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Study of a class of nonlinear heterogeneous diffusion with mixed phases under $L^\infty-$ data

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Abstract. In this paper we investigate a class of nonlinear degenerate parabolic equations involving heterogeneous (p, q)-Laplacian operators and subject to Dirichlet boundary conditions. These equations model complex diffusion phenomena with mixed-phase behavior in heterogeneous media. Our aim is to establish existence and uniqueness results for weak solutions under minimal regularity assumptions on the source term f, without requiring any control at infinity. The main difficulties stem from the degeneracy of the operator, the non-standard (p, q)-growth conditions, and the discontinuity of material phases. To overcome these challenges, we develop a variational framework based on Orlicz–Sobolev space theory and employ a generalized version of the Minty–Browder theorem to ensure the surjectivity of the nonlinear operator. Our approach yields new energy estimates, compactness results in non-reflexive settings, and stability under L^∞-perturbations of the data. This work provides a rigorous mathematical foundation for analyzing nonlinear diffusion problems in complex and irregular environments.

Keywords: double phase operator; weak solution; existence; semi-discretization; Rothe's method, weighted Sobolev space.

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1 Introduction

This work focuses on analyzing the existence and uniqueness of weak solutions for a class of nonlinear degenerate parabolic equations characterized by heterogeneous diffusion and mixed-phase structures. The model incorporates diffusion terms of both $p-$Laplacian and $q-$Laplacian types, combined with a nonlinear drift component θ and a reaction function $g(u)$. The mathematical formulation of the problem is as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - div(\mathcal{B}_{p,q}(x, \nabla u, \theta(u))) + g(u) = f(x,u) \quad \text{in} \quad Q_T, \\ u = 0 \quad \text{on} \quad \Sigma_T, \\ u(.,0) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $Q_T := \Omega \times]0,T[, \Sigma_T = \partial\Omega \times]0,T[$. Here and in the sequel, we will assume that $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded domain with a connected Lipschitz boundary $\partial\Omega$ and T is a finite positive time. The operator $\mathcal{B}_{p,q}$ is defined as follows :

$$\mathcal{B}_{p,q}(x, \nabla u, \theta(u)) := |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u)) + a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u)).$$

The problem is subject to homogeneous Dirichlet boundary conditions and an initial condition defined on Ω, and θ is continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$, the datum f is in L^∞ and $a : \Omega \to \mathbb{R}$ is an Lipschitz continuous map, $a(x) \geq 0$ for all $x \in \overline{\Omega}$.

In recent years, partial differential equations (PDEs) are essential in mathematical modeling, capturing a broad spectrum of phenomena across physics, biology, and engineering. These equations involve functions of multiple variables and their partial derivatives, governing essential processes like heat transfer, fluid flow, wave dynamics, and quantum mechanics. Based on their characteristics, PDEs are categorized as elliptic, parabolic, or hyperbolic, each associated with distinct physical interpretations. The study of PDEs relies on analytical and numerical approaches to determine solution properties such as existence, uniqueness, and regularity, often employing advanced mathematical frameworks like functional analysis, variational methods, and distribution theory. As a fundamental area of applied mathematics, PDEs remain a dynamic field of research, with ongoing advancements in nonlinear models, fractional calculus, and complex system applications, In the following, we have examples of this phenomenon.

- **Model 1. Fluid flow through porous media [19]**. This model is governed by the following equation,

$$\frac{\partial \theta}{\partial t} - div(|\nabla \varphi(\theta) - k(\theta)e|^{p-2}(\nabla \varphi(\theta) - k(\theta)e) = 0,$$

  where θ is the volumetric content of moisture, $\varphi(\theta)$ is the hydraulic conductivity; $k(\theta)$ is the hydrostatic potential, e is the unit vector in the vertical direction.

- **Model 2. The magneto-quasi-static approximation. [19]** The system describes how a physical, chemical, or biological quantity $u(x,t)$—such as the concentration of a chemical substance or the density of a population—changes over **space** (x) and **time** (t), driven by **nonlinear diffusion** and **reaction processes**. The governing equation is:

$$\frac{\partial u}{\partial t} - div\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) + R(u) = f(x,t),$$

where

- $u(x,t)$: the unknown scalar field (e.g., concentration, density).
- $\frac{\partial u}{\partial t}$: the time derivative, representing the rate of change at a fixed point.
- $-div(\mathcal{B}_{p,q}(x,\nabla u))$: the divergence of a flux $\mathcal{B}_{p,q}$, representing net diffusion.
- $\mathcal{B}_{p,q}(x,\nabla u) = |\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u$: the total flux, driven by the gradient ∇u.
- p,q: nonlinearity exponents $(1 < p \leq q)$. They define the diffusion modes:
  * the p-**Laplacian** term $(|\nabla u|^{p-2}\nabla u)$ governs standard diffusion.
  * the q-**Laplacian** term $(|\nabla u|^{q-2}\nabla u)$ becomes dominant in high-gradient regions, enabling faster "supra-diffusion."
- $a(x)$: a non-negative spatial coefficient that modulates the strength of the q-diffusion term across the domain.
- $R(u)$: the reaction term, representing local growth or decay (e.g., $R(u) = u(1-u)$ for logistic growth).
- $f(x,t)$: the source/sink term, representing an external forcing function.

The study of weak solutions became particularly important in the context of elliptic PDEs. In the 1950s and 1960s, De Giorgi and Nash independently proved regularity results for weak solutions of elliptic equations. Their work showed that even if a solution is not classically smooth, it can still possess important regularity properties. This notion serves as a foundational framework for studying nonlinear elliptic and parabolic problems, encompassing scenarios with both constant and variable exponents and under different boundary conditions, be it Dirichlet or Neumann.

In this paper, by using the Rothe's method, we want to prove the existence question of weak solutions to nonlinear parabolic problem (1.1) to this end, we employ Rothe's method, beginning with here a time discretization of this continuous problem (1.1) and by using Euler forward scheme we show existence and uniqueness of weak solutions to the discretized problem. secondly , we shall construct a sequence derived from the weak solution of the discretized problem, establishing its convergence to a solution of the nonlinear parabolic problem (1.1) . The advantage of our approach lies in its capability not only to ascertain the existence of weak solutions to the problem (1.1) but also to compute numerical approximations effectively. It is worth noting that the Euler forward scheme has been employed by various authors in the examination of time discretization for nonlinear parabolic problems; refer to works [1, 2, 4, 13, 20, 21] for specific details.

Recently, when $\theta \equiv 0$, $a(.) = 0$ and $\omega(x) \equiv 1$, Bhuvaneswari, Lingeshwaran and Balachandran proved the existence of weak solution for the degenerate p-Laplacian parabolic by using semi-discretization process in [2], and where $p(.)$

is a variable exponent Sanchón and Urbano proved the existence and uniqueness of entropy solutions of $p(x)$-Laplace equation with L^1 data in [21], also, Chao Zhang in [22] study the entropy solutions for nonlinear elliptic problem with variable exponents by using proprieties of Orlicz spaces, then in [13] Jamea, Lamrani and El Hachimi, studied the existence of entropy solutions to a class of nonlinear parabolic problem and for that they using Rothe time-discretization method and some result of variable exponent Sobolev spaces. Moreover, Khaleghi and Razani investigated the existence and multiplicity of weak solution for an elliptic problem involving $p(.)$-Laplacian operator under Steklov boundary, condition, then Sabri and Jamea for the same and when p constant they studied the existence and uniqueness of entropy solutions of this equation in [20] and we have other example in [8] and [7].
The double phase case has been studied by several authors, often considering various scenarios. For example, the following cases have been extensively analyzed by Chems Eddine, Ouannasser and Ragusa in [10], Moujane, El Ouaarabi and Allalou in [15,16] and Zuo, Allalou and Raji in [3,17,23]; Fang, Rădulescu and Zhang in [11].

2 Preliminaries and notations

In the present section, we give some definitions, notations and results which well be used in this work. Let φ function from $\Omega \times \mathbb{R}^+$ to $\mathbb{R}^+$ defined by

$$\varphi(x,y) = y^p + a(x)\, y^q,$$

where a and p, q verify condition (H_1), see [11]

(H_1) $a : \Omega \to \mathbb{R}$ is a Lipschitz continuous and $p > \frac{Nq}{N+q-1}$ i.e., $\left(\frac{q}{p} < 1 + \frac{q-1}{N}\right)$.

The function φ is a generalized N-function and

$$\varphi(x, 2y) = 2^p \varphi(x,y).$$

Now, we define the Musielak–Orlicz space $L^\varphi(\Omega)$ by

$$L^\varphi(\Omega) = \left\{ v : \Omega \to \mathbb{R} : v \text{ is measurable and} \int_\Omega \varphi(x,|v|)dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|v\|_\varphi = \inf\left\{ \lambda > 0, \int_\Omega \varphi\Big(x, \frac{|v|}{\lambda}\Big) dx \le 1 \right\}.$$

The Sobolev space corresponds to the L^φ space is

$$W^{1,\varphi}(\Omega) = \big\{ v \in L^\varphi(\Omega) \ \text{ such that } \nabla v \in L^\varphi(\Omega) \big\}$$

with the norm

$$\|v\|_{1,\varphi} = \|v\|_\varphi + \|\nabla v\|_\varphi.$$

Theorem 1 [[9,11]].

i) If $q \neq N$ for all $r \in [1, q*]$, we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$, where

$$q^* = \begin{cases} \frac{Nq}{N-q}, & \text{if} \quad q < N, \\ +\infty, & \text{if} \quad q \geq N. \end{cases}$$

ii) If $q = N$ for all $r \in [1, +\infty[$, we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$.

iii) If $q \leq N$ for all $r \in [1, q*]$, we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$ compactly.

iv) If $q > N$, $W^{1,\varphi}(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly.

v) $W^{1,\varphi}(\Omega) \hookrightarrow L^q(\Omega)$.

We define now the weighted Lebesgue space $L^q_a(\Omega)$ by

$$L^q_a(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \text{ is measurable and } \|v\|_{q,a} = \int_\Omega a(x)|v|^q dx < \infty \right\}.$$

On the space $L^\varphi(\Omega)$ we consider the function $\varrho_\varphi : L^\varphi(\Omega) \to \mathbb{R}^+$ defined by

$$\varrho_\varphi(v) = \int_\Omega \varphi\left(x, \frac{|v|}{\lambda}\right) dx = \int_\Omega [|v|^p + a(x)|v|^q] dx.$$

The connection between ϱ_φ and $\|.\|_\varphi$ is established by the next result.

Proposition 1 [[6,9]].
Let u be an element of $L^\varphi(\Omega)$. The following assertions hold:

i) $\|u\|_\varphi < 1$(respectively $>, = 1$) $\Leftrightarrow \varrho_\varphi(u) < 1$(respectively $>, = 1$),

ii) If $\|u\|_\varphi < 1$, then $\|u\|^p_\varphi \leq \varrho_\varphi(u) \leq \|u\|^q_\varphi$,

iii) If $\|u\|_\varphi > 1$, then $\|u\|^q_\varphi \leq \varrho_\varphi(u) \leq \|u\|^p_\varphi$,

iv) $\|u\|_\varphi \to 0 \Leftrightarrow \varrho_\varphi(u) \to 0$ and $\|u\|_\varphi \to \infty \Leftrightarrow \varrho_\varphi(u) \to \infty$.

Definition 1. Let $\varphi : [0, \infty) \to [0, \infty]$. We denote by φ^* the conjugate function of φ which is defined, for $u \geq 0$, by

$$\varphi^*(u) := \sup_{t \geq 0}(tu - \varphi(t)).$$

Proposition 2 [[6,9]]. *For any functions $u \in L^\varphi(\Omega)$, $v \in L^{\varphi^*}(\Omega)$, and under the assumption that hypothesis (H_1) be satisfied, we have:*

$$\int_\Omega |uv| dx \leq 2\|u\|_\varphi \|v\|_{\varphi^*}.$$

In the following of this paper, the space $W^{1,\varphi}_0(\Omega)$ denote the closure of C^∞_0 in $W^{1,\varphi}(\Omega)$ with respect the norm $\|.\|_{1,\varphi}$ (see [12]).

Proposition 3 [[6, 9]]. *The spaces* $(L^{\varphi}(\Omega), \|.\|_{\varphi})$ *and* $(W^{1,\varphi}(\Omega), \|.\|_{1,\varphi})$ *are separable and uniformly convex (hence reflexive) Banach spaces.*

We have

$$L^p(\Omega) \hookrightarrow L^{\varphi}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_a(\Omega).$$

Proposition 4 [[7]]. *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a bounded set, and let* $u \in W^{1,\varphi}(\Omega)$

$$\|u\|_{\varphi} \leq C_0 \|\nabla v\|_{\varphi},$$

C_0 *is a strictly positive constant depends on the exponent diam*(Ω) *and the dimension* N.

Proposition 5 [[8]]. *Let* $1 < p < +\infty$. *There exist two positive constants* μ_p *and* ρ_p *such that for every* $x, y \in \mathbb{R}^N$, *it holds that*

$$\begin{aligned} \mu_p(|x| + |y|)^{p-2}|x - y|^2 &\leq \langle |x|^{p-2}x - |y|^{p-2}y; x - y\rangle \\ &\leq \rho_p(|x| + |y|)^{p-2}|x - y|^2. \end{aligned}$$

Proposition 6 [[5]]. *Suppose that* $\Omega \subset \mathbb{R}^N$ *be a bounded set. Then, for all* $u \in W_0^{1,\varphi}(\Omega)$, *the inequality*

$$\|u\|_{\infty} \leq C' \|\nabla u\|_{\varphi},$$

is satisfied where the constant C' *depends on the exponent* $p, q, meas(\Omega)$ *and the dimension* N.

DEFINITION 2. Given a constant $k > 0$; we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s, & \text{if } |s| < k, \\ k, & \text{if } s > k, \\ -k, & \text{if } s < -k. \end{cases}$$

Lemma 1 [[2]]. *For* $\xi, \eta \in \mathbb{R}^N$ *and* $1 < p < \infty$, *we have:*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\eta - \xi),$$

where a dot denote the Euclidean scalar product in $\mathbb{R}^N$.

Lemma 2 [[2]]. *For* $a > 0, b > 0$ *and* $1 \leq p < \infty$ *we have*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

Lemma 3 [[18]]. *Let* p *and* p' *be two real numbers such that* $p > 1$, $p > 1$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *There existed a positive constant* m *such that*

$$|((|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)|^{p'} \leq m\{(\xi - \eta)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\}^{\frac{\beta}{2}}\{\xi^p + \eta^{p'}\}^{1-\frac{\beta}{2}},$$

for all $\xi, \eta \in \mathbb{R}^N$, $\beta = 2$, *if* $1 < p \leq 2$, *and* $\beta = p'$ *if* $p > 2$.

Definition 3 [[14]]. Let Y be a reflexive Banach space and let A be an operator from Y to its dual $Y^{'}$. We say that A is *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0 \qquad\qquad \forall u, v \in Y.$$

Theorem 2 [[14]]. *Let Y be a reflexive real Banach space and $A : Y \to Y^{'}$ be a bounded operator, hemi-continuous, coercive and monotone on space Y . Then, the equation $Au = v$ has at least one solution $u \in Y$ for each $v \in Y^{'}$.*

3 Proof of main result

Our intention in this section is to prove the main result, which is divided into three steps. First, we discretize the continuous problem (1.1) using the Euler forward scheme and investigate the existence and uniqueness of weak solutions to the discretized problem. In the second step, we establish stability results for the discrete weak solutions. Finally, using the Rothe method, we construct a sequence of functions and demonstrate that this sequence converges to a weak solution of the nonlinear degenerate parabolic problem (1.1). To achieve this, we need certain key assumptions that help us properly study our problem. In other words, we assume the following:

(H_2) θ is a function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\theta(0) = 0$ and for all real numbers x, y we have

a) $\|\theta(x) - \theta(y)\| < \lambda_0|x - y|$.

b) λ_0 is a real constant such that $0 < \lambda_0 < C_0$.

(H_3) $f \in L^\infty(Q_T)$ and $u_0 \in L^\infty(\Omega)$.

3.1 The semi-discrete problem

By Euler forward scheme, we discretize the problem (1.1) , we obtain the following problems

$$U_n - \tau div(\mathcal{B}_{p,q}(x, \nabla U_n, \theta(U_n))) + \tau|U_n|^{p-2}U_n = \tau f_n + U_{n-1}, \qquad (3.1)$$
$$U_n = 0, \quad \text{on} \quad \partial\Omega, \quad U_0 = u_0, \quad \text{in} \quad \Omega.$$

where

1. $N\tau = T,\ \ 0 < \tau < 1,\ \ 1 \leq n \leq N,\ \ t_n = n\tau.$

2) $f_n(.) = \dfrac{1}{\tau}\displaystyle\int_{t_{n-1}}^{t_n} f(s, .)ds \quad \text{in} \quad \Omega.$

Definition 4. A weak solution to the discretized problem (3.1) is a sequence $(U_n)_{0\leq n\leq N}$ such that $U_0 = u_0$ and for n in $\{1, 2, ..., N\}$, U_n is defined by induction as an unique weak solution to the problem:

$$\begin{cases} u - \tau div(\mathcal{B}_{p,q}(x, \nabla u, \theta(u))) + \tau|u|^{p-2}u = \tau f_n + U_{n-1}, \\ u = 0, \quad \text{on} \quad \partial\Omega, \end{cases}$$

i.e., for all $n \in \{1;2;...;N\}$, $U_n \in W_0^{1,\varphi}(\Omega)$ and for all $\varphi \in W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$, $\tau > 0$, we have

$$\begin{cases} \int_\Omega U_n \varphi dx + \tau \int_\Omega \mathcal{B}_{p,q}(x, \nabla U_n, \theta(U_n)) \nabla\varphi dx \\ \qquad + \tau \int_\Omega |U_n|^{p-2} U_n \varphi dx = \int_\Omega (\tau f_n + U_{n-1}) \varphi dx, \\ u = 0, \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.2}$$

Theorem 3. *Let hypotheses (H_1)–(H_3) be satisfied. If $(U_n)_{0\le n\le N}$ is an unique weak solution of discretized problem* (3.1)*, then for all $n = 1, ..., N$, we have $U_n \in L^\infty(\Omega)$.*

Proof. Let U_n be a weak solution of (3.1). For $n = 1$, we take $\varphi = U_n$ in inequality (3.2), we get

$$\begin{aligned} \|U_n\|_2^2 &+ \tau \int_\Omega |\nabla U_n - \theta(U_n)|^{p-2} (\nabla U_n - \theta(U_n)) \nabla U_n dx \\ &+ \tau \int_\Omega a(x) |\nabla U_n - \theta(U_n)|^{q-2} (\nabla U_n - \theta(U_n)) \nabla U_n dx \\ &+ \tau \int_\Omega |U_n|^p dx = \int_\Omega (\tau f_n + U_{n-1}) U_n dx, \end{aligned} \tag{3.3}$$

where

$$\int_\Omega U_n U_n dx = \|U_n\|_2^2, \quad \int_\Omega |U_n|^{p-2} U_n U_n dx = \int_\Omega |U_n|^p dx.$$

On the other hand and by using Hölder's inequality

$$\begin{aligned} \int_\Omega (\tau f_n + U_{n-1}) U_n dx &= \int_\Omega \tau f_n U_n dx + \int_\Omega U_{n-1} U_n dx \\ &\le \tau \|f_n\|_\infty \|U_n\|_1 + \|U_{n-1}\|_\infty \|U_n\|_1 \le \left(\tau \|f_n\|_\infty + \|U_{n-1}\|_\infty\right) \|U_n\|_1. \end{aligned}$$

Now, we define A and B by

$$\begin{aligned} \langle AU_n, U_n \rangle &= \int_\Omega |\nabla U_n - \theta(U_n)|^{p-2} (\nabla U_n - \theta(U_n)) \nabla U_n dx + \int_\Omega |U_n|^p dx, \\ \langle BU_n, U_n \rangle &= \int_\Omega a(x) |\nabla U_n - \theta(U_n)|^{q-2} (\nabla U_n - \theta(U_n)) \nabla U_n dx. \end{aligned}$$

In [14], we prove $A + B$ is a coercive operator, then we have

$$\langle (A+B) U_n, U_n \rangle \ge M \|U_n\|_{1,\varphi} \ge M(1 + C_0) \|\nabla U_n\|_\varphi. \tag{3.4}$$

By (3.3) and (3.4) we have

$$\|U_n\|_2^2 + \tau M(1 + C_0) \|\nabla U_n\|_\varphi \le (\tau \|f_n\|_\infty + \|U_{n-1}\|_\infty) \|U_n\|_1.$$

We have $\|U_n\|_2^2 \ge 0$, so,

$$\tau M(1 + C_0) \|\nabla U_n\|_\varphi \le (\tau \|f_n\|_\infty + \|U_{n-1}\|_\infty) \|U_n\|_1. \tag{3.5}$$

On the other hand we take $\varphi = sign(U_n)$ in inequality (3.2), we get

$$\int_\Omega U_n sign(U_n)dx + \tau \int_\Omega \mathcal{B}_{p,q}(x, \nabla U_n, \theta(U_n))\nabla sign(U_n)dx$$
$$+ \ \tau \int_\Omega |U_n|^{p-2}U_n sign(U_n)dx = \int_\Omega (\tau f_n + U_{n-1})\, sign(U_n)dx.$$

It holds that

$$\int_\Omega U_n sign(U_n)dx = \|U_n\|_1, \quad \nabla(sign(U_n)) = 0,$$
$$\int_\Omega |U_n|^{p-2}U_n sign(U_n)dx = \int_\Omega |U_n|^{p-1} \geq 0.$$

Therefore, we obtain

$$\begin{aligned}\|U_n\|_1 \leq &\int_\Omega (\tau f_n + U_{n-1})sign(U_n)dx \\ \leq &\tau\|f_n\|_\infty \int_\Omega sign(U_n)dx + \|U_{n-1}\|_\infty \int_\Omega sign(U_n)dx \\ \leq &\tau\|f_n\|_\infty \int_\Omega 1dx + \|U_{n-1}\|_\infty \int_\Omega 1dx \leq (\tau\|f_n\|_\infty + \|U_{n-1}\|_\infty)meas(\Omega).\end{aligned} \tag{3.6}$$

Using (3.5) and (3.6) then we have

$$\begin{aligned}\tau M(1+C_0)\|\nabla U_n\|_\varphi &\leq (\tau\|f_n\|_\infty + \|U_{n-1}\|_\infty)\|U_n\|_1 \\ &\leq (\tau\|f_n\|_\infty + \|U_{n-1}\|_\infty) \times (\tau\|f_n\|_\infty + \|U_{n-1}\|_\infty)meas(\Omega) \\ &\leq (\tau\|f_n\|_\infty + \|U_{n-1}\|_\infty)^2 meas(\Omega).\end{aligned}$$

By Proposition 6 we have

$$\|U_n\|_\infty \leq C'\|\nabla U_n\|_\varphi.$$

Then,

$$\tau\frac{M}{C'}\|U_n\|_\infty \quad \leq \quad (\tau\|f_n\|_\infty + \|U_{n-1}\|_\infty)^2 meas(\Omega).$$

Finally, $U_n \in L^\infty(\Omega)$ and by induction, we deduce in the same manner that $U_n \in L^\infty(\Omega)$; $\forall n = 1, \ldots, N$. □

Theorem 4. *Assume that hypotheses* (H_1)–(H_3) *be satisfied. Then the discretized problem* (3.1) *has a unique weak solution* $(U_n)_{0\leq n\leq N}$.

Proof. **Existence part:** For $n = 1$, we rewrite the discretized problem (3.1) as

$$\begin{cases} u - div(\mathcal{B}_{p,q}(x, \nabla u, \theta(u))) + |u|^{p-2}u = h, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.7}$$

where $u = U_1$ and $h = \tau f_1 + u_0$, and with the hypothesis (H_3) we have $h \in L^\infty$. And the variational problem of last equation is

$$\begin{cases} \int_\Omega u\varphi dx + \tau \int_\Omega \left(\mathcal{B}_{p,q}(x, \nabla u, \theta(u))\right) \nabla\varphi dx \\ \qquad +\tau \int_\Omega |u|^{p-2} u\varphi dx = \int_\Omega h\varphi dx, \quad \text{in} \quad \Omega, \\ u = 0, \quad \text{on} \quad \partial\Omega. \end{cases}$$

Let T be an operator defined by:

$$\begin{array}{rcl} T : W_0^{1,\varphi}(\Omega) & \longrightarrow & (W_0^{1,\varphi}(\Omega))' \\ u & \longmapsto & T(u) := A(u) + B(u) + C(u) - L, \end{array}$$

where $(W_0^{1,\varphi}(\Omega))'$ is the dual space of $W_0^{1,\varphi}(\Omega)$, and for $u, \varphi \in W_0^{1,\varphi}(\Omega)$

$$\begin{aligned} \langle Au, \varphi \rangle &= \int_\Omega |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u))\nabla\varphi dx + \int_\Omega |u|^{p-2} u\varphi dx, \\ \langle Bu, \varphi \rangle &= \int_\Omega |\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u))\nabla\varphi dx, \\ \langle Cu, \varphi \rangle &= \int_\Omega u\varphi dx, \quad \langle L, \varphi \rangle = \int_\Omega h\varphi dx. \end{aligned}$$

By Theorem 2, in [14] the operator $A + B$ and L is continue, coercive, hemi-continue and of type (M) and $B = id$ then the operator T is continue, coercive, hemi-continue and of type (M) and by Theorem 2 the problem (3.7) has a unique weak solution U_1. By induction, using the same argument above, we prove that the problem (3.7) has a weak solution $(U_n)_{0 \le n \le N}$ and $U_n \in L^\infty(\Omega)$ for all $n = 1, \ldots, N$.

Uniqueness part:
Let u and v be two weak solutions of the problem (3.7). For the solution u, we take $\varphi = u - v$ as test function and for the solution v we take $\varphi = v - u$ as test function in the Equation (3.1), then we have

$$\begin{aligned} \int_\Omega u(u - v)dx + \tau\langle Au, u - v\rangle + \tau\langle Bu, u - v\rangle &= \langle L, u - v\rangle, \\ \int_\Omega v(v - u)dx + \tau\langle Av, v - u\rangle + \tau\langle Bu, u - v\rangle &= \langle L, v - u\rangle. \end{aligned}$$

By summing up the two above equalities we get

$$\int_\Omega (u - v)^2 dx + \tau\langle Au - Av, u - v\rangle + \tau\langle Bu, u - v\rangle = 0.$$

The operator A and B is monotone so $\langle Au - Av, u - v\rangle \ge 0$ and $\langle Bu - Bv, u - v\rangle$. This implies that $\|u - v\|_2^2 = 0$. Finally,

$$u = v \quad \text{in} \quad \Omega.$$

□

3.2 A priori estimates

In this section, we give some a priori estimates for the discrete weak solution $(U_n)_{1\leq n\leq N}$ which will be used to derive the convergence results for the Euler forward scheme.

Lemma 4. *Let hypothesis (H_1)–(H_4) be satisfied. Then, there exist positive constants depending on the data f, u_0, and on N such that for all $n = 1, \dots, N$, we have:*

1. $\|U_n\|_2^2 \leq C(T, \tau,_0, f)$.
2. $\sum_{i=1}^{n} \|U_i - U_{i-1}\|_2^2 \leq C(T, \tau, u_0, f, N)$.
3. $\sum_{i=1}^{n} \|U_i - U_{i-1}\|_1 \leq C(T, \tau, u_0, f, N)$.
4. $\tau \sum_{i=1}^{n} \|U_i\|_{W_0^{1,\varphi}}^{p^-} \leq C(T, \tau, u_0, f, N, P^-)$.

Proof. For (1) and (2), let $1 \leq i \leq N$, we take $\varphi = U_i$ as test function in the Equation (3.2) we obtain

$$\int_\Omega (U_i - U_{i-1})U_i dx + \tau\langle AU_i, U_i\rangle + \tau\langle BU_i, U_i\rangle \quad = \quad \int_\Omega \tau f U_i dx.$$

Then,

$$\int_\Omega (U_i - U_{i-1})U_i dx + \tau\langle (A+B)U_i, U_i\rangle \quad \leq \quad \tau\|f\|_\infty \|U_i\|_1.$$

And by the coercivity of the operator $A + B$ we have

$$\int_\Omega (U_i - U_{i-1})U_i dx \quad \leq \quad \tau\|f\|_\infty \|U_i\|_1. \tag{3.8}$$

With the aid of the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ from (3.8) we obtain

$$\frac{1}{2}\|U_i\|_2^2 - \frac{1}{2}\|U_{i-1}\|_2^2 + \|U_i - U_{i-1}\|_2^2 \quad \leq \quad \tau C_2 \|U_i\|_1. \tag{3.9}$$

Now, summing (3.9) from $i = 1$ to n and using the (3.6), we get

$$\frac{\|U_n\|_2^2 - \|U_0\|_2^2}{2} + \sum_{i=1}^{n} \|U_i - U_{i-1}\|_2^2 \quad \leq \quad \tau C_3.$$

Hence, the stability result (1) and (2) are then proved.
For (3), we have

$$\|U_i - U_{i-1}\|_1 \leq \sqrt{meas(\Omega)}\ \ \|U_i - U_{i-1}\|_2,$$

then,

$$\sum_{i=1}^{n} \|U_i - U_{i-1}\|_1 \leq C(f, u_0, \Omega). \tag{3.10}$$

For (4), we pose $I_0 = \{i \in 1, 2, \dots N, \quad \text{such that} \quad \|U_i\|_{1,\varphi} \leq 1\}$.

$$\tau \sum_{i=1}^{n} \|U_i\|_{W_0^{1,\varphi}}^{p^-} = \tau \sum_{i \in I_0} \|U_i\|_{W_0^{1,\varphi}}^{p^-} + \tau \sum_{i \notin I_0} \|U_i\|_{W_0^{1,\varphi}}^{p^-} \leq T + \tau C_4 \varrho_\varphi(u).$$

And using (3.10) we have (4). □

3.3 Weak solution of the continuous problem

Let us introduce the following piecewise linear extension (called Rothe function), such that $\forall t \in]t_{n-1}, t_n], n = 1, \dots, N$, we have

$$\begin{cases} u_N(0) = u_0, \\ u_N(t) = U_{n-1} + (U_n - U_{n-1}) \frac{t - t_n}{\tau}, \end{cases} \qquad \text{in } \Omega.$$

And the following piecewise constant function

$$\begin{cases} \overline{u}_N(0) = u_0, \\ \overline{u}_N(t) = U_n, \end{cases} \qquad \text{in } \Omega.$$

We have by Theorem 4 that for any $N \in \mathbb{N}$. The weak solution $(U_n)_{1 \leq n \leq N}$ of problems (4) is unique, thus, the two sequences $(u_N)_{N \in \mathbb{N}}$ and $(\overline{u}_N)_{N \in \mathbb{N}}$ are uniquely defined.

Lemma 5. *Let hypotheses* (H_1)–(H_3) *be satisfied and* $\overline{u}_N$, u_N *has a unique weak solution of problem* (3.1), *then for all* $N \in \mathbb{N}$, *we have*

$$1. \|\overline{u}_N - u_N\|_{L^2(Q_T)}^2 \leq \frac{1}{N} C(T, u_0, f), \quad 2. \|\overline{u}_N\|_{L^\infty(0,T,L^2(\Omega))}^2 \leq C(T, u_0, f),$$
$$3. \|u_N\|_{L^\infty(0,T,L^2(\Omega))}^2 \leq C(T, u_0, f), \quad 4. \|u_N\|_{L^p(0,T,W_0^{1,\varphi})} \leq C(T, u_0, f),$$
$$5. \left\|\frac{\partial u_N}{\partial t}\right\|_{L^2(Q_T)}^2 \leq C(T, u_0, f), \quad 6. \|\overline{u}_N\|_{L^p(0,T,W_0^{1,\varphi})} \leq C.$$

Proof. For (1), we have

$$\|\overline{u}_N - u_N\|_{L^2(Q_T)}^2 = \int_0^T \int_\Omega \|\overline{u}_N - u_N|^2 dx dt$$
$$\leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |U_i - U_{i-1}|^2 \left(\frac{t_i - t}{\tau}\right)^2 dx dt = \frac{\tau}{3} \sum_{i=1}^{N} \|U_i - U_{i-1}\|_2^2.$$

By using (2) of Lemma 4, we get

$$\|\overline{u}_N - u_N\|_{L^2(Q_T)}^2 \leq \frac{1}{N} C(u_0, T, f).$$

For (5), we have for $n = 1, \dots, N$ and $t \in (t_{n-1}; t_n]$

$$\frac{\partial u_N}{\partial t} = \frac{(U_i - U_{i-1})}{\tau}.$$

This implies that

$$\left\|\frac{\partial u_N}{\partial t}\right\|^2_{L^2(Q_T)} = \int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 dxdt$$
$$= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \frac{1}{\tau} \|U_i - U_{i-1}\|^2_{L^2(\Omega)} dt = \frac{1}{\tau} \sum_{i=1}^N \|U_i - U_{i-1}\|^2_{L^2(\Omega)}.$$

Then, we apply the result (2) of Lemma 4 then we obtain the result (5). For (2),(3),(4) and (6) we use (2),(3) and (4) of Lemma 4 and we will get the result. □

Lemma 6. *Let hypotheses* (H_1)*–*(H_3) *be satisfied and* $\overline{u}_N$ *solution of* (3.1) *for* $N \in \mathbb{N}$*, the sequence* $(\overline{u}_N)_{N\in\mathbb{N}}$ *converges in measure and a.e. in* Q_T*.*

Proof. Let ε and r be positive real numbers and let $N; M \in \mathbb{N}$, we have the following inclusion we have by Lemma 5, the sequence $(\overline{u}_N)_{N\in\mathbb{N}}$ is bounded in space $L^\varphi(Q_T)$. and then there exists a subsequence, still denoted by $(\overline{u}_N)_{N\in\mathbb{N}}$ such that $(\overline{u}_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^\varphi(Q_T)$. Therefore, there exists $N_0 \in \mathbb{N}$ such that for all N; $M \geq N_0$, we have

$$meas\{|\overline{u}_N - \overline{u}_M| > r\} \quad < \quad \varepsilon.$$

Consequently, $(\overline{u}_N)_{N\in\mathbb{N}}$ converges in measure and there exists a measurable function u in Q_T, such that

$$\overline{u}_N \to u \ a.e \quad in \quad Q_T.$$

Now, using the two results (2) and (3) of Lemma 5, the sequences $(u_N)_{N\in\mathbb{N}}$ and $(\overline{u}_N)_{N\in\mathbb{N}}$ are uniformly bounded in $L^\infty(0, T, L^2(\Omega))$, therefore, there exist two elements $L^\infty(0, T, L^2(\Omega))$ such that

$$\begin{aligned} \overline{u}_N \quad &\to \quad u \text{ weakly in } \ L^\infty(0, T, L^2(\Omega)), \\ u_N \quad &\to \quad v \text{ weakly in } L^\infty(0, T, L^2(\Omega)). \end{aligned}$$

And from the result (1) of Lemma 5, it follows that $u \equiv v$. Furthermore, by Lemma 5 we have that

$$\begin{aligned} \frac{\partial u_N}{\partial t} \quad &\to \quad \frac{\partial u}{\partial t} \quad \text{in} \quad L^2(Q_T), \\ \overline{u}_N \quad &\to \quad u \quad \text{in } \ L^\varphi(0, T, W_0^{1,\varphi}(\Omega)). \end{aligned}$$

By (5) of Lemma 5, we have the sequence $(\nabla\overline{u}_N)_{N\in\mathbb{N}}$ is bounded in $L^\varphi(Q_T)$ Then, there exists a subsequence, still denoted $(\nabla\overline{u}_N)_{N\in\mathbb{N}}$ such that

$$\nabla\overline{u}_N \text{ converges weakly to an element v in } L^{\varphi}(Q_T).$$

And by Lemma 6 we have $\overline{u}_N \to u$ in $L^{\varphi}(Q_T)$. Hence, it follows that

$$\overline{u}_N \to u \quad \text{weakly in} \quad L^{\varphi}(Q_T). \; \theta(\overline{u}_N) \to \theta(u) \quad \text{weakly in} \quad L^{\varphi}(Q_T).$$

Then,

$$|\nabla\overline{u}_N - \theta(\overline{u}_N)|^{p-2}(\nabla\overline{u}_N - \theta(\overline{u}_N)) \to$$
$$|\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u)) \text{ weakly in } \; L^{\varphi}(Q_T)$$

and

$$a(x)|\nabla\overline{u}_N - \theta(\overline{u}_N)|^{q-2}(\nabla\overline{u}_N - \theta(\overline{u}_N)) \to$$
$$a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u)) \text{ weakly in } \; L^{\varphi}(Q_T).$$

On the other hand, we have by Lemma 5 and Aubin–Simon's compactness result that $u_N \to u$ in $C(0,T,L^2(\Omega))$. Now, we prove that the limit function u is a weak solution of problem (1.1). Firstly, we have $u_N(0) = U_0 = u_0$ for all $N \in \mathbb{N}$, then $u(0,.) = u_0$. Secondly, let $\varphi \in C^1(Q_T)$, we rewrite (5) in the forms of

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial u_N}{\partial t}\varphi dxdt \; &+ \int_0^T\int_\Omega |\nabla u_N - \theta(u_N)|^{p-2}(\nabla u_N - \theta(u_N))\nabla\varphi dxdt \\ &+ \int_0^T\int_\Omega a(x)|\nabla u_N - \theta(u_N)|^{q-2}(\nabla u_N - \theta(u_N))\nabla\varphi dxdt \\ &+ \int_0^T\int_\Omega |u_N|^{p-2}u_N\varphi dxdt = \int_0^T\int_\Omega f_N\varphi dxdt, \qquad (3.11)\end{aligned}$$

where $f_N(x,t) = f_n(x)$, $\forall t \in]t_{n-1}, t_n]$, $n = 1, \ldots, N$. Taking limits as $N \to \infty$ in (14) and using the above results, we deduce that u is a weak solution of the nonlinear parabolic problem (1.1).

Uniqueness part. Let u and v be two weak solutions of the problem (1.1). For the solution u , we take $\varphi = u - v$ as test function and for the solution v we take $\varphi = v - u$ as test function in the Equation (3.11), then we have

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial u}{\partial t}(u-v)dxdt \; &+ \int_0^T\int_\Omega |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u))\nabla(u-v)dxdt \\ &+ \int_0^T\int_\Omega |\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u))\nabla(u-v)dxdt \\ &+ \int_0^T\int_\Omega |u|^{p-2}(u-v)dxdt = \int_0^T\int_\Omega f(u-v)dxdt\end{aligned}$$

and

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial v}{\partial t}(v-u)dxdt \; &+ \int_0^T\int_\Omega |\nabla v - \theta(v)|^{p-2}(\nabla v - \theta(v))\nabla(v-u)dxdt \\ &+ \int_0^T\int_\Omega a(x)|\nabla v - \theta(v)|^{q-2}(\nabla v - \theta(v))\nabla(v-u)dxdt \\ &+ \int_0^T\int_\Omega |v|^{p-2}(v-u)dxdt = \int_0^T\int_\Omega f(v-u)dxdt.\end{aligned}$$

In other words,

$$\int_0^T\int_\Omega \frac{\partial v}{\partial t}(v-u)dxdt + \int_0^T\int_\Omega \mathcal{B}_{p,q}(x,\nabla u,\theta(u))\nabla(v-u)dxdt + \int_0^T\int_\Omega |v|^{p-2}(v-u)dxdt = \int_0^T\int_\Omega f(v-u)dxdtx.$$

By summing up the two above equalities, and using the same arguments in the proof of uniqueness part of Theorem 4, we conclude that

$$\text{u} = \text{v} \qquad \text{a.e} \quad \text{in } \Omega.$$

□

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