frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2 \widetilde{\zeta} \, dx' dz_3 + 2\mu_{\text{eff}} \varepsilon^2 \int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2 \widetilde{\zeta} \, dx' dz_3$$
$$+ \int_\omega b \widetilde{\zeta} \, dx'. \tag{4.18}$$

- First and second terms in the left-hand side of (4.18). By using convergences (4.7) and (4.8), we get

$$
\varepsilon^2 \int_\Omega k \nabla_\varepsilon \widetilde{T}^\varepsilon \cdot \nabla_\varepsilon \widetilde{\zeta} \, dx' dz_3
$$
$$
= \varepsilon^2 \int_\Omega k \nabla_{x'} \widetilde{T}^\varepsilon \cdot \nabla_{x'} \widetilde{\psi}(x') \, \widetilde{\phi}(z_3) \, dx' dz_3 + \int_\Omega k \partial_{z_3} \widetilde{T}^\varepsilon \, \partial_{z_3} \widetilde{\phi}(z_3) \, \widetilde{\psi}(x') \, dx' dz_3
$$
$$
= \int_\Omega k \partial_{z_3} \widetilde{T}^\star \, \partial_{z_3} \widetilde{\zeta} \, dx' dz_3 + O_\varepsilon.
$$

- First term in the right-hand side of (4.18). From the strong convergence (4.17), we deduce

$$
\frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\varepsilon|^2 \widetilde{\zeta} \, dx' dz_3 = \frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\star|^2 \widetilde{\zeta} \, dx' dz_3 + O_\varepsilon.
$$

- Second term in the right-hand side of (4.18). From the strong convergence (4.17), we deduce

$$
2\mu_{\text{eff}} \varepsilon^2 \int_\Omega |\mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2 \widetilde{\zeta} \, dx' dz_3 = 2\mu_{\text{eff}} \int_\Omega |\varepsilon \mathbb{D}_\varepsilon[\widetilde{\mathbf{u}}^\varepsilon]|^2 \widetilde{\zeta} \, dx' dz_3
$$
$$
= 2\mu_{\text{eff}} \int_\Omega |\partial_{z_3}[\widetilde{\mathbf{u}}^\star]|^2 \widetilde{\zeta} \, dx' dz_3 + O_\varepsilon.
$$

Therefore, passing to the limit in (4.18) as $\varepsilon \to 0$, by previous convergences, we deduce the following limit variational formulation

$$
\int_\Omega k \partial_{z_3} \widetilde{T}^\star \, \partial_{z_3} \widetilde{\zeta} \, dx' dz_3
$$
$$
= \frac{\mu}{K} \int_\Omega |\widetilde{\mathbf{u}}^\star|^2 \widetilde{\zeta} \, dx' dz_3 + 2\mu_{\text{eff}} \int_\Omega |\partial_{z_3}[\widetilde{\mathbf{u}}^\star]|^2 \widetilde{\zeta} \, dx' dz_3 + \int_\omega b \widetilde{\zeta} \, dx',
$$

which by density holds for ever $\widetilde{\zeta} \in V^q_{z_3, \Gamma_1}$, and so, by taking into account (4.14), it is equivalent to $(4.11)_{3,5,6}$. $\square$

Remark 3. We observe that if we consider the scaling for b_ε of order smaller than $\mathcal{O}(\varepsilon^{-1})$ in (2.6), then, we derive thermal homogeneous boundary condition $\partial_{z_3} \widetilde{T}^* = 0$ on Γ_0 in (4.11).

Finally, by solving the limit model obtained in Theorem 1, we derive the expressions for $\widetilde{\mathbf{u}}^\star$, $\widetilde{p}^\star$ and $\widetilde{T}^\star$.

Corollary 2. We have the following expressions for $\widetilde{\mathbf{u}}^\star$, $\widetilde{p}^\star$ and $\widetilde{T}^\star$:

- Velocity $\widetilde{\mathbf{u}}^\star \in V^2_{z_3, \Gamma_0 \cup \Gamma_1}$ is given by

$$
\widetilde{\mathbf{u}}^\star(x', z_3) = \frac{K}{\mu} \left(A_1^\star(x') e^{M z_3} + A_2^\star(x') e^{-M z_3} + 1 \right) (\mathbf{f}'(x') - \nabla_{x'} \widetilde{p}^\star(x')),
$$
(4.19)

  where $M = \sqrt{\frac{\mu}{K \mu_{\text{eff}}}}$ and

$$
A_1^*(x') = -\frac{1 - e^{-M h(x')}}{e^{M h(x')} - e^{-M h(x')}}, \quad A_2^*(x') = \frac{1 - e^{M h(x')}}{e^{M h(x')} - e^{-M h(x')}}. \quad (4.20)
$$

- Pressure $p^\star \in L_0^2(\omega) \cap H^1(\omega)$ is the unique solution of the Reynolds equation:

$$\operatorname{div}_{x'}\left(\frac{K}{\mu}\left(\frac{2}{M}\frac{e^{Mh(x')}-e^{-Mh(x')}-2}{e^{Mh(x')}-e^{-Mh(x')}}-h(x')\right)(\nabla_{x'}\widetilde{p}^\star - \mathbf{f}')\right)=0 \qquad (4.21)$$

in ω, with the following boundary condition on $\partial\omega$

$$\left(\frac{K}{\mu}\left(\frac{2}{M}\frac{e^{Mh(x')}-e^{-Mh(x')}-2}{e^{Mh(x')}-e^{-Mh(x')}}-h(x')\right)(\nabla_{x'}\widetilde{p}^\star-\mathbf{f}')\right)\cdot n = 0.$$

- The temperature $\widetilde{T}^\star \in V_{z_3,\Gamma_1}^q$ is given by

$$\widetilde{T}^\star(x',z_3) = -\frac{K}{k\mu}\left(V_1^\star(x',z_3) - V_1^\star(x',h(x'))\right)|\mathbf{f}'(x') - \nabla_{x'}\widetilde{p}^\star(x')|^2$$
$$-\frac{\mu_{\text{eff}}K^2M^2}{k\mu^2}\left(V_2^\star(x',z_3) - V_2^\star(x',h(x'))\right)|\mathbf{f}'(x') - \nabla_{x'}\widetilde{p}^\star(x')|^2$$
$$-\frac{b}{k}(z_3 - h(x'))|\mathbf{f}'(x') - \nabla_{x'}\widetilde{p}^\star(x')|^2,$$

$$(4.22)$$

with

$$V_1^\star(x',z_3) = \frac{1}{4M^2}\left(A_1^\star(x')^2\left(e^{2Mz_3}-1\right) + A_2^\star(x')^2\left(e^{-2Mz_3}-1\right)\right)$$
$$+\frac{2}{M^2}\left(A_1^\star(x')(e^{Mz_3}-1) + A_2^\star(x')(e^{Mz_3}-1)\right)$$
$$+\left(\frac{1}{2} + A_1^\star(x')A_2^\star(x')\right)z_3^2 - \frac{1}{2M}\left(A_1^\star(x')^2 - A_2^\star(x')^2\right)z_3$$
$$-\frac{2}{M}(A_1^\star(x') - A_2^\star(x'))z_3, \qquad (4.23)$$

$$V_2^\star(x',z_3) = \frac{1}{2}\left((A_1^\star)^2\left(e^{2Mh(x')}-e^{2Mz_3}\right) - (A_2^\star)^2\left(e^{-2Mh(x')}-e^{-2Mz_3}\right)\right)$$
$$+ A_1^\star A_2^\star(h(x')-z_3)^2. \qquad (4.24)$$

Proof.   Expression for $\widetilde{\mathbf{u}}^\star$ is the solution of

$$\begin{cases} -\mu_{\text{eff}}\partial_{z_3}^2\widetilde{\mathbf{u}}^\star + \dfrac{\mu}{K}\hat{\mathbf{u}}^\star = \mathbf{f}'(x') - \nabla_{x'}\widetilde{p}^\star(x') & \text{in } \Omega, \\ \hspace{6.5cm} \widetilde{\mathbf{u}}^\star = 0 & \text{on } \Gamma_0 \cup \Gamma_1, \end{cases}$$

which is a second-order linear differential equation with respect to the variable z_3 with homogeneous boundary conditions. The computations to derive (4.19)–(4.20) can be found in [20, Equations (3.3)–(3.6)], so we omit it.

According to the expression of $\widetilde{\mathbf{u}}^\star$ and divergence condition $(4.11)_2$, after computing $\int_0^{h(x')}\widetilde{\mathbf{u}}^\star\,dz_3$ and according to the expressions of $A_1^\star(x')$ and $A_2^\star(x')$, we deduce that $p^\star$ is the unique solution of (4.21).

The temperature is solution of Equation $(4.11)_{3,5,6}$ given by

$$
\begin{cases}
\partial_{z_3}^2 \hat{T}^\star = -\dfrac{\mu}{Kk}|\widetilde{\mathbf{u}}^\star|^2 - \dfrac{\mu_{\text{eff}}}{k}|\partial_{z_3}\widetilde{\mathbf{u}}^\star|^2 & \text{in } \Omega, \\[2mm]
\widetilde{T}^\star = 0 & \text{on } \Gamma_1, \\[1mm]
\partial_{z_3}\widetilde{T}^\star = -\dfrac{b}{k} & \text{on } \Gamma_0,
\end{cases}
$$

which is a second-order linear differential equation with respect to z_3 with Robin conditions. The solution is given by

$$
\begin{aligned}
\widetilde{T}^\star(x', z_3) = & -\frac{\mu}{Kk}\int_0^{z_3}\int_0^\tau |\widetilde{\mathbf{u}}^\star|^2\, ds\, d\tau - \frac{\mu_{\text{eff}}}{k}\int_0^{z_3}\int_0^\tau |\partial_{z_3}\widetilde{\mathbf{u}}^\star|^2 ds\, d\xi \\
& + B_1(x')z_3 + B_2(x'),
\end{aligned}
$$

with $B_1(x') = -\frac{b}{k}$ and

$$
B_2(x') = \frac{bh(x')}{k} + \frac{\mu}{Kk}\int_0^{h(x')}\int_0^\tau |\widetilde{\mathbf{u}}^\star|^2\, ds\, d\tau + \frac{\mu_{\text{eff}}}{k}\int_0^{h(x')}\int_0^\tau |\partial_{z_3}\widetilde{\mathbf{u}}^\star|^2 ds\, d\tau.
$$

Let us compute $\int_0^{z_3}\int_0^\tau |\widetilde{\mathbf{u}}^\star|^2\, ds\, d\tau$. We observe that

$$
\begin{aligned}
& \int_0^\tau \left(A_1^\star(x')e^{Ms} + A_2^\star(x')e^{-Ms} + 1\right)^2 ds \\
= & \frac{1}{2M}\left((A_1^\star)^2 e^{2M\tau} - (A_2^\star)^2 e^{-2M\tau}\right) + (1 + 2A_1^\star A_2^\star)\tau \\
& + \frac{2}{M}\left(A_1^\star e^{M\tau} - A_2^\star e^{-M\tau}\right) - \frac{1}{2M}\left((A_1^\star)^2 - (A_2^\star)^2\right) - \frac{2}{M}(A_1^\star - A_2^\star)
\end{aligned}
$$

and so, $V_1^\star(x', z_3) = \int_0^{z_3}\int_0^\tau \left(A_1^\star(x')e^{Ms} + A_2^\star(x')e^{-Ms} + 1\right)^2 ds\, d\tau$ is given by (4.23). This implies that

$$
\int_{z_3}^{h(x')}\int_0^\tau |\widetilde{\mathbf{u}}^\star|^2\, ds\, d\tau = \frac{K^2}{\mu^2}V_1^\star(x', y_3)|\mathbf{f}'(x') - \nabla_{x'}\widetilde{p}^\star(x')|^2.
$$

Finally, let us compute $\int_0^{z_3}\int_0^\tau |\partial_{z_3}\widetilde{\mathbf{u}}^\star|^2\, ds\, d\tau$. We observe that

$$
\partial_{z_3}\mathbf{u}^\star(x', z_3) = \frac{KM}{\mu}\left(A_1^\star(x')e^{Mz_3} + A_2^\star(x')e^{-Mz_3}\right)(\mathbf{f}'(x') - \nabla_{x'}p^\star),
$$

and, according to previous computations, we deduce

$$
\int_{z_3}^{h(x')}\int_0^\tau |\partial_{z_3}\mathbf{u}^\star|^2\, ds d\tau = \frac{K^2 M^2}{\mu^2}V_2^\star(x')|\mathbf{f}'(x') - \nabla_{x'}p^\star|^2,
$$

with $V_2^\star(x', z_3) = \int_{z_3}^{h(x')}\int_0^\tau \left(A_1^\star(x')e^{Ms} + A_2^\star(x')e^{-Ms}\right)^2 ds\, d\tau$ given by (4.24). From the above computations, we get (4.22). $\quad\square$

5   Concluding remarks

This study deals with the limit analysis of steady-state Darcy-Brinkman in a thin domain with a one-way coupling via a steady-state heat equation with particular focus on viscous dissipation. Using asymptotic methods with respect to the domain thickness, we propose a simplified model that accounts for both mechanical and thermal effects. The main obstacle is the nonlinear coupling via the viscous dissipation, which necessitates strong convergence for the velocity field and its derivatives. It should be emphasized that the effective model has been rigorously justified by proving the convergence results and can be used as an useful check for numerical simulations.

We remark that we can consider the Darcy-Lapwood-Brinkman equation which takes into account the viscous shear and inertia effects by incorporating the convective inertial term $\frac{\rho}{\phi^2}(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon$ in $(2.3)_1$, where ρ is the fluid density and ϕ is the porosity. Namely, it reads

$$\begin{cases} -2\mu_{\text{eff}}\operatorname{div}(\mathbb{D}[\mathbf{u}^\varepsilon]) + \frac{\mu}{K_\varepsilon}\mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}^\varepsilon + \frac{\rho}{\phi^2}(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon, \\ \operatorname{div}(\mathbf{u}^\varepsilon) = 0. \end{cases} \tag{5.1}$$

The corresponding thermodynamic closure of the problem (5.1) is obtained by considering the heat equation

$$-k\Delta T^\varepsilon = (\mathbf{u}^\varepsilon \cdot \nabla)T^\varepsilon + \frac{\mu}{K_\varepsilon}|\mathbf{u}^\varepsilon|^2 + 2\mu_{\text{eff}}|\mathbb{D}[\mathbf{u}^\varepsilon]|^2, \tag{5.2}$$

where we have introduced the convective term $(\mathbf{u}^\varepsilon \cdot \nabla)T^\varepsilon$ in $(2.3)_3$ as well. We remark that, at main order, due to the *a priori* estimates in thin domain, the convective terms would vanish, and to take into account them a further asymptotic development (lower order terms) is necessary. We believe that the analysis presented here can be extended to study the problem (5.1)–(5.2) and this will be the subject of our future work.

We also remark that other boundary conditions could be considered. For example, following [3], we can consider the following boundary conditions for velocity. Consider $\mathbf{g} = (g_1, g_2, g_3)$ such that

$$\int_{\partial\Omega^\varepsilon} \mathbf{g} \cdot \mathbf{n}\, d\sigma = 0,$$

such that

- No-slip condition: $\mathbf{u}^\varepsilon = \mathbf{g} = 0$ on Γ_1^ε.

- The velocity is known and parallel to the ω-plane: $\mathbf{u}^\varepsilon = \mathbf{g} = 0$ with $g_3 = 0$ on Γ_ℓ^ε.

- There is a no flux condition across ω so that $u_3^\varepsilon = g_3 = 0$. The tangential velocity on ω is unknown and satisfies the Tresca friction law with a friction coefficient k^ε:

$$\left.\begin{array}{l} |\sigma_t^\varepsilon| < k^\varepsilon \Rightarrow u_t^\varepsilon = s \\ |\sigma_t^\varepsilon| = k^\varepsilon \Rightarrow \exists\lambda \geq 0 \text{ such that } u_t^\varepsilon = s - \lambda\sigma_t^\varepsilon \end{array}\right\} \quad \text{on } \omega.$$

We believe that our study can be adapted to the case of a flow of a non-Newtonian fluid through a thin three-dimensional domain given by the following system

$$
\begin{cases}
-2\mu_{\text{eff}}\,\text{div}(|\mathbb{D}[\mathbf{u}^\varepsilon]|^{r-2}\mathbb{D}[\mathbf{u}^\varepsilon]) + \nabla p^\varepsilon = \mathbf{f}^\varepsilon, \\
\text{div}(\mathbf{u}^\varepsilon) = 0, \\
-k\Delta T^\varepsilon = 2\mu_{\text{eff}}|\mathbb{D}[\mathbf{u}^\varepsilon]|^r,
\end{cases} \tag{5.3}
$$

with boundary conditions (2.4)–(2.5). Also, the $\frac{\mu}{K_\varepsilon}\mathbf{u}^\varepsilon$ in $(2.3)_1$ could be replaced in the non-Newtonian setting by a drag force function in the spirit of [9], that is, a function $\mathcal{G} : \mathbb{R}^3 \to \mathbb{R}^3$ continuous, strictly monotone, satisfying an homogeneity condition $\mathcal{G}(\lambda\xi) = |\lambda|^{r-2}\lambda\mathcal{G}(\xi)$, $\lambda \in \mathbb{R}$, and the following condition

$$
m|\xi|^{r-1} \leq |\mathcal{G}(\xi)| \leq M|\xi|^{r-1},
$$

for some constants $m, M > 0$. Including this term in $(5.3)_1$ and the corresponding one in the viscous dissipation Φ in $(5.3)_3$, we would consider the generalization of the present study to the non-Newtonian setting, which will be a subject of our future work.

Acknowledgements

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