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Weak solutions to degenerate $p(t)$-Laplacian elliptic equations involving $(q, r(t))$ double phase Hardy terms

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Abstract. This paper is devoted to establishing novel existence criteria for weak solutions to a class of weighted quasilinear degenerate elliptic equations featuring double phase Hardy-type singular coefficients. These types of problems are rarely discussed in variable exponent Sobolev spaces in previous work. We prove the existence of at least one and at least two weak solutions via variational methods and critical point theory, under appropriate assumptions on the weight function and the nonlinearity.

Keywords: weighted Sobolev space; degenerate $p(t)$-Laplacian operators; $r(t)$-Hardy terms; variational methods.

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1 Introduction

Elliptic equations featuring Hardy potentials are fundamental to modeling physical, mathematical, and engineering systems that exhibit singular behavior and critical phenomena. The introduction of a singularity, particularly at the origin, significantly complicates the analytical properties of the differential operator and sensitively influences the solution's behavior. This is underpinned by the classical Hardy inequality, which guarantees that for $1 < p < N$, a bounded domain $\Omega \in \mathbb{R}^N$ with smooth boundary and a function ξ in $W^{1,p}(\mathbb{R}^N)$ or $W^{1,p}(\Omega)$, the weighted function $\xi/|t|$ remains integrable in $L^p(\Omega)$. This principle has been extended to more general settings involving variable exponents. Specifically, for a non-negative continuous function $r(t) \in C(\overline{\Omega})$, the integrability of $|\xi|^{e(t)}/|t|^{r(t)}$ over Ω can be established under certain conditions on $e(t)$, giving rise to the $r(t)$-type Hardy potential. The solvability of elliptic equations with such potentials has become an active area of research, as evidenced by numerous recent studies (see, e.g., [1, 12, 14, 15, 22, 23]). In parallel, degenerate elliptic equations with nonlinear weights are employed to model a variety of complex nonlinear processes. The nonlinear weight function is used to portray the relationship between the critical current density and the magnetic field. In recent years, p-Laplacian elliptic equations have been studied extensively, (see, e.g., [16, 17, 18, 19, 20]). The emergence of a singular weighted function $\omega(t)$ in the p-Laplacian operator or the $p(t)$-Laplacian operator, that is $\mathrm{div}(\omega(t)|\nabla\xi|^{p-2}\nabla\xi)$ or $\mathrm{div}(\omega(t)|\nabla\xi|^{p(t)-2}\nabla\xi)$, is called the degenerate p-Laplacian operator or the degenerate $p(t)$-Laplacian operator. A major analytical challenge arises when the weight function $\omega(t)$ is singular or fails to be bounded away from zero, as these conditions lead to degenerate or singular equations. The challenge cannot be addressed within the framework of the standard Sobolev spaces $W^{1,p}(\Omega)$ or $W^{1,p(t)}(\Omega)$. Instead, the framework of weighted Sobolev spaces, specifically $W^{1,p}(\omega,\Omega)$ or $W^{1,p(t)}(\omega,\Omega)$, must be adopted to deal with these issues effectively. A detailed discussion can be found in [4].

The aim of this paper is to study the existence of weak solutions to the following weighted $p(t)$-Laplacian quasilinear elliptic equations with double phrase Hardy potentials

$$\begin{cases} -\Delta_{p(t),a(t,\xi)}\xi+|\xi|^{p(t)-2}\xi+\dfrac{b(t)|\xi|^{q-2}\xi}{|t|^{q}}+\dfrac{c(t)|\xi|^{e(t)-2}\xi}{|t|^{r(t)}}=\lambda f(t,\xi) \text{ in } \Omega, \\ \xi=0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω denotes an open bounded subset in $\mathbb{R}^N (N\geq 3)$ with smooth boundary $\partial\Omega, 0\in\overline{\Omega}$, $\Delta_{p(t),a(t,\xi)}\xi=\mathrm{div}(a(t,\xi)|\nabla\xi|^{p(t)-2}\nabla\xi)$ is the degenerate $p(t)$-Laplacian operator, $a(t,\xi)=\omega(t)g(\xi)$, $g(\xi)$ is a continuous function satisfying

$$a_1\leq g(\xi)\leq a_2, \quad a.e.\ \xi\in\mathbb{R}, \tag{1.2}$$

in which a_1,a_2 are positive constants, $\omega>0$ is measurable and satisfying (ω) $\omega^{-h(t)}\in L^1(\Omega)$, for any $h(t)\in C(\overline{\Omega}),\omega\in L^1_{loc}(\Omega)$, and

$$h(t)\in(\frac{N}{p(t)},+\infty)\cap[\frac{1}{p(t)-1},+\infty),$$

$1<p(t)<+\infty, 1<q<p_h(t)<N$ with $p_h(t)=\frac{h(t)p(t)}{h(t)+1}$, $0<b(t),c(t)\in L^\infty(\Omega)$, $0\leq r(t)\in C(\overline{\Omega})$, $p(t),e(t)\in C(\overline{\Omega})$ with $1\leq e(t)<\frac{N-r(t)}{N}p_h^*(t)$, $\lambda>0$ is a parameter, the Carathéodory function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying

$$(f_1)\quad |f(t,\xi)|\leq M_1(t)+M_2|\xi|^{s(t)-1}, \quad a.e.\ (t,\xi)\in\Omega\times\mathbb{R},$$

where $M_1(t)>0, M_1(t)\in L^{\frac{s(t)}{s(t)-1}}(\Omega)$ with $1<s(t)<p_h^*(t)=\frac{Np_h(t)}{N-p_h(t)}$, and M_2 is a positive constant.

The primary objective of this paper is to establish new existence criteria for at least one and at least two weak solutions to the elliptic equation (1.1) under general assumptions on the weight function $a(t,\xi)$ and the nonlinear term $f(t,\xi)$. Our approach employs variational methods and critical point theory applied to the energy functional associated with the elliptic equation (1.1), enabling us to establish the existence of bounded, nontrivial weak solutions within precisely characterized intervals.

2 Basic notations and technical preliminaries

Let

$$\begin{aligned} C_+(\overline{\Omega}) =& \{p(t) | p(t) \in C(\overline{\Omega}),\ p(t) > 1,\ \forall t \in \overline{\Omega}\}, \\ p^- =& \text{ess inf}_{t\in\overline{\Omega}} p(t), \quad p^+ = \text{ess sup}_{t\in\overline{\Omega}} p(t). \end{aligned}$$

For $l > 0$, $p(t) \in C_+(\overline{\Omega})$, denote $l^{\hat{p}} = \max\{l^{p^+},\ l^{p^-}\}$.
Set

$$\begin{aligned} L^{p(t)}(\Omega) =& \Big\{\xi : \Omega \to \mathbb{R} \text{ measurable} \Big| \int_\Omega |\xi(t)|^{p(t)}\, dt < \infty \Big\}, \\ L^{p(t)}(\omega, \Omega) =& \Big\{\xi : \Omega \to \mathbb{R} \text{ measurable} \Big| \int_\Omega \omega(t) |\xi(t)|^{p(t)}\, dt < \infty \Big\} \end{aligned}$$

with corresponding norms

$$\|\xi\|_{L^{p(t)}(\Omega)} = \|\xi\|_{p(t)} = \inf \Big\{ \eta > 0 \Big| \int_\Omega |\frac{\xi(t)}{\eta}|^{p(t)}\, dt \le 1 \Big\},$$

and

$$\|\xi\|_{L^{p(t)}(\omega(t),\Omega)} = \|\xi\|_{(p(t),\omega(t))} = \inf \Big\{ \eta > 0 \Big| \int_\Omega \omega(t) |\frac{\xi(t)}{\eta}|^{p(t)}\, dt \le 1 \Big\}.$$

Now, we define the variable exponent Sobolev space

$$W^{1,p(t)}(\Omega) = \Big\{ \xi \in L^{p(t)}(\Omega) \Big| |\nabla \xi| \in L^{p(t)}(\Omega) \Big\},$$

with the norm

$$\|\xi\|_{W^{1,p(t)}(\Omega)} = \||\nabla \xi|\|_{p(t)} + \|\xi\|_{p(t)}.$$

The weighted Sobolev space is

$$W^{1,p(t)}(\omega, \Omega) = \left\{ \xi \in L^{p(t)}(\Omega) \Big| \omega^{\frac{1}{p(t)}} |\nabla \xi| \in L^{p(t)}(\Omega) \right\},$$

and we denote by $W_0^{1,p(t)}(\omega, \Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(t)}(\omega, \Omega)$ with respect to the norm:

$$\|\xi\| = \inf \left\{ \eta > 0 \Big| \int_\Omega \left(\omega(t) \left| \frac{\nabla \xi(t)}{\eta} \right|^{p(t)} + \left| \frac{\xi(t)}{\eta} \right|^{p(t)} \right) dt \le 1 \right\}.$$

Lemma 1. *[8] If $k_1(t), k_2(t) \in C_+(\overline{\Omega})$ satisfy $k_1(t) \leq k_2(t)$ almost everywhere in $t \in \Omega$, then $W^{1,k_2(t)}(\Omega) \hookrightarrow W^{1,k_1(t)}(\Omega)$ is continuous.*

Proposition 1. *[9] If $p(t) \in C_+(\overline{\Omega})$ and $\xi, \xi_n \in L^{p(t)}(\Omega)$, there holds*

$$\min\left\{\|\xi\|_{p(t)}^{p^-}, \|\xi\|_{p(t)}^{p^+}\right\} \leq \int_\Omega |\xi(t)|^{p(t)} dt \leq \max\left\{\|\xi\|_{p(t)}^{p^-}, \|\xi\|_{p(t)}^{p^+}\right\}.$$

Proposition 2. *[6] If $p(t) \in C_+(\overline{\Omega})$, $z(t)$ is a positive measurable function on Ω, then for any $\xi \in L^{p(t)}(z(t), \Omega)$ there holds*

$$\min\left\{\|\xi\|_{(p(t),z(t))}^{p^-}, \|\xi\|_{(p(t),z(t))}^{p^+}\right\} \leq \int_\Omega z(t)|\xi(t)|^{p(t)} dt \leq \max\left\{\|\xi\|_{(p(t),z(t))}^{p^-}, \|\xi\|_{(p(t),z(t))}^{p^+}\right\}.$$

On the basis of Propositions 1 and 2, we can derive the following lemma.

Lemma 2. *Let*

$$\rho_\omega(\xi) = \int_\Omega (|\xi(t)|^{p(t)} + \omega(t)\left|\nabla\xi(t)\right|^{p(t)}) dt.$$

For any $\xi \in W^{1,p(t)}(\omega, \Omega), p(t) \in C_+(\overline{\Omega})$, there holds

$$\min\left\{\|\xi\|^{p^-}, \|\xi\|^{p^+}\right\} \leq \rho_\omega(\xi) \leq \max\left\{\|\xi\|^{p^-}, \|\xi\|^{p^+}\right\}.$$

If condition (ω) holds, then $W^{1,p(t)}(\omega, \Omega)$ forms a reflexive and separable Banach space(see [11]). Moreover, Theorem 2.11 of [13] establishes that under condition (ω), the embedding

$$W^{1,p(t)}(\omega, \Omega) \hookrightarrow W^{1,p_h(t)}(\Omega) \tag{2.1}$$

is continuous, where

$$p(t) > p_h(t) = \frac{p(t)h(t)}{h(t)+1}.$$

By Proposition 2.7, Proposition 2.8 in [7] and (2.1), the embedding

$$W^{1,p(t)}(\omega, \Omega) \hookrightarrow L^{r(t)}(\Omega)$$

is continuous, where

$$1 \leq r(t) \leq p_h^*(t) = \frac{Np_h(t)}{N - p_h(t)} = \frac{Np(t)h(t)}{Nh(t) + N - p(t)h(t)}.$$

Furthermore, the embedding is compact when $1 \leq r(t) < p_h^*(t)$.

Lemma 3. *[6] Assume that $0 \in \overline{\Omega}$, $\partial\Omega$ possesses the cone property and $p_h, r, e \in C(\overline{\Omega}), 0 \leq r(t) < N, \forall t \in \overline{\Omega}$. If*

$$1 \leq e(t) < \frac{N - r(t)}{N} p_h^*(t), \forall t \in \overline{\Omega},$$

then the embedding

$$W^{1,p_h(t)}(\Omega) \hookrightarrow L^{e(t)}(|t|^{-r(t)}, \Omega)$$

is compact.

Lemma 4. *There exists a positive constant* $\tilde{c}$ *such that*

$$\int_\Omega \frac{|\xi|^{e(t)}}{|t|^{r(t)}} dt \le \tilde{c}\left(\|\xi\|^{e^-} + \|\xi\|^{e^+}\right), \; \forall \xi \in W_0^{1,p(t)}(\omega, \Omega).$$

Proof. Taking Proposition 2 and Lemma 3 into account, for $\forall \xi \in W_0^{1,p(t)}(\omega, \Omega)$ there exists $\tilde{c_1} > 0$ such that

$$\begin{aligned}\int_\Omega \frac{|\xi|^{e(t)}}{|t|^{r(t)}} dt &\le \|\xi\|^{e^+}_{(e(t),|t|^{-r(t)})} + \|\xi\|^{e^-}_{(e(t),|t|^{-r(t)})} \\ &\le \tilde{c_1}(\|\xi\|^{e^+}_{W^{1,p_h(t)}(\Omega)} + \|\xi\|^{e^-}_{W^{1,p_h(t)}(\Omega)}).\end{aligned}$$

Since $W_0^{1,p(t)}(\omega, \Omega) \hookrightarrow W_0^{1,p_h(t)}(\Omega)$, we can find a constant $\tilde{c}_2 > 0$ satisfying

$$\|\xi\|^{e^+}_{W^{1,p_h(t)}(\Omega)} + \|\xi\|^{e^-}_{W^{1,p_h(t)}(\Omega)} \le \tilde{c_2}(\|\xi\|^{e^+} + \|\xi\|^{e^-}),$$

and thus taking $\tilde{c} = \tilde{c}_1 \cdot \tilde{c}_2$ yields the desired inequality. □

The functional $\mathcal{I}_\lambda \colon W_0^{1,p(t)}(\omega, \Omega) \to \mathbb{R}$ is given by

$$\mathcal{I}_\lambda(\xi) = \Phi(\xi) - \lambda\Psi(\xi),$$

where

$$\Phi(\xi) = \int_\Omega \frac{a(t,\xi)}{p(t)} |\nabla \xi|^{p(t)} dt + \int_\Omega \frac{1}{p(t)} |\xi|^{p(t)} dt + \frac{1}{q}\int_\Omega \frac{b(t)|\xi|^q}{|t|^q} dt + \int_\Omega \frac{c(t)|\xi|^{e(t)}}{e(t)|t|^{r(t)}} dt,$$

$$\Psi(\xi) = \int_\Omega F(t,\xi) dt, \; F(t,\xi) = \int_0^\xi f(t,\tau) d\tau, \; \forall (t,\xi) \in \Omega \times \mathbb{R}.$$

A direct calculation shows that Φ and Ψ are continuously Gâteaux differentiable with derivatives

$$\begin{aligned}\langle \Phi'(\xi), v\rangle = &\int_\Omega a(t,\xi)|\nabla\xi|^{p(t)-2}\nabla\xi\nabla v dt + \int_\Omega |\xi|^{p(t)-2}\xi v dt \\ &+ \int_\Omega \frac{b(t)|\xi|^{q-2}\xi v}{|t|^q} dt + \int_\Omega \frac{c(t)|\xi|^{e(t)-2}\xi v}{|t|^{r(t)}} dt,\end{aligned}$$

and

$$\langle \Psi'(\xi), v\rangle = \int_\Omega f(t,\xi) v dt, \quad \forall \xi, v \in W_0^{1,p(t)}(\omega, \Omega).$$

It is easy to obtain $\frac{\langle \Phi'(\xi),\xi\rangle}{\|\xi\|} \to \infty$ as $\|\xi\| \to \infty$, thus Φ' is coercive.

$\xi \in W_0^{1,p(t)}(\omega, \Omega)$ is called a weak solution to the elliptic equation (1.1) if

$$\langle \mathcal{I}'_\lambda(\xi), v\rangle = 0, \quad \forall v \in W_0^{1,p(t)}(\omega, \Omega).$$

The subsequent lemmas provide essential technical tools for our analysis.

Lemma 5. *(Hölder-type inequality [7]) If $u,v \geq 1$ are measurable functions on Ω and*

$$\frac{1}{u(t)} + \frac{1}{v(t)} = 1, \qquad a.e.\ t \in \Omega.$$

For any $f \in L^{u(t)}(\Omega)$ and $g \in L^{v(t)}(\Omega)$, there holds

$$\int_\Omega f(t)g(t)dt \leq 2\|f\|_{u(t)}\|g\|_{v(t)}. \tag{2.2}$$

Lemma 6. *[5] If $u(t)$ and $v(t)$ are measurable functions satisfying $u(t) \in L^\infty(\Omega)$ and $1 \leq u(t)v(t) \leq \infty$ for almost every $t \in \Omega$. For $\xi \in L^{v(t)}(\Omega)$ with $\xi \neq 0$, we have*

$$\min\left\{\|\xi\|^{u^-}_{u(t)v(t)}, \|\xi\|^{u^+}_{u(t)v(t)}\right\} \leq \||\xi|^{u(t)}\|_{v(t)} \leq \max\left\{\|\xi\|^{u^-}_{u(t)v(t)}, \|\xi\|^{u^+}_{u(t)v(t)}\right\}.$$

Lemma 7. *Φ' is of type (S_+), that is, if $\xi_n \rightharpoonup \xi$ in $W_0^{1,p(t)}(\omega,\Omega)$, and $\overline{\lim}_{n\to\infty}\langle\Phi'(\xi_n) - \Phi'(\xi), \xi_n - \xi\rangle \leq 0$, then $\xi_n \to \xi$ in $W_0^{1,p(t)}(\omega,\Omega)$.*

Proof. Assume $\xi_n \rightharpoonup \xi$ in $W_0^{1,p(t)}(\omega,\Omega)$, with

$$\overline{\lim}_{n\to\infty}\langle\Phi'(\xi_n) - \Phi'(\xi), \xi_n - \xi\rangle \leq 0. \tag{2.3}$$

Firstly, we claim

$$\lim_{n\to\infty}\int_\Omega (a(t,\xi_n) - a(t,\xi))|\nabla\xi|^{p(t)-2}\nabla\xi(\nabla\xi_n - \nabla\xi)dt = 0. \tag{2.4}$$

In fact, by the continuity of $g(\xi)$, Lemma 5 and Lemma 6, for $\forall\varepsilon > 0$, we have

$$\begin{aligned}
&\left|\int_\Omega (a(t,\xi_n) - a(t,\xi))|\nabla\xi|^{p(t)-2}\nabla\xi(\nabla\xi_n - \nabla\xi)dt\right| \\
&\leq \left|\int_\Omega \omega(t)(g(\xi_n) - g(\xi))|\nabla\xi|^{p(t)-1}|\nabla\xi_n - \nabla\xi|dt\right| \\
&< \varepsilon\left|\int_\Omega (\omega(t)^{\frac{1}{p(t)}}|\nabla\xi|)^{p(t)-1}\omega(t)^{\frac{1}{p(t)}}|\nabla\xi_n - \nabla\xi|dt\right| \\
&\leq 2\varepsilon\|(\omega(t)^{\frac{1}{p(t)}}|\nabla\xi|)^{p(t)-1}\|_{\frac{p(t)}{p(t)-1}}\|\omega(t)^{\frac{1}{p(t)}}|\nabla\xi_n - \nabla\xi|\|_{p(t)} \\
&\leq 2\varepsilon(\|(\omega(t)^{\frac{1}{p(t)}}|\nabla\xi|)\|^{p^+-1}_{p(t)} + \|(\omega(t)^{\frac{1}{p(t)}}|\nabla\xi|)\|^{p^--1}_{p(t)})\|\omega(t)^{\frac{1}{p(t)}}|\nabla\xi_n - \nabla\xi|\|_{p(t)} \\
&\to 0\ (n\to\infty),
\end{aligned}$$

which leads to the desired result. □

Secondly, in view of (2.2) of [21], for any $\eta,\theta \in \mathbb{R}^N$, there exists $C_p > 0$ such that

$$(|\eta|^{p-2}\eta - |\theta|^{p-2}\theta)(\eta - \theta) \geq C_p|\eta - \theta|^p, \text{ if } p \geq 2,$$

and

$$(|\eta|^{p-2}\eta - |\theta|^{p-2}\theta)(\eta - \theta) \geq \frac{C_p|\eta - \theta|^2}{(|\eta| + |\theta|)^{2-p}}, \text{ if } 1 < p < 2.$$

While

$$\begin{aligned}
&\langle \Phi'(\xi_n) - \Phi'(\xi), \xi_n - \xi\rangle \\
&= \int_\Omega (a(t,\xi_n)|\nabla\xi_n|^{p(t)-2}\nabla\xi_n - a(t,\xi)|\nabla\xi|^{p(t)-2}\nabla\xi)(\nabla\xi_n - \nabla\xi)dt \\
&\quad + \int_\Omega (|\xi_n|^{p(t)-2}\xi_n - |\xi|^{p(t)-2}\xi)(\xi_n - \xi)dt \\
&\quad + \int_\Omega \Big(\frac{b(t)|\xi_n|^{q-2}}{|t|^q}\xi_n(\xi_n - \xi) - \frac{b(t)|\xi|^{q-2}}{|t|^q}\xi(\xi_n - \xi)\Big)dt \\
&\quad + \int_\Omega \Big(\frac{c(t)|\xi_n|^{e(t)-2}}{|t|^{r(t)}}\xi_n(\xi_n - \xi) - \frac{c(t)|\xi|^{e(t)-2}}{|t|^{r(t)}}\xi(\xi_n - u)\Big)dt \\
&= \int_\Omega a(t,\xi_n)(|\nabla\xi_n|^{p(t)-2}\nabla\xi_n - |\nabla\xi|^{p(t)-2}\nabla\xi)(\nabla\xi_n - \nabla\xi)dt \\
&\quad + \int_\Omega (a(t,\xi_n) - a(t,\xi))|\nabla\xi|^{p(t)-2}\nabla\xi(\nabla\xi_n - \nabla\xi)dt \\
&\quad + \int_\Omega (|\xi_n|^{p(t)-2}\xi_n - |\xi|^{p(t)-2}\xi)(\xi_n - \xi)dt \\
&\quad + \int_\Omega \Big(\frac{b(t)|\xi_n|^{q-2}}{|t|^q}\xi_n(\xi_n - \xi) - \frac{b(t)|\xi|^{q-2}}{|t|^q}\xi(\xi_n - \xi)\Big)dt \\
&\quad + \int_\Omega \Big(\frac{c(t)|\xi_n|^{e(t)-2}}{|t|^{r(t)}}\xi_n(\xi_n - \xi) - \frac{c(t)|\xi|^{e(t)-2}}{|t|^{r(t)}}\xi(\xi_n - u)\Big)dt,
\end{aligned} \tag{2.5}$$

thus combine (2.3), (2.4) and (2.5), we get

$$\overline{\lim}_{n\to\infty} \int_\Omega a(t,\xi_n)(|\nabla\xi_n|^{p(t)-2}\nabla\xi_n - |\nabla\xi|^{p(t)-2}\nabla\xi)(\nabla\xi_n - \nabla\xi)dt \le 0.$$

Further, by (1.2) one has

$$\overline{\lim}_{n\to\infty} \int_\Omega \omega(t)(|\nabla\xi_n|^{p(t)-2}\nabla\xi_n - |\nabla\xi|^{p(t)-2}\nabla\xi)(\nabla\xi_n - \nabla\xi)dt \le 0,$$

then $\xi_n \to \xi$ in $W_0^{1,p(t)}(\omega,\Omega)$ according to Lemma 3.2 in [10].

Lemma 8. *Φ' is a homeomorphism.*

Proof. The strict monotonicity of Φ' ensures injectivity. Since Φ' is coercive, it is surjective and thus admits an inverse mapping $(\Phi')^{-1}$.

Set $\tilde{f}_n, \tilde{f} \in (W_0^{1,p(t)}(\omega,\Omega))^*$ with $\tilde{f}_n \to \tilde{f}$. Set $\xi_n = (\Phi')^{-1}(\tilde{f}_n)$ and $\xi = (\Phi')^{-1}(\tilde{f})$, so that $\Phi'(\xi_n) = \tilde{f}_n$ and $\Phi'(\xi) = \tilde{f}$. The coercivity of Φ' implies boundedness of $\{\xi_n\}$. Without loss of generality, assume $\xi_n \rightharpoonup \xi_0$, which yields

$$\lim_{n\to\infty} (\Phi'(\xi_n) - \Phi'(\xi), \xi_n - \xi_0) = \lim_{n\to\infty} (\tilde{f}_n - \tilde{f}, \xi_n - \xi_0) = 0.$$

Since Φ' is of type (S_+), we have $\xi_n \to \xi_0$, and thus $\Phi'(\xi_n) \to \Phi'(\xi_0)$. Combining this with $\Phi'(\xi_n) \to \Phi'(\xi)$, we obtain $\Phi'(\xi) = \Phi'(\xi_0)$. The injectivity of Φ' then implies $\xi = \xi_0$ and $\xi_n \to \xi$, hence $(\Phi')^{-1}(\tilde{f}_n) \to (\Phi')^{-1}(\tilde{f})$, establishing continuity of $(\Phi')^{-1}$. □

Lemma 9. *$\Psi' : W_0^{1,p(t)}(\omega,\Omega) \to (W_0^{1,p(t)}(\omega,\Omega))^*$ is compact.*

Proof. By condition (f_1) and the compact embedding

$$W_0^{1,p(t)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{s(t)}(\Omega), \quad \text{for } 1 \le s(t) < p_h^*(t),$$

the derivative Ψ' is compact.

Indeed, let $\xi_n \rightharpoonup \xi$ in $W_0^{1,p(t)}(\omega,\Omega)$. By the compact embedding, there is a subsequence (denoted by $\{\xi_n\}$) such that $\xi_n \to \xi$ in $L^{s(t)}(\Omega)$. Since f is a Carathéodory function satisfying (f_1), the Nemytskii operator $N_f(\xi)(t) = f(t,\xi(t))$ is continuous from $L^{s(t)}(\Omega)$ into $L^{\frac{s(t)}{s(t)-1}}(\Omega)$. Hence,

$$N_f(\xi_n) \to N_f(\xi) \quad \text{in } L^{\frac{s(t)}{s(t)-1}}(\Omega).$$

Now, for any $v \in W_0^{1,p(t)}(\omega,\Omega)$, using Hölder's inequality (Lemma 5), we obtain

$$\begin{aligned} |\langle \Psi'(\xi_n) - \Psi'(\xi), v\rangle| &= \left|\int_\Omega \big(f(t,\xi_n) - f(t,\xi)\big) v dt\right| \le \int_\Omega |f(t,\xi_n) - f(t,\xi)||v|dt \\ &\le 2\|v\|_{L^{s(t)}(\Omega)} \|N_f(\xi_n) - N_f(\xi)\|_{L^{\frac{s(t)}{s(t)-1}}(\Omega)} \\ &\le 2c_s\|v\|\|N_f(\xi_n) - N_f(\xi)\|_{L^{\frac{s(t)}{s(t)-1}}(\Omega)}, \end{aligned}$$

where c_s is the embedding constant for $W_0^{1,p(t)}(\omega,\Omega) \hookrightarrow L^{s(t)}(\Omega)$. Therefore,

$$\|\Psi'(\xi_n) - \Psi'(\xi)\|_{(W_0^{1,p(t)}(\omega,\Omega))^*} \to 0,$$

which shows that Ψ' is completely continuous, hence compact. □

3 Main results

This section is devoted to establishing the existence of at least one or at least two nontrivial weak solutions for the elliptic equation (1.1).

Theorem 1. *Assume that (f_1) holds, and there exist constants $\mu > \max\{p^+, q, e^+\}$ and $R > 0$ such that*

$$(f_2) \quad 0 < \mu F(t,\xi) \le f(t,\xi)\xi, \ \forall\, t \in \Omega, \ |\xi| \ge R,$$

then the elliptic equation (1.1) admits at least two weak solutions for all $\lambda \in (0,\lambda_0)$, where

$$\lambda_0 = \left(2c_sC_1\|M_1\|_{\frac{s(t)}{s(t)-1}}(p^+)^{\frac{1}{p^-}} + C_1M_2c_s^{s^+}(p^+)^{\frac{s^+}{p^-}} + C_1M_2c_s^{s^-}(p^+)^{\frac{s^-}{p^-}}\right)^{-1},$$

c_s denotes the embedding constant for $W_0^{1,p(t)}(\omega,\Omega) \hookrightarrow L^{s(t)}(\Omega), 1 < s(t) < p_h^*(t)$, and C_1 will appear in the following proof process.

Proof. Notice that

$$\begin{aligned}\Phi(\xi)=&\int_\Omega \frac{a(t,\xi)}{p(t)}|\nabla\xi|^{p(t)}dt+\int_\Omega\frac{1}{p(t)}|\xi|^{p(t)}dt+\frac{1}{q}\int_\Omega\frac{b(t)|\xi|^q}{|t|^q}dt+\int_\Omega\frac{c(t)|\xi|^{e(t)}}{e(t)|t|^{r(t)}}dt\\ &\geq\frac{1}{p^+}\min\{a_1,1\}\cdot\rho_\omega(\xi)\geq\frac{1}{p^+}\min\{a_1,1\}\cdot\min\left\{\|\xi\|^{p^+},\|\xi\|^{p^-}\right\},\end{aligned}$$

which shows that Φ is bounded from below.

Let $\{\xi_n\}\subset W_0^{1,p(t)}(\omega,\Omega)$ such that $\{I_\lambda(\xi_n)\}$ is bounded, and $I'_\lambda(\xi_n)\to 0$, as $n\to+\infty$, Thus there exists $M_0>0$ independent of n such that

$$|I_\lambda(\xi_n)|\leq M_0,$$

and for n large enough, one has

$$|I'_\lambda(\xi_n)\xi_n|\leq\|I'_\lambda(\xi_n)\|_{(W_0^{1,p(t)}(\omega,\Omega))^*}\|\xi_n\|\leq\|\xi_n\|, \tag{3.1}$$

thus, when $|\xi_n|\geq R$, one has

$$\begin{aligned}\mu I_\lambda(\xi_n)-I'_\lambda(\xi_n)\xi_n=&\mu\int_\Omega\frac{a(t,\xi_n)}{p(t)}|\nabla\xi_n|^{p(t)}dt+\mu\int_\Omega\frac{1}{p(t)}|\xi_n|^{p(t)}dt\\ &+\frac{\mu}{q}\int_{\mathbb{R}^N}\frac{b(t)|\xi_n|^q}{|t|^q}dt+\mu\int_\Omega\frac{c(t)|\xi|^{e(t)}}{e(t)|t|^{r(t)}}dt-\lambda\mu\int_\Omega F(t,\xi_n)dt\\ &-\int_\Omega a(t,\xi_n)|\nabla\xi_n|^{p(t)}dt-\int_\Omega|\xi_n|^{p(t)}dt-\int_\Omega\frac{b(t)|\xi_n|^q}{|t|^q}dt-\int_\Omega\frac{c(t)|\xi_n|^{e(t)}}{|t|^{r(t)}}dt\\ &+\lambda\int_\Omega f(t,\xi_n)\xi_n dt\\ \geq&(\frac{\mu}{p^+}-1)\min\{a_1,1\}\cdot\rho_\omega(\xi_n)+(\frac{\mu}{q}-1)\int_\Omega\frac{b(t)|\xi_n|^q}{|t|^q}dt\\ &+(\frac{\mu}{e^+}-1)\int_\Omega\frac{c(t)|\xi_n|^{e(t)}}{|t|^{r(t)}}dt\\ \geq&(\frac{\mu}{p^+}-1)\min\{a_1,1\}\cdot\min\{\|\xi_n\|^{p^+},\|\xi_n\|^{p^-}\}.\end{aligned}$$

Combining with (3.1), we get

$$\mu M_0+\|\xi_n\|\geq I_\lambda(\xi_n)-I'_\lambda(\xi_n)\xi_n\geq\left(\frac{\mu}{p^+}-1\right)\min\{a_1,1\}\cdot\min\{\|\xi_n\|^{p^+},\|\xi_n\|^{p^-}\},$$

which implies boundedness of $\{\xi_n\}$ since $\mu>\max\{p^+,q,e^+\}$.

Without loss of generality, assume $\xi_n\rightharpoonup\xi$. The compactness of Ψ' (Lemma 9) gives $\Psi'(\xi_n)\to\Psi'(\xi)$. Since $I'_\lambda(\xi_n)=\Phi'(\xi_n)-\lambda\Psi'(\xi_n)\to 0$, we have $\Phi'(\xi_n)\to\lambda\Psi'(\xi_n)$. The fact Φ' is of type (S_+) and the homeomorphism property of Φ' (Lemma 8) yields $\xi_n\to\xi$, establishing that I_λ satisfies the Palais-Smale condition.

To show I_λ is unbounded below, we first establish the Hardy-type inequality

$$\int_\Omega \frac{|\xi|^q}{|t|^q} dt \le \frac{1}{H} \int_\Omega |\nabla \xi|^q dt, \quad \forall \xi \in W_0^{1,p(t)}(\omega, \Omega), \tag{3.2}$$

for some $H > 0$. The classical Hardy inequality [1, Lemma 2.1] states that for $1 < h < N$

$$\int_\Omega \frac{|\xi|^h}{|t|^h} dt \le \left(\frac{h}{N-h}\right)^h \int_\Omega |\nabla \xi|^h dt, \quad \forall \xi \in W_0^{1,h}(\Omega).$$

This implies (3.2) holds for any $\xi \in W_0^{1,q}(\Omega)$ when $1 < q < N$, and the continuous embeddings

$$W_0^{1,p(t)}(\omega, \Omega) \hookrightarrow W_0^{1,p_h(t)}(\Omega) \hookrightarrow W_0^{1,q}(\Omega),$$

obtained from (2.1), Lemma 1, and $1 < q < p_h(t) < N$, ensure its validity for $\xi \in W_0^{1,p(t)}(\omega, \Omega)$.

Secondly, from (f_2), there exist positive constants α and β such that

$$F(t,\xi) \ge \alpha |\xi|^\mu - \beta, \ \forall \ t \in \overline{\Omega}, |\xi| \ge R.$$

Thus, for some fixed $\tilde{\xi} \in W_0^{1,p(t)}(\omega, \Omega) \setminus \{0\}$, when $m > 1$, combining Lemma 2, Lemma 4, (1.2) with (3.2), one has

$$\begin{aligned}
I_\lambda(m\tilde{\xi}) = &\int_\Omega \frac{a(t,\xi)}{p(t)} |m \nabla \tilde{\xi}|^{p(t)} dt + \int_\Omega \frac{1}{p(t)} |m\tilde{\xi}|^{p(t)} dt + \frac{m^q}{q} \int_\Omega \frac{b(t)|\tilde{\xi}|^q}{|t|^q} dt \\
&+ \int_\Omega \frac{c(t)|m\tilde{\xi}|^{e(t)}}{e(t)|t|^{r(t)}} dt - \lambda \int_\Omega F(t, m\tilde{\xi}) dt \\
\le &\frac{m^{p^+}}{p^-} \max\{a_2, 1\} \cdot (\|\tilde{\xi}\|^{p^-} + \|\tilde{\xi}\|^{p^+}) + \frac{m^q \|b\|_\infty}{qH} \int_\Omega |\nabla \tilde{\xi}|^q dt \\
&+ \frac{m^{e^+} \|c\|_\infty}{e^-} (\|\tilde{\xi}\|^{e^-} + \|\tilde{\xi}\|^{e^+}) - \lambda \beta |\Omega| - \lambda \alpha m^\mu \int_\Omega |\tilde{\xi}|^\mu dt.
\end{aligned}$$

Since $\mu > \max\{p^+, q, e^+\}$, we conclude $I_\lambda(m\tilde{\xi}) \to -\infty$ as $m \to +\infty$, proving I_λ is unbounded from below.

From (f_1), there exists $C_1 > 0$ such that

$$|F(t,\xi)| \le C_1(M_1(t)|\xi| + M_2 |\xi|^{s(t)}), \quad a.e. \ (t,\xi) \in \Omega \times \mathbb{R}. \tag{3.3}$$

For $\xi \in \Phi^{-1}((-\infty, 1])$, we have $\Phi(\xi) \le 1$ and $\|\xi\| \le (p^+)^{\frac{1}{p^-}}$. Thus via (3.3) and Hölder-type inequality (2.2), we have

$$\begin{aligned}
&\sup_{\xi\in\Phi^{-1}((-\infty,1])}\Psi(\xi)=\sup_{\xi\in\Phi^{-1}((-\infty,1])}\int_\Omega F(t,\xi)dt\\
&\le C_1\sup_{\|\xi\|\le(p^+)^{\frac{1}{p^-}}}\int_\Omega(M_1(t)|\xi|+M_2|\xi|^{s(t)})dt\\
&\le C_1\sup_{\|\xi\|\le(p^+)^{\frac{1}{p^-}}}(2\|M_1\|_{\frac{s(t)}{s(t)-1}}\|\xi\|_{s(t)}+M_2(\|\xi\|^{s^+}_{s(t)}+\|\xi\|^{s^-}_{s(t)}))\\
&\le C_1\sup_{\|\xi\|\le(p^+)^{\frac{1}{p^-}}}(2c_s\|M_1\|_{\frac{s(t)}{s(t)-1}}\|\xi\|+M_2(c_s^{s^+}\|\xi\|^{s^+}+c_s^{s^-}\|\xi\|^{s^-}))\\
&\le 2c_sC_1\|M_1\|_{\frac{s(t)}{s(t)-1}}(p^+)^{\frac{1}{p^-}}+C_1M_2c_s^{s^+}(p^+)^{\frac{s^+}{p^-}}+C_1M_2c_s^{s^-}(p^+)^{\frac{s^-}{p^-}},
\end{aligned}$$

where c_s is the embedding constant for $W_0^{1,p(t)}(\omega,\Omega)\hookrightarrow L^{s(t)}(\Omega)$ with $1<s(t)<p_h^*(t)$.

All hypotheses of [2, Theorem 3.2] are now verified, ensuring the elliptic equation (1.1) has at least two distinct weak solutions, one of which might be trivial. □

To establish the existence of two nontrivial solutions, we proceed as follows. Define the function:

$$\rho(t)=\sup\{\rho>0\mid B(t,\rho)\subseteq\Omega\},\quad \forall t\in\Omega.$$

There exists $t_0\in\Omega$ such that $B(t_0,d)\subseteq\Omega$, where $d=\sup_{t\in\Omega}\rho(t)$.

Theorem 2. *Assume that* (f_1) *and* (f_2) *hold, with* $F(t,\xi)\ge 0$ *for all* $(t,\xi)\in B(t_0,d)\times[0,\delta]$. *Suppose there exist constants* C_δ *and* γ *satisfying* $C_\delta<\min\left\{\frac{\gamma^{p^-}}{p^+},\frac{\gamma^{p^+}}{p^{+p^+/p^-}}\right\}$ *such that*

$$\frac{2c_s\gamma C_1\|M_1\|_{\frac{s(t)}{s(t)-1}}+C_1M_2(\gamma c_s)^{s^+}+C_1M_2(\gamma c_s)^{s^-}}{\min\{\frac{\gamma^{p^-}}{p^+},\frac{\gamma^{p^+}}{p^{+p^+/p^-}}\}}=\frac{1}{\lambda_2}<\frac{1}{\lambda_1}=\frac{ess\ inf_{B(t_0,\frac{d}{2})}F(t,\delta)|B(t_0,\frac{d}{2})|}{C_\delta}.$$

Then, the elliptic equation (1.1) *admits at least two nontrivial solutions for all* $\lambda\in(\lambda_1,\lambda_2)$.

Proof. It is easy to verify that $\inf_X\Phi=\Phi(0)=\Psi(0)=0$. Define the test function

$$\bar\xi(t)=\begin{cases}0, & t\in\overline{\Omega}\setminus\bar B(t_0,d),\\ \frac{2\delta}{d}(d-|t-t_0|), & t\in B(t_0,d)\setminus\bar B(t_0,\frac{d}{2}),\\ \delta, & t\in B(t_0,\frac{d}{2}),\end{cases}$$

thus,

$$\Psi(\bar{\xi}) = \int_\Omega F(t,\bar{\xi})dt \geq \int_{B(t_0,\frac{d}{2})} F(t,\delta)dt \geq |B(t_0,\frac{d}{2})| \text{ ess inf}_{t\in B(t_0,\frac{d}{2})}F(t,\delta). \tag{3.4}$$

Direct calculation shows $\bar{\xi}(t) < \delta, \nabla\bar{\xi}(t) = \frac{2\delta}{d}$. Let

$$\frac{a_2}{p^-}(\frac{2\delta}{d})^{\hat{p}}\|\omega\|_{L^1(\Omega)} + \frac{1}{p^-}\delta^{\hat{p}}|B(t_0,d)| = M.$$

By (1.2), the Hardy inequality (3.2) and Lemma 4, one has

$$\begin{aligned}
\Phi(\bar{\xi}) &= \int_\Omega \frac{a(t,\xi)}{p(t)}|\nabla\bar{\xi}|^{p(t)}dt + \int_\Omega \frac{1}{p(t)}|\bar{\xi}|^{p(t)}dt \\
&\quad + \frac{1}{q}\int_\Omega \frac{b(t)|\bar{\xi}|^q}{|t|^q}dt + \int_\Omega \frac{c(t)|\bar{\xi}|^{e(t)}}{e(t)|t|^{r(t)}}dt \\
&\leq \frac{a_2}{p^-}(\frac{2\delta}{d})^{\hat{p}}\|\omega\|_{L^1(\Omega)} + \frac{1}{p^-}\delta^{\hat{p}}|B(t_0,d)| + \frac{\|b\|_\infty}{qH}(\frac{2\delta}{d})^q)|B(t_0,d)| \\
&\quad + \frac{\|c\|_\infty\tilde{c}}{e^-}(M^{\frac{1}{p^-}} + M^{\frac{1}{p^+}})^{e^+} + \frac{\|c\|_\infty\tilde{c}}{e^-}(M^{\frac{1}{p^-}} + M^{\frac{1}{p^+}})^{e^-} \\
&= M + \frac{\|c\|_\infty\tilde{c}}{e^-}(M^{\frac{1}{p^-}} + M^{\frac{1}{p^+}})^{e^+} + \frac{\|c\|_\infty\tilde{c}}{e^-}(M^{\frac{1}{p^-}} + M^{\frac{1}{p^+}})^{e^-} \\
&\quad + \frac{\|b\|_\infty}{qH}(\frac{2\delta}{d})^q)|B(t_0,d)| := C_\delta < \min\Big\{\frac{\gamma^{p^-}}{p^+}, \frac{\gamma^{p^+}}{p^{+p^+/p^-}}\Big\}.
\end{aligned} \tag{3.5}$$

Set $r = \min\{\frac{\gamma^{p^-}}{p^+}, \frac{\gamma^{p^+}}{p^{+p^+/p^-}}\}$, so, $0 < \Phi(\bar{\xi}) < r$. For $\xi \in \Phi^{-1}((-\infty,r])$, we have $\Phi(\xi) \leq r$, and

$$\|\xi\| \leq \max\{(rp^+)^{\frac{1}{p^-}}, (rp^+)^{\frac{1}{p^+}}\} \leq p^{+\frac{1}{p^-}} \max\{r^{\frac{1}{p^-}}, r^{\frac{1}{p^+}}\} = \gamma.$$

Thus via (3.3) and Hölder-type inequality (2.2), we have

$$\begin{aligned}
&\sup_{\xi\in\Phi^{-1}((-\infty,r])}\Psi(\xi) = \sup_{\xi\in\Phi^{-1}((-\infty,r])}\int_\Omega F(t,\xi)dt \\
&\leq C_1 \sup_{\|\xi\|\leq\gamma}\int_\Omega (M_1(t)|\xi| + M_2|\xi|^{s(t)})dt \\
&\leq C_1 \sup_{\|\xi\|\leq\gamma} (2\|M_1\|_{\frac{s(t)}{s(t)-1}}\|\xi\|_{s(t)} + M_2(\|\xi\|^{s^+}_{s(t)} + \|\xi\|^{s^-}_{s(t)})) \\
&\leq C_1 \sup_{\|\xi\|\leq\gamma} (2c_s\|M_1\|_{\frac{s(t)}{s(t)-1}}\|\xi\| + M_2(c_s^{s^+}\|\xi\|^{s^+} + c_s^{s^-}\|\xi\|^{s^-})) \\
&\leq 2c_s\gamma C_1\|M_1\|_{\frac{s(t)}{s(t)-1}} + C_1M_2(\gamma c_s)^{s^+} + C_1M_2(\gamma c_s)^{s^-}.
\end{aligned} \tag{3.6}$$

Thus (3.4), (3.5) and (3.6) leads to

$$\frac{\sup_{\xi\in\Phi^{-1}((-\infty,r])}\Psi(\xi)}{r} \le \frac{2c_s\gamma C_1\|M_1\|_{\frac{s(t)}{s(t)-1}} + C_1M_2(\gamma c_s)^{s^+} + C_1M_2(\gamma c_s)^{s^-}}{\min\{\frac{\gamma^{p^-}}{p^+}, \frac{\gamma^{p^+}}{p^{+p^+/p^-}}\}}$$
$$< \frac{\text{ess inf}_{B(t_0,\frac{d}{2})}F(t,\delta)|B(t_0,\frac{d}{2})|}{C_\delta} \le \frac{\Psi(\bar{\xi})}{\Phi(\bar{\xi})}.$$

Therefore, all conditions of [3, Theorem 2.1] are satisfied, guaranteeing at least two nontrivial solutions to the elliptic equation (1.1). □

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