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Solvability of two-dimensional nonlinear singular Volterra integral equations with fractional order in Banach space and approximation of the solution of it

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Abstract. In this article, existence and uniqueness of the solution for two-dimensional non-linear singular Volterra integral equations with fractional orders in a Banach space is discussed by utilizing the concept of the measure of non-compactness and fixed-point theorem. In fact, this kind of equations is a generalization of two-dimensional Riemann-Liouville fractional non-linear integral equations. To approximate the solution of the above problem, we use modified homotopy perturbation with the help of Adomian polynomials. To validity of the derived results, we introduce an example in the field of singular non-linear integral equations. Hence, a semi-analytic solution for given example is obtained ensuring satisfactory accuracy. Also, to ensure the effectiveness of the proposed method the results are compared with some other works.

Keywords: measure of non-compactness; two dimensional non-linear singular integral equation; fractional order; iterative algorithm.

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1 Introduction and preliminaries

The investigation of two-dimensional non-linear singular integral equations of Volterra type is of significant importance in various fields. These equations, characterized by complexity and non-linearity, play an important role in understanding real-world phenomena, such as the conductor-like screening model, quantum chemistry, the kinetic theory of gases, the free electron laser, and the chemistry resolvent, as seen in [5, 10, 11, 16, 20, 24]. Integral equations with a singular kernel or a fractional form are a significant part of nonlinear analysis, which can be seen in [1, 9, 13, 27, 29]. By addressing these challenges, we not only contribute to the development of mathematical analysis but also gain valuable insights that can impact many fields. For the solvability of nonlinear integral equations, fixed-point theory and measure of non-compactness play an essential role [3, 7, 14, 21, 22, 30]. Additionally, the existence of solutions to n-nonlinear, 2D Volterra, product-type fractional, Hadamard fractional, and n-product functional integral equations is studied using Petryshyn's fixed point theorem in [2, 8, 12, 17, 18]. In [31], the existence and uniqueness of the solution of Hammerstein functional integral equations by extending Burton's method is discussed. In this article, we demonstrate the existence and uniqueness of solution in a Banach space with a measure of non-compactness, for the two-dimensional non-linear singular Volterra integral equation with fractional order,

$$\begin{aligned} u(s,\tau) \quad &= g(s,\tau) \quad +h(s,\tau,u(s,\tau))\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\ &\quad \times m\xi^{m-1}n\omega^{n-1}k(f(s,\tau,\xi,\omega))u(\xi,\omega)d\xi d\omega, \end{aligned} \tag{1.1}$$

where $0<\gamma,\theta\le 1, s,\tau\in[0,1]$ and $m,n>0$, and $\Gamma(\gamma)$ is defined as the integral of $\tau^{\gamma-1}e^{-\tau}$ from zero to infinity. The function $h(s,\tau,u)$ is produced by the superposition operator H. Specifically, $(Hu)(s,\tau)=h(s,\tau,u(s,\tau))$ is defined on $([0,1]\times[0,1])\longrightarrow\Re$. We establish the existence of non-decreasing solutions for equation (1.1) within the set of all continuous functions on $C([0,1]\times[0,1])$.

We will assume A is non-empty and a subset of $\mathbb{B}$ throughout the context of this article, where $(\mathbb{B},\|\cdot\|)$ represents a real Banach space in one- and two-dimensional cases. Moreover, we consider $\mathfrak{M}_{\mathbb{B}}$ to be a non-empty family of bounded subsets of $\mathbb{B}$, and $\mathfrak{N}_{\mathbb{B}}$ as a subfamily comprising all relatively compact sets.

Definition 1. [6] A mapping $\pounds:\mathfrak{M}_{\mathbb{B}}\to\mathbb{R}^+$ is considered a measure of non-compactness in $\mathbb{B}$ if it fulfills the following conditions:

(1^0) Family $ker\pounds=\{A\in\mathfrak{M}_{\mathbb{B}}:\pounds(A)=0\}$ is nonempty and $ker\pounds\subset\mathfrak{N}_{\mathbb{B}}$,

(2^0) $A\subset B\Rightarrow\pounds(A)\le\pounds(B)$, ($3^0$) $\pounds(\bar{A})=\pounds(A)$, [($4^0$)] $\pounds(ConvA)=\pounds(A)$,

(5^0) $\pounds(\lambda A+(1-\lambda)B)\le\lambda\pounds(A)+(1-\lambda)\pounds(B)$ for $\lambda\in[0,1]$,

(6^0) If $\{A_n\}$ is a sequence of closed sets from m_E such that $A_{n+1}\subset A_n$ for $n\in\mathbb{N}$ and if $\lim_{n\longrightarrow\infty}\pounds(A_n)=0$, then the set $A_\infty=\bigcap_{n=1}^\infty A_n$ is nonempty.

2 A generalization of Darbo fixed-point theorem

To establish a generalization of the Darbo fixed-point theorem [6], we use a type of contraction which was applied in [25]. From now on, we suppose that functions $G,\Psi,\vartheta:[0,+\infty)\to[0,+\infty)$ satisfy the following conditions:

(i) $G\in C[0,+\infty)$ and $G(0)=0<G(s)$, for all $s>0$;

(ii) $\vartheta(s)<\Psi(s)$, for all $s>0$ and $\vartheta(0)=\Psi(0)=0$;

(iii) $\vartheta(s),\Psi(s)\in C[0,+\infty)$; (iv) Ψ is increasing.

Moreover, consider $\mathbb{G}=\{G: G$ satisfies condition (i)$\}$ and $\Sigma=\{(\Psi,\vartheta):\Psi$ and ϑ satisfy conditions (ii), (iii), and (iv)$\}$. The following definition and theorem describe a generalization of a (Ψ,G,ϑ)-contractive mapping, employing the measure of non-compactness and its application [25].

DEFINITION 2. Let $\varrho \neq \varnothing$ be a subset of $\mathbb{B}$, and $\varsigma : \varrho \to \varrho$ be a mapping. We define ς as a generalized (Ψ, G, ϑ)-contractive mapping if, for any $0 < a < b < \infty$, there exist $0 < \rho_{ab} < 1$, $G \in \mathbb{G}$, and $(\Psi, \vartheta) \in \Sigma$, such that for all $A \subseteq \varrho$ and an arbitrary measure of non-compactness $\pounds$, the following holds:

$$a \leq G(\pounds(A)) \leq b \Longrightarrow \Psi(G(\pounds(\varsigma A))) \leq \rho_{ab}\vartheta(G(\pounds(A))).$$

Theorem 1. *Consider $\varrho \neq \varnothing$, a closed, bounded, convex subset of $\mathbb{B}$, and $\varsigma : \varrho \to \varrho$ as a generalized (Ψ, G, ϑ)-contractive continuous mapping. It follows that ς has at least one fixed point in ϱ.*

Proof. In [25] the proof is done in Banach space $\mathbb{B} = C([0,1])$ with the standard norm. Because it is similar to the proof in Banach space $\mathbb{B} = C([0,1] \times [0,1])$ equipped with the standard norm, we therefore omit it. □

As an application of a generalization of the Darbo fixed-point theorem, we prove the existence of a solution for two-dimensional non-linear singular integral equations in the Banach space $\mathbb{B} = C([0,1] \times [0,1])$ with the standard norm: $\|u\| = \max\{|u(s,\tau)| : s,\tau \geq 0\}$. Therefore, assume that $A \neq \varnothing$ is a bounded subset of $C([0,1] \times [0,1])$, and for $u \in A$ and $\epsilon \geq 0$, let us set

$$\begin{aligned} \Pi(u,\epsilon) &:= \sup\{|u(s,\tau) - u(\xi,\omega)| : \\ &\quad s,\tau,\xi,\omega \in [0,1], |s-\xi| \leq \epsilon\,, |\tau-\omega| \leq \epsilon\}, && (2.1)\\ \Pi(A,\epsilon) &:= \sup\{\Pi(u,\epsilon) : u \in A\}, \qquad \Pi_0(A) := \lim_{\epsilon\to 0} \Pi(A,\epsilon), && (2.2) \end{aligned}$$

and we define

$$\begin{aligned} &J(u) := \sup\{|u(\xi,\omega) - u(s,\tau)| - [u(\xi,\omega) - u(s,\tau)] : s,\tau,\xi,\omega \in [0,1], s \leq \xi, \tau \leq \omega\},\\ &J(A) := \sup\{J(u) : u \in A\}. \qquad (2.3) \end{aligned}$$

It is evident that all functions within A are non-decreasing on $[0,1] \times [0,1]$ if and only if $J(A) = 0$. Let us introduce $\pounds$ on $\mathfrak{M}_{C([0,1]\times[0,1])}$ by this form:

$$\pounds(A) := \Pi_0(A) + J(A). \qquad (2.4)$$

Following [6], it is not difficult to show that the function $\pounds$ is a measure of non-compactness on $C([0,1] \times [0,1])$.

Now, let's examine Equation (1.1) based on the following conditions:

(b_1) $g : [0,1] \times [0,1] \to \mathbb{R}^+$ is a continuous, non-decreasing, and nonnegative function;

(b_2) $h : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function in s, τ, u such that $h([0,1] \times [0,1] \times \mathbb{R}^+) \subseteq \mathbb{R}^+$. Additionally, there exists a continuous non-decreasing function $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\vartheta(0) = 0$. For each $s > 0$, it holds that $\vartheta(s) < s$, such that

$$|h(s,\tau,u) - h(s,\tau,z)| \leq \vartheta(|u-z|), \forall s,\tau \in [0,1], \forall u,z \in \mathbb{R},$$

also ϑ is superadditive, so that $\vartheta(s) + \vartheta(\xi) \leq \vartheta(s+\xi)$ for all $s,\xi \in \mathbb{R}^+$;

(b_3) In Equation (1.1), the operator H satisfies the condition $J(Hu)\leq\vartheta(J(u))$ for any nonnegative function u, where ϑ is introduced in (b_2);

(b_4) $f:[0,1]\times[0,1]\times[0,1]\times[0,1]\to\mathbb{R}$ is continuous and non-decreasing with respect to each variable separately;

(b_5) $k:\mathrm{Im}f\to\mathbb{R}^+$ is a non-decreasing continuous function on the compact set $\mathrm{Im}f$;

(b_6) With hypotheses $M_1=\max\{|g(s,\tau)|:s,\tau\in[0,1]\}$ and $M_2=\max\{|h(s,\tau,0)|:s,\tau\in[0,1]\}$, the following inequality

$$M_1\Gamma(\gamma+1)\Gamma(\theta+1)+(\vartheta(r)+M_2)\,\|k\|\,r\leq\Gamma(\gamma+1)\Gamma(\theta+1)r,$$

has a solution $r_0>0$, where $\lambda=\frac{\|k\|r_0}{\Gamma(\gamma+1)\Gamma(\theta+1)}<1$.

Theorem 2. *Under conditions (b_1)–(b_6), the Equation (1.1) has at least one non-decreasing solution as $u=u(\xi,\omega)\in C([0,1]\times[0,1])$.*

Proof. Regarding Equation (1.1), operators G and ς on $C([0,1]\times[0,1])$ are defined in the forms:

$$\begin{aligned}(Gu)(s,\tau)=&\int_0^\tau\int_0^s\frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}\\&\times k(f(s,\tau,\xi,\omega))u(\xi,\omega)d\xi d\omega,\\(\varsigma u)(s,\tau)=&g(s,\tau)+h(s,\tau,u(\xi,\omega))(Gu)(s,\tau).\end{aligned}\tag{2.5}$$

At first, it is shown that G is a self-map on $C([0,1]\times[0,1])$. Let $\epsilon>0$ be given, and suppose $u\in C([0,1]\times[0,1])$ and $s_1,s_2,\tau_1,\tau_2\in[0,1]$. Without loss of generality, let $s_2\geq s_1,\tau_2\geq\tau_1$ and $|s_2-s_1|\leq\epsilon,|\tau_2-\tau_1|\leq\epsilon$. Then,

$$\begin{aligned}&|(Gu)(s_2,\tau_2)-(Gu)(s_1,\tau_1)\,|\\
=&\Big|\int_0^{\tau_2}\int_0^{s_2}\frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_2,\tau_2,\xi,\omega))u(\xi,\omega)d\xi d\omega\\
&-\int_0^{\tau_1}\int_0^{s_1}\frac{(s_1^m-\xi^m)^{\gamma-1}(\tau_1^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_1,\tau_1,\xi,\omega))u(\xi,\omega)d\xi d\omega\Big|\\
\leq&\Big|\int_0^{\tau_2}\int_0^{s_2}\frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_2,\tau_2,\xi,\omega))u(\xi,\omega)d\xi d\omega\\
&-\int_0^{\tau_2}\int_0^{s_2}\frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma),\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_1,\tau_1,\xi,\omega))u(\xi,\omega)d\xi d\omega\Big|\\
&+\Big|\int_0^{\tau_2}\int_0^{s_2}\frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_1,\tau_1,\xi,\omega))u(\xi,\omega)d\xi d\omega\\
&-\int_0^{\tau_1}\int_0^{s_1}\frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma),\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_1,\tau_1,\xi,\omega))u(\xi,\omega)d\xi d\omega\Big|\\
&+\Big|\int_0^{\tau_1}\int_0^{s_1}\frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_1,\tau_1,\xi,\omega))u(\xi,\omega)d\xi d\omega\\
&-\int_0^{\tau_1}\int_0^{s_1}\frac{(s_1^m-\xi^m)^{\gamma-1}(\tau_1^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}k(f(s_1,\tau_1,\xi,\omega))u(\xi,\omega)d\xi d\omega\Big|\\
&\leq\int_0^{\tau_2}\int_0^{s_2}\frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}\end{aligned}$$

$$|k(f(s_2,\tau_2,\xi,\omega)) - k(f(s_1,\tau_1,\xi,\omega))||u(\xi,\omega)|d\xi d\omega$$
$$+ \int_{\tau_1}^{\tau_2}\int_{s_1}^{s_2} \frac{(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1}n\omega^{n-1}$$
$$|k(f(s_1,\tau_1,\xi,\omega))||u(\xi,\omega)|d\xi d\omega$$
$$+ \int_0^{\tau_1}\int_0^{s_1} \frac{|(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}-(s_1^m-\xi^m)^{\gamma-1}(\tau_1^n-\omega^n)^{\theta-1}|}{\Gamma(\gamma)\Gamma(\theta)}$$
$$\times m\xi^{m-1}n\omega^{n-1}|k(f(s_1,\tau_1,\xi,\omega))||u(\xi,\omega)|d\xi d\omega.$$

If we set $\Pi_{kof}(\epsilon,.)=\sup\{|k(f(s,\tau,\xi,\omega))-k(f(s',\tau',\xi,\omega))|\}$, such that $s,s',\tau,\tau',\xi\ ,\omega\in[0,1]$, $|s-s'|\le\epsilon$, and $|\tau-\tau'|\le\epsilon$, then, from (b_3) we have,

$$|(Gu)(s_2,\tau_2)-(Gu)(s_1,\tau_1)|$$
$$\le\frac{\|u\|\,\Pi_{kof}(\epsilon,.)}{\Gamma(\gamma)\Gamma(\theta)}\int_0^{\tau_2}\int_0^{s_2}(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}m\xi^{m-1}n\omega^{n-1}d\xi d\omega$$
$$+\frac{\|u\|\,\|k\|}{\Gamma(\gamma)\Gamma(\theta)}\int_{\tau_1}^{\tau_2}\int_{s_1}^{s_2}(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}m\xi^{m-1}n\omega^{n-1}d\xi d\omega$$
$$+\frac{\|u\|\,\|k\|}{\Gamma(\gamma)\Gamma(\theta)}\int_0^{\tau_2}\int_0^{s_1}\Big[(s_1^m-\xi^m)^{\gamma-1}(\tau_1^n-\omega^n)^{\theta-1}$$
$$-(s_2^m-\xi^m)^{\gamma-1}(\tau_2^n-\omega^n)^{\theta-1}\Big]m\xi^{m-1}n\omega^{n-1}d\xi d\omega$$
$$\le\frac{\|u\|\,\Pi_{kof}(\epsilon,0)}{\Gamma(\gamma)\Gamma(\theta)}\frac{s_2^{m\gamma}}{\gamma}\frac{\tau_2^{n\theta}}{\theta}+\frac{\|u\|\,\|k\|}{\Gamma(\gamma)\Gamma(\theta)}\frac{(s_2^m-s_1^m)^\gamma}{\gamma}\frac{(\tau_2^n-\tau_1^n)^\theta}{\theta}$$
$$+\frac{\|u\|\,\|k\|}{\Gamma(\gamma)\Gamma(\theta)}[\frac{(s_2^m-s_1^m)^\gamma(\tau_2^n-\tau_1^n)^\theta}{\gamma\theta}+\frac{s_1^{m\gamma}}{\gamma}\frac{\tau_1^{n\theta}}{\theta}-\frac{s_2^{m\gamma}}{\gamma}\frac{\tau_2^{n\theta}}{\theta}]$$
$$\le\frac{\|u\|\,\Pi_{kof}(\epsilon,.)}{\Gamma(\theta+1)\Gamma(\gamma+1)}+\frac{2\,\|u\|\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}(s_2^m-s_1^m)^\gamma(\tau_2^n-\tau_1^n)^\theta. \tag{2.6}$$

Clearly, given the uniform continuity of the function $k\circ f$ on the set $[0,1]\times[0,1]\times[0,1]\times[0,1]$, we can get $\omega_{k\circ f}(\epsilon,0)\to 0$ as $\epsilon\to 0$. Thus, $Gu\in C([0,1]\times[0,1])$, and consequently, $\varsigma u\in C([0,1]\times[0,1])$. Furthermore, we have

$$|(Gu)(s,\tau)|\le\int_0^\tau\int_0^s\frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}m\xi^{m-1}n\omega^{n-1}$$
$$|k(f(s,\tau,\xi,\omega))u(\xi,\omega)|d\xi d\omega,$$
$$\le\frac{\|k\|\,\|u\|}{\Gamma(\gamma)\Gamma(\theta)}\int_0^\tau\int_0^s(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}\,m\xi^{m-1}n\omega^{n-1}d\xi d\omega$$
$$\le\frac{\|k\|\,\|u\|}{\Gamma(\gamma+1)\Gamma(\theta+1)} \tag{2.7}$$

for all $s,\tau\in[0,1]$. Accordingly

$$\begin{aligned}|(\varsigma u)(s,\tau)| &\le |g(s,\tau)|+|h(s,\tau,u)||Gu(\xi,\omega)|\\ &\le M_1+[|h(s,\tau,u)-h(s,\tau,0)|+|h(s,\tau,0)|]\frac{\|k\|\,\|u\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}\\ &\le M_1+(\vartheta(\|u\|)+M_2)\frac{\|k\|\,\|u\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}.\end{aligned}$$

For this reason,

$$\|\varsigma u\| \leq M_1 + (\vartheta(\|u\|) + M_2)\frac{\|k\| \, \|u\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}.$$

Therefore, if $\|u\| \leq r_0$ then by assumption (b_6) we have,

$$\|\varsigma u\| \leq M_1 + (\vartheta(r_0) + M_2)\frac{\|k\| \, r_0}{\Gamma(\gamma+1)\Gamma(\theta+1)} \leq r_0.$$

Thus, the operator ς maps the ball $B_{r_0} \subset C([0,1] \times [0,1])$ into itself. To demonstrate the continuity of ς on B_{r_0}, assume that $\{u_n\}$ is a sequence in B_{r_0} such that $u_n \to u$. We need to show that $\varsigma u_n \to \varsigma u$; specifically, for all $s, \tau \in [0,1]$,

$$\begin{aligned}
&|(Gu_n)(s,\tau) - (Gu)(s,\tau)| \\
&= \Big| \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \, m\xi^{m-1} n\theta^{n-1} k(f(s,\tau,\xi,\omega)) u_n(\xi,\omega) d\xi d\omega \\
&- \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \, m\xi^{m-1} n\omega^{n-1} k(f(s,\tau,\xi,\omega)) u(\xi,\omega) d\xi d\omega \Big| \\
&\leq \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \, m\xi^{m-1} n\theta^{n-1} |k(f(s,\tau,\xi,\omega))| \\
&\quad \times |u_n(\xi,\omega) - u(\xi,\omega)| d\xi d\omega \leq \frac{1}{\Gamma(\gamma)\Gamma(\theta)} \, \|k\| \, \|u_n - u\| \, \frac{s^{m\gamma}\tau^{n\theta}}{\gamma\theta}.
\end{aligned}$$

Since $s^{m\gamma}\tau^{n\theta} \leqslant 1$, then

$$\|Gu_n - Gu\| \leq \frac{\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)} \left\| u_n - u \right\|.$$

As, from (b_2) and (b_6) we conclude that,

$$\begin{aligned}
&|(\varsigma u_n)(s,\tau) - (\varsigma u)(s,\tau)| = |h(s,\tau,u_n(s,\tau))(Gu_n)(s,\tau) \\
&- h(s,\tau,u(s,\tau))(Gu)(s,\tau)| \leq |h(s,\tau,u_n(s,\tau))(Gu_n)(s,\tau) \\
&- h(s,\tau,u(s,\tau))(Gu_n)(s,\tau)| + |h(s,\tau,u(s,\tau))(Gu_n)(s,\tau) \\
&- h(s,\tau,u(s,\tau))(Gu)(s,\tau)| \leq |h(s,\tau,u_n(s,\tau)) - h(s,\tau,u(s,\tau))| \\
&\times |(Gu_n)(s,\tau)| + |h(s,\tau,u(s,\tau)) - h(s,\tau,0) + h(s,\tau,0)| \\
&\times |(Gu_n)(s,\tau) - (Gu)(s,\tau)| \leq \vartheta(|u_n(s,\tau) - u(s,\tau)|) \\
&\times \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \theta^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \, m\xi^{m-1} n\omega^{n-1} \\
&\times |k(f(s,\tau,\xi,\omega))||u_n(\xi,\omega) - u(\xi,\omega)| d\xi d\omega + (\vartheta(|u(s,\tau)|) + M_2) \\
&\times \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \theta^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \, m\xi^{m-1} n\omega^{n-1}
\end{aligned}$$

$$\times |k(f(s,\tau,\xi,\omega))||u_n(\xi,\omega)-u(\xi,\omega)|d\xi d\omega.$$

It follows that

$$\|\varsigma u_n-\varsigma u\| \leq \vartheta(\|u_n-u\|)\frac{\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}\|u_n\| + (\vartheta(\|u\|)+M_2)\frac{\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}\|u_n-u\|.$$

Thus, ς is continuous on B_{r_0}. We present

$$\widetilde{B}_{r_0}=\{u\in B_{r_0}: u(s,\tau)\geq 0,\ for\ s,\tau\in[0,1]\}\subseteqq B_{r_0}.$$

Clearly, $\widetilde{B}_{r_0}\neq\varnothing$ is closed, bounded, and convex. By assumptions (b_1), (b_2), and (b_5), if $u(s,\tau)\geq 0$, then $(\varsigma u)(s,\tau)\geq 0$ for all $s,\tau\in[0,1]$; as a result, ς projects $\widetilde{B}_{r_0}$ into itself. Additionally, ς is continuous on $\widetilde{B}_{r_0}$. Suppose $A\neq\varnothing$ is a subset of $\widetilde{B}_{r_0}$, and let $\epsilon>0$ and $s_1,\ s_2,\ \tau_1,\ \tau_2\in[0,1]$ be such that $|s_2-s_1|\leq\epsilon$ and $|\tau_2-\tau_1|\leq\epsilon$. Using (2.1), (2.6), (2.7), (b_2), and (b_7), and also for simplicity, assuming $s_2\geq s_1$ and $\tau_2\geq\tau_1$, then we have,

$$\begin{aligned}
&|(\varsigma u)(s_2,\tau_2)-(\varsigma u)(s_1,\tau_1)|=|g(s_2,\tau_2)+h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2)\\
&-g(s_1,\tau_1)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)|\leq|g(s_2,\tau_2)-g(s_1,\tau_1)|\\
&+|h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_2,\tau_2))(Gu)(s_2,\tau_2)|\\
&+|h(s_1,\tau_1,u(s_2,\tau_2))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)|\\
&+|h(s_1,\tau_1 u(s_1,\tau_1))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)|\\
&\leq|g(s_2,\tau_2)-g(s_1,\tau_1)|+|h(s_2,\tau_2,u(s_2,\tau_2))\\
&-h(s_1,\tau_1,u(s_2,\tau_2))||(Gu)(s_2,\tau_2)|+|h(s_1,\tau_1,u(s_2,\tau_2))\\
&-h(s_1,\tau_1,u(s_1,\tau_1))||(Gu)(s_2,\tau_2)|+|h(s_1,\tau_1,u(s_1,\tau_1))\\
&-h(s,\tau,0)+h(s,\tau,0)||(Gu)(s_2,\tau_2)-(Gu)(s_1,\tau_1)|\\
&\leq\Pi(g,\epsilon)+\rho_{r_0}(h,\epsilon)\frac{\|u\|\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}\\
&+\vartheta(|u(s_2,\tau_2)-u(s_1,\tau_1)|)\frac{\|u\|\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}+(\vartheta(\|u\|)+M_2)\\
&\Big[\frac{\|u\|\,\Pi_{kof}(\epsilon,0)}{\Gamma(\theta+1)\Gamma(\gamma+1)}+\frac{2\,\|u\|\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}(s_2^m-s_1^m)^\gamma(\tau_2^n-\tau_1^n)^\theta\Big],
\end{aligned}$$

where we introduce

$$\rho_{r_0}(h,\epsilon)=\sup\{|h(s,\tau,u)-h(s',\tau',u)|: s,s',\tau,\tau'\in[0,1],\ u\in[0,r_0]\times[0,r_0],|s-s'|\leq\epsilon\ ,|\tau-\tau'|\leq\epsilon\},$$

and employing the mean value theorem ($|s_2^m-s_1^m|^\gamma\leq m^\gamma|s_2-s_1|^\gamma$, $|\tau_2^n-\tau_1^n|^\theta\leq n^\theta|\tau_2-\tau_1|^\theta$) in the last inequality, we derive that,

$$|(\varsigma u)(s_2,\tau_2)-(\varsigma u)(s_1,\tau_1)|\leq\Pi(g,\epsilon)+\rho_{r_0}(h,\epsilon)\frac{\|u\|\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}$$

$$+\vartheta(|u(s_2,\tau_2)-u(s_1,\tau_1)|)\frac{\|u\|\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}$$
$$+(\vartheta(\|u\|)+M_2)[\frac{\|u\|\,\Pi_{kof}(\epsilon,0)}{\Gamma(\gamma+1)\Gamma(\theta+1)}+\frac{2\,\|u\|\,\|k\|\,(m\epsilon)^{\gamma}(n\epsilon)^{\theta}}{\Gamma(\gamma+1)\Gamma(\theta+1)}], \tag{2.8}$$

Thus, from (2.1) and (2.8),

$$\Pi(\varsigma u,\epsilon)\leq \Pi(g,\epsilon)+\rho_{r_0}(h,\epsilon)\frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}+\vartheta(\Pi(u,\epsilon))\frac{r_0\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}$$
$$+(\vartheta(r_0)+M_2)[\frac{r_0\Pi_{kof}(\epsilon,0)}{\Gamma(\theta+1)\Gamma(\gamma+1)}+\frac{2r_0\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}(m\epsilon)^{\gamma}(n\epsilon)^{\theta}].$$

Taking the supremum on $u\in A$, we conclude that

$$\Pi(\varsigma A,\epsilon)\leq \Pi(g,\epsilon)+\rho_{r_0}(h,\epsilon)\frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}+\vartheta(\Pi(A,\epsilon))\frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}$$
$$+(\vartheta(r_0)+M_2)[\frac{r_0\Pi_{kof}(\epsilon,0)}{\Gamma(\theta+1)\Gamma(\gamma+1)}+\frac{2r_0\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}(m\epsilon)^{\gamma}(n\epsilon)^{\theta}].$$

As g is continuous, and h and $k\circ f$ are uniformly continuous on $[0,1]\times[0,1]\times[0,r_0]$ and $[0,1]\times[0,1]\times[0,1]\times[0,1]$ respectively, it follows that when $\epsilon\to 0$, then $\Pi(g,\epsilon)\to 0$, $\rho_{r_0}(h,\epsilon)\to 0$, $\Pi_{k\circ f}(\epsilon,0)\to 0$, and moreover, by (2.2), we have:

$$\Pi_0(\varsigma A)\leq \frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}\,\vartheta(\Pi_0(A)). \tag{2.9}$$

Assuming $u\in A$ and $s_1,s_2,\tau_1,\tau_2\in[0,1]$ such that $s_1<s_2$ and $\tau_1<\tau_2$, and with regard to (2.3), (2.5), (b_2), (b_7), and the non-decreasing functions g and Gu, then,

$$\begin{aligned}
&|(\varsigma u)(s_2,\tau_2)-(\varsigma u)(s_1,\tau_1)|-[(\varsigma u)(s_2,\tau_2)-(\varsigma u)(s_1,\tau_1)]\\
=\;&|g(s_2,\tau_2)+h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2)-g(s_1,\tau_1)\\
&-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)|\\
&-[g(s_2,\tau_2)+h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2)-g(s_1,\tau_1)\\
&-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)]\\
\leq\;&\{|g(s_2,\tau_2)-g(s_1,\tau_1)|-[g(s_2,\tau_2)-g(s_1,\tau_1)]\}\\
&+|h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)|\\
&+|h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)|\\
&-[h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)]\\
&+[h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)]\\
\leq\;&J(g)+\Big\{|h(s_2,\tau_2,u(s_2,\tau_2))-h(s_1,\tau_1,u(s_1,\tau_1))|\\
&-\Big[h(s_2,\tau_2,u(s_2,\tau_2))-h(s_1,\tau_1,u(s_1,\tau_1))\Big]\Big\}(Gu)(s_2,\tau_2)\\
&+h(s_1,\tau_1,u(s_1,\tau_1))\Big\{|(Gu)(s_2,\tau_2)-(Gu)(s_1,\tau_1)|
\end{aligned}$$

$$
\begin{aligned}
&\quad -\Big[(Gu)(s_2,\tau_2)-(Gu)(s_1,\tau_1)\Big]\Big\}\\
&\leq \quad J(g)+J(Hu)\frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}+(\vartheta(\|u\|)+M_2)J(Gu)\\
&= \quad J(Hu)\frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}.
\end{aligned}
$$

Furthermore, from (b_4) it concludes that,

$$
J(\varsigma u)\leq J(Hu)\frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}\leq \vartheta(J(u))\frac{r_0\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)},
$$

and consequently by (2.3),

$$
J(\varsigma A)\leq \frac{r_0\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}\vartheta(J(A)). \tag{2.10}
$$

Regarding the definition of $\pounds$ in (2.4), and also from (2.9) and (2.10), we have:

$$
\begin{aligned}
\pounds(\varsigma A) &= \Pi_0(\varsigma A)+J(\varsigma A)\\
&\leq \frac{r_0\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}\,\vartheta(\Pi_0(A))+\frac{r_0\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}\vartheta(J(A))\\
&\leq \frac{r_0\,\|k\|}{\Gamma(\theta+1)\Gamma(\gamma+1)}(\vartheta(\Pi_0(A)+J(A)))\leq \lambda\vartheta(\pounds(A)).
\end{aligned}
$$

The proof is completed by Theorem 1 for $G(s)=\Psi(s)=s$, the aforementioned inequality, and the condition $\lambda=\frac{r_0\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}<1$. □

Theorem 3. *Under conditions of Theorem 2 and condition* $\frac{(M_2+2r_0)\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}<1$, *then Equation* (1.1) *has unique non-decreasing solution as* $u=u(\xi,\omega)\in C([0,1]\times[0,1])$.

Proof. According to the proving of Theorem 2, let $u_1(\xi,\omega),u_2(\xi,\omega)\in B_{r_0}\subset C([0,1]\times[0,1])$ are two different solutions of the Equation (1.1), thus from Equation (1.1) and applying (b_2) and (b_6) we have,

$$
\begin{aligned}
&|u_1(s,\tau)-u_2(s,\tau)\,|\\
= &\Big|h(s,\tau,u_1(s,\tau))\int_0^\tau\int_0^s\frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,m\xi^{m-1}n\omega^{n-1}\\
\times\ &k(f(s,\tau,\xi,\omega))u_1(\xi,\omega)d\xi d\omega\\
&-h(s,\tau,u_2(s,\tau))\int_0^\tau\int_0^s\frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,m\xi^{m-1}n\omega^{n-1}\\
\times\ &k(f(s,\tau,\xi,\omega))u_2(\xi,\omega)d\xi d\omega\Big|\\
\leq &\Big|h(s,\tau,u_1(s,\tau))\int_0^\tau\int_0^s\frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,m\xi^{m-1}n\omega^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad k(f(s,\tau,\xi,\omega))\Big(u_1(\xi,\omega)-u_2(\xi,\omega)\Big)d\xi d\omega\Big| \\
&+ \quad \Big|\Big(h(s,\tau,u_1(s,\tau))-h(s,\tau,u_2(s,\tau))\Big)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\
&\quad\quad m\xi^{m-1}n\omega^{n-1}k(f(s,\tau,\xi,\omega))u_2(\xi,\omega)d\xi d\omega\Big| \\
&\le \quad \Big[|h(s,\tau,u_1)-h(s,\tau,0)|+|h(s,\tau,0)|\Big]\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\
&\quad\quad m\xi^{m-1}n\omega^{n-1}|k(f(s,\tau,\xi,\omega))|\;|u_1(\xi,\omega)-u_2(\xi,\omega)|d\xi d\omega \\
&\quad\quad +\Big|h(s,\tau,u_1(s,\tau))-h(s,\tau,u_2(s,\tau))\Big|\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\
&\quad\quad m\xi^{m-1}n\omega^{n-1}|k(f(s,\tau,\xi,\omega))|\;|u_2(\xi,\omega)|d\xi d\omega \\
&\le \quad (\vartheta(|u_1|)+M_2)\,\|k\|\;\;\|u_1-u_2\|\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\
&\quad\quad m\xi^{m-1}n\omega^{n-1}d\xi d\omega \\
&\quad\quad +\vartheta(|u_1-u_2|)\,\|k\|\;\;\|u_2\|\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\
&\quad\quad m\xi^{m-1}n\omega^{n-1}d\xi d\omega \\
&\le \quad (\|u_1\|+M_2+\|u_2\|\,\|k\|\;\;\|u_1-u_2\|\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\
&\quad\quad m\xi^{m-1}n\omega^{n-1}d\xi d\omega \\
&\le \quad (M_2+2r_0)\,\|k\|\;\;\|u_1-u_2\|\,\frac{1}{\Gamma(\gamma)\Gamma(\theta)}\frac{s^{m\gamma}\tau^{n\theta}}{\gamma\theta}.
\end{aligned}
$$

□

Since $s^{m\gamma}\tau^{n\theta}\leqslant 1$, then

$$
\|u_1-u_2\|\le\frac{(M_2+2r_0)\,\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}\,\|u_1-u_2\|\,,
$$

consequently from $\frac{(M_2+2r_0)\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}<1$ concludes that $u_1=u_2$.

Regarding the Riemann-Liouville fractional integral [19] and Theorem 2, we derive some results as follows.

Corollary 1. If Theorem 2's conditions are satisfied, then there is at least one solution for integral equations with fractional order in $C([0,1]\times[0,1])$. Some examples of these would be:

(i) For $m=n=1$, this implies two-dimensional non-linear Riemann-Liouville (or Volterra) fractional integral equations with orders γ and θ, as follows:

$$
\begin{aligned}
u(s,\tau) &= g(s,\tau)+h(s,\tau,u(s,\tau)) \\
&\quad\times\int_0^\tau\int_0^s \frac{(s-\xi)^{\gamma-1}(\tau-\omega)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,k(f(s,\tau,\xi,\omega))u(\xi,\omega)\,d\xi d\omega.
\end{aligned}
$$

(ii) For $m = n = 1$ and $h(s, \tau, u(s, \tau)) = 1$, we obtain two-dimensional Riemann-Liouville fractional integral equations of the second kind with orders γ and θ:

$$u(s,\tau) = g(s,\tau) + \int_0^\tau \int_0^s \frac{(s-\xi)^{\gamma-1}(\tau-\omega)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} k(f(s,\tau,\xi,\omega))u(\xi,\omega)\, d\xi d\omega.$$

(iii) For $m = n = 1, k = I$ and $g(s, \tau) = 0$, this introduces two-dimensional non-linear Riemann-Liouville fractional integral equations of the first kind with orders γ and θ:

$$u(s,\tau) = h(s,\tau,u(s,\tau)) \int_0^\tau \int_0^s \frac{(s-\xi)^{\gamma-1}(\tau-\omega)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} f(s,\tau,\xi,\omega)u(\xi,\omega)\, d\xi d\omega.$$

(iv) For $m = n = 1, k = I, h(s, \tau, u(s, \tau)) = 1$ and $g(s, \tau) = 0$, this introduces two-dimensional Riemann-Liouville fractional integral equations of the first kind with orders γ and θ:

$$u(s,\tau) = \int_0^\tau \int_0^s \frac{(s-\xi)^{\gamma-1}(\tau-\omega)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} f(s,\tau,\xi,\omega)u(\xi,\omega)\, d\xi d\omega.$$

According to the above process, integral equation (1.1) is a general form of several kinds of Riemann-Liouville fractional integral equations.

Corollary 2. If the conditions of Theorem 3 are satisfied, then there is a unique solution to the integral equations of fractional order in Corollary 1.

3 Approximating the solution of Equation (1.1)

The solvability of Equation (1.1) is demonstrated in Section 2. In existence and uniqueness theorems, it is important that the solution be nontrivial (non zero). This issue becomes evident when finding an approximate solution to the above equation. Based on a topological notion, a homotopy is created, and some of its applications are provided in [23]. The homotopy perturbation approach has been modified in [15, 26]. The Adomian decomposition method is used in [26, 28] for nonlinear problems. In order to solve Equation (1.1), we use modified homotopy perturbation and Adomian decomposition methods. Taking into account the general form of Equation (1.1),

$$A(u(s,\tau)) - g(s,\tau) = 0, \qquad (s,\tau) \in \Pi = [0,1] \times [0,1],$$

where g represents a known function, and the nonlinear operator A is divided into two operators, $\aleph_1$ and $\aleph_2$, which may be linear or nonlinear. Additionally, the function g is split into two functions, g_1 and g_2, so that

$$\aleph_1(u(s,\tau)) - g_1(s,\tau) + \aleph_2(u(s,\tau)) - g_2(s,\tau) = 0,$$

and in this way, a modified homotopy perturbation is introduced by

$$H(y(s,\tau),q) = \aleph_1(y(s,\tau)) - g_1(s,\tau) + q\big(\aleph_2(y(s,\tau)) - g_2(s,\tau)\big) = 0, \quad (3.1)$$

where $0 \le q \le 1$ is an embedding parameter and y is an approximation of u. By varying q from 0 to 1, this leads from $\aleph_1(y(s,\tau)) = g_1(s,\tau)$ to $A(y(s,\tau)) - g(s,\tau) = 0$. This signifies that for $q = 1$ in (3.1), a solution of Equation (1.1) is obtained. We regard the above solution as a series:

$$u(s,\tau) \simeq y(s,\tau) = \sum_{i=0}^{\infty} q^i y_i(s,\tau), \qquad u(s,\tau) = \lim_{q\to 1} y(s,\tau). \tag{3.2}$$

To solve Equation (1.1) for $s,\tau \in [0,1]$, we reformulate it as follows:

$$\begin{aligned} u(s,\tau) - h(s,\tau,u(s,\tau)) \int_0^{\tau}\int_0^{s} \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \\ \times\, m\xi^{m-1} n\omega^{n-1} k(f(s,\tau,\xi,\omega))u(\xi,\omega)\, d\xi d\omega - g(s,\tau) = 0, \end{aligned}$$

where $g(s,\tau)$ is a known function and $u \in C([0,1]\times[0,1])$. The operators $\aleph_1$ and $\aleph_2$ and also the function g can be given as

$$\begin{cases} \aleph_1(y(s,\tau)) = y(s,\tau), \\ \aleph_2(y(s,\tau)) = -h(s,\tau,y(s,\tau)) \displaystyle\int_0^{\tau}\int_0^{s} \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1} \\ \qquad k(f(s,\tau,\xi,\omega))y(\xi,\omega)\, d\xi d\omega, \\ g(s,\tau) = g_1(s,\tau) + g_2(s,\tau). \end{cases} \tag{3.3}$$

Replacing (3.2) and (3.3) in (3.1), we can derive

$$\begin{aligned} \Big(\sum_{i=0}^{\infty} q^i y_i(s,\tau) - g_1(s,\tau)\Big) + q\Big(- h(s,\tau,\sum_{i=0}^{\infty} q^i y_i(s,\tau)) \\ \times \int_0^{\tau}\int_0^{s} \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1} \\ \times\, k(f(s,\tau,\xi,\omega)) \sum_{i=0}^{\infty} q^i y_i(\xi,\omega)\, d\xi d\omega - g(s,\tau) + g_1(s,\tau)\Big) = 0. \end{aligned} \tag{3.4}$$

To solve the nonlinear part of (3.4), we use Adomian polynomials as follows:

$$\begin{aligned} h(s,\tau,\sum_{i=0}^{\infty} q^i y_i(s,\tau)) = \sum_{j=0}^{\infty} q^j A_j(s,\tau), \\ A_j(s,\tau) = \frac{1}{j!}\Big(\frac{d^j}{dq^j} h(s,\tau,\sum_{i=0}^{\infty} q^i y_i(s,\tau))\Big)_{q=0}, \quad j = 0,1,2,\ldots. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\Big(\sum_{i=0}^{\infty} q^i y_i(s,\tau) - g_1(s,\tau)\Big) + q\Big(- \sum_{j=0}^{\infty} q^j A_j(s,\tau)$$

$$\times \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1}$$
$$\times k(f(s,\tau,\xi,\omega)) \sum_{i=0}^{\infty} q^i y_i(\xi,\omega)\, d\xi d\omega - g(s,\tau) + g_1(s,\tau)\Big) = 0. \qquad (3.6)$$

Rearranging (3.6) with respect to the powers of q, an iterative algorithm is obtained as follows:

Algorithm 1.

$$y_0(s,\tau) = g_1(s,\tau),$$
$$y_1(s,\tau) = A_0(s,\tau) \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1}$$
$$\times k(f(s,\tau,\xi,\omega)) y_0(\xi,\omega) d\xi d\omega + g(s,\tau) - g_1(s,\tau),$$
$$y_j(s,\tau) = \sum_{i=0}^{j-1} A_i(s,\tau) \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1}$$
$$\times k(f(s,\tau,\xi,\omega)) y_{j-1-i}(\xi,\omega) d\xi d\omega, \quad j = 2, 3, \ldots.$$

The convergence of the Algorithm 1 is established by a lemma and a theorem.

Lemma 1. *Algorithm 1 is equivalent to a recursive relation:*

$$u_n(s,\tau) = g(s,\tau) - \aleph_2(u_{n-1}(s,\tau)), \qquad (3.7)$$

where operators $\aleph_1$ *and* $\aleph_2$ *and also* $g(s,\tau)$ *are given by* (3.3).

Proof. Corresponding to (3.2), the approximate solution of Equation (1.1) for $q \to 1$ is given by

$$u_n(s,\tau) = \sum_{i=0}^{n} y_i(s,\tau). \qquad (3.8)$$

Substituting $y_i's$ from Algorithm 1 into (3.8) leads to:

$$u_n(s,\tau) = \sum_{i=0}^{n} y_i(s,\tau) = g_1(s,\tau)$$
$$+A_0(s,\tau) \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1} k(f(s,\tau,\xi,\omega))$$
$$\times y_0(\xi,\omega) d\xi d\omega + g(s,\tau) - g_1(s,\tau)$$
$$+A_0(s,\tau) \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1} k(f(s,\tau,\xi,\omega))$$
$$\times y_1(\xi,\omega) d\xi d\omega + A_1(s,\tau)$$
$$\times \int_0^\tau \int_0^s \frac{(s^m - \xi^m)^{\gamma-1}(\tau^n - \omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1} k(f(s,\tau,\xi,\omega))$$
$$\times y_0(\xi,\omega) d\xi d\omega$$

$$\vdots$$

$$+\sum_{i=0}^{n-1} A_i(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1}n\omega^{n-1}$$
$$\times k(f(s,\tau,\xi,\omega))y_{n-1-i}(\xi,\omega)d\xi d\omega$$
$$= g(s,\tau)+\sum_{i=0}^{n-1} A_i(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1}$$
$$\times n\omega^{n-1}k(f(s,\tau,\xi,\omega))\sum_{j=0}^{n-1-i} y_j(\xi,\omega)d\xi d\omega$$
$$= g(s,\tau)-\aleph_2(\sum_{i=0}^{n-1} y_i(s,\tau)) = g(s,\tau)-\aleph_2(u_{n-1}(s,\tau)).$$

□

In the above process, we used forms (3.6) and (1) to introduce $\aleph_2(\sum_{i=0}^{n-1} y_i(s,\tau))$. Assuming that the recursive relation (3.7) holds, we obtain Algorithm 1 by induction. Let $u_0(s,\tau) = y_0(s,\tau)$. Then, for $n = 1$ in (3.7) and applying (3.3) and (3.5), we have

$$u_1(s,\tau) = g(s,\tau)-\aleph_2(u_0(s,\tau)) = g(s,\tau)-\aleph_2(y_0(s,\tau)) \tag{3.9}$$
$$= g(s,\tau)+h(s,\tau,y_0(s,\tau))\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}$$
$$\times\, m\xi^{m-1}n\omega^{n-1}k(f(s,\tau,\xi,\omega))y_0(\xi,\omega)d\xi d\omega$$
$$= g_1(s,\tau)+A_0(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}$$
$$\times\, m\xi^{m-1}n\omega^{n-1}k(f(s,\tau,\xi,\omega))y_0(\xi,\omega)d\xi d\omega + g(s,\tau)-g_1(s,\tau).$$

Because $u_1(s,\tau) = y_0(s,\tau)+y_1(s,\tau)$, by choosing $y_0(s,\tau) = g_1(s,\tau)$, the remaining terms of (3.9) correspond to $y_1(s,\tau)$. Thus, the induction test holds. We assume that the induction hypothesis holds for $(n-1)$; then, for n, we have

$$u_n(s,\tau) = g(s,\tau)-\aleph_2(u_{n-1}(s,\tau)) = g(s,\tau)-\aleph_2(\sum_{i=0}^{n-1} y_i(s,\tau))$$
$$= \quad g(s,\tau)+\sum_{i=0}^{n-1} A_i(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1}n\omega^{n-1}$$
$$\times k(f(s,\tau,\xi,\omega))\sum_{j=0}^{n-1-i} y_j(\xi,\omega)d\xi d\omega$$
$$= \quad g_1(s,\tau)+A_0(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1}n\omega^{n-1}$$
$$\times k(f(s,\tau,\xi,\omega))y_0(\xi,\omega)d\xi d\omega + \; g(s,\tau)-g_1(s,\tau)$$

$$\begin{aligned}
&\vdots\\
&+\sum_{i=0}^{n-2} A_i(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1}\\
&\quad\times k(f(s,\tau,\xi,\omega))y_{n-2-i}(\xi,\omega)d\xi d\omega\\
&+\sum_{i=0}^{n-1} A_i(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1}\\
&\quad\times k(f(s,\tau,\xi,\omega))y_{n-1-i}(\xi,\omega)d\xi d\omega.
\end{aligned}$$

From $u_n(s,\tau) = \sum_{i=0}^{n-1} y_i(s,\tau) + y_n(s,\tau)$ and the induction hypothesis, we can write

$$\begin{aligned}
y_n(s,\tau) = &\sum_{i=0}^{n-1} A_i(s,\tau)\int_0^\tau\int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1}\\
&\times k(f(s,\tau,\xi,\omega))\, y_{n-1-i}(\xi,\omega)\, d\xi d\omega,
\end{aligned}$$

and the proof is completed.

In Section 2, we proved that the solution space of Equation (1.1) is the Banach space $C([0,1]\times[0,1], \|\cdot\|_\infty)$. Therefore, the following theorem is stated for the convergence of Algorithm 1.

Theorem 4. *Let $u_n(s,\tau) = \sum_{i=0}^{n} y_i(s,\tau)$ for $n\in\mathbb{N}$, and for $y_i(s,\tau)\neq 0$, assume that $\|y_i(s,\tau)\|_\infty \le \lambda\|y_{i-1}(s,\tau)\|_\infty$, where $0\le\lambda<1$, $i\in\mathbb{N}$. Then,*

(i) $\{u_n\}_{n=1}^\infty$ is a convergent sequence in the Banach space $(C([0,1]\times[0,1]), \|\cdot\|_\infty)$.

(ii) $\lim_{n\to\infty} u_n(s,\tau) = \sum_{i=0}^{\infty} y_i(s,\tau) = u^*(s,\tau)$ satisfies the recursive relation (3.7) and the general form of Equation (1.1).

Proof. (i) Similar to [24], the sequence $\{u_n\}_{n=1}^\infty$ is easily shown to be a Cauchy sequence in the Banach space, and the proof is complete.
(ii) From (i), there exists $u^*\in C([0,1]\times[0,1])$ such that

$$\lim_{n\to\infty} u_n(s,\tau) = \lim_{n\to\infty}\sum_{i=0}^{n} y_i(s,\tau) = \sum_{i=0}^{\infty} y_i(s,\tau) = u^*(s,\tau).$$

Therefore, from (3.7), it follows that

$$\begin{aligned}
u^*(s,\tau) &= \lim_{n\to\infty} u_n(s,\tau) = g(s,\tau) - \lim_{n\to\infty}\aleph_2(u_{n-1}(s,\tau))\\
&= g(s,\tau) - \lim_{n\to\infty}\aleph_2\Big(\sum_{i=0}^{n-1} y_i(s,\tau)\Big) = g(s,\tau)
\end{aligned}$$

$$
\begin{aligned}
&- \lim_{n\to\infty} \sum_{i=0}^{n-1} A_i(s,\tau) \int_0^\tau \int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1} \\
&\times k(f(s,\tau,\xi,\omega)) \sum_{j=0}^{n-1-i} y_j(\xi,\omega)\, d\xi d\omega \\
&= g(s,\tau) - \sum_{i=0}^{\infty} A_i(s,\tau) \int_0^\tau \int_0^s \frac{(s^m-\xi^m)^{\gamma-1}(\tau^n-\omega^n)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} m\xi^{m-1} n\omega^{n-1} \\
&\times k(f(s,\tau,\xi,\omega)) \sum_{j=0}^{\infty} y_j(\xi,\omega)\, d\xi d\omega \\
&= g(s,\tau) - \aleph_2\Big(\sum_{i=0}^{\infty} y_i(s,\tau)\Big) = g(s,\tau) - \aleph_2(u^*(s,\tau)).
\end{aligned}
$$

Since $\aleph_1$ is defined as the identity operator in (3.3), it follows that $u^*(s,\tau)$ satisfies the general form of Equation (1.1). Thus, $u^*(s,\tau)$ is a solution of Equation (1.1). □

4 Application

Let us examine an example by employing Theorems 2 and 3 for Equation (1.1) in the special case where $m = n = 1$ and $\gamma = \theta = \frac{1}{2}$.

Example 1. Consider a two-dimensional non-linear singular Volterra integral equation with fractional order, where $s, \tau \in [0,1]$.

$$
\begin{aligned}
u(s,\tau) &= \frac{1}{7}s^2\tau^2 + \frac{3s\tau|u(s,\tau)|}{7(1+s+\tau)(1+|u(s,\tau)|)} \\
&\times \int_0^\tau \int_0^s \frac{4\xi\omega[\frac{1}{50}(s+\xi)(\tau+\omega)+\frac{1}{5}]}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(s^2-\xi^2)^{\frac{1}{2}}(\tau^2-\omega^2)^{\frac{1}{2}}} u(\xi,\omega) d\xi d\omega.
\end{aligned}
\tag{4.1}
$$

Thus, $g(s,\tau) = \frac{1}{7}s^2\tau^2$ satisfies assumption (b_1), with $M_1 = \frac{1}{7}$. The function $h(s,\tau,u) = \frac{3s\tau}{7(1+s+\tau)} \frac{|u(s,\tau)|}{1+|u(s,\tau)|}$ satisfies hypothesis (b_2) with the assumption $\vartheta(s) = \frac{3}{7}s$:

$$
|h(s,\tau,u) - h(s,\tau,z)| \le \frac{3}{7}|u-z| = \vartheta(|u-z|), \ \forall u,z \in \mathbb{R} \quad s,\tau \in [0,1].
$$

In addition, H satisfies (b_3). Consider an arbitrary non-negative function $u \in C([0,1]\times[0,1])$ and $s_1, s_2, \tau_1, \tau_2 \in [0,1]$ ($s_1 \le s_2$ and $\tau_1 \le \tau_2$); then

$$
\begin{aligned}
&|(Hu)(s_2,\tau_2) - (Hu)(s_1,\tau_1)| - [(Hu)(s_2,\tau_2) - (Hu)(s_1,\tau_1)] \\
&= |h(s_2,\tau_2,u(s_2,\tau_2)) - h(s_1,\tau_1,u(s_1,\tau_1))| \\
&- [h(s_2,\tau_2,u(s_2,\tau_2)) - h(s_1,\tau_1,u(s_1,\tau_1))] \\
&= |\frac{3s_2\tau_2}{7(1+s_2+\tau_2)} \frac{u(s_2,\tau_2)}{1+u(s_2,\tau_2)} - \frac{3s_1\tau_1}{7(1+s_1+\tau_1)} \frac{u(s_1,\tau_1)}{1+u(s_1,\tau_1)}|
\end{aligned}
$$

$$
\begin{aligned}
&-\Big[\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}\frac{u(s_2,\tau_2)}{1+u(s_2,\tau_2)}-\frac{3s_1\tau_1}{7(1+s_1+\tau_1)}\frac{u(s_1,\tau_1)}{1+u(s_1,\tau_1)}\Big]\\
&\leq\Big|\frac{3s_2\tau_2u(s_2,\tau_2)}{7(1+s_2+\tau_2)(1+u(s_2,\tau_2))}-\frac{3s_2\tau_2u(s_1,\tau_1)}{7(1+s_2+\tau_2)(1+u(s_1,\tau_1))}\Big|\\
&+\Big|\frac{3s_2\tau_2u(s_1,\tau_1)}{7(1+s_2+\tau_2)(1+u(s_1,\tau_1))}-\frac{3s_1\tau_1u(s_1,\tau_1)}{7(1+s_1+\tau_1)(1+u(s_1,\tau_1))}\Big|\\
&-\Big[\frac{3s_2\tau_2u(s_2,\tau_2)}{7(1+s_2+\tau_2)(1+u(s_2,\tau_2))}-\frac{3s_2\tau_2u(s_1,\tau_1)}{7(1+s_2+\tau_2)(1+u(s_1,\tau_1))}\Big]\\
&+\Big[\frac{3s_2\tau_2u(s_1,\tau_1)}{7(1+s_2+\tau_2)(1+u(s_1,\tau_1))}-\frac{3s_1\tau_1u(s_1,\tau_1)}{7(1+s_1+\tau_1)(1+u(s_1,\tau_1))}\Big]\\
&\leq\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}\Big|\frac{u(s_2,\tau_2)}{1+u(s_2,\tau_2)}-\frac{u(s_1,\tau_1)}{1+u(s_1,\tau_1)}\Big|\\
&+\frac{u(s_1,\tau_1)}{1+u(s_1,\tau_1)}\Big|\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}-\frac{3s_1\tau_1}{7(1+s_1+\tau_1)}\Big|\\
&-\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}\Big[\frac{u(s_2,\tau_2)}{1+u(s_2,\tau_2)}-\frac{u(s_1,\tau_1)}{1+u(s_1,\tau_1)}\Big]\\
&-\frac{u(s_1,\tau_1)}{1+u(s_1,\tau_1)}\Big[\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}-\frac{3s_1\tau_1}{7(1+s_1+\tau_1)}\Big]\\
&=\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}\Big\{\Big|\frac{u(s_2,\tau_2)-u(s_1,\tau_1)}{(1+u(s_2,\tau_2))(1+u(s_1,\tau_1))}\Big|\\
&-\Big[\frac{u(s_2,\tau_2)-u(s_1,\tau_1)}{(1+u(s_2,\tau_2))(1+u(s_1,\tau_1))}\Big]\Big\}+\frac{u(s_1,\tau_1)}{1+u(s_1,\tau_1)}\\
&\Big\{\Big|\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}-\frac{3s_1\tau_1}{7(1+s_1+\tau_1)}\Big|-\Big[\frac{3s_2\tau_2}{7(1+s_2+\tau_2)}-\frac{3s_1\tau_1}{7(1+s_1+\tau_1)}\Big]\Big\}\\
&=\frac{3}{7}\{|u(s_2,\tau_2)-u(s_1,\tau_1)|-[u(s_2,\tau_2)-u(s_1,\tau_1)]\}\leq\frac{3}{7}J(u)=\vartheta(J(u)).
\end{aligned}
$$

Thus, $J(Hu)\leq\vartheta(J(u))$. Also, function $f(s,\tau,\xi,\omega)=\frac{1}{10}(s+\xi)(\tau+\omega)$ satisfies (b_4). Let $k:[0,\frac{2}{5}]\to\mathbb{R}^+$ and $k(f)=\frac{1}{5}f+\frac{1}{5}$; then k satisfies (b_5) with $\|k\|=\frac{7}{25}$.

According to the example, the relation of assumption (b_6) is converted to $25\Gamma(1.5)\Gamma(1.5)+21r^2\leq 175\Gamma(1.5)\Gamma(1.5)r$, for which a solution of the above inequality is $r_0=1$, and $\lambda=\dfrac{\|k\|\ r_0}{\Gamma(1.5)\Gamma(1.5)}=\frac{28}{25\pi}<1$.

Thus, Theorem 2 ensures that Equation (4.1) possesses a non-decreasing solution. Moreover, since $\frac{(M_2+2r_0)\|k\|}{\Gamma(\gamma+1)\Gamma(\theta+1)}=\frac{(0+2)\frac{7}{25}}{\Gamma(1.5)\Gamma(1.5)}<1$, it follows that the solution is unique.

5 Numerical results and comparing with some other works

In Section 4, existence and uniqueness the solution of Equation (4.1) is demonstrated in Banach space $C([0,1]\times[0,1])$. We use Algorithm 1 to approximate the solution of this equation,

$$g(s,\tau)=\frac{1}{7}s^2\tau^2,\ \ h(s,\tau,u)=\frac{3s\tau}{7(1+s+\tau)}\frac{|u(s,\tau)|}{1+|u(s,\tau)|},$$
$$\text{Singular part of kernel}=\frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(s^2-\xi^2)^{\frac{1}{2}}(\tau^2-\omega^2)^{\frac{1}{2}}},$$
$$k(f(s,\tau,\xi,\omega))=\frac{1}{50}(s+\xi)(\tau+\omega)+\frac{1}{5}.$$

Because in Equation (4.1) the function $g(s,\tau)=\frac{1}{7}s^2\tau^2$, a choice of $g_1(s,\tau)$ as a part of function g can be $g_1(s,\tau)=g(s,\tau)=\frac{1}{7}s^2\tau^2$ or $g_1(s,\tau)=0$. Therefore, from (3.5) some terms of the Algorithm 1 are given by:

$$\begin{aligned}
y_0(s,\tau) &= g_1(s,\tau)=\frac{1}{7}s^2\tau^2,\\
A_0(s,\tau) &= \frac{3s\tau}{7(1+s+\tau)}\frac{|y_0(s,\tau)|}{1+|y_0(s,\tau)|}=\frac{3s\tau}{7(1+s+\tau)}\frac{s^2\tau^2}{7+s^2\tau^2},\\
y_1(s,\tau) &= A_0(s,\tau)\int_0^\tau\int_0^s\frac{k(f(s,\tau,\xi,\omega))}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(s^2-\xi^2)^{\frac{1}{2}}(\tau^2-\omega^2)^{\frac{1}{2}}}y_0(\xi,\omega)d\xi d\omega\\
&\quad +g(s,\tau)-g_1(s,\tau)=\frac{7s^6\tau^6(10240+(32+9\pi)^2s\tau)}{3292800\pi(1+s+\tau)(7+s^2\tau^2)}.
\end{aligned}$$

Also,

$$\begin{aligned}
&y_0(s,\tau)=g_1(s,\tau)=0,\ \ A_0(s,\tau)=\frac{3s\tau}{7(1+s+\tau)}\frac{|y_0(s,\tau)|}{1+|y_0(s,\tau)|}=0,\\
&y_1(s,\tau)=A_0(s,\tau)\int_0^\tau\int_0^s\frac{k(f(s,\tau,\xi,\omega))}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(s^2-\xi^2)^{\frac{1}{2}}(\tau^2-\omega^2)^{\frac{1}{2}}}y_0(\xi,\omega)d\xi d\omega\\
&\qquad +g(s,\tau)-g_1(s,\tau)=\frac{1}{7}s^2\tau^2,\\
&A_1(s,\tau)=\frac{3s\tau}{7(1+s+\tau)}\frac{|y_1(s,\tau)|}{(1+|y_0(s,\tau)|)^2}=\frac{3s^3\tau^3}{49(1+s+\tau)},\ \ y_2(s,\tau)=0,\\
&A_2(s,\tau)=\frac{3s\tau(y_2(s,\tau)(1+y_0(s,\tau))-y_1^2(s,\tau))}{7(1+s+\tau)(1+y_0(s,\tau))^3}=\frac{-3s^5\tau^5}{343(1+s+\tau)},\\
&y_3(s,\tau)=\sum_{i=0}^{3-1}A_i(s,\tau)\int_0^\tau\int_0^s\frac{k(f(s,\tau,\xi,\omega))}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(s^2-\xi^2)^{\frac{1}{2}}(\tau^2-\omega^2)^{\frac{1}{2}}}\\
&\qquad \times y_{3-1-i}(\xi,\omega)d\xi d\omega=\frac{7s^6\tau^6(10240+(32+9\pi)^2s\tau)}{3292800\pi(1+s+\tau)(7+s^2\tau^2)}.
\end{aligned}$$

As is observed, two kinds of starting points for Algorithm 1 lead to the similar terms to approximate the solution of Equation (4.1), such as

$$\frac{7s^6\tau^6(10240+(32+9\pi)^2s\tau)}{3292800\pi(1+s+\tau)(7+s^2\tau^2)}.$$

Thus, we can get an approximate solution by series (3.2) as follows:

$$u(s,\tau) \simeq \sum_{i=0}^{3} q^i y_i(s,\tau) = \frac{1}{7}s^2\tau^2 + \frac{7s^6\tau^6(10240 + (32+9\pi)^2 s\tau)}{3292800\pi(1+s+\tau)(7+s^2\tau^2)}. \quad (5.1)$$

For the validity and accuracy of (5.1) as an approximation of the solution of Equation (4.1), we substitute it in Equation (4.1), where the absolute errors are shown in Table 1. To illustrate the ability of the proposed convergent iterative Algorithm 1, we compare it with some other methods.

Table 1. Absolute errors to approximate solution of Equation (4.1) by MHPM.

(s,τ)	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0	0	0	0	0	0
0.2	0	6.7×10^{-13}	1.5×10^{-13}	3.5×10^{-12}	3.1×10^{-11}	1.7×10^{-10}
0.4	0	1.5×10^{-13}	3.5×10^{-11}	8.3×10^{-10}	7.7×10^{-9}	4.2×10^{-8}
0.6	0	3.5×10^{-12}	8.3×10^{-10}	1.9×10^{-8}	1.8×10^{-7}	1.0×10^{-6}
0.8	0	3.1×10^{-11}	7.7×10^{-9}	1.8×10^{-7}	1.7×10^{-6}	9.7×10^{-6}
1.0	0	1.7×10^{-10}	4.2×10^{-8}	1.0×10^{-6}	9.7×10^{-6}	5.4×10^{-5}

Example 2. Consider the two-dimensional Volterra integral equation with fractional order which was solved by Bernstein polynomials with the operational matrix in [4], and Euler polynomials with Gauss-Jacobi quadrature in [32].

$$u(s,\tau) - \frac{1}{\Gamma(\nu)\Gamma(\theta)} \int_0^s \int_0^\tau (s-\xi)^{\nu-1}(\tau-\omega)^{\theta-1}\sqrt{s\tau}\ \xi\omega\ u(\xi,\omega)\, d\omega d\xi = g(s,\tau), \quad (5.2)$$

where $s,\tau \in [0,1]$, $g(s,\tau) = s^3(\tau^2-\tau) - \frac{1}{60}s^{\frac{11}{2}}\tau^{\frac{7}{2}}(3\tau-4)$ and the exact solution is $u(s,\tau) = s^3(\tau^2-\tau)$ for $\nu = \theta = 1$.

At first, we consider the numerical results obtained by [4] and [32] in Table 2.

For comparing our results with the above-mentioned references, the absolute errors is computed at the similar points of Table 2. Moreover, Algorithm 1 with regards to Equation (5.2) can be written as:

Algorithm 2.

$$\begin{cases} y_0(s,\tau) = g_1(s,\tau), \\ y_1(s,\tau) = \frac{1}{\Gamma(\nu)\Gamma(\theta)} \int_0^s \int_0^\tau (s-\xi)^{\nu-1}(\tau-\omega)^{\theta-1}\sqrt{s\tau}\ \xi\omega\ y_0(\xi,\omega)\, d\omega d\xi, \\ \qquad + g(s,\tau) - g_1(s,\tau) \\ y_j(s,\tau) = \frac{1}{\Gamma(\nu)\Gamma(\theta)} \int_0^s \int_0^\tau (s-\xi)^{\nu-1}(\tau-\omega)^{\theta-1}\sqrt{s\tau}\xi\omega y_{j-1}(\xi,\omega)\, d\omega d\xi, j \geq 2. \end{cases}$$

Because the function $g(s,\tau)$ is known, choosing it or a part of it as a starting point of Algorithm 2 can be suitable. We will show the effectiveness of these initial suggestions to approach the approximate solutions with high accuracy. Furthermore, since the proposed method is a semi-analytic technique and the convergence of the algorithm is guaranteed by Theorem 4, the exact solution can be found in some cases.

Table 2. Absolute errors for Example 2 with varying ν and θ.

		In [4]		
$s=\tau$	$\nu=0.8$ $\theta=0.8$	$\nu=0.8$ $\theta=0.95$	$\nu=0.95$ $\theta=0.8$	$\nu=0.95$ $\theta=0.95$
0.1	1.4e-03	1.2e-03	1.0e-03	9.0e-04
0.2	3.1e-03	2.6e-03	2.3e-03	1.9e-04
0.3	4.8e-03	4.1e-03	3.6e-03	3.0e-04
0.4	6.6e-03	5.5e-03	4.9e-03	4.1e-04
0.5	8.1e-03	6.9e-03	6.0e-03	5.1e-04
		In [32]		
$s=\tau$	$\nu=0.8$ $\theta=0.8$	$\nu=0.8$ $\theta=0.95$	$\nu=0.95$ $\theta=0.8$	$\nu=0.95$ $\theta=0.95$
0.1	1.0e-06	4.9e-07	6.7e-07	2.1e-07
0.2	1.0e-06	2.3e-07	8.3e-07	1.4e-07
0.3	6.7e-06	2.9e-06	4.1e-06	1.2e-06
0.4	1.5e-05	8.9e-06	6.3e-06	2.5e-06
0.5	8.5e-05	5.3e-05	3.6e-05	1.5e-05

Case 1. Choosing a part of function $g(s,\tau)$ for starting point of Algorithm 2: $y_0(s,\tau) = g_1(s,\tau) = g(s,\tau) = s^3(\tau^2-\tau) - \frac{1}{60}s^{\frac{11}{2}}\tau^{\frac{7}{2}}(3\tau-4)$. For example, for $\nu = 0.8, \theta = 0.8$ and $\nu = 1, \theta = 1$, the approximate solutions are given by:

$$u_{1,0.8,0.8}(s,\tau) \approx \sum_{i=0}^{1} y_i(s,\tau) = s^3(\tau^2-\tau) - 0.0166667 s^{5.5}\tau^{3.5}(-4+3\tau)$$
$$+ s^{5.3}(-0.119426\tau^{3.3} + 0.0942835\tau^{4.3}) + s^{7.8}(0.0034861\tau^{5.8} - 0.0022825\tau^{6.8}),$$
$$u_{1,1,1}(s,\tau) \approx \sum_{i=0}^{1} y_i(s,\tau) = s^3(\tau^2-\tau) + 0.0000311 s^{6.5}(52-33\tau)\tau^{4.5}(s\tau)^{1.5}.$$

In Table 3, the absolute errors for different values of ν and θ are shown. Obviously, for ν and θ closer to 1, the absolute errors decrease and the approximate solutions converge to the exact solution.

Case 2. Selecting a part of function $g(s,\tau)$ as a starting point: $y_0(s,\tau) = g_1(s,\tau) = -\frac{1}{60}s^{\frac{11}{2}}\tau^{\frac{7}{2}}(3\tau-4)$. For example,

$$u_{1,.8,.95}(s,\tau) \approx \sum_{i=0}^{1} y_i(s,\tau) = s^3(\tau^2-\tau)$$
$$+\tau(s^{\frac{11}{2}}(0.0666667 - 0.05\tau)\tau^{\frac{5}{2}} + s^{7.8}(0.160346 - 0.102547\tau)\tau^{\frac{99}{20}}).$$

The absolute errors for different values of ν and θ are shown in Table 4.

Table 3. Absolute errors for different values of ν and θ (Case 1).

$s=\tau$	$\nu=0.8$ $\theta=0.8$	$\nu=0.8$ $\theta=0.95$	$\nu=0.95$ $\theta=0.8$	$\nu=0.95$ $\theta=0.95$	$\nu=1$ $\theta=1$
0.1	0	0	0	0	0
0.2	3.6×10^{-10}	1.9×10^{-10}	1.4×10^{-10}	5.2×10^{-11}	1.8×10^{-16}
0.3	6.8×10^{-8}	3.8×10^{-8}	2.9×10^{-8}	1.0×10^{-8}	7.3×10^{-13}
0.4	1.6×10^{-6}	9.8×10^{-6}	7.3×10^{-7}	2.8×10^{-7}	1.2×10^{-10}
0.5	1.6×10^{-5}	1.0×10^{-5}	7.3×10^{-6}	2.9×10^{-6}	5.6×10^{-9}

Table 4. Absolute errors for different values of ν and θ (Case 2).

$s=\tau$	$\nu=0.8$ $\theta=0.8$	$\nu=0.8$ $\theta=0.95$	$\nu=0.95$ $\theta=0.8$	$\nu=0.95$ $\theta=0.95$	$\nu=1$ $\theta=1$
0.1	0	0	0	0	0
0.2	1.2×10^{-10}	1.2×10^{-10}	1.2×10^{-10}	1.2×10^{-10}	1.2×10^{-10}
0.3	3.2×10^{-8}	3.2×10^{-8}	3.2×10^{-8}	3.2×10^{-8}	3.2×10^{-8}
0.4	1.0×10^{-6}	1.0×10^{-6}	1.0×10^{-6}	1.0×10^{-6}	1.0×10^{-6}
0.5	1.2×10^{-5}	1.2×10^{-5}	1.2×10^{-5}	1.2×10^{-5}	1.2×10^{-5}

Case 3. Choosing a part of function $g(s,\tau)$ for starting point, such as $y_0(s,\tau)=g_1(s,\tau)=s^3(\tau^2-\tau)$. For example,

$$u_{1,.95..95}(s,\tau)\approx\sum_{i=0}^{1}y_i(s,\tau)=s^3(\tau^2-\tau)$$
$$+s^{5.45}(-0.0772644+0.0586818\tau)\tau^{\frac{69}{20}}+s^{\frac{11}{2}}(0.0666667-0.05\tau)\tau^{\frac{7}{2}},$$
$$u_{1,1,1}(s,\tau)\approx\sum_{i=0}^{1}y_i(s,\tau)=s^3(\tau^2-\tau)=\text{exact solution.}$$

The absolute errors are illustrated by varying ν and θ in Table 5, where for ν and θ closer to 1, the approximate solutions converge to the exact solution.

Table 5. Absolute errors for different values of ν and θ (Case 3).

$s=\tau$	$\nu=0.8$ $\theta=0.8$	$\nu=0.8$ $\theta=0.95$	$\nu=0.95$ $\theta=0.8$	$\nu=0.95$ $\theta=0.95$	$\nu=1$ $\theta=1$
0.1	0	0	0	0	0
0.2	3.6×10^{-10}	1.9×10^{-10}	1.4×10^{-10}	5.2×10^{-11}	0
0.3	6.8×10^{-8}	3.8×10^{-8}	2.9×10^{-8}	1.0×10^{-8}	0
0.4	1.6×10^{-6}	9.8×10^{-7}	7.3×10^{-7}	2.8×10^{-7}	0
0.5	1.6×10^{-5}	1.0×10^{-5}	7.4×10^{-6}	2.9×10^{-6}	0

Although the methods used in [4] and [32] are interesting and their accuracy is acceptable, the proposed method has a higher accuracy (comparing Table 2 with Tables 3–5). For more illustration, we show the convergence of the numerical results to the exact solution by assuming $s = \tau$ in Case 1 in Figure 1 (the Cases 2 and 3 are similar, so we omit them).

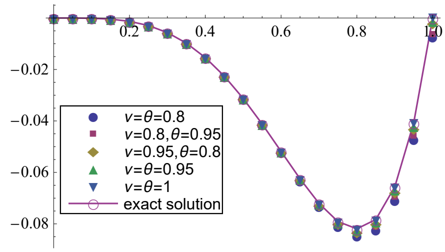

Figure 1. Convergence of numerical results for $s = \tau$ (Case 1).

6 Conclusions

We investigated the solvability of a two-dimensional non-linear singular Volterra integral equation with fractional orders in a Banach space. To guarantee the proposed theorems, we present an application of them as a non-linear singular integral equation with two variables. Moreover, the convergence of the iterative algorithm is proved via some lemma and theorem, and numerical results is obtained by the algorithm compared with some other methods.

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