ISSN: 1392-6292/ eISSN: 1648-3510

VILNIUS TECH
Vilnius Gediminas Technical University

MATHEMATICAL MODELLING and ANALYSIS

2026
Volume 31
Issue 3
Pages 561–582
https://doi.org/10.3846/mma.2026.23999

Two regularization methods for identifying the initial value of Caputo–Hadamard time-fractional diffusion equation

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Article History:

- received May 21, 2025
- revised February 9, 2026
- accepted March 5, 2026

Abstract. In this paper, the inverse problem of identifying the unknown initial value for time fractional diffusion equation with Caputo-Hadamard derivative is considered. This problem is ill-posed and two regularization methods are used to solve it. Firstly, we prove that this problem is ill-posed. Secondly, the conditional stability result and the optimal error bound are given. Then, the error estimates of the Quasi-boundary regularization method and the fractional Landweber iterative regularization method under a priori and a posteriori regularization parameter selection rules are given respectively. Finally, numerical examples are given to illustrate the effectiveness of two regularization methods.

Keywords: Caputo-Hadamard derivative; initial value identification; quasi-boundary regularization method; fractional Landweber iterative regularization method.

AMS Subject Classification: 35R25; 47A52; 35R30.

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1 Introduction

Fractional derivatives and fractional differential equations are widely used in many scientific fields, such as biology, physics, chemistry and so on [9, 12]. We know that so far fractional integrals and derivatives have several forms, such as Riemann-Liouville, Grünwald-Letnikov, Riesz, Caputo, and Hadamard derivatives and/or integrals [9, 12]. Moreover, most researchers' work is based on Riemann-Liouville and Caputo type fractional differential equations.

In 1892, Hadamard proposed another effective and useful integral-differential [6]. Riemann-Liouville fractional derivative is formally a fractional power $(\frac{d}{dx})^\alpha$ of the differentiation $\frac{d}{dx}$ and is invariant with respect to translation on the whole axis. Hadamard proposed a construction of fractional differentiation, which is a fractional power of the type $(x\frac{d}{dx})^\alpha$. This construction is well suited to the case of the half-axis and is invariant relative to dilation [6]. In [1], the authors summarize the latest development of fractional differential equations, integrodifferential equations, and inequalities involving Hadamard derivatives and integrals. The study of Hadamard derivative and Hadamard type fractional differential equation is also worthy of in-depth study. It has been found to be widely used in practical problem related to mechanics and engineering, such as in fracture analysis [1], the Lomnitz logarithmic creep law of special substances [3], and the dynamics of Hadamard fractional differential system is studied [18].

In 2012, Jarad et al. proposed Caputo-Hadamard derivative, which was modified by Caputo derivative [8]. Reference [4] studies the existence and uniqueness of solutions of Caputo-Hadamard fractional differential equation. At present, the research on differential equations with Caputo-Hadamard fractional derivatives is mainly in the numerical aspect [11,19], and there is basically no research on its inverse problem.

In this paper, we study the following inverse problem [19]

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}u(x,t) - \Delta u(x,t) = f(x), & x \in \Omega,\ t \in (a,T],\ 0<\alpha<1, \\ u(x,a) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (a,T], \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \in R$ is a bounded domain with smooth boundary $\partial\Omega$ and ${}_{CH}D^{\alpha}_{a,t}$ is the Caputo-Hadamard fractional derivative of the order $\alpha(0<\alpha<1)$, which is defined as follows:

$$ {}_{CH}D^{\alpha}_{a,t}u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (\log\frac{t}{w})^{-\alpha}\delta u(x,w)dw,\ 0<\alpha<1, \tag{1.2}$$

where $\delta = t\frac{d}{dt}$, $\Gamma(\cdot)$ is a Gamma function, and a is a positive constant.

In problem (1.1), the source term $f(x)$ is known, and the initial value $\varphi(x)$ is unknown. The final value $u(x,T)=g(x)$ is the given data. This paper uses the final value $g(x)$ to identify the initial value $\varphi(x)$. Assuming that the exact data $g(x)$ and the measurement data $g^{\delta}(x)$ satisfy

$$\|g^{\delta}(\cdot) - g(\cdot)\| \le \delta, \tag{1.3}$$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm and $\delta>0$ is the measurement error.

For the inverse problem (1.1), we choose two regularization methods to solve it. One is the Quasi-boundary regularization method [2], and the other is the fractional Landweber iterative regularization method [25].

The Quasi-boundary regularization method is also called nonlocal boundary value method. In 1994, Clark et al. proposed the Quasi-reversibility method. They used the quasi-boundary-value method to perturb the final value condition, thus forming a nonlocal boundary value problem. By solving this problem, a regularization solution of the original problem can be obtained [2]. Reference [7] modified the parabolic equation by nonlocal boundary value problem method. Reference [10] considered a parabolic equation with time-dependent coefficient. They used a modified regularization method to solve this ill-posed problem. References [15, 21] used a modified Quasi-boundary regularization method to identify source term and initial value simultaneously for fractional pseudo-parabolic equation with involution. Reference [23] used a modified Quasi-boundary regularization method to solve the nonlinear backward heat equation. Reference [13] applied the quasi-boundary regularization method to identify the unknown source term in a stochastic Caputo-Hadamard time-fractional diffusion equation.

In 2017, Xiong et al. studied an modified Landweber iteration method through the gradient flow equation obtained from a weighted least square function. Compared with the standard Landweber method, this method greatly reduces the iteration step size [25]. There has been some research on the fractional Landweber iterative regularization method. In [27], the Tikhonov-Landweber iterative regularization method is used to solve the source term identification problem of time-fractional diffusion wave equations in a spherically symmetric domain. Reference [29] used the fractional Landweber iterative regularization method to identify the source term of time-space fractional diffusion equation. Reference [30] studied the fractional Landweber iterative regularization method to solve the inverse problem of non-homogeneous time-fractional diffusion equations in cylindrical domain. Reference [28] used the fractional Landweber iterative regularization method to solve an inverse source problem for the Sobolev equation with fractional Laplacian. Reference [14] used the Landweber iterative regularization method to identify the initial value of time fractional telegraph equation. Reference [26] used the fractional Landweber iterative regularization method to identify the source term for Caputo-Hadamard type time-fractional diffusion-wave equation.

The paper is organized as follows: Section 2 presents some preliminary knowledge. In Section 3, we find the solution of problem (1.1), analyze its ill-posedness, and give the conditional stability result. In Section 4, we give the optimal error bound. In Section 5, the Quasi-boundary regularization method is used to solve the problem (1.1), and the error estimation is given. In Section 6, the fractional Landweber iterative regularization method is used to solve the problem (1.1), and the error estimation is given. In Section 7, two numerical examples are given to verify the effectiveness of the two regularization methods. Finally, in Section 8, the simple conclusion is given.

2 Preliminary

In this section, we give some preliminary theoretical results.

Definition 1. Without loss of generality, we assume a set of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$, where the corresponding eigenfunctions group $\{X_n\}_{n=1}^{\infty}$ forms an orthonormal basis in group $L^2(\Omega)$. Therefore, the eigenvalues of the spatial fractional operator $-\Delta$ in this paper are denoted as λ_n, and its eigenfunctions are denoted as $X_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy

$$\begin{cases} -\Delta X_n(x) = \lambda_n X_n(x), & \text{in } \Omega, \\ X_n(x) = 0, & \text{on } \partial\Omega . \end{cases}$$

DEFINITION 2. For any $p > 0$, we define the space

$$D((-\Delta)^p) = \left\{ \phi \in L^2(\Omega) \,\middle|\, \sum_{n=1}^{\infty} \lambda_n^p |(\phi, X_n)|^2 < \infty \right\},$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and $D((-\Delta)^p)$ is a Hilbert space with the norm

$$\|\phi\|_{D((-\Delta)^p)} := \Big(\sum_{n=1}^{\infty} \lambda_n^p |(\phi, X_n)|^2 \Big)^{\frac{1}{2}}.$$

DEFINITION 3. [9] The Mittag-Leffler function is defined as follow

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in C,$$

where $\alpha > 0$ and $\beta \in R$ are arbitrary constants.

Lemma 1. *[11] If $\lambda > 0$, then the following equation holds*

$$\int_a^{\infty} e^{-s \log \frac{t}{a}} (\log \frac{t}{a})^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm\lambda(\log \frac{t}{a})^{\alpha}) \frac{dt}{t} = \frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)^{k+1}},\ Re(s) > |\lambda|^{\frac{1}{\alpha}},$$

where $E_{\alpha,\beta}^{(m)}(y) := \frac{d^m}{dy^m} E_{\alpha,\beta}(y)$.
Lemma 1 means that the Laplace transformation of $(\log \frac{t}{a})^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm\lambda(\log \frac{t}{a})^{\alpha})$ is $\frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)^{k+1}}$.

Lemma 2. *[16] Assume that $0 < \alpha_0 < \alpha_1 < 1$. Then there exist positive constants C_-, $C^- > 0$ depending only on α_0, α_1 such that for all $\alpha \in [\alpha_0, \alpha_1]$, we obtain*

$$\frac{C_-}{\Gamma(1-\alpha)} \frac{1}{1-z} \le E_{\alpha,1}(z) \le \frac{C^-}{\Gamma(1-\alpha)} \frac{1}{1-z},\ \ z \le 0.$$

Lemma 3. *[5, 20] For $0 < \alpha < 1$, $z > 0$, we have $0 \le E_{\alpha,1}(-z) < 1$. Moreover, $E_{\alpha,1}(-z)$ is completely monotonic, that is*

$$(-1)^n \frac{d^n}{dz^n} E_{\alpha,1}(-z) \ge 0,\ z \ge 0,\ n \ge 0.$$

Lemma 4. *For any λ_n that satisfies $0 < \lambda_1 \le \lambda_n$, there are positive constants C_1 and C_2 that depends on α, a, T and λ_1 such that*

$$\frac{C_1}{\lambda_n} \le E_{\alpha,1}(-\lambda_n \Big(\log \frac{T}{a} \Big)^{\alpha}) \le \frac{C_2}{\lambda_n},$$

where $C_1 = \frac{C_-}{\Gamma(1-\alpha)} \frac{1}{\frac{1}{\lambda_1} + (\log \frac{T}{a})^{\alpha}}$, $C_2 = \frac{C^-}{\Gamma(1-\alpha)} \frac{1}{(\log \frac{T}{a})^{\alpha}}$.

Proof. From Lemma 2, we obtain

$$E_{\alpha,1}(-\lambda_n\Big(\log\frac{T}{a}\Big)^{\alpha}) \le \frac{C^-}{\Gamma(1-\alpha)}\frac{1}{1+\lambda_n(\log\frac{T}{a})^{\alpha}} \le \frac{C^-}{\Gamma(1-\alpha)}\frac{1}{(\log\frac{T}{a})^{\alpha}}\frac{1}{\lambda_n} = \frac{C_2}{\lambda_n},$$

$$E_{\alpha,1}(-\lambda_n\Big(\log\frac{T}{a}\Big)^{\alpha}) \ge \frac{C_-}{\Gamma(1-\alpha)}\frac{1}{1+\lambda_n(\log\frac{T}{a})^{\alpha}} \ge \frac{C_-}{\Gamma(1-\alpha)}\frac{1}{\frac{1}{\lambda_1}+(\log\frac{T}{a})^{\alpha}}\frac{1}{\lambda_n}$$
$$= C_1/\lambda_n.$$

We complete the proof of Lemma 4. □

Lemma 5. *For any $p>0$, $0<\mu<1$ and $0<\lambda_1\le s$, and a_1 is a positive constant, the following inequality holds*

$$F_1(s) = \frac{\mu s^{1-\frac{p}{2}}}{a_1+\mu s} \le \begin{cases} C_{11}\mu^{\frac{p}{2}}, & 0<p<2, \\ C_{12}\mu, & p\ge 2, \end{cases}$$

where $C_{11} := \frac{1}{2}p^{\frac{p}{2}}a_1^{-\frac{p}{2}}(2-p)^{\frac{2-p}{2}}$, $C_{12} := \frac{1}{a_1}\lambda_1^{1-\frac{p}{2}}$.

Proof. When $0<p<2$,

$$F_1^{'}(s) = \frac{\mu s^{-\frac{p}{2}}}{(a_1+\mu s)^2}\left(\frac{2-p}{2}b_2 - \frac{p}{2}\mu s\right).$$

If s_1 satisfies equation $F_1^{'}(s_1)=0$, then we can easily get $s_1 = \frac{2-p}{p\mu}a_1$. Therefore,

$$F_1(s) \le F_1(s_1) = \mu\Big(\frac{2-p}{p\mu}a_1\Big)^{\frac{2-p}{2}}\frac{1}{a_1+\mu\frac{2-p}{p\mu}a_1} =: C_{11}\mu^{\frac{p}{2}}.$$

When $p\ge 2$,

$$F_1(s) = \frac{\mu s^{1-\frac{p}{2}}}{a_1+\mu s} \le \frac{\mu s^{1-\frac{p}{2}}}{a_1} \le \frac{1}{a_1}\lambda_1^{1-\frac{p}{2}}\mu.$$

We complete the proof of Lemma 5. □

Lemma 6. *For any $p>0$, $0<\mu<1$ and $0<\lambda_1\le s$, and a_2 is a positive constant, the following inequality holds*

$$F_2(s) = \frac{\mu s^{\frac{2-p}{4}}}{a_2+\mu s} \le \begin{cases} C_{13}\mu^{\frac{p+2}{4}}, & 0<p<2, \\ C_{14}\mu, & p\ge 2, \end{cases}$$

where $C_{13} := \frac{1}{4}(p+2)^{\frac{p+2}{4}}(a_2)^{-\frac{p+2}{4}}(2-p)^{\frac{2-p}{4}}$, $C_{14} := \frac{1}{a_2}\lambda_1^{\frac{2-p}{4}}$.

Proof. The proof is similar to Lemma 5. □

Lemma 7. *For any $p>0$, $0<\gamma<1$ and $0<\lambda_1\le s$, b_1 and m are positive constants, the following inequality holds*

$$G(s) = (1-\frac{b_1}{s^{\gamma+1}})^m s^{-\frac{p}{2}} \le C_{21}(\frac{1}{m+1})^{\frac{p}{2(\gamma+1)}},$$

where $C_{21} := (\frac{p}{b_1})^{\frac{p}{2(\gamma+1)}}$.

Proof. The proof is similar to Lemma 5. □

Lemma 8. *For any $p > 0$, $0 < \gamma < 1$ and $0 < \lambda_1 \le s$, b_2 and m are positive constants, the following inequality holds*

$$G_1(s) = (1 - \frac{b_2}{s^{\gamma+1}})^{m-1} s^{-(1+\frac{p}{2})} \le C_{22}(\frac{1}{m})^{\frac{p+2}{2(\gamma+1)}},$$

where $C_{22} := (\frac{p+2}{2b_2\gamma})^{\frac{p+2}{2(\gamma+1)}}$.

Proof. The proof is similar to Lemma 5. □

The relevant proofs of the stability and uniqueness of the direct problem of Problem (1.1) can be referred to in [22]. These proofs will not be elaborated on excessively in this paper; next, the ill-posedness of the inverse problem will be analyzed.

3 The solution, the ill-posed analysis and the result of conditional stability

The solution of the Problem (1.1) is obtained by using the separated variable method, the Laplace transformation and the inverse Laplace transformation of the Mittag-Leffler function

$$u(x,t)=\sum_{n=1}^{\infty}\Big(E_{\alpha,1}(-\lambda_n(\log\frac{t}{a})^{\alpha})\varphi_n+(\log\frac{t}{a})^{\alpha}E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{t}{a})^{\alpha})f_n\Big)X_n(x),$$

where $\varphi_n = (\varphi(x), X_n(x))$ and $f_n = (f(x), X_n(x))$ are the Fourier coefficient.

Using $u(x,T) = g(x)$, we have

$$g(x)=\sum_{n=1}^{\infty}\Big(E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\varphi_n+(\log\frac{T}{a})^{\alpha}E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{T}{a})^{\alpha})f_n\Big)X_n(x).$$

So,

$$g_n = E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\varphi_n + (\log\frac{T}{a})^{\alpha}E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{T}{a})^{\alpha})f_n,$$

where $g_n = (g(x), X_n(x))$ is the Fourier coefficient.
Denote

$$h_n = g_n - (\log\frac{T}{a})^{\alpha}E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{T}{a})^{\alpha})f_n,$$

then we have

$$E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\varphi_n = h_n. \tag{3.1}$$

We define the operator $K : \varphi(\cdot) \longrightarrow h(\cdot)$, then the Problem (1.1) can be converted to the operator equation: $K\varphi(x) = h(x)$. K is a bounded self-adjoint compact operator, its singular value is $\{\sigma_n\}_{n=1}^{\infty}$, $\sigma_n = E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})$. From (3.1), we have

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{h_n}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})} X_n(x). \tag{3.2}$$

When $n \to \infty$, $\lambda_n \to \infty$, $\frac{1}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})} \to \infty$, so from formula (3.2), the small perturbation of $h^{\delta}(x)$ will cause a great change in the initial value $\varphi(x)$. Therefore, the inverse problem is ill-posed.

Next, we give the conditional stability result. Firstly, we give a priori bound condition for the exact solution $\varphi(x)$

$$\|\varphi(\cdot)\|_{D((-\Delta)^p)} = \Big(\sum_{n=1}^{\infty} \lambda_n^p |(\varphi, X_n)|^2\Big)^{\frac{1}{2}} \le E, \tag{3.3}$$

where E and p are both positive constants.

Theorem 1. *[26] If $\varphi(x)$ satisfies the priori bound condition $\|\varphi(\cdot)\|_{D((-\Delta)^p)} \le E$, we have*

$$\|\varphi(\cdot)\| \le (\frac{1}{C_1})^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \|h(\cdot)\|^{\frac{p}{p+2}}, \quad p > 0.$$

4 Preliminary results and optimal error bound

4.1 Preliminary

Let X and Y be infinite dimensional Hilbert spaces, $K : X \to Y$ is a linear injective operator, and its range is $R(K)$ and is not closed. Consider the following inverse problem

$$Kx = y, \tag{4.1}$$

where $x \in X$ and $y \in Y$. Suppose $y^{\delta} \in Y$ is the data with measurement error and satisfies $\|y^{\delta} - y\| \le \delta$, where $\delta > 0$ is the measurement error. Any operator $R : Y \to X$ can be considered as an effective method to solve (4.1), and Ry^{δ} is taken as the approximate solution of Equation (4.1).

Let $M \in X$ be a bounded set. We define the worst-case error as [24]

$$\Delta(\delta, R) := \sup\{\|Ry^{\delta} - x\| \mid x \in M, y^{\delta} \in Y, \|Kx - y^{\delta}\| \le \delta\}. \tag{4.2}$$

We define the optimal error bound as

$$w(\delta) := \inf_R \Delta(\delta, R).$$

If set $M = M_{\phi,E}$ is the optimal result of a set of elements satisfying some source conditions, i.e.,

$$M_{\phi,E} := \{x \in X \mid x = [\phi(K^*K)]^{\frac{1}{2}} v, \|v\| \le E\}, \tag{4.3}$$

here, operator $\phi(K^*K)$ is defined by spectral representation as [17]

$$\phi(K^*K) = \int_0^a \phi(\lambda) dE_\lambda,$$

where $K^*K = \int_0^a \lambda dE_\lambda$ represents the spectral decomposition of operator K^*K, $\{E_\lambda\}$ represents the unit decomposition of operator K^*K, and a is a constant satisfying $\|K^*K\| \leq a$. If $K : L^2(R) \to L^2(R)$ is a product operator, then $Kx(s) = \alpha(s)x(s)$, and the operator function $\phi(K^*K)$ has the following form

$$\phi(K^*K)x(s) = \phi(|\alpha(s)|^2)x(s). \tag{4.4}$$

A method called R_0 [24]:

- If $\Delta(\delta, R_0) = w(\delta, E)$ holds, it is optimal on set $M_{\phi,E}$;
- If $\Delta(\delta, R_0) \leq Cw(\delta, E)(C \geq 1)$ holds, the order is optimal on set $M_{\phi,E}$.

In order to show the optimal error bound of the worst-case error $\Delta(\delta, E)$ defined in (4.2), assume that the function ϕ in (4.4) satisfies the following assumption:

ASSUMPTION 1. [24] Let $\phi(\lambda) : (0, a] \to (0, \infty)$ in Equation (4.4) be a continuous function with the following properties

- $\lim_{\lambda\to 0} \phi(\lambda) = 0$;
- $\phi(\lambda)$ increases strictly monotonically on $(0, a]$;
- $\rho(\lambda) = \lambda\phi^{-1}(\lambda) : (0, \phi(a)] \to (0, a\phi(a)]$ is convex.

Based on the above assumption, the following theorem gives the general formula of the optimal error bound.

Theorem 2. *[24] Let $M_{\phi,E}$ be given by* (4.3)*, Assumption 1 holds, and $\frac{\delta^2}{E^2} \in \sigma(K^*K\phi(K^*K))$, where $\sigma(K^*K))$ is the spectrum of operator K^*K, then*

$$w(\delta, M_{\phi,E}) = E\sqrt{\rho^{-1}(\delta^2/E^2)}.$$

We can get the optimal error bound from Theorem 2. This is a good result, but there are two difficulties when we use it. Firstly, the convexity of ρ is sometimes destroyed, so it is difficult for us to test its convexity. Secondly, even for a small δ, $\frac{\delta^2}{E^2}$ may not belong to $\sigma(K^*K\phi(K^*K))$. For example, when K is a compact operator.

We will give the following two theorems to solve these two difficulties.

Theorem 3. *[24] If ρ is not necessarily convex, then*

- $E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})} \leq w(\delta, M_{\phi,E}) \leq \sqrt{2}E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}$ *for* $\frac{\delta^2}{E^2} \in \sigma(K^*K\phi(K^*K))$;
- $w(\delta, M_{\phi,E}) \leq \sqrt{2}E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}$ *for* $\frac{\delta^2}{E^2} \notin \sigma(K^*K\phi(K^*K))$.

Theorem 4. *[24] Let K^*K be compact and $\lambda_1 > \lambda_2 > \cdots$ be the ordered eigenvalues of K^*K. If there is a constant $k > 0$ such that $\phi(\lambda_{i+1}) \geq k\phi(\lambda_i)$ for any $i \in N$, there is*

$$w(\delta, M_{\phi,E}) \leq \sqrt{k}E\sqrt{\rho^{-1}(\delta^2/E^2)},$$

where $\delta \in (0, \delta_1]$ and $\delta_1 = E\sqrt{\lambda_1\phi(\lambda_1)}$.

If Theorems 3 and 4 are satisfied, we can easily obtain the optimal error bound of Problem (1.1).

4.2 Optimal error bound of Problem (1.1)

In this section, we give the optimal error bound of Problem (1.1), and the noise data g^δ given by (1.2) is used to identify the optimal possible worst-case error of $\varphi(x)$ given by Equation (3.2), where $\varphi(x) \in M_{p,E}$, $M_{p,E}$ is expressed as follows:

$$\varphi(x) \in M_{p,E} = \{\varphi(x) \in L^2(\Omega) \mid \|\varphi(x)\|_{D((-\Delta)^p)} \leq E, p > 0\}, \tag{4.5}$$

where $\|\varphi(x)\|_{D((-\Delta)^p)}$ is the norm on Hilbert space $D((-\Delta)^p)$.
We use the linear operator $K : L^2(\Omega) \to L^2(\Omega)$ to express the Problem (1.1) as an operator equation

$$K\varphi(x) = h(x), \tag{4.6}$$

where $h(x) = \sum_{n=1}^{\infty} h_n X_n(x)$. By (3.2), we have

$$K\varphi(x) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\varphi_n X_n(x),$$

where $\varphi(x) = \sum_{n=1}^{\infty} \varphi_n X_n(x)$. So, $K : L^2(\Omega) \to L^2(\Omega)$ can be written as

$$K = E_{\alpha,1}(-\lambda_n(\log(T/a))^\alpha).$$

Because K is a linear self-adjoint operator, we can get

$$KK^* = K^*K = (E_{\alpha,1}(-\lambda_n(\log(T/a))^\alpha))^2.$$

Next, we convert Equation (4.5) into the equivalent form of Equation (4.3), and find the form of special function $\phi(\lambda)$ through the equivalence of the new form.

Proposition 1. *Consider the operator equation in Equation (4.6). Then, the set $M_{p,E}$ in Equation (4.5) is equivalent to the general source set $M_{\phi,E}$ in Equation (4.3):*

$$M_{p,E} := \{\varphi \in L^2(\Omega) \mid \|\phi(K^*K)]^{-\frac{1}{2}}\varphi\| \leq E\},$$

where $\phi = \phi(\lambda)$ is given by the following parameter form

$$\begin{cases} \lambda(l) = (E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha))^2, \\ \phi(l) = l^{-p}, \end{cases} \quad \lambda_1 \leq l < \infty, \tag{4.7}$$

where $l = \lambda_n$.

Proposition 2. *The function $\phi(\lambda)$ defined by (4.7) is continuous and has the following properties.*

(i) $\lim_{\lambda\to 0}\phi(\lambda)=0$*;*

(ii) $\phi(\lambda)$ *is strictly monotonically increasing;*

(iii) $\rho(\lambda)=\lambda\phi^{-1}(\lambda)$ *is strictly monotonic and has the following parameter form*

$$\begin{cases}\lambda(l)=l^{-p},\\ \rho(l)=l^{-p}(E_{\alpha,1}(-l(\log\frac{T}{a})^{\alpha}))^2,\end{cases}\qquad \lambda_1\le l<\infty;$$

(iv) $\rho^{-1}(\lambda)$ *is strictly monotonic and has the following parameter form*

$$\begin{cases}\lambda(l)=l^{-p}(E_{\alpha,1}(-l(\log\frac{T}{a})^{\alpha}))^2,\\ \rho^{-1}(l)=l^{-p},\end{cases}\qquad \lambda_1\le l<\infty;$$

(v) inverse function $\rho^{-1}(\lambda)$ *satisfies*

$$\rho^{-1}(\lambda)=(\Gamma(1-\alpha)(\log\frac{T}{a})^{\alpha})^{\frac{2p}{p+2}}\lambda^{\frac{p}{p+2}}(1+o(1)),\ \lambda\to 0. \tag{4.8}$$

Proof. (i), (ii), (iii) and (iv) are obviously established, so their proof is omitted. In order to prove (v), we only need to prove that $\lim_{\lambda\to 0}F(\lambda)=1$, where $F(\lambda)=\rho^{-1}(\lambda)/((\Gamma(1-\alpha)(\log\frac{T}{a})^{\alpha})^{\frac{2p}{p+2}}\lambda^{\frac{p}{p+2}})$. From reference [4], we know that Mittag-Leffler function has the following property

$$E_{\alpha,\beta}=-\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma(\beta-\alpha k)}+o(|z|^{-1-n}), \tag{4.9}$$

where $0<\alpha<2$, β is an arbitrary constant, μ is any real number and satisfies $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$. When $n=1$ in (4.9), we can get

$$E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})=-\frac{(-l(\log\frac{T}{a})^{\alpha})^{-1}}{\Gamma(1-k)}+o(\frac{1}{l})=\frac{1}{\Gamma(1-k)l(\log\frac{T}{a})^{\alpha}}+o(\frac{1}{l}).$$

Then,

$$\begin{aligned}\lim_{\lambda\to 0}F(\lambda)&=\lim_{l\to\infty}\frac{l^{-p}}{(\Gamma(1-\alpha)(\log\frac{T}{a})^{\alpha})^{\frac{2p}{p+2}}(l^{-p}E^2_{\alpha,1}(-l(\log\frac{T}{a})^{\alpha}))^{\frac{p}{p+2}}}\\ &=\lim_{l\to\infty}\frac{1}{(\Gamma(1-\alpha)(\log\frac{T}{a})^{\alpha}lE_{\alpha,1}(-l(\log\frac{T}{a})^{\alpha}))^{\frac{2p}{p+2}}}\\ &=\lim_{l\to\infty}\frac{1}{(\Gamma(1-\alpha)(\log\frac{T}{a})^{\alpha}l(\frac{1}{\Gamma(1-\alpha)(\log\frac{T}{a})^{\alpha}l}+o(\frac{1}{l})))^{\frac{2p}{p+2}}}=1.\end{aligned}$$

□

Theorem 5. *Assumptions* (1.3) *and* (4.5) *hold, the following results are true for Problem* (1.1).

(*i*) *For* $\frac{\delta^2}{E^2} \in \sigma(K^*K\phi(K^*K))$ *and* $\delta \to 0$, *we have*

$$CE^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}} \leq w(\delta, M_{p,E}) \leq \sqrt{2}CE^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{4.10}$$

(*ii*) *For* $\frac{\delta^2}{E^2} \notin \sigma(K^*K\phi(K^*K))$ *and* $\delta \to 0$, *we have*

$$\left(2/5\right)^{\frac{p}{2}} CE^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}} \leq w(\delta, M_{p,E}) \leq \sqrt{2}CE^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{4.11}$$

where $C = (\Gamma(1-\alpha)(\log\frac{T}{a})^{\alpha})^{\frac{p}{p+2}}$.

Proof. (*i*). From (4.8), we can obtain

$$E\sqrt{\rho^{-1}\left(\delta^2/E^2\right)} = (\Gamma(1-\alpha)\left(\log(T/a)\right)^{\alpha})^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.$$

Combining with Theorem 3, we can get (4.10).

(ii). By (4.7), we have

$$\frac{\phi(\lambda_{i+1})}{\phi(\lambda_i)} = \frac{(1+i^2)^p}{(1+(i+1)^2)^p} \geq \left(\frac{2}{5}\right)^p.$$

Combining with Theorem 4 and taking $k = \left(\frac{2}{5}\right)^p$, we can get (4.11). □

5 The Quasi-boundary regularization method and the error estimation

In this section, we will use the Quasi-boundary regularization method to solve the Problem (1.1), and give the error estimates of this method under a priori and a posteriori regularization parameter selection rules.

For the Quasi-boundary regularization method, we use $u(x,T)+\mu u(x,a) = g(x)$ instead of $u(x,T) = g(x)$ in (1.1) to obtain the regularization solution, that is, to solve the following equation

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}u^{\mu,\delta}(x,t) - \Delta u^{\mu,\delta}(x,t) = f(x), & x \in \Omega,\ t \in (a,T],\ 0<\alpha<1, \\ u^{\mu,\delta}(x,a) = \varphi^{\mu,\delta}(x), & x \in \Omega, \\ u^{\mu,\delta}(x,t) = 0, & x \in \partial\Omega,\ t \in (a,T], \\ u^{\mu,\delta}(x,T) + \mu u^{\mu,\delta}(x,a) = g^{\delta}(x), & x \in \Omega, \end{cases} \tag{5.1}$$

where μ is the regularization parameter.

Using the method of separating variables, we get the solution of Problem (5.1) as follows

$$u^{\mu,\delta}(x,t) = \sum_{n=1}^{\infty}\Big(E_{\alpha,1}(-\lambda_n(\log\frac{t}{a})^{\alpha})\varphi_n^{\mu,\delta} + (\log\frac{t}{a})^{\alpha}E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{t}{a})^{\alpha})f_n\Big) \times X_n(x). \tag{5.2}$$

According to $u^{\mu,\delta}(x,T)+\mu u^{\mu,\delta}(x,a)=g^{\delta}(x)$ and the above formula (5.2), we have

$$g^{\delta}(x) \\ =\sum_{n=1}^{\infty}\Big((E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu)\varphi_n^{\mu,\delta}+(\log\frac{T}{a})^{\alpha}E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{T}{a})^{\alpha})f_n\Big)X_n(x).$$

So,

$$\varphi^{\mu,\delta}(x)=\sum_{n=1}^{\infty}\frac{h_n^{\delta}}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}X_n(x), \tag{5.3}$$

where $h_n^{\delta}=g_n^{\delta}-(\log\frac{T}{a})^{\alpha}E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{T}{a})^{\alpha})f_n$.

5.1 The error estimation under a priori parameter choice rule

Next, we give the error convergence estimate of $\|\varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\|$ under a priori regularization parameter.

Theorem 6. *If both the priori bound condition* (3.3) *and the error assumption* (1.3) *hold, we have*
(1) If $0<p<2$ and the regularization parameter $\mu=(\frac{\delta}{E})^{\frac{2}{p+2}}$ is selected, then we have

$$\|\varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\|\le(1+C_{11})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.$$

(2) If $p\ge 2$ and the regularization parameter $\mu=(\frac{\delta}{E})^{\frac{1}{2}}$ is selected, then we have

$$\|\varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\|\le(1+C_{12})E^{\frac{1}{2}}\delta^{\frac{1}{2}},$$

where $C_{11}:=\frac{1}{2}p^{\frac{p}{2}}(C_1)^{-\frac{p}{2}}(2-p)^{\frac{2-p}{2}}$, $C_{12}:=\frac{1}{C_1}\lambda_1^{1-\frac{p}{2}}$.

Proof. By the triangle inequality, we obtain

$$\|\varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\|\le\|\varphi^{\mu,\delta}(\cdot)-\varphi^{\mu}(\cdot)\|+\|\varphi^{\mu}(\cdot)-\varphi(\cdot)\|. \tag{5.4}$$

We first give an estimate of the first term of (5.4). Through (5.3) and (1.3), we have

$$\|\varphi^{\mu,\delta}(\cdot)-\varphi^{\mu}(\cdot)\|^2=\Big\|\sum_{n=1}^{\infty}\frac{1}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}(h_n^{\delta}-h_n)X_n(x)\Big\|^2 \\ =\sum_{n=1}^{\infty}\Big(\frac{1}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}\Big)^2(h_n^{\delta}-h_n)^2 \\ \le\sup_{n\ge1}\Big(\frac{1}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}\Big)^2\sum_{n=1}^{\infty}(h_n^{\delta}-h_n)^2\le\Big(\frac{\delta}{\mu}\Big)^2.$$

So, there is

$$\|\varphi^{\mu,\delta}(\cdot)-\varphi^{\mu}(\cdot)\|\le\delta/\mu. \tag{5.5}$$

Next, we give an estimate of the second term of (5.4). Using (3.2), (3.3) and Lemma 5, we can deduce

$$
\begin{aligned}
&\|\varphi^{\mu}(\cdot)-\varphi(\cdot)\|^2\\
&=\Big\|\sum_{n=1}^{\infty}\frac{1}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}h_nX_n(x)-\sum_{n=1}^{\infty}\frac{h_n}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})}X_n(x)\Big\|^2\\
&=\Big\|\sum_{n=1}^{\infty}\frac{-\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}\frac{1}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})}h_nX_n(x)\Big\|^2\\
&=\sum_{n=1}^{\infty}\left(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}\right)^2\varphi_n^2\\
&=\sup_{n\geq1}\Big|\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}\lambda_n^{-\frac{p}{2}}\Big|^2\sum_{n=1}^{\infty}\varphi_n^2\lambda_n^p\\
&=\sup_{n\geq1}\Big|\frac{\mu\lambda_n^{1-\frac{p}{2}}}{C_1+\mu\lambda_n}\Big|^2E^2=\sup_{n\geq1}\big(A_1(n)\big)^2E^2,
\end{aligned}
$$

where $A_1(n)=\mu\lambda_n^{1-\frac{p}{2}}/(C_1+\mu\lambda_n)$. Let $s=\lambda_n$ and Lemma 8, then we have

$$
A_1(n)=\frac{\mu s^{1-\frac{p}{2}}}{C_1+\mu s}\leq\begin{cases}C_{11}\mu^{\frac{p}{2}}, & 0<p<2,\\ C_{12}\mu, & p\geq2,\end{cases}
$$

where $C_{11}:=\frac{1}{2}p^{\frac{p}{2}}(C_1)^{-\frac{p}{2}}(2-p)^{\frac{2-p}{2}}$, $C_{12}:=\frac{1}{C_1}\lambda_1^{1-\frac{p}{2}}$. Therefore, we obtain

$$
\|\varphi^{\mu}(\cdot)-\varphi(\cdot)\|\leq\begin{cases}C_{11}\mu^{\frac{p}{2}}E, & 0<p<2,\\ C_{12}\mu E, & p\geq2.\end{cases}\tag{5.6}
$$

Combining (5.5) with (5.6), we obtain

$$
\|\varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\|\leq\frac{\delta}{\mu}+\begin{cases}C_{11}\mu^{\frac{p}{2}}E, & 0<p<2,\\ C_{12}\mu E, & p\geq2.\end{cases}
$$

By choosing the regularization parameters $\mu=(\frac{\delta}{E})^{\frac{2}{p+2}}$ $(0<p<2)$ and $\mu=(\frac{\delta}{E})^{\frac{1}{2}}(p\geq2)$, we can get

$$
\|\varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\|\leq\begin{cases}(1+C_{11})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, & 0<p<2,\\ (1+C_{12})E^{\frac{1}{2}}\delta^{\frac{1}{2}}, & p\geq2.\end{cases}
$$

The proof of Theorem 6 is completed. □

5.2 The error estimation under a posteriori parameter choice rule

In this section, we will choose the regularization parameter through the Morozov's discrepancy principle. This discrepancy principle is as follows

$$
\|\mu(K+\mu)^{-1}(K\varphi^{\mu,\delta}(x)-h^{\delta}(x))\|=\tau_1\delta,\tag{5.7}
$$

where $\tau_1>1$ is a constant, and $\|h^{\delta}\|\geq\tau_1\delta$.

Lemma 9. *Let* $\rho(\mu) := \|\mu(K+\mu)^{-1}(K\varphi^{\mu,\delta}(x) - h^\delta(x))\|$, *we obtain*
a) $\rho(\mu)$ *is a continuous function;* *b)* $\lim_{\mu\to 0}\rho(\mu) = 0$; *c)* $\lim_{\mu\to+\infty}\rho(\mu) = \|h^\delta(\cdot)\|$; *d)* $\rho(\mu)$ *is a strictly monotone increasing function for any* $\mu \in (0,+\infty)$.

Proof. The above conclusions are the direct inference of expression (5.8)

$$\rho(\mu) = \Big(\sum_{n=1}^{\infty}\Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu}\Big)^4 (h_n^\delta)^2\Big)^{\frac{1}{2}}. \tag{5.8}$$

□

Theorem 7. *If both the priori bound condition* (3.3) *and the error assumption* (1.3) *hold. The regularization parameter* $\mu > 0$ *is given by the* (5.7), *we obtain*
(1)If $0 < p < 2$, *we have the following error estimate*

$$\|\varphi^{\mu,\delta}(\cdot) - \varphi(\cdot)\| \le \Big(\big(\frac{\tau_1+1}{C_1}\big)^{\frac{p}{p+2}} + \big(\frac{C_{13}^2}{\tau_1-1}\big)^{\frac{4}{p+2}}\Big)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.$$

(2)If $p \ge 2$, *we have the following error estimate*

$$\|\varphi^{\mu,\delta}(\cdot) - \varphi(\cdot)\| \le \Big(\big(\frac{\tau_1+1}{C_1}\big)^{\frac{p}{p+2}} + \big(\frac{C_{14}^2}{\tau_1-1}\big)^{\frac{1}{2}}\Big)E^{\frac{1}{2}}\delta^{\frac{1}{2}},$$

where $C_{13} := \frac{1}{4}(p+2)^{\frac{p+2}{4}}(C_1)^{-\frac{p+2}{4}}(2-p)^{\frac{2-p}{4}}$ *and* $C_{14} := \frac{1}{C_1}\lambda_1^{1/2-p/4}$.

Proof. With the triangular inequality, we have

$$\|\varphi^{\mu,\delta}(\cdot) - \varphi(\cdot)\| \le \|\varphi^{\mu,\delta}(\cdot) - \varphi^{\mu}(\cdot)\| + \|\varphi^{\mu}(\cdot) - \varphi(\cdot)\|. \tag{5.9}$$

We first give an estimate of the first term of (5.9), which applies (5.5)

$$\|\varphi^{\mu,\delta}(\cdot) - \varphi^{\mu}(\cdot)\| \le \delta/\mu. \tag{5.10}$$

Using (5.7) and (1.3), we obtain

$$\begin{aligned}
\tau_1\delta &= \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu}\Big)^2 h_n^\delta X_n(x)\Big\| \\
&= \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu}\Big)^2 (h_n^\delta - h_n + h_n) X_n(x)\Big\| \\
&\le \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu}\Big)^2 (h_n^\delta - h_n) X_n(x)\Big\| \\
&\quad + \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu}\Big)^2 h_n X_n(x)\Big\| \\
&\le \delta + \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu}\Big)^2 h_n X_n(x)\Big\| \le \delta + J.
\end{aligned}$$

Applying a priori boundary condition (3.3), there is

$$\begin{aligned}
J &= \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu} \Big)^2 h_n X_n(x) \Big\| \\
&= \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu} \Big)^2 E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) \varphi_n X_n(x) \Big\| \\
&= \Big(\sum_{n=1}^{\infty} \Big(\frac{\mu}{E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu} \Big)^4 \Big(E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) \Big)^2 (\varphi_n)^2 \Big)^{\frac{1}{2}} \\
&\le \sup_{n\ge 1} \Big| \frac{\mu}{E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu} \Big(E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) \Big)^{\frac{1}{2}} \lambda_n^{-\frac{p}{4}} \Big|^2 (\sum_{n=1}^{\infty} \varphi_n^2 \lambda_n^p)^{\frac{1}{2}} \\
&\le \sup_{n\ge 1} \Big| (C_2)^{\frac{1}{2}} \frac{\mu \lambda_n^{\frac{1}{2}-\frac{p}{4}}}{C_1 + \mu\lambda_n} \Big|^2 E = C_2 \sup_{n\ge 1} (A_2(n))^2 E,
\end{aligned}$$

where $A_2(n) = \mu \lambda_n^{\frac{2-p}{4}} / (C_1 + \mu\lambda_n)$.
Let $s = \lambda_n$, and applying Lemma 6, we obtain

$$A_2(n) = \frac{\mu s^{\frac{2-p}{4}}}{C_1 + \mu s} \le \begin{cases} C_{13} \mu^{\frac{p+2}{4}}, & 0 < p < 2, \\ C_{14} \mu, & p \ge 2, \end{cases}$$

where $C_{13} := \frac{1}{4}(p+2)^{\frac{p+2}{4}} (C_1)^{-\frac{p+2}{4}} (2-p)^{\frac{2-p}{4}}$, $C_{14} := \frac{1}{C_1} \lambda_1^{\frac{1}{2}-\frac{p}{4}}$.

So there is

$$(\tau_1 - 1)\delta \le \begin{cases} C_2 C_{13}^2 \mu^{\frac{p+2}{2}} E, & 0 < p < 2, \\ C_2 C_{14}^2 \mu^2 E, & p \ge 2. \end{cases}$$

Therefore,

$$\frac{1}{\mu} \le \begin{cases} (\frac{C_2 C_{13}^2}{\tau_1 - 1})^{\frac{2}{p+2}} (\frac{E}{\delta})^{\frac{2}{p+2}}, & 0 < p < 2, \\ (\frac{C_2 C_{14}^2}{\tau_1 - 1})^{\frac{1}{2}} (\frac{E}{\delta})^{\frac{1}{2}}, & p \ge 2. \end{cases} \tag{5.11}$$

Substituting (5.11) to (5.10), we have

$$\|\varphi^{\mu,\delta}(\cdot) - \varphi^{\mu}(\cdot)\| \le \frac{\delta}{\mu} \le \begin{cases} (\frac{C_2 C_{13}^2}{\tau_1 - 1})^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2, \\ (\frac{C_2 C_{14}^2}{\tau_1 - 1})^{\frac{1}{2}} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \ge 2. \end{cases} \tag{5.12}$$

Here, we estimate the second term of formula (5.9). From (5.7), we have

$$\begin{aligned}
&\|K(\varphi^{\mu}(x) - \varphi(x))\| \\
&= \Big\| \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) \Big(\sum_{n=1}^{\infty} \frac{1}{E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu} h_n X_n(x) \\
&- \sum_{n=1}^{\infty} \frac{h_n}{E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha})} X_n(x) \Big) \Big\| = \Big\| \sum_{n=1}^{\infty} \frac{\mu}{E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu} h_n X_n(x) \Big\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \sum_{n=1}^{\infty} \frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}(h_n - h_n^{\delta} + h_n^{\delta})X_n(x)\Big\| \\
&\le \Big\| \sum_{n=1}^{\infty} \frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu}(h_n - h_n^{\delta})X_n(x)\Big\| \\
&+ \Big\| \sum_{n=1}^{\infty} \frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu} h_n^{\delta} X_n(x)\Big\| \le \delta + \tau_1\delta = (\tau_1+1)\delta,
\end{aligned}
$$

$$
\begin{aligned}
\|\varphi^{\mu}(x) - \varphi(x)\|_{D((-\Delta)^p))} &= \Big\| \sum_{n=1}^{\infty} \frac{1}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu} h_n X_n(x) \\
&- \sum_{n=1}^{\infty} \frac{h_n}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})} X_n(x)\Big\|_{D((-\Delta)^p))} \\
&= \Big\| \sum_{n=1}^{\infty} \frac{\mu}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu} \frac{h_n}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})} X_n(x)\Big\|_{D((-\Delta)^p))} \\
&\le \Big\| \sum_{n=1}^{\infty} \varphi_n X_n(x)\Big\|_{D((-\Delta)^p))} \le E.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\|\varphi^{\mu}(x) - \varphi(x)\| &\le \Big(\frac{1}{C_1}\Big)^{\frac{p}{p+2}} \|K(\varphi_{\mu}(x) - \varphi(x))\|^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \\
&\le \Big(\frac{1}{C_1}\Big)^{\frac{p}{p+2}} ((\tau_1+1)\delta)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} = \Big(\frac{\tau_1+1}{C_1}\Big)^{\frac{p}{p+2}} \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}.
\end{aligned}
\tag{5.13}
$$

Combining (5.9), (5.12) with (5.13), we have

$$
\|\varphi^{\mu,\delta} - \varphi(x)\| \le \Big(\frac{\tau_1+1}{C_1}\Big)^{\frac{p}{p+2}} \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}} + \begin{cases} (\frac{C_{13}^2}{\tau_1-1})^{\frac{4}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0<p<2, \\ (\frac{C_{14}^2}{\tau_1-1})^{\frac{1}{2}} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \ge 2. \end{cases}
$$

The proof of Theorem 7 is completed. □

Remark 1. From Theorem 5, 6 and 7, we can deduce that the error estimation of Quasi-boundary regularization method can reach the optimal order when $p < 2$ under a priori and a posteriori regularization parameter selection rules, and the highest order is $\frac{1}{2}$.

6 The Fractional Landweber iterative regularization method and the error estimation

In this section, we will use the fractional Landweber iterative regularization method to solve the problem (1.1), and give the error estimates under a priori and a posteriori regularization parameter.

For the ill-posed problem (1.1), we use the fractional Landweber iterative regularization method to obtain its regularization solution, which use the operator equation $\varphi = (I - b(K^*K)^{\frac{\gamma+1}{2}})\varphi + b(K^*K)^{\frac{\gamma-1}{2}}K^*h$ to replace the equation $K\varphi = h$, and we can obtain the following iterative form

$$\varphi^{0,\delta} = 0, \varphi^{m,\delta} = (I - b(K^*K)^{\frac{\gamma+1}{2}})\varphi^{m-1,\delta} + b(K^*K)^{\frac{\gamma-1}{2}}K^*h^\delta,$$

where m is the iterative step number and the regularization parameter. The cofficient b is called the relaxation factor and satisfies $0 < b < \frac{1}{\|K\|^{\gamma+1}}$. K^* is the adjoint operator of K. We define the operator $R_m : L^2(\Omega) \to L^2(\Omega)$ as follows:

$$R_m = b\sum_{n=1}^{m-1}(I - b(K^*K)^{\frac{\gamma+1}{2}})^n(K^*K)^{\frac{\gamma-1}{2}}K^*.$$

By taking the singular value of operator K, we obtain the fractional Landweber regularization solution with error as follows:

$$\varphi^{m,\delta}(x) = \sum_{n=1}^{\infty}\frac{1-\Big(1-b\big(E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^{\gamma+1}\Big)^m}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)}h_n^\delta X_n(x), \tag{6.1}$$

and the regularization solution without error as follows:

$$\varphi^{m}(x) = \sum_{n=1}^{\infty}\frac{1-\Big(1-b\big(E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^{\gamma+1}\Big)^m}{E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)}h_n X_n(x),$$

where $h^\delta(x) = g^\delta(x) - \sum_{n=1}^{\infty}(\log\frac{T}{a})^\alpha E_{\alpha,1+\alpha}(-\lambda_n(\log\frac{T}{a})^\alpha)f_nX_n(x)$.

6.1 The error estimation of under a priori parameter choice rule

Next, we give the error estimate of $\|\varphi^{m,\delta}(\cdot) - \varphi(\cdot)\|$ under a priori regularization parameter.

Theorem 8. *If both a priori bound condition* (3.3) *and the error assumption* (1.3) *hold, the regularization parameter* $m = \left[(\frac{E}{\delta})^{\frac{2(\gamma+1)}{p+2}}\right]$ *is selected, then we have error estimate*

$$\|\varphi^{m,\delta}(\cdot) - \varphi(\cdot)\| \le (b^{\frac{1}{\gamma+1}} + C_{21})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}},$$

where $C_{21} := \left(\frac{p}{b}\right)^{\frac{p}{2(\gamma+1)}}(C_1)^{-\frac{p}{2}}$.

Proof. The proof is similar to Theorem 6, here we omit the proof of Theorem 6. □

6.2 The error estimation under a posteriori parameter choice rule

In this section, we will give the error estimate of $\|\varphi^{m,\delta}(\cdot) - \varphi(\cdot)\|$ under a posteriori regularization parameter. We will use the following Morozov's discrepancy principle to select the regularization parameter.

Let $\tau_2 > 1$ be a given constant and the regularization parameter m satisfies

$$\|K\varphi^{m,\delta}(x) - h^\delta(x)\| \le \tau_2\delta, \tag{6.2}$$

when $m = m(\delta)$ appears for the first time, the iteration stops, $\|h^\delta\| \ge \tau_2\delta$.

Theorem 9. *If both the priori bound condition* (3.3) *and the assumption* (1.3) *hold, and* $\varphi^{m,\delta}$ *is given by* (6.1). *The selection of the regularization parameter* m *is given by* (6.2), *we have the following error estimate*

$$\|\varphi^{m,\delta}(\cdot) - \varphi(\cdot)\| \le C_{23}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}},$$

where $C_{23} = (b^{\frac{1}{\gamma+1}}(\frac{C_2C_{22}}{\tau_2-1})^{\frac{2}{p+2}} + (\frac{\tau_2+1}{C_1})^{\frac{p}{p+2}})$.

Proof. The proof is similar to Theorem 7, here we omit the proof of Theorem 9. □

Remark 2. From Theorem 5, 8 and 9, we can see that the error estimates obtained by the fractional Landweber iterative regularization method under a prior regularization parameter and a posterior regularization parameter selection rules are order optimal for all $p > 0$.

7 Numerical experiments

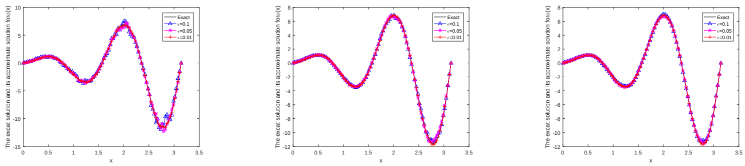
In this section, we give two numerical examples to verify the effectiveness and stability of the Quasi-boundary regularization method and the fractional Landweber regularization method. Let $\Omega = (0,\pi)$, $a = 1$, $T = 5$. Consider a one-dimensional problem

$$\begin{cases} {}_{CH}D^\alpha_{a,t}u(x,t) - \Delta u(x,t) = f(x), & x \in (0,\pi),\ t \in (1,5],\ 0<\alpha<1, \\ u(x,1) = \varphi(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = 0, & t \in (1,5], \\ u(x,5) = g(x), & x \in (0,\pi). \end{cases}$$

Next, we give the validity and stability of the fractional Landweber iterative regularization method and the Quasi-boundary regularization method based on the posteriori regularization parameter selection rule. Set $\tau_1 = 1.1$ in formula (5.7) and $\tau_2 = 1.01$ in formula (6.2). Let $f(x) = \sin(x)$, $M = 100$, $N = 50$. Here we give one numerical example.

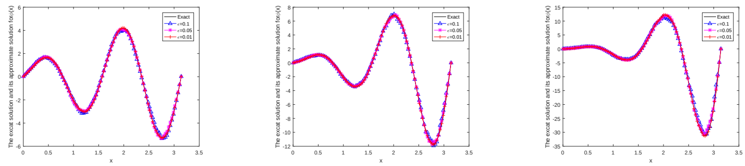
Example 1. Consider the smooth function

$$\varphi(x) = x^\alpha e^{x^\alpha}\sin(4x),\ x \in [0,\pi].$$



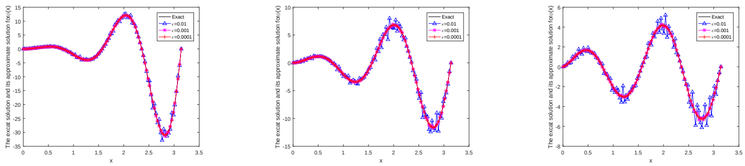
a) b) c)

Figure 1. The comparison between the exact solution $\varphi(x)$ and its Quasi-boundary regularization approximation solution $\varphi^{\mu,\delta}(x)$ of 1. (a) $\alpha = 0.3$, (b) $\alpha = 0.6$, (c) $\alpha = 0.9$.



a) b) c)

Figure 2. The comparison between the exact solution $\varphi(x)$ and its fractional Landweber iterative regularization solution $\varphi^{m,\delta}(x)$ of Example 1 with $\alpha = 0.6$. (a) $\gamma = 0.3$, (b) $\gamma = 0.6$, (c) $\gamma = 0.9$.



a) b) c)

Figure 3. The comparison between the exact solution $\varphi(x)$ and its fractional Landweber iterative regularization solution $\varphi^{m,\delta}(x)$ of Example 1 with $\gamma = 0.6$. (a) $\alpha = 0.3$, (b) $\alpha = 0.6$, (c) $\alpha = 0.9$.

Figure 1 describes the relationship between the exact solution $\varphi(x)$ and the Quasi-boundary regularization solution $\varphi^{\mu,\delta}(x)$ of the smooth function Example 1 with $\alpha = 0.3, 0.6, 0.9$ for the noise level $\epsilon = 0.01, 0.001, 0.0001$. From Figure 1, we can see that with the increase of α, the curve simulated by the Quasi-boundary regularization method becomes more unsmooth.

Figure 2 describes the relationship between the exact solution $\varphi(x)$ and the fractional Landweber regularization solution $\varphi^{m,\delta}(x)$ of the smooth function Example 1 with $\gamma = 0.3, 0.6, 0.9$ for the noise level $\epsilon = 0.1, 0.05, 0.01$ when $\alpha = 0.6$. We can also see from Figure 2 that the larger the γ, the smoother the image and the better the effect.

Figure 3 describes the relationship between the exact solution $\varphi(x)$ and the fractional Landweber regularization solution $\varphi^{m,\delta}(x)$ of the smooth function Example 1 with $\alpha = 0.3, 0.6, 0.9$ for noise level $\epsilon = 0.1, 0.05, 0.01$ when $\gamma = 0.6$. From Figure 3, we can see that as α increases, the error between the fractional landeweber regularization solution $\varphi^{m,\delta}(x)$ and the exact solution $\varphi(x)$ increases, and the number of iteration steps required decreases. To sum up, the smaller the noise levels ϵ and α, the better the numerical effect and the smaller the relative error. In addition, for the fractional Landweber iterative regularization method, the greater the order γ, the better the numerical effect, and the more iterative steps are required.

8 Conclusions

In this paper, we study the inverse initial value problem of Caputo-Hadamard time-fractional diffusion equation, which is ill-posed. In order to solve this ill-posed problem, we use two regularization methods to restore the stability of the solution. From the error estimation formula, we can see that the Quasi-boundary regularization method has saturation effect, and the fractional Landweber iterative regularization method has no saturation effect. Finally, two numerical examples are given to illustrate the effectiveness of the two methods. Through image comparison, we can see that when γ is larger, the numerical effect of the fractional Landweber iterative regularization method is better, requiring more iteration steps. Moreover, we can see that only when γ is large, the numerical effect of the fractional Landweber iterative regularization method is better than that of the Quasi-boundary regularization method. In addition, we can see that the Quasi-boundary regularization method has better numerical result at the inflection point of non-smooth function than the fractional Landweber iterative regularization method.

Acknowledgements

The project is supported by the National Natural Science Foundation of China (No.12461083), 2025 National College student Innovation Training Program of Lanzhou University of Technology(No. DC20250976).

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