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Equivalents of the Riemann hypothesis involving the Gram points

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Abstract. The Riemann hypothesis (RH) on zeros of the zeta-function $\zeta(s)$, $s = \sigma + it$, states that all zeros of $\zeta(s)$ in the strip $0 < \sigma < 1$ lie on the line $\sigma = 1/2$. Several equivalents of RH are known. In the paper, we obtain equivalents of RH in terms of self-approximation of $\zeta(s)$ by shifts $\zeta(s + iht_k)$, $k \in \mathbb{N}$, where $\{t_k, k \in \mathbb{N}\}$ is the sequence of the Gram points.

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1 Introduction

Let $s = \sigma + it$ denote a complex variable and $\mathbb{P}$ the set of all prime numbers. The Riemann zeta-function $\zeta(s)$ on the half plane $\sigma > 1$ is defined by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

or by the Euler product

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - 1/p^s\right)^{-1}, \tag{1.1}$$

and has the analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. L. Euler was the first to study the function $\zeta(s)$, though only for real values of s. Notably, identity (1.1) is due to Euler. B. Riemann, unlike Euler, discovered, that the function $\zeta(s)$ is an important analytic object for complex s. He proved the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\big((1-s)/2\big)\zeta(1-s) \tag{1.2}$$

for all $s\in\mathbb{C}$, where $\Gamma(s)$ is the Euler gamma-function, gave analytic continuation for $\zeta(s)$, proposed the application of $\zeta(s)$ for investigation of the function

$$\pi(x)=\sum_{p\leqslant x,\ p\in\mathbb{P}}1,\quad x\to\infty,$$

and stated important conjectures [25]. Riemann's method to estimate the function $\pi(x)$ is related to the zeros of the function $\zeta(s)$. From Equation (1.2), it follows that $\zeta(s)=0$ for $s=-2k, k\in\mathbb{N}$. These zeros of $\zeta(s)$ are called trivial. Let $N(T)$ denote the number of zeros of $\zeta(s)$ lying in the rectangle $\{s\in\mathbb{C}:0<\sigma<1,\ 0<t\leqslant T\}$. Riemann conjectured [25] that

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T).$$

The latter formula was proved by H. von Mangoldt [28]. Thus, $\zeta(s)$ has infinitely many zeros lying in the strip $\{s\in\mathbb{C}:0<\sigma<1\}$. These zeros of $\zeta(s)$ are called non-trivial. Riemann conjectured that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma=1/2$. This conjecture, called the Riemann hypothesis (RH), is one the most important problems of mathematics. RH was included in the list of Hilbert problems [14,15] (Problem 8), it is among the seven most important Millennium problems of mathematics [1]. One can easily obtain that $\zeta(s)\neq 0$ for $\sigma\geqslant 1$.

The Riemann method for the asymptotics of the function $\pi(x)$ is based on using the zero-free regions lying in the half-plane $\sigma\leqslant 1$. J. Hadamard [13] and C.-J. de la Valée Poussin [8,9,10] independently developed Riemann's ideas. They obtained that there is a constant $c>0$ such that $\zeta(s)\neq 0$ for

$$\sigma>1-\frac{c}{\log(|t||+2)}.$$

From this, they derived that

$$\pi(x)=\int_2^x\frac{\mathrm{d}u}{\log u}+O(xe^{-c_1\sqrt{\log x}}),\quad x\to\infty,$$

with a certain constant $c_1>0$. It is known that the RH implies the estimate

$$\pi(x)=\int_2^x\frac{\mathrm{d}u}{\log u}+O(\sqrt{x}\log x),\quad x\to\infty. \tag{1.3}$$

On the other hand, the latter estimate implies RH [27]. Thus, (1.3) is one of the oldest equivalents of RH. There are many other equivalents of RH stated in terms of various estimates, positivity of some functions, self-approximation, etc., see [5] and [6]. The present paper is focuses on approximation properties of the function $\zeta(s)$.

Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. By $\mathcal{K}$ denote the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the set of continuous non-vanishing on K functions that are analytic inside of K. Let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. It is well known that functions of the set $H_0(K)$, uniformly on K, are approximated by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. This property of $\zeta(s)$ is called universality, and is stated as follows:

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\Big\} > 0.$$

Moreover, "lim inf" can be replaced by "lim" for all but at most countably many $\varepsilon > 0$.

The first assertion above is the improved version of the Voronin universality theorem and can be found in [2, 11, 18, 19, 26], while the second assertion was proved in [23].

Based on universality of $\zeta(s)$, B. Bagchi proved [3] the following equivalent of RH: the RH is equivalent to the assertion that, for every $K \in \mathcal{K}$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon\right\} > 0.$$

In [20], the latter result was supplemented by the following statement: the RH is equivalent to the assertion that, for every $K \in \mathcal{K}$, the limit

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon\right\}$$

exists and is positive for all at most but at most countably many $\varepsilon > 0$.

The Bagchi criterion [3] inspired several works on the so-called strong recurrence of $\zeta(s)$. For example, in [24], it was obtained that for any real $d \neq 0$ and for any $\varepsilon > 0$ and $K \in \mathcal{K}$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \zeta(s+id\tau)| < \varepsilon\right\} > 0.$$

In [21], the latter criteria of RH were extended by using generalized shifts of $\zeta(s)$. Return to the functional equation (1.2). The product $g(s) = \pi^{-s/2}\Gamma(s/2)$ is an important ingredient of (1.2). Denote by $\theta(t)$ the increment of the argument of the function $g(s)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. It is known [16, 22] that the equation

$$\theta(t) = \pi(\tau - 1)$$

with $\tau \geqslant 0$ has the unique solution t_τ, and this solution is called the Gram function. This name comes back to J.-P. Gram who considered [12] t_τ with $\tau = k$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, in connection with imaginary parts of non-trivial zeros of $\zeta(s)$. The numbers τ_k, $k \in \mathbb{N}_0$ are called Gram points. In [21], the criteria of RH was given in terms of self approximation by shifts $\zeta(s + it_\tau)$.

Theorem 1. *[21]. The RH holds if and only if, for every $K \in \mathcal{K}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T]:\ \sup_{s\in K} |\zeta(s+it_\tau) - \zeta(s)| < \varepsilon\right\} > 0.$$

Theorem 2. *The RH holds if and only if, for every $K \in \mathcal{K}$ the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T]:\ \sup_{s\in K} |\zeta(s+it_\tau) - \zeta(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorems 1 and 2 are closely connected and inspired by universality theorem for $\zeta(s)$ in terms of shifts $\zeta(s+it_\tau)$ ([21], Theorem 3): suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(H)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T]:\ \sup_{s\in K} |\zeta(s+it_\tau) - f(s)| < \varepsilon\right\}.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T]:\ \sup_{s\in K} |\zeta(s+it_\tau) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The latter theorem is of continuous type: τ takes arbitrary values from the interval $[0,T]$. There exists its discrete analogue using the shifts $\zeta(s+iht_k)$, $h > 0$, with Gram points $t_k, k \in \mathbb{N}_0$. Denote by $\#A$ the cardinality of a set A, and let N run over the set $\mathbb{N}$. Then the following statement has been obtained ([17], Theorems 1.1 and 1.2).

Proposition 1. *Suppose that $h > 0$ is a fixed number, $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N:\ \sup_{s\in K} |\zeta(s+iht_k) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N:\ \sup_{s\in K} |\zeta(s+iht_k) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The aim of this paper is the discrete versions of Theorems 1 and 2.

Theorem 3. *The RH is true if and only if, for every $K \in \mathcal{K}$, $h > 0$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N:\ \sup_{s\in K} |\zeta(s+iht_k) - \zeta(s)| < \varepsilon\right\} > 0.$$

Theorem 4. *The RH is true if and only if, for every $K \in \mathcal{K}$, $h > 0$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + iht_k) - \zeta(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

We will prove Theorems 3 and 4 in Section 3. Section 2 is devoted to discrete limit theorems on weak convergence of probability measures in the space of analytic functions.

2 Weak convergence

Denote by $H(D)$ the space of analytic functions on the strip D endowed with the topology of uniform convergence on compacta. Recall a metric in $H(D)$ which induces its topology. There is a sequence $\{K_l : l \in \mathbb{N}\} \subset D$ of compact subsets such that [7]

$$D = \bigcup_{1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and every compact set $K \subset D$ lies in some set K_l. Then, for $g_1, g_2 \in H(D)$, taking

$$d(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|},$$

we have the desired metric.

Clearly, the sets K_l can be chosen with connected complements. For example, we can take closed rectangles.

Let $\mathbb{X}$ be an arbitrary topological space. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of $\mathbb{X}$, i.e., the σ-field generated by open sets of $\mathbb{X}$. We will consider probability measures defined on $(H(D), \mathcal{B}(H(D)))$. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Recall that P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{w} P$) if, for every real continuous bounded function f on $\mathbb{X}$, the relation

$$\lim_{n\to\infty} \int_{\mathbb{X}} f \mathrm{d}P_n = \int_{\mathbb{X}} f \mathrm{d}P$$

holds.

Several equivalents of weak convergence of probability measures are known. Denote by ∂A the boundary of a $A \subset \mathbb{X}$. Recall that the set A is said to be a continuity set of P if $P(\partial A) = 0$. For our aims, the following statement will be useful.

Lemma 1. *(A part of Theorem 2.1 of [4]). The statements*

(i)

$$P_n \xrightarrow[n\to\infty]{w} P;$$

(ii) For every open set $G \subset \mathbb{X}$

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

(iii) For every continuity set A of P

$$\lim_{n\to\infty} P_n(A) \geqslant P(A)$$

are equivalent.

In order to state a limit theorem for $\zeta(s)$ in the space $H(D)$, we need some notation. Define the Cartesian product

$$\Omega = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

With the product topology and pointwise multiplication, the set Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote elements of Ω by $\omega = (\omega(p) : p \in \mathbb{P})$ and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element

$$\zeta(s, \omega) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Notice that the latter infinite product, for almost all $\omega \in \Omega$, is uniformly convergent on compact sets of the strip D, see Theorem 5.1.7 of [19].

Denote by P_ζ the distribution of the random element $\zeta(s, \omega)$, i.e.,

$$P_\zeta(A) = m_H\{\omega \in \Omega : \zeta(s, \omega) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

and, on $(H(D), \mathcal{B}(H(D)))$, define the probability measure

$$P_{N,h}(A) = \frac{1}{N}\#\{1 \leqslant k \leqslant N : \zeta(s + iht_k) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then the following limit theorem is true.

Theorem 5. *([17], Theorem 3.1). Suppose that $h > 0$ is fixed. Then the relation $P_{N,h} \xrightarrow[N\to\infty]{w} P_\zeta$ holds.*

In [17], Theorem 5 is applied for discrete universality of the function $\zeta(s)$ by using the shifts $\zeta(s + iht_k)$. We recall the main points of this proof.

First the measure

$$Q_{N,h}(A) = \frac{1}{N}\#\{1 \leqslant k \leqslant N : (p^{-iht_k} : p \in \mathbb{P}) \in A\}, \quad A \in \mathcal{B}(\Omega),$$

is considered, and, using the Fourier transform method as well as the uniform distribution modulo 1 of the sequence $\{ht_k : k \in \mathbb{N}\}$, the limit relation

$$Q_{N,h} \xrightarrow[N\to\infty]{w} m_H \tag{2.1}$$

is obtained.

The next step deals with the absolutely convergent Dirichlet series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{\exp\{-(m/n)^{\theta}\}}{m^s}, \quad n \in \mathbb{N},$$

with a fixed $\theta > 1/2$. Let the map $u_n : \Omega \to H(D)$ be given by

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{\exp\{-(m/n)^{\theta}\}\omega(m)}{m^s}, \quad \omega(m) = \prod_{p^l \mid m,\ p^{l+1}\nmid m} \omega^l(p), \quad m \in \mathbb{N}.$$

Then (2.1) implies that, for

$$P_{N,h}(A) = \frac{1}{N}\#\{1 \leqslant k \leqslant N : \zeta_n(s + iht_k) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

the limit relation

$$P_{N,h} \xrightarrow[N\to\infty]{w} m_H u_n^{-1} \stackrel{\text{def}}{=} P_n \tag{2.2}$$

holds, where $m_H u_n^{-1}(A) = m_H(u_n^{-1}A)$ for $A \in \mathcal{B}(H(D))$.

An important part of the proof is the approximation of $\zeta(s)$ by $\zeta_n(s)$ in the mean, i.e., that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \sum_{k=1}^{N} d(\zeta(s + iht_k), \zeta_n(s + iht_k)) = 0. \tag{2.3}$$

Finally, it is proved that the probability measure P_n is tight, i.e., that for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset H(D)$ such that

$$P_n(K_\varepsilon) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This, the relations (2.2) and (2.3) together with Theorem 4.2 of [4] prove Theorem 5.

One more ingredient for the proofs of Theorems 3 and 4 is the support of the measure P_ζ. Recall that the support of P_ζ is a minimal closed $S_\zeta \subset H(D)$ such that $P_\zeta(S_\zeta) = 1$. The set S_ζ consists of all functions $g \in H(D)$ such that, for every open neighbourhood G of g, the inequality $P_\zeta(G) > 0$ is satisfied.

Set

$$S = \{g \in H(D) : g(s) \neq 0 \text{ on } D, \text{ or } g(s) \equiv 0\}.$$

Proposition 2. *([2, 19]). The support of the measure P_ζ is the set S.*

3 Proofs of Theorems 3 and 4

Proof. (Proof of Theorem 3). *Necessity.* Suppose that RH is true. Then $\zeta(s) \neq 0$ on the strip D. Hence, $\zeta(s) \in H_0(K)$ for all $K \in \mathcal{K}$. Therefore, by Proposition 1,

$$\liminf_{N\to\infty} \frac{1}{N} \#\{1 \leqslant k \leqslant N : \sup_{s\in\mathcal{K}} |\zeta(s+iht_k) - \zeta(s))| < \varepsilon\} > 0 \tag{3.1}$$

for all K, $h > 0$ and $\varepsilon > 0$.

Inequality (3.1) also follows from Theorem 5 and Proposition 2. Since $\zeta(s) \neq 0$ for $s \in D$, we have $\zeta(s) \in S$. Therefore, for every compact set K and $\varepsilon > 0$, the set

$$G_\varepsilon(K) = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - \zeta(s)| < \varepsilon \right\}$$

is an open neighborhood of the element $\zeta(s)$ of the support of the measure P_ζ in view of Proposition 2. Hence, by a property of the support,

$$P_\zeta(G_\varepsilon(K)) > 0.$$

Thus, Theorem 5 and (ii) of Lemma 1 yield

$$\liminf_{N\to\infty} P_{N,h}(G_\varepsilon(K)) \geqslant P_\zeta(G_\varepsilon(K)) > 0,$$

and the definitions of $P_{N,h}$ and $G_\varepsilon(K)$ prove inequality (3.1).

Sufficiency. We will show that inequality (3.1) implies the RH. Suppose, on the contrary, that the RH is not valid. Then $\zeta(s)$ has zeroes lying in D, thus, $\zeta(s) \notin S$, and, by Proposition 2, $\zeta(s)$ is not an element of the support of the measure P_ζ. Therefore, by a support property, there exists an open neighborhood $\mathcal{G}$ of $\zeta(s)$ such that

$$P_\zeta(\mathcal{G}) = 0. \tag{3.2}$$

There exists $\delta > 0$ such that

$$\mathcal{G}_\delta = \{g \in H(D) : d(g(s), \zeta(s)) < 2\delta\} \subset \mathcal{G}.$$

We will show that there exists a set $K \in \mathcal{K}$ and $\varepsilon > 0$ such that the set $G_\varepsilon(K)$ satisfies the inclusion $G_\varepsilon(K) \subset \mathcal{G}_\delta$. For this, we use the sets from the definition of the metric d. Let l_0 be such that

$$\sum_{l>l_0} 2^{-l} < \delta. \tag{3.3}$$

By a property of the sequence $\{K_l : l \in \mathbb{N}\}$, the inclusion $K_l \subset K_{l_0}$ is true for all $l = 1, \dots, l_0$. This remark and (3.3), for $g \in G_\varepsilon(K_{l_0})$, give

$$\begin{aligned} d(g(s), \zeta(s)) &= \left(\sum_{l=1}^{l_0} + \sum_{l>l_0} \right) 2^{-l} \frac{\sup_{s\in K_l} |g(s) - \zeta(s)|}{1 + \sup_{s\in K_l} |g(s) - \zeta(s)|} \\ &< \varepsilon \sum_{l=1}^{l_0} 2^{-l} + \sum_{l>l_0} 2^{-l} < 2\delta \end{aligned}$$

for $\varepsilon < \delta$. Hence, we have an inclusion $G_\varepsilon(K_{l_0}) \subset \mathcal{G}_\delta$ for all $0 < \varepsilon < \delta$. This implies, for $0 < \varepsilon < \delta$, the inclusion $G_\varepsilon(K_{l_0}) \subset \mathcal{G}$, thus, in view of (3.2),

$$P_\zeta(G_\varepsilon(K_{l_0})) = 0 \tag{3.4}$$

for $0 < \varepsilon < \delta$.

The boundary $\partial G_\varepsilon(K_{l_0})$ of the set $G_\varepsilon(K_{l_0})$ belongs to the set

$$\Big\{g \in H(D) : \sup_{s \in K_l} |g(s) - \zeta(s)| = \varepsilon\Big\}.$$

Therefore, the boundaries $\partial G_{\varepsilon_1}(K_{l_0})$ and $\partial G_{\varepsilon_2}(K_{l_0})$ do not intersect for $\varepsilon_1 \neq \varepsilon_2$. Hence, there are at most countably many values of ε such that

$$P_\zeta(\partial G_\varepsilon(K_{l_0})) > 0.$$

Actually, for $m \in \mathbb{N} \setminus \{1\}$, there are at most m sets $\partial G_{\varepsilon_m}(K_{l_0})$ such that

$$P_\zeta(\partial G_{\varepsilon_m}(K_{l_0})) > 1/m.$$

Thus, the set

$$A \stackrel{\text{def}}{=} \bigcup_{m=2}^{\infty} \left\{\varepsilon_m > 0 : P_\zeta(\partial G_{\varepsilon_m}(K_{l_0})) > \frac{1}{m}\right\}$$

is at most countable. Since

$$\{\varepsilon > 0 : P_\zeta(\partial G_\varepsilon(K_{l_0})) > 0\} = A,$$

we have the desired assertion. The latter remark implies that $P_\zeta(\partial G_\varepsilon(K_{l_0})) = 0$ for all but at most countably many $\varepsilon > 0$, in other words, the set $G_\varepsilon(K_{l_0})$ is the continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. Therefore, there exists $\varepsilon_1 \in (0, \delta)$ such that the set $G_{\varepsilon_1}(K_{l_0})$ is a continuity of the measure P_ζ, and, by (3.4), satisfies

$$P_\zeta(G_{\varepsilon_1}(K_{l_0})) = 0.$$

Hence, application of Theorem 5 and (iii) of Lemma 1 give

$$\lim_{N\to\infty} P_{N,h}(G_{\varepsilon_1}(K_{l_0})) = P_\zeta(G_{\varepsilon_1}(K_{l_0})) = 0.$$

Thus,

$$\liminf_{N\to\infty} \frac{1}{N} \#\Big\{1 \leqslant k \leqslant N : \sup_{s \in K_{l_0}} |\zeta(s + iht_k) - \zeta(s)| < \varepsilon_1\Big\} = 0,$$

and this contradicts inequality (3.1). This contradiction shows that RH follows. □

Proof. (Proof of Theorem 4). *Necessity.* Suppose that the RH is true. Then we have that $\zeta(s) \in H_0(K)$ with every $K \in \mathcal{K}$. Therefore, the second statement of Proposition 1 shows that, for every K, the limit

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+iht_k) - \zeta(s)| < \varepsilon\right\} \tag{3.5}$$

exists and is positive for all but at most countably many $\varepsilon > 0$. This also can be derived by using Theorem 5 and Proposition 2. Preserving the above notation, in view of Proposition 2, we have $P_\zeta(G_\varepsilon(K)) > 0$ for every compact set $K \subset D$. Moreover, as in the proof of Theorem 3, we obtain that the set $G_\varepsilon(K)$ is a continuity set for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 5 and (iii) of Lemma 1, the limit (3.5) exists and equals to

$$P_\zeta(G_\varepsilon(K)) > 0$$

for all but at most countably many $\varepsilon > 0$.

Sufficiency. Suppose that the limit (3.5) exists and is positive for all but at most countably many $\varepsilon > 0$. We will prove that the RH is valid. If, on the contrary, RH is not valid, then, as in the proof of Theorem 3, we obtain that there exists $\delta > 0$ and the set K_{l_0} such that equality (3.4) is true for all $0 < \varepsilon < \delta$. Moreover, the set $G_\varepsilon(K_{l_0})$ is a continuity set for all but at most countably many $\varepsilon > 0$. From these remarks, Theorem 5 and (iii) of Lemma 1, we found that

$$\lim_{N\to\infty} P_{N,h}(G_\varepsilon(K_{l_0})) = P_\zeta(G_\varepsilon(K_{l_0})) = 0$$

for all but at most countably many $0 < \varepsilon < \delta$. However, this contradicts the positivity of the limit (3.5) for all but at most countably many $\varepsilon > 0$. This contradiction proves the validity of the RH. The theorem is proved. □

Remark 1. Statements analogical to Theorems 3 and 4 for shifts $\zeta(s + i\varphi(k))$ with a function $\varphi(t)$ such that

$$\frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \zeta(s+i\varphi(k)) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

converge weakly to P_ζ as $N \to \infty$, are valid.

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