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On shifts of periodic zeta-function in short intervals

Marius Grigaliūnas[a], Antanas Laurinčikas[a] and Darius Šiaučiūnas[b] ✉

[a] *Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Lithuania*
[b] *Institute of Regional Development, Šiauliai Academy, Vilnius University, Lithuania*

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Abstract. The periodic zeta-function $\zeta(s;\mathfrak{a})$, $s=\sigma+it$, $\mathfrak{a}=\{a_m\in\mathbb{C}:m\in\mathbb{N}\}$, in the half-plane $\sigma>1$ is defined by Dirichlet series with periodic coefficients a_m, and has the meromorphic continuation to the whole complex plane. The function $\zeta(s;\mathfrak{a})$ is a generalization of the Riemann zeta-function and Dirichlet L-functions. In the paper, using only the periodicity of the sequence $\mathfrak{a}$, we obtain that the shifts $\zeta(s+i\tau;\mathfrak{a})$, $\tau\in\mathbb{R}$, approximate a certain class of analytic functions, defined in the strip $\{s\in\mathbb{C}:1/2<\sigma<1\}$. For $T^{23/70}\leqslant H\leqslant T^{1/2}$, the set of such shifts has a positive lower density in the interval $[T,T+H]$, $T\to\infty$. The case of positive density is also discussed. For the proof, the mean square estimate in short intervals for the Hurwitz zeta-function, and probabilistic limit theorems are applied.

Keywords: approximation of analytic functions; Dirichlet series; Hurwitz zeta-function; periodic zeta-function; weak convergence.

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✉ Corresponding author. E-mail: darius.siauciunas@sa.vu.lt

1 Introduction

Denote by $\mathbb{C}$ the set of all complex numbers, by $s=\sigma+it$ a complex variable, and let $\mathfrak{a}=\{a_m\in\mathbb{C}:m\in\mathbb{N}\}$. The series of the form

$$\sum_{m=1}^{\infty}\frac{a_m}{m^s} \tag{1.1}$$

are called (ordinary) Dirichlet series. A convergence region of (1.1) is a half-plane $\sigma>\sigma_c$. This means that there is a number $\sigma_c\in\mathbb{R}$ such that the series (1.1) is convergent for $\sigma>\sigma_c$, and divergent for $\sigma<\sigma_c$. The convergence on the line $\sigma=\sigma_c$ must be considered separately. Notice that there are Dirichlet series convergent for all s, and also there exist series divergent for all s.

In what follows, we will deal with Dirichlet series, therefore, we recall some classical facts of them. Denote by $f(s)$ the sum of the series (1.1) for $\sigma > \sigma_c$. Let $s_0 = \sigma_0 + it_0$, $\sigma_0 > \sigma_c$. Then the series (1.1) is uniformly convergent in any closed region lying in the half-plane $\sigma > \sigma_0$. Hence, the function $f(s)$ is analytic in the half-plane $\sigma > \sigma_c$.

Almost periodic analytic functions are represented by Dirichlet series. This class is not wide. However, Dirichlet series are widely used in analytic number theory because they are generating functions of arithmetic sequences, and by Dirichlet series very important zeta- and L-functions are defined. Among them, the famous Riemann zeta-function $\zeta(s)$, for $\sigma > 1$, given by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and Dirichlet L-functions $L(s,\chi)$ with Dirichlet characters χ defined by

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

Also, more general the so-called periodic zeta-functions are defined by Dirichlet series. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic zeta-function $\zeta(s;\mathfrak{a})$, for $\sigma > 1$, is defined by

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Since the periodic sequence is bounded, the latter series is absolutely convergent for $\sigma > 1$. Thus, the function $\zeta(s;\mathfrak{a})$ is analytic in the half-plane $\sigma > 1$. For analytic continuation of $\zeta(s;\mathfrak{a})$ to the region $\sigma \leqslant 1$, the Hurwitz zeta-function is applied. Let $0 < \alpha \leqslant 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s,\alpha)$, for $\sigma > 1$, is given by the Dirichlet series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

It is well known, see, for example, [25], that the function $\zeta(s,\alpha)$, similarly the functions $\zeta(s)$ and $L(s,\chi)$, has analytic continuation to the whole complex plane, except for the point $s = 1$ which is a simple pole with residue 1. The periodicity of $\mathfrak{a}$, for $\sigma > 1$, implies the equality

$$\zeta(s;\mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, l/q\right). \tag{1.2}$$

Therefore, by the properties of $\zeta(s,\alpha)$, the function $\zeta(s;\mathfrak{a})$ is analytic in the entire complex plane, except for the point $s = 1$ which is a simple pole with residue

$$r \stackrel{\text{def}}{=} \frac{1}{q} \sum_{l=1}^{q} a_l$$

provided $r \neq 0$. If $r = 0$, then the function $\zeta(s; \mathfrak{a})$ is entire.

In this paper, we are interested in approximation of some classes of analytic functions by shifts $\zeta(s + i\tau; \mathfrak{a})$, $\tau \in \mathbb{R}$. Recall that the first result of such a type was obtained for the functions $\zeta(s)$ and $L(s, \chi)$ in [29], and is called their universality. We observe that Dirichlet L-functions lie in the class of periodic zeta-functions with completely multiplicative coefficients. Their universality was widely studied in works by S.M. Voronin [28], S.M. Gonek [6], B. Bagchi [3], Ł. Pańkowski [?] and others. Moreover, a lot of universality results were obtained for zeta-functions with more general sequences $\{a_m\}$ including the Selberg-Steuding class [8,13,26], Matsumoto zeta-functions [12], zeta-functions of cusp forms [18], etc. For results and a wider bibliography, we recommend the excellent survey [24] prepared by Kohji Matsumoto, see also [11].

For the statements of results, we use some specific notation. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ denote the class of compact subsets of D with connected complements, $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K that are analytic inside of K, and $H_0(K)$ the subclass of $H(K)$ of non-vanishing on K functions. Moreover, let

$$\mathfrak{m}_T(\dots) = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \dots \right\},$$

where meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and in place of dots a condition satisfying by τ is to be written. The first universality theorem for the function $\zeta(s; \mathfrak{a})$ with non-multiplicative $\mathfrak{a}$ has been proven in [26], see also [2]. Suppose that $q > 2$, a_m is not a multiple of Dirichlet character modulo q, and $a_m = 0$ for $(m, q) > 1$. Let $K \in \mathcal{K}$, $g(s) \in H(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \mathfrak{m}_T \left(\sup_{s \in K} |g(s) - \zeta(s + i\tau; \mathfrak{a})| < \varepsilon \right) > 0. \tag{1.3}$$

As it follows from [22], in the above case the sequence $\mathfrak{a}$ is not a multiplicative. The universality of $\zeta(s; \mathfrak{a})$ with prime period q has been discussed in [14]. The first universality theorem for $\zeta(s; \mathfrak{a})$ with a multiplicative sequence $\mathfrak{a}$ ($a_{m_1 m_2} = a_{m_1} a_{m_2}$ for all $(m_1, m_2) = 1$) has been obtained in [19]. Let $K \in \mathcal{K}$ and $g(s) \in H_0(K)$. Then, for every $\varepsilon > 0$, inequality (1.3) holds. Later many people continued these studies in the field, see, for example, [9,10,27].

Approximation of analytic functions by discrete shifts $\zeta(s + ikh; \mathfrak{a})$ with $h > 0$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is possible as well. For example, assuming the multiplicativity of $\mathfrak{a}$ and linear independence over $\mathbb{Q}$ of the set (possible, multiset) $\{(h \log p : p \in \mathbb{P}), 2\pi\}$, $\mathbb{P}$ is the set of all prime numbers, it was obtained [23], that for every $K \in \mathcal{K}$, $g(s) \in H_0(K)$ and $\varepsilon > 0$, the set

$$\left\{ 0 \leqslant k \leqslant N : \sup_{s \in K} |g(s) - \zeta(s + ikh; \mathfrak{a})| < \varepsilon \right\}$$

has a positive lower density.

In the above cited works, the density of the set of approximating shifts is considered in interval $[0, T]$ of length T. For consideration approximating values of τ (or k), it is more convenient to deal with intervals as short as possible. In analytic number theory, the distribution of some objects, for example, prime numbers, zeros of the Riemann zeta-function, etc., see [7,15], are studied in the so-called short intervals, i. e., in intervals $[T, T+H]$ with $H = o(T)$ as $T \to \infty$. Results obtained in short intervals contain more information on investigated things.

Our aim is to show that the periodic zeta-functions have good properties with respect to approximation of analytic functions by shifts $\zeta(s + i\tau; \mathfrak{a})$ in short intervals.

Let

$$\mathbf{m}_{T,H}(\dots) = \frac{1}{H}\text{meas}\left\{\tau \in [T, T+H]; \dots\right\},$$

and let $H(D)$ stand for the space of analytic functions on D endowed with the topology of uniform convergence on compacta. In this topology, a sequence $\{f_n(s)\} \subset H(D)$ converges to $f(s) \in H(D)$ as $n \to \infty$ if, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \sup_{s\in K} |f_n(s) - f(s)| = 0.$$

Theorem 1. *Suppose that $T^{23/70} \leqslant H \leqslant T^{1/2}$. Then there exists a non-empty closed set $F_{\mathfrak{a}} \subset H(D)$, such that for every compact set $K \subset D$, function $g(s) \in F_{\mathfrak{a}}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \mathbf{m}_{T,H}\left(\sup_{s\in K} |g(s) - \zeta(s + i\tau; \mathfrak{a})| < \varepsilon\right) > 0,$$

and the limit

$$\lim_{T\to\infty} \mathbf{m}_{T,H}\left(\sup_{s\in K} |g(s) - \zeta(s + i\tau; \mathfrak{a})| < \varepsilon\right)$$

exists and positive for all but at most countably many $\varepsilon > 0$.

Theorem 1 is the first result on periodic zeta-functions in short intervals. In the case of the Riemann zeta-function, an universality theorem in short intervals with $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$ has been proven in [16]. The lower bound for H till $T^{1273/4053}$ was improved in [1]. Recently, Y. Lee and Ł. Pańkowski proved in [21] for discs under the Riemann hypothesis that one can take $H = (\log T)^B$ with some $B > 0$ depending on a disc. We notice that Theorem 1 is only the first step in characterization of approximation by shifts $\zeta(s+i\tau; \mathfrak{a})$ in short intervals. A big problem of identification of the set $F_{\mathfrak{a}}$ remains open, obviously, it is closely connected to the sequence $\mathfrak{a}$.

Notice that Theorem 1, in view of (2.1), remains true for $H = T$. We expect that it is also true in the range $T^{1/2} < H < T$. On the other hand, we stress that the priority of our investigation is the lower bound of H.

Theorem 1 will be proved in Section 4. Section 2 is devoted to some estimates of the mean type. In Section 3, we prove a limit theorem on weakly convergent probability measures in the space $H(D)$.

2 Some estimates in short intervals

Let $\delta > 1/2$ be a fixed number, and $\kappa_n(m) = \exp\{-(m/n)^\delta\}$. Define the series

$$\zeta_n(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \kappa_n(m)}{m^s}, \quad n \in \mathbb{N}.$$

Then, the latter series converges absolutely in every half-plane $\sigma > \widehat{\sigma}$ with arbitrary $\widehat{\sigma}$. In this section, we consider approximation of $\zeta(s;\mathfrak{a})$ by $\zeta_n(s;\mathfrak{a})$ in the mean in short intervals. For this, we need mean square estimates in short intervals.

Lemma 1. [7]. *Let (κ, λ) be an exponential pair and $1/2 < \sigma < 1$ fixed. Then, for $T^{(\kappa+\lambda+1-2\sigma)/(2(\kappa+1))}(\log T)^{(2+\kappa)/\kappa+1} \leqslant H \leqslant T$, $1+\lambda-\kappa \geqslant 2\sigma$, we have uniformly in H*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \, \mathrm{d}t \ll_\sigma H.$$

Lemma 2. *For $T^{23/70} \leqslant H \leqslant T$ and fixed $1/2 < \sigma \leqslant 31/52$, uniformly in H*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \, \mathrm{d}t \ll_\sigma H.$$

Proof. The lemma follows from Lemma 1 by taking the exponent pair (9/26, 7/13) [5]. □

Lemma 3. *For $T^{23/70} \leqslant H \leqslant T^\sigma$ and fixed $1/2 < \sigma \leqslant 31/52$, uniformly in H*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it,\alpha)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} H.$$

Proof. We repeat the arguments used in the proof of Theorem 2 of [20] with the exponent pair $(9/26, 7/13)$ in place of $(11/30, 16/30)$. □

We also expect that the lemma remains true in the less interesting case $T^\sigma < H \leqslant T$, however, there arise some problems related to the approximate functional equation for $\zeta(s,\alpha)$. For $H = T$, the lemma is valid because, for fixed $\sigma > 1/2$,

$$\int_{T-T}^{T+T} |\zeta(\sigma+it,\alpha)|^2 \, \mathrm{d}t = \int_{0}^{2T} |\zeta(\sigma+it,\alpha)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} T \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} \ll_{\sigma,\alpha} T, \tag{2.1}$$

see, for example, Theorem 3.3.1 of [17].

Lemma 4. *For $T^{23/70} \leqslant H \leqslant T^\sigma$ and fixed $1/2 < \sigma \leqslant 31/52$, uniformly in H*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it;\mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma,\mathfrak{a}} H.$$

Proof. In view of equality (1.2), we have

$$\zeta(s;\mathfrak{a}) = \frac{a_q}{q^s}\zeta(s) + \frac{1}{q^s}\sum_{l=1}^{q-1} a_l \zeta\left(s, \frac{l}{q}\right).$$

Therefore,

$$|\zeta(s;\mathfrak{a})|^2 \ll_{\mathfrak{a}} |\zeta(s)|^2 + \sum_{l=1}^{q-1}\left|\zeta\left(s, \frac{l}{q}\right)\right|^2.$$

Hence, the lemma follows by Lemmas 2 and 3. □

For the statement of the main result of the section, we recall metric d in the space $H(D)$ inducing its topology. Thus, for $f_1, f_2 \in H(D)$,

$$d(f_1, f_2) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s\in K_j} |f_1(s) - f_2(s)|}{1 + \sup_{s\in K_j} |f_1(s) - f_2(s)|}.$$

Here $\{K_j : j \in \mathbb{N}\}$ is a sequence of compact embedded sets such that

$$\bigcup_{j=1}^{\infty} K_j = D,$$

and every compact set $K \subset D$ lies in some set K_j.

Theorem 2. *Suppose that* $T^{23/70} \leqslant H \leqslant T^{1/2}$. *Then, for all* $\mathfrak{a}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H}\int_T^{T+H} d\big(\zeta_n(s+i\tau;\mathfrak{a}), \zeta(s+i\tau;\mathfrak{a})\big)\,\mathrm{d}\tau = 0.$$

Proof. By the definition of the metric d, it suffices to show that, for all compact sets $K \subset D$, the equality

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H}\int_T^{T+H} \sup_{s\in K}\big|\zeta_n(s+i\tau;\mathfrak{a}) - \zeta(s+i\tau;\mathfrak{a})\big|\,\mathrm{d}\tau = 0 \tag{2.2}$$

holds. Let us fix a compact set $K \subset D$ and a periodic sequence $\mathfrak{a}$. Thus, there exists a number $0 < \delta_1 \leqslant 5/52$ such that all $s = \sigma + it \in K$ lie in the strip $1/2 + 2\delta_1 \leqslant \sigma \leqslant 1 - \delta_1$. Suppose that δ is from the definition of $\kappa_n(m)$, and $\delta = 1/2 + \delta_1$. Then the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(z)a^{-z}\,\mathrm{d}z = \mathrm{e}^{-a}, \quad a, b > 0,$$

where $\Gamma(s)$ is the Euler gamma-function, implies the representation

$$\zeta_n(s;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \zeta(s+z;\mathfrak{a}) w_n(z)\,\mathrm{d}z \tag{2.3}$$

for $s \in K$, where $w_n(z) = \delta^{-1}\Gamma(\delta^{-1}z)n^z$.

Let $\delta_2 = 1/2 + \delta_1 - \sigma$. Then $\delta_2 < 0$ and $\delta_2 > -\delta$. This, properties of the function $\Gamma(s)$, (2.3), and the residue theorem give

$$\zeta_n(s;\mathfrak{a}) - \zeta(s;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\delta_2-i\infty}^{\delta_2+i\infty} \zeta(s+z;\mathfrak{a}) w_n(z)\,\mathrm{d}z + r w_n(1-s).$$

Hence, for $s \in K$,

$$\begin{aligned}\zeta_n(s;\mathfrak{a}) - \zeta(s;\mathfrak{a}) =& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \zeta\left(\frac{1}{2}+\delta_1-\sigma+\sigma+it+iu;\mathfrak{a}\right) \\ & \times w_n\left(\frac{1}{2}+\delta_1-\sigma+iu\right)\mathrm{d}u + r w_n(1-s).\end{aligned}$$

Therefore, shifting $t+u$ to u, we obtain

$$\begin{aligned}\sup_{s\in K}\left|\zeta_n(s+i\tau;\mathfrak{a}) - \zeta(s+i\tau,\mathfrak{a})\right| \ll \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\delta_1+i\tau+iu;\mathfrak{a}\right)\right| \\ \times \sup_{s\in K}\left|w_n\left(\frac{1}{2}+\delta_1-s+iu\right)\right|\mathrm{d}u + |r|\sup_{s\in K}|w_n(1-s-i\tau)|. \qquad (2.4)\end{aligned}$$

For the gamma-function $\Gamma(s)$, the estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c>0, \qquad (2.5)$$

is valid uniformly in σ in any fixed finite interval. Thus, by the definition of $w_n(z)$, for $s \in K$,

$$\begin{aligned} w_n\left(1/2+\delta_1-s+iu\right) &\ll_{\delta_1} n^{1/2+\delta_1-\sigma}\exp\left\{-\frac{c}{\delta_1}|u-t|\right\} \\ &\ll_K n^{-\delta_1}\exp\{-c_1|u|\}, \quad c_1>0. \qquad (2.6)\end{aligned}$$

Moreover, by (1.2) and the estimate [25],

$$\zeta(\sigma+it,\alpha) \ll_{\sigma,\alpha} |t|^{1/2}, \quad \sigma \geqslant \frac{1}{2},\ |t| \geqslant 2,$$

we have

$$\zeta(\sigma+it,\mathfrak{a}) \ll_{\sigma,\mathfrak{a}} |t|^{1/2}, \quad \sigma \geqslant \frac{1}{2},\ |t| \geqslant 2.$$

Hence, for $\tau \geqslant T$,

$$\begin{aligned}&\left(\int_{-\infty}^{-\log^2 T} + \int_{\log^2 T}^{\infty}\right)\left|\zeta\left(\frac{1}{2}+\delta_1+i\tau+iu;\mathfrak{a}\right)\right|\sup_{s\in K}\left|w_n\left(\frac{1}{2}+\delta_1-s+iu\right)\right|\mathrm{d}u \\ &\ll_{\mathfrak{a},K} n^{-\delta_1}\left(\int_{-\infty}^{-\log^2 T} + \int_{\log^2 T}^{\infty}\right)(\tau+|u|)^{1/2}\exp\{-c_1|u|\}\,\mathrm{d}u \\ &\ll_{\mathfrak{a},K} n^{-\delta_1}(\tau+1)^{1/2}\exp\{-c_2\log^2 T\}, \quad c_2>0.\end{aligned}$$

Therefore, by (2.4),

$$\begin{aligned} I &\stackrel{\text{def}}{=} \frac{1}{H}\int_T^{T+H} \sup_{s\in K} |\zeta_n(s+i\tau;\mathfrak{a})-\zeta(s+i\tau;\mathfrak{a})|\,\mathrm{d}\tau \\ &\ll_{\mathfrak{a},K} \int_{-\log^2 T}^{\log^2 T} \left(\frac{1}{H}\int_T^{T+H} \left|\zeta\left(\frac{1}{2}+\delta_1+i\tau+iu;\mathfrak{a}\right)\right|\mathrm{d}\tau\right) \\ &\times \sup_{s\in K}\left|w_n\left(1/2+\delta_1-s+iu\right)\right|\mathrm{d}u + \frac{1}{H}\int_T^{T+H} \sup_{s\in K}|w_n(1-s-i\tau)|\,\mathrm{d}\tau \\ &+ n^{-\delta_1}\exp\{-c_2\log^2 T\}\frac{1}{H}\int_T^{T+H}(\tau+1)^{1/2}\,\mathrm{d}\tau \stackrel{\text{def}}{=} J_1+J_2+J_3. \end{aligned} \tag{2.7}$$

Trivially, we have

$$J_3 \ll_{\mathfrak{a},K} n^{-\delta_1}\exp\{-c_3\log^2 T\}, \quad c_3>0. \tag{2.8}$$

For estimation of J_1, we apply Lemma 4. Since $0<\delta_1\leqslant 5/52$, the inequality $1/2+\delta_1\leqslant 31/52$ is true, and, for $|u|\leqslant \log^2 T$,

$$T^{23/70}\leqslant H+|u|\leqslant T^{1/2}+\log^2 T \ll T^{1/2+\delta_1}.$$

Thus, Lemma 4 is applicable, and we find

$$\begin{aligned} &\frac{1}{H}\int_T^{T+H}\left|\zeta\left(1/2+\delta_1+i\tau+iu;\mathfrak{a}\right)\right|\mathrm{d}\tau \\ &\leqslant \left(\frac{1}{H}\int_{T-H-|u|}^{T+H+|u|}\left|\zeta\left(1/2+\delta_1+i\tau;\mathfrak{a}\right)\right|^2\mathrm{d}\tau\right)^{1/2} \\ &\ll_{\mathfrak{a},K}\left(\frac{1}{H}(H+|u|)\right)^{1/2} \ll_{\mathfrak{a},K} (|u|+1)^{1/2}. \end{aligned}$$

This together with (2.6) and (2.7) yields

$$J_1 \ll_{\mathfrak{a},K} n^{-\delta_1}\int_{-\log^2 T}^{\log^2 T}(|u|+1)^{1/2}\exp\{-c_1|u|\}\,\mathrm{d}u \ll_{\mathfrak{a},K} n^{-\delta_1}. \tag{2.9}$$

Using once more (2.5), we obtain that, for $s\in K$,

$$w_n(1-s-i\tau)\ll n^{1-\sigma}\exp\{-c|t+\tau|\}\ll_K n^{1/2-2\delta_1}\exp\{-c_4|\tau|\}, \quad c_4>0.$$

Therefore, by (2.7),

$$J_2 \ll_{\mathfrak{a},K} \frac{n^{1/2-2\delta_1}}{H}\int_T^{T+H}\exp\{-c_4|\tau|\}\,\mathrm{d}\tau \ll_{\mathfrak{a},K} n^{1/2-2\delta_1}\exp\{-c_4 T\}.$$

This, and (2.7)–(2.9) show the bound

$$I \ll_{\mathfrak{a},K} n^{-\delta_1}+n^{1/2-2\delta_1}\exp\{-c_4T\}+n^{-\delta_1}\exp\{-c_3\log^2 T\}.$$

Taking $T\to\infty$ and then $n\to\infty$, we find $\lim_{n\to\infty}\limsup_{T\to\infty} I=0$. Thus, (2.2) is valid, and this proves the theorem. □

3 Weak convergence

Let $\mathbb{X}$ be a topological space with its Borel σ-field $\mathcal{B}(\mathbb{X})$, and P and P_n, $n \in \mathbb{N}$, probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. By the definition, P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$), if, for every real bounded continuous function f on $\mathbb{X}$,

$$\lim_{n\to\infty} \int_{\mathbb{X}} f \,\mathrm{d}P_n = \int_{\mathbb{X}} f \,\mathrm{d}P. \tag{3.1}$$

For $A \subset \mathbb{X}$, denote by ∂A the boundary of the set A. If $P(\partial A) = 0$, then A is called a continuity set of P. Then (3.1) is equivalent to

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G)$$

for every open set $G \subset \mathbb{X}$, or

$$\lim_{n\to\infty} P_n(A) = P(A)$$

for every continuity set A of P [4].

In this section, we deal with weak convergence for some probability measures on $(H(D), \mathcal{B}(H(D)))$ defined by means of the function $\zeta(s; \mathfrak{a})$ is short intervals. More precisely, we will study

$$P_{T,H,\mathfrak{a}}(A) = \mathbf{m}_{T,H}(\zeta(s + i\tau; \mathfrak{a}) \in A), \quad A \in \mathcal{B}(H(D)),$$

for $T^{23/70} \leqslant H \leqslant T^{1/2}$ as $T \to \infty$. The main result of the section is the following statement.

Theorem 3. *Suppose that $T^{23/70} \leqslant H \leqslant T^{1/2}$. Then, on $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P_{\mathfrak{a}}$ such that $P_{T,H,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\mathfrak{a}}$.*

Proof of Theorem 3 is quite standard, however, for a convenience of the readers, we will present it in almost full form.

We start with a case of certain topological group. Denote by $\mathbb{P}$ the set of all prime numbers, and define the torus

$$\mathbb{T} = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

With the product topology and operation of pointwise multiplication, the set $\mathbb{T}$ is a compact topological group. Let $\mathfrak{t} = (t(p) : p \in \mathbb{P})$ be elements of $\mathbb{T}$. For $A \in \mathcal{B}(\mathbb{T})$, set

$$Q^{\mathbb{T}}_{T,H}(A) = \mathbf{m}_{T,H}\left(\left(p^{-i\tau} : p \in \mathbb{P}\right) \in A\right).$$

Lemma 5. *Suppose that $H \to \infty$ as $T \to \infty$. Then, on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, there exists a probability measure $Q^{\mathbb{T}}$ such that $Q^{\mathbb{T}}_{T,H} \xrightarrow[T\to\infty]{\mathrm{w}} Q^{\mathbb{T}}$.*

Proof. Consider the Fourier transform $f_{T,H}$ of $Q^{\mathbb{T}}_{T,H}$ as $T \to \infty$. The characters of $\mathbb{T}$ are of the form

$$\prod_{p\in\mathbb{P}}^{*} t^{k_p}(p),$$

where the star means that only a finite number of the integer numbers k_p are non-zeros. Therefore, denoting $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, we have

$$\begin{aligned} f_{T,H}(\underline{k}) &= \int_{\mathbb{T}} \left(\prod_{p\in\mathbb{P}}^{*} t^{k_p}(p) \right) \mathrm{d}Q^{\mathbb{T}}_{T,H} = \frac{1}{H} \int_T^{T+H} \left(\prod_{p\in\mathbb{P}}^{*} p^{-ik_p\tau}(p) \right) \mathrm{d}\tau \\ &= \frac{1}{H} \int_T^{T+H} \exp\left\{ -i\tau \sum_{p\in\mathbb{P}}^{*} k_p \log p \right\} \mathrm{d}\tau. \end{aligned} \tag{3.2}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers, from (3.2), we have

$$f_{T,H}(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} = \underline{0}, \\ \dfrac{\exp\left\{-iT \sum^{*}_{p\in\mathbb{P}} k_p \log p\right\} - \exp\left\{-i(T+H) \sum^{*}_{p\in\mathbb{P}} k_p \log p\right\}}{iH \sum^{*}_{p\in\mathbb{P}} k_p \log p}, & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Hence,

$$\lim_{T\to\infty} f_{t,H}(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} = \underline{0}, \\ 0, & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and the proof is complete. □

The next lemma concerns the weak convergence of

$$P_{T,H,n,\mathfrak{a}}(A) \stackrel{\text{def}}{=} \mathbf{m}_{T,H}\left(\zeta_n(s+i\tau;\mathfrak{a}) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Lemma 6. *Suppose that $H \to \infty$ as $T \to \infty$. Then, on $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P_{n,\mathfrak{a}}$ such that $P_{T,H,n,\mathfrak{a}} \xrightarrow[T\to\infty]{\text{w}} P_{n,\mathfrak{a}}$.*

Proof. We apply Lemma 5 and the weak convergence preservation principle under continuous mappings, see Section 5 of [4]. Define a mapping $u_{n,\mathfrak{a}} : \mathbb{T} \to H(D)$ by

$$u_{n,\mathfrak{a}}(\mathfrak{t}) = \sum_{m=1}^{\infty} \frac{a_m \kappa_n(m) t(m)}{m^s},$$

where, for $m \in \mathbb{N}$,

$$t(m) = \prod_{p^l | m,\ p^{l+1} \nmid m} t^l(p).$$

The series defining $u_{n,\mathfrak{a}}$ is absolutely convergent for $\sigma > \widehat{\sigma}$ with every finite $\widehat{\sigma}$. Therefore, the mapping $u_{n,\mathfrak{a}}$ is continuous. Moreover,

$$\begin{aligned} u_{n,\mathfrak{a}}\left((p^{-i\tau} : p \in \mathbb{P})\right) &= \sum_{m=1}^{\infty} \frac{a_m \kappa_n(m) \prod_{\substack{p^l|m \\ p^{l+1}\nmid m}} p^{-li\tau}}{\prod_{\substack{p^l|m \\ p^{l+1}\nmid m}} p^{ls}} = \sum_{m=1}^{\infty} \frac{a_m \kappa_n(m)}{m^{s+i\tau}} \\ &= \zeta_n(s+i\tau;\mathfrak{a}). \end{aligned}$$

Hence,

$$P_{T,H,n,\mathfrak{a}}(A) = \mathbf{m}_{T,H}\left(u_{n,\mathfrak{a}}\left((p^{-i\tau} : p \in \mathbb{P})\right) \in A\right)$$
$$= \mathbf{m}_{T,H}\left((p^{-i\tau} : p \in \mathbb{P}) \in u_{n,\mathfrak{a}}^{-1}A\right) = Q_{T,H}^{\mathbb{T}}(u_{n,A}^{-1}A)$$

for all $A \in \mathcal{B}(H(D)$. Thus, we have $P_{T,H,n,\mathfrak{a}} = Q_{T,H}^{\mathbb{T}} u_{n,\mathfrak{a}}^{-1}$, where, for all $A \in \mathcal{B}(H(D))$,

$$Q_{T,H}^{\mathbb{T}} u_{n,\mathfrak{a}}^{-1}(A) = Q_{T,H}^{\mathbb{T}}(u_{n,\mathfrak{a}}^{-1}A).$$

Therefore, Theorem 5.1 of [4], Lemma 5 and continuity of $u_{n,\mathfrak{a}}$ imply the relation

$$P_{T,H,n,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathrm{w}} Q^{\mathbb{T}} u_{n,\mathfrak{a}}^{-1},$$

where $Q^{\mathbb{T}}$ is the limit measure in Lemma 5. Thus, $P_{n,\mathfrak{a}} = Q^{\mathbb{T}} u_{n,\mathfrak{a}}^{-1}$. $\square$

Now we consider the measure $P_{n,\mathfrak{a}}$ as $n \to \infty$.

Lemma 7. *The measure $P_{n,\mathfrak{a}}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K_\varepsilon \subset H(D)$ such that $P_{n,\mathfrak{a}}(K) > 1 - \varepsilon$, for all $n \in \mathbb{N}$.*

Proof. In virtue of absolute convergence of the series $\zeta_n(s;\mathfrak{a})$, it follows that, for $\sigma > 1/2$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\zeta_n(\sigma+it;\mathfrak{a})|^2\,\mathrm{d}t = \sum_{m=1}^{\infty} \frac{|a_m|^2\kappa_n^2(m)}{m^{2\sigma}}.$$

Thus, by Cauchy integral formula, with a certain $\sigma_1 > 1/2$

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_T^{2T} \sup_{s\in K_j} |\zeta_n(s+i\tau;\mathfrak{a})|^2\,\mathrm{d}\tau \ll_{K_j} \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma_1}} \leqslant R_j < \infty, \quad (3.3)$$

where K_j are compact sets from the definition of the metric d.

On a certain probability space $(\Omega, \mathcal{B}, P)$, define a random variable θ_T which is uniformly distributed in the interval $[T, 2T]$. Introduce the $H(D)$-valued random element

$$y_{T,n,\mathfrak{a}} = y_{T,n,\mathfrak{a}}(s) = \zeta_n(s+i\theta_T;\mathfrak{a}).$$

Moreover, let $y_{n,\mathfrak{a}}(s)$ be $H(D)$-valued random element with the distribution $P_{n,\mathfrak{a}}$. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then, in view of Lemma 6,

$$y_{T,n,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} y_{n,\mathfrak{a}}.$$

Hence,

$$\sup_{s\in K_j} \left|y_{T,n,\mathfrak{a}}(s)\right| \xrightarrow[T\to\infty]{\mathcal{D}} \sup_{s\in K_j} \left|y_{n,\mathfrak{a}}(s)\right|. \quad (3.4)$$

Fix $\varepsilon > 0$, and set $M_j = 2^{-j}\varepsilon^{-1}R_j^2$, $j \in \mathbb{N}$. Then, (3.3) and (3.4) imply

$$\begin{aligned}\limsup_{T\to\infty} P\Big\{\sup_{s\in K_j}\big|y_{T,n,\mathfrak{a}}(s)\big| \geqslant M_j\Big\} &= P\Big\{\sup_{s\in K_j}\big|y_{n,\mathfrak{a}}(s)\big| \geqslant M_j\Big\}\\ = \mathbf{m}_{T,T}\Big(\sup_{s\in K_j}\big|\zeta_n(s+i\tau;\mathfrak{a})\big| \geqslant M_j\Big) &\leqslant \frac{1}{TM_j}\int_T^{2T}\sup_{s\in K_j}\big|\zeta_n(s+i\tau;\mathfrak{a})\big|\,\mathrm{d}\tau\\ \leqslant \frac{1}{M_j}\left(\frac{1}{T}\int_T^{2T}\sup_{s\in K_j}\big|\zeta_n(s+i\tau;\mathfrak{a})\big|^2\,\mathrm{d}\tau\right)^{1/2} &\leqslant \frac{\varepsilon}{2^j}. \qquad (3.5)\end{aligned}$$

Let

$$K_\varepsilon = \Big\{f \in H(D) : \sup_{s\in K_j}|f(s)| \leqslant M_j, j \in \mathbb{N}\Big\}.$$

The set K_ε is uniformly bounded on compact subsets of $H(D)$, hence, it is compact. Moreover, by (3.5),

$$P\left\{y_{n,\mathfrak{a}} \in K_\varepsilon\right\} = 1 - P\left\{y_{n,\mathfrak{a}} \notin K_\varepsilon\right\} \geqslant 1 - \varepsilon\sum_{j=1}^{\infty}\frac{1}{2^j} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Since $P_{n,\mathfrak{a}}$ is the distribution of $y_{n,\mathfrak{a}}$, this shows that

$$P_{n,\mathfrak{a}}(K_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. The lemma is proven. □

To pass from the measure $P_{T,H,n,\mathfrak{a}}$ to $P_{T,H,\mathfrak{a}}$, we need one more lemma.

Lemma 8. *Let x_{nk}, x_k and y_n, $n, k \in \mathbb{N}$ be $H(D)$-valued random elements on arbitrary probability space $(\Omega, \mathcal{B}, P)$. Suppose that*

$$x_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} x_k, \qquad x_k \xrightarrow[k\to\infty]{\mathcal{D}} x,$$

and, for every $\varepsilon > 0$,

$$\lim_{k\to\infty}\limsup_{n\to\infty} P\{d(x_{nk}, y_n) \geqslant \varepsilon\} = 0.$$

Then, $y_n \xrightarrow[n\to\infty]{\mathcal{D}} x$.

Proof. Since the space $H(D)$ is separable, the lemma is a particular case of Theorem 4.2 of [4]. □

Proof. [Proof of Theorem 3] Let the random variable $\theta_{T,H}$ be defined on the probability space $(\Omega, \mathcal{B}, P)$, and distributed uniformly in the interval $[T, T+H]$. Define the $H(D)$-valued random elements

$$\begin{aligned}X_{T,H,n,\mathfrak{a}} &= X_{T,H,n,\mathfrak{a}}(s) = \zeta_n(s + i\theta_{T,H};\mathfrak{a}),\\ X_{T,H,\mathfrak{a}} &= X_{T,H,\mathfrak{a}}(s) = \zeta(s + i\theta_{T,H};\mathfrak{a}),\end{aligned}$$

and let $X_{n,\mathfrak{a}}$ have the distribution $P_{n,\mathfrak{a}}$. By Lemma 6,

$$X_{T,H,n,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\mathfrak{a}}. \tag{3.6}$$

Since, by Lemma 7, the measure $P_{n,\mathfrak{a}}$ is tight, in virtue of the Prokhorov theorem, Theorem 6.1 of [4], it is relatively compact. Therefore, there is a sequence $n_l \to \infty$ and a probability measure $P_{\mathfrak{a}}$ on $(H(D), \mathcal{B}(H(D)))$ such that $P_{n_l,\mathfrak{a}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\mathfrak{a}}$. Hence,

$$X_{n_l,\mathfrak{a}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\mathfrak{a}}. \tag{3.7}$$

Now, we apply Lemma 5. For every $\varepsilon > 0$,

$$P\left\{d\left(X_{T,H,n,\mathfrak{a}}(s), X_{T,H,\mathfrak{a}}(s)\right) \geqslant \varepsilon\right\} = \mathbf{m}_{T,H}\left\{d\left(\zeta_n(s+i\tau;\mathfrak{a}), \zeta(s+i\tau;\mathfrak{a})\right) \geqslant \varepsilon\right\}$$
$$\leqslant \frac{1}{H\varepsilon}\int_T^{T+H} d\left(\zeta_n(s+i\tau;\mathfrak{a}), \zeta(s+i\tau;\mathfrak{a})\right)\,\mathrm{d}\tau.$$

Therefore, Theorem 2 yields

$$\lim_{l\to\infty}\limsup_{T\to\infty} P\left\{d\left(X_{T,H,n_l,\mathfrak{a}}(s), X_{T,H,\mathfrak{a}}(s)\right) \geqslant \varepsilon\right\} = 0.$$

The latter equality, and relations (3.6) and (3.7) show that Lemma 8 is applicable to random elements $X_{T,H,n_l,\mathfrak{a}}$, $X_{n_l,\mathfrak{a}}$ and $X_{T,H,\mathfrak{a}}$. Thus, by Lemma 8 and (3.7), we obtain

$$X_{T,H,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\mathfrak{a}},$$

what is equivalent to $P_{T,H,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\mathfrak{a}}$. The theorem is proven. □

4 Approximation results

In this section, we prove Theorem 1. It is a simple consequence of Theorem 3. We recall that the support of the measure $P_{\mathfrak{a}}$ is a minimal closed set $S_{\mathfrak{a}} \subset H(D)$ such that $P_{\mathfrak{a}}(S_{\mathfrak{a}}) = 1$. The set consists of all elements $f \in H(D)$ satisfying $P_{\mathfrak{a}}(G_f) > 0$ for any open neighbourhood G_f of f.

Proof. [Proof of Theorem 1] We take $F_{\mathfrak{a}} = S_{\mathfrak{a}}$. Then $F_{\mathfrak{a}}$ is non-empty closed set because $P_{\mathfrak{a}}(F_{\mathfrak{a}}) = 1$. If $g \in F_{\mathfrak{a}}$, then,

$$G_{g,\varepsilon} = \left\{f \in H(D) : \sup_{s\in K} |f(s) - g(s)| < \varepsilon\right\}$$

is an open neighborhood of an element g of the support of the measure $P_{\mathfrak{a}}$. Hence, by a support property,

$$P_{\mathfrak{a}}(G_{g,\varepsilon}) > 0. \tag{4.1}$$

Therefore, by Theorem 3 and the equivalent of weak convergence in terms of open sets, we have

$$\liminf_{T\to\infty} P_{T,H,\mathfrak{a}}(G_{g,\varepsilon}) \geqslant P_{\mathfrak{a}}(G_{g,\varepsilon}) > 0.$$

This, and the definitions of $P_{T,H,\mathfrak{a}}$ and $G_{g,\varepsilon}$ give the first statement of the theorem.

For the proof of the second statement of the theorem, we apply the equivalent of weak convergence in terms of continuity sets. Observe, that the set $G_{g,\varepsilon}$ is a continuity set of the measure $P_{\mathfrak{a}}$ for all but at most countably many $\varepsilon > 0$. Actually, the boundary $\partial G_{g,\varepsilon}$ lies in the set

$$\Big\{ f \in H(D) : \sup_{s\in K} |f(s) - g(s)| = \varepsilon \Big\},$$

therefore, $\partial G_{g,\varepsilon_1} \cap \partial G_{g,\varepsilon_2} = \varnothing$ for positive ε_1, ε_2, $\varepsilon_1 \neq \varepsilon_2$. Hence, it follows that $P_{\mathfrak{a}}(\partial G_{g,\varepsilon}) > 0$ for at most countably many $\varepsilon > 0$, i.e., $P_{\mathfrak{a}}(\partial G_{g,\varepsilon}) = 0$ for all but at most countably many $\varepsilon > 0$.

Now, in view of Theorem 3, the above remark, the equivalent of weak convergence in terms of continuity sets and (4.1) imply the limit

$$\lim_{T\to\infty} P_{T,H,\mathfrak{a}}(G_{g,\varepsilon}) = P_{\mathfrak{a}}(G_{g,\varepsilon}) > 0$$

for all but at most countably many $\varepsilon > 0$. This proves the second statement of the theorem. The proof is complete. □

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References

[1] J. Andersson, R. Garunkštis, R. Kačinskaitė, K. Nakai, Ł. Pańkowski, A. Sourmelidis, R. Steuding, J. Steuding and S. Wananiyakul. Notes on universality in short intervals and exponential shifts. *Lith. Math. J.*, **64**(2):125–137, 2024. https://doi.org/10.1007/s10986-024-09631-5.

[2] B. Bagchi. *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series.* PhD Thesis, Indian Statistical Institute, Calcutta, 1981.

[3] B. Bagchi. A joint universality theorem for Dirichlet *L*-functions. *Math. Z.*, **181**(3):319–334, 1982. https://doi.org/10.1007/BF01161980.

[4] P. Billingsley. *Convergence of Probability Measures.* Willey, New York, 1968.

[5] J. Bourgain and N. Watt. Decoupling for perturbed cones and the mean square of $|\zeta(1/2 + it)|$. *Int. Math. Res. Not.*, **2018**(17):5219–5296, 2018. https://doi.org/10.1093/imrn/rnx009.

[6] S.M. Gonek. *Analytic Properties of Zeta and L-Functions.* PhD Thesis, University of Michigan, Ann Arbor, 1979.

[7] A. Ivič. *The Riemann Zeta-Function.* The theory of the Riemann zeta-function with applications, John Wiley & Sons, Inc., New York, 1985.

[8] R. Kačinskaitė. On discrete universality in the Selberg–Steuding class. *Sib. Math. J.*, **63**(2):277–285, 2022. https://doi.org/10.1134/S0037446622020069.

[9] R. Kačinskaitė and B. Kazlauskaitė. Two results related to the universality of zeta-functions with periodic coefficients. *Results Math.*, **73**(3):article. no. 95, 2018. https://doi.org/10.1007/s00025-018-0856-z.

[10] R. Kačinskaitė and A. Laurinčikas. The joint distribution of periodic zeta-functions. *Studia Sci. Math. Hungar.*, **48**(2):257–279, 2011. https://doi.org/10.1556/sscmath.48.2011.2.1162.

[11] R. Kačinskaitė and K. Matsumoto. A survey of the theory of mixed joint universality for zeta-functions. In K. Kurşungöz and A. Zeytin(Eds.), *Number Theory: Proceedings of the Journées Arithmétiques, 2019, XXXI, Istanbul University*, pp. 47–60, Berlin, Boston, 2022. De Gruyter. https://doi.org/10.1515/9783110761115-004.

[12] R. Kačinskaitė, K. Matsumoto and Ł. Pańkowski. On mixed joint discrete universality for a class of zeta-functions: one more case. *Taiwanese J. Math.*, **27**(2):221–236, 2023. https://doi.org/10.11650/tjm/220804.

[13] R. Kačinskaitė and B. Togobickij. A modification of the mixed joint universality theorem for a class of zeta functions. *Mathematics*, **10**(19):article no. 3536, 2022. https://doi.org/10.3390/math10193536.

[14] J. Kaczorowski. Some remarks on the universality of periodic L-functions. In R. Steuding and J. Steuding(Eds.), *New directions in value-distribution theory of zeta and L-functions*, pp. 113–120, Aachen, 2009. Shaker Verlag.

[15] A.A. Karatsuba and S.M. Voronin. *The Riemann Zeta-Function.* Walter de Gruyter, Berlin, 1992. https://doi.org/10.1515/9783110886146.

[16] A. Laurinčikas. Universality of the Riemann zeta-function in short intervals. *J. Number Theory*, **204**:279–295, 2019. https://doi.org/10.1016/j.jnt.2019.04.006.

[17] A. Laurinčikas and R. Garunkštis. *The Lerch Zeta-Function.* Kluwer Academic Publishers, Dordrecht, Boston, London, 2002. https://doi.org/10.1007/978-94-017-6401-8.

[18] A. Laurinčikas and K. Matsumoto. The universality of zeta-functions attached to certain cusp forms. *Acta Arith.*, **98**(4):345–359, 2001. https://doi.org/10.4064/aa98-4-2.

[19] A. Laurinčikas and D. Šiaučiūnas. Remarks on the universality of the periodic zeta-function. *Math. Notes*, **80**(3–4):532–538, 2006. https://doi.org/10.1007/s11006-006-0171-y.

[20] A. Laurinčikas and D. Šiaučiūnas. The mean square of the Hurwitz zeta-function in short intervals. *Axioms*, **13**(8):article no. 510, 2024. https://doi.org/10.3390/axioms13080510.

[21] Y. Lee and Ł. Pańkowski. Universality of the zeta function in short intervals. *Ramanujan J.*, **68**(1):article no. 5, 2025. https://doi.org/10.1007/s11139-025-01144-0.

[22] D. Leitmann and D. Wolke. Periodische und multiplikative zahlentheoretische Funktionen. *Monatsh. Math.*, **81**:279–289, 1976. https://doi.org/10.1007/BF01387755.

[23] R. Macaitienė, M. Stoncelis and D. Šiaučiūnas. A weighted discrete universality theorem for periodic zeta-functions. In A. Dubickas, A. Laurinčikas, E. Manstavičius and G. Stepanauskas(Eds.), *Anal. Probab. Methods Number Theory, Proc. 6th Int. Conf., Palanga, Lithuania, 11–17 September 2016*, pp. 97–107, Vilnius, 2017. VU Leidykla.

[24] K. Matsumoto. A survey on the theory of universality for zeta and *L*-functions. In M. Kaneko, S. Kanemitsu and J. Liu(Eds.), *Number Theory: Plowing and Starring Through High Wawe Forms, Proc. 7th China-Japan Semin. (Fukuoka 2013)*, volume 11 of *Number Theory and Appl.*, pp. 95–144, New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai, 2015. World Scientific Publishing Co. https://doi.org/10.1142/9789814644938_0004

[25] K. Prachar. *Distribution of Prime Numbers.* Mir, Moscow, 1967. (in Russian)

[26] J. Steuding. *Value-Distribution of L-Functions.* Lecture Notes Math. vol. 1877, Springer, Berlin, Heidelberg, 2007. https://doi.org/10.1007/978-3-540-44822-8.

[27] M. Tekorė. *Universality of Periodic Zeta-Functions with Multiplicative Coefficients.* PhD Thesis, Vilnius University, Vilnius, 2024. https://doi.org/10.15388/vu.thesis.709.

[28] S.M. Voronin. The functional independence of Dirichlet *L*-functions. *Acta Arith.*, **27**:493–503, 1975. https://doi.org/10.4064/aa-27-1-493-503. (in Russian)

[29] S.M. Voronin. Theorem on the “universality” of the Riemann zeta-function. *Izv. Akad. Nauk SSSR, Ser. Matem.*, **39**:475–486, 1975. https://doi.org/10.1070/IM1975v009n03ABEH001485. (in Russian)