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Periodic travelling waves of a forced Cahn-Hilliard equation

Pedro J. Torres

Departamento de Matematica Aplicada, Universidad de Granada, 18071 Granada, Spain

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Abstract. We prove analytically the existence of a uniparametric family of periodic travelling waves for a Cahn-Hilliard equation with an external forcing term modelling the phase separation of a binary mixture of fluids with thermal diffusion. Some quantitative estimates on the solutions are derived.

Keywords: Cahn-Hilliard equation; phase separation; travelling wave; periodic solution; a priori estimates.

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✉ Corresponding author. E-mail: ptorres@ugr.es

1 Introduction and main result

The Cahn-Hilliard equation [1] is a widely accepted model for the dynamics of the variation of concentrations of the components of a binary mixture of fluids. The dynamics of phase separation can be altered by external forces, like an external heating of the fluid, known as the Ludwig-Sorret effect or thermal diffusion. Following [4, 6, 7], we consider a Cahn-Hillard equation with a travelling spatially periodic forcing term of the form

$$\frac{\partial C}{\partial t} = D\partial_{xx}(C^3 - C - \gamma\partial_{xx}C) + f_0 k\cos[k(x - vt)], \tag{1.1}$$

where all the parameters are positive and have different physical meaning, D is the mobility parameter, k is the wave number, $f_0 k$ is the amplitude, and v is the velocity of the external force, which models a travelling temperature modulation. The scalar field $C(x,t)$ measures phase separation, with $C(x,t) = 0$ indicating a locally well-mixed state (total homogenization), while $C(x,t) \neq 0$ means a higher concentration of one of the components relative to the other. Our purpose is to perform a complete analytical study of the existence of spatially-periodic travelling wave solutions of the form

$$C(x,t) = u(\eta), \qquad \eta = x - vt, \qquad u(\eta + L) = u(\eta), \tag{1.2}$$

where $L = 2\pi/k$ is the minimal period of the external forcing.

Introducing the ansatz (1.2) into (1.1), we obtain the fourth-order ordinary differential equation

$$-v\frac{du}{d\eta} = D\frac{d^2}{d\eta^2}\left(u^3 - u - \gamma\frac{d^2u}{d\eta^2}\right) + f_0 k\cos(k\eta). \tag{1.3}$$

Integrating in η and denoting the derivative in η by $'$, we arrive to the third-order equation

$$\gamma D u''' = D(3u^2 - 1)u' + vu + C_0 + f_0\sin(k\eta),$$

where C_0 is a constant of integration. If there exists an L-periodic solution u, an integration of this equation on a whole period $[0, L]$ shows that $C_0 = 0$ if $v = 0$. Thus, by convenience we can write $C_0 = -vc_0$ without loss of generality, where c_0 is a new arbitrary constant, and the former equation reads

$$\gamma D u''' = D(3u^2 - 1)u' + v(u - c_0) + f_0\sin(k\eta), \tag{1.4}$$

in perfect analogy with Equation (5) from [6]. If $v \neq 0$ and u is an L-periodic solution, integrating over $[0, L]$ gives

$$c_0 = \frac{1}{L}\int_0^L u(\eta)d\eta := \langle u\rangle.$$

Thus, fixing c_0, we are prescribing the mean value of the eventual periodic solutions of (1.3).

Our main result is some kind of universal existence result.

Theorem 1. *For any value of $c_0, v \in \mathbb{R}$ and any value of the positive parameters $\gamma, D, f_0, k > 0$, Equation (1.4) has at least one L-periodic soluton.*

The obvious consequence is that for any prescribed c_0, Equation (1.1) has a periodic travelling wave with mean value c_0, whatever is the value of the involved parameters. The proof is developed in Section 3. For $v \neq 0$, it relies on Schaefer's fixed point theorem, which is one of the most simple and elegant fixed point theorems for operators defined on a Banach space. For the case of a standing wave forcing, $v = 0$, it is possible to reduce the equation one order more and to use the method of upper and lower solutions, as it is done in Section 5. In the course of the proof, some a priori bounds on the solutions are derived, providing quantitative information about the location of the travellling waves. Due to its independent interest for the physical model, the computations are done separately in Section 2. The paper finishes with some further remarks in Section 5.

2 A priori estimates for the case $v \neq 0$

We are going to embed Equation (1.4) into a one-parametric family of equations given by

$$\gamma D u''' - vu' = \lambda\left[D(3u^2 - 1)u' - vc_0 + f_0\sin(k\eta)\right], \tag{2.1}$$

where $\lambda \in [0,1]$. The reason has to do with Schaefer's fixed point theorem and will be explained in the next section. It is evident that (1.4) corresponds to (2.1) for $\lambda = 1$.

The objective of this section is to find uniform bounds (not depending on λ) for any eventual L-periodic solution of (2.1) and its derivative. Denote by $\|.\|_2$ the usual L^2-norm. For brevity, let us denote the forcing term by $p(\eta) = f_0 \sin(k\eta)$. Given u an L-periodic solution, multiplying (2.1) by u and integrating on $[0,L]$, we get

$$v\|u\|_2^2 = \lambda v c_0^2 - \lambda \int_0^L p(\eta)u(\eta)d\eta.$$

Here, we have used that $c_0 = \langle u \rangle$. Now, taking absolute values and using Cauchy-Schwarz inequality on the integral,

$$|v|\|u\|_2^2 \leq \lambda |v|\, c_0^2 + \lambda \left| \int_0^L p(\eta)u(\eta)d\eta \right| \leq |v|\, c_0^2 + \|p\|_2 \|u\|_2 .$$

Such quadratic inequality is easily solvable, giving

$$\|u\|_2^2 \leq \frac{1}{2|v|} \left(\|p\|_2 + \sqrt{\|p\|_2^2 + 4c_0^2|v|^2} \right) =: C_1. \tag{2.2}$$

Now, multiplying (2.1) by u' and integrating by parts on the left-hand side, we obtain

$$\begin{aligned} \gamma D \|u''\|_2^2 &= \lambda D \int_0^L (1-3u^2)(u')^2 d\eta - \lambda \int_0^L p(\eta)u'(\eta)d\eta \\ &\leq D \|u'\|_2^2 + \|p\|_2 \|u'\|_2 , \end{aligned} \tag{2.3}$$

where again Cauchy-Schwartz inequality has been used. Observe that, by the periodicity of the solution,

$$\|u'\|_2^2 = \int_0^L u'^2 d\eta = -\int_0^L u(\eta)u''(\eta)d\eta \leq \|u\|_2 \|u''\|_2 .$$

Inserting this inequality into (2.3), we arrive to

$$\gamma D \|u''\|_2^2 \leq D\|u\|_2 \|u''\|_2 + \|p\|_2 \|u\|_2^{1/2} \|u''\|_2^{1/2} ,$$

and using (2.2), to

$$\gamma D \|u''\|_2^2 \leq D C_1 \|u''\|_2 + \|p\|_2\, C_1^{1/2} \|u''\|_2^{1/2} . \tag{2.4}$$

Since the left-hand side is quadratic in $\|u''\|_2$ and the right-hand side increases linearly with respect to $\|u''\|_2$, it is sure that there exists $C_2 > 0$ (not depending on λ) such that

$$\|u''\|_2 \leq C_2.$$

Such constant C_2 can be computed explicitly. To this aim, let us call $H = \|u''\|_2^{1/2}$ and look to (2.4) as the cubic inequality

$$\gamma D H^3 \leq D C_1 H + \|p\|_2 C_1^{1/2}.$$

Thus, we are compelled to study the cubic polynomial in H

$$H^3 - \frac{C_1}{\gamma} H - \frac{\|p\|_2 C_1^{1/2}}{\gamma D} = 0.$$

A basic analysis shows that there exists a unique positive root, and it is explicitly given by Cardano's formula

$$H = \sqrt[3]{\frac{\|p\|_2 C_1^{1/2}}{2\gamma D} - \sqrt{\frac{\|p\|_2^2 C_1}{4\gamma^2 D^2} - \frac{C_1^3}{27\gamma^3}}} + \sqrt[3]{\frac{\|p\|_2 C_1^{1/2}}{2\gamma D} + \sqrt{\frac{\|p\|_2^2 C_1}{4\gamma^2 D^2} - \frac{C_1^3}{27\gamma^3}}}. \quad (2.5)$$

Then,

$$C_2 = H^2.$$

We are now capable of determining a bound for the derivative of the solutions. Observe that u is a bounded and regular function, then it must have critical points. Take $\eta_0 \in [0, L]$ such that $u'(\eta_0) = 0$, then by Cauchy-Schwarzt inequality,

$$|u'(\eta)| \leq \left| \int_{\eta_0}^{\eta} u''(s) ds \right| \leq \sqrt{L} \|u''\|_2 \leq \sqrt{L} C_2,$$

for all $\eta \in [0, L]$.

Finally, considering that $c_0 = \langle u \rangle$, by the Mean Value Theorem for integrals, we can take $\tilde{\eta}_0 \in [0, L]$ such that $u(\tilde{\eta}_0) = c_0$. Then,

$$|u(\eta)| \leq \left| \int_{\eta_0}^{\eta} u'(s) ds + u(\tilde{\eta}_0) \right| \leq L\sqrt{L} C_2 + c_0,$$

for all $\eta \in [0, L]$.

To finish this section, let us note that the above computations are independent of the concrete shape of the forcing term $p(\nu)$, provided that it has a zero mean value. In the particular case of $p(\eta) = f_0 \sin(k\eta)$ considered in [4, 6, 7], an elemental computation provides

$$\|p\|_2 = f_0 \sqrt{\pi / k}.$$

3 Proof of the main result for the case $v \neq 0$

Denote by C_L^n the space of L-periodic functions with continuous derivatives up to order n. C_L^1 is a Banach space endowed with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$, where $\|.\|_\infty$ stands for the usual uniform norm.

The first step is to reformulate the problem as a fixed point problem for an adequate operator. Let us write (1.4) as

$$\mathcal{L}u = \mathcal{N}u, \tag{3.1}$$

where

$$\mathcal{L}u := u''' - \frac{v}{\gamma D}u, \quad \mathcal{N}u = \frac{1}{D}(3u^2 - 1)u' - \frac{vc_0}{\gamma D} + \frac{f_0}{\gamma D}\sin(k\eta).$$

If $v \neq 0$, $\mathcal{L} : C_L^3 \to C_L$ is an invertible operator by the classical Fredholm's alternative. In fact, $\mathcal{L}^{-1}$ can be written explicitly as a convolution operator with an associated Green's function. Then, Equation (3.1) is written as the fixed-point problem

$$u = \mathcal{A}u,$$

where $\mathcal{A} = \mathcal{L}^{-1}\mathcal{N} : C_L^1 \to C_L^1$. More explicitly, the operator $\mathcal{A}$ can be written as

$$\mathcal{A}u(\eta) = \int_0^L G(\eta, s)\left[\frac{1}{D}(3u(s)^2 - 1)u'(s) - \frac{vc_0}{\gamma D} + \frac{f_0}{\gamma D}\sin(ks)\right] ds,$$

where $G : [0, L] \times [0, L] \to \mathbb{R}$ is the Green's function associated to the operator $\mathcal{L}$ with periodic conditions. Then, the continuity of $\mathcal{A}$ follows from the Dominated Convergence Theorem, while the compactness is a consequence of Ascoli-Arzela Theorem.

To obtain a fixed point of the operator $\mathcal{A}$, we will use the following result.

Theorem 2 [Schaefer]. *Let X be a Banach space and $\mathcal{A} : X \to X$ be a continuous and compact mapping such that the set*

$$\{u \in X : u = \lambda\mathcal{A}u \text{ for some } 0 \le \lambda \le 1\} \tag{3.2}$$

is bounded. Then $\mathcal{A}$ has a fixed point.

This is a very well-known consequence of the Leray-Schauder degree and can be found in all the monographies dedicated to topological degree from the point of view of Nonlinear Analysis (see for example [5]). In our case $X = C_L^1$ and it is easy to check that $u = \lambda\mathcal{A}u$ is equivalent to Equation (2.1). Due to the estimates derived in Section 2, we know that the set (3.2) is bounded, so the result applies directly and the proof is done.

4 The case $v = 0$

The stationary case $v = 0$ presents some peculiarities that require separate treatment. Note that in this case, Equation (1.4) can be integrated again to get

$$\gamma D u'' = D(u^3 - u) - \frac{f_0}{k}\cos(k\eta) + h_0, \tag{4.1}$$

where h_0 is a constant of integration. Now, this constant is no longer the mean value of an eventual periodic solution, nevertheless

$$h_0 = \frac{D}{L}\int_0^L \left(u - u^3\right) d\eta.$$

It is clear that any possible L-periodic solution of (4.1) is also a solution of (1.3). Hence, h_0 will play a similar role as c_0 in identifying a uni-parametric family of solutions of (1.3).

Equation (4.1) is recognized as a periodically forced Duffing equation. Due to its interest as a model in different fields, there is a vast bibliography dedicated to its study (see [3] and the references therein). There are a wide variety of analytical and numerical methods available to study its dynamics, which in general can be quite complex, including the appearance of KAM scenarios. Here we perform an ad-hoc analysis based on the concept of upper and lower solutions (see [2, Section 1.1]).

Let us define α as the lowest real root of the cubic polynomial

$$P_-(u) := D(u^3 - u) + \frac{f_0}{k} + h_0 = 0,$$

and β as the highest real root of

$$P_+(u) = D(u^3 - u) - \frac{f_0}{k} + h_0 = 0.$$

Suppose that a local maximum of a solution of (4.1) is attained at η_*. Then,

$$0 \geq \gamma D u''(\eta_*) = D(u^3 - u) - \frac{f_0}{k}\cos(k\eta_*) + h_0 \geq P_+(u(\eta_*)),$$

so in consequence, $u(\eta_*) \leq \beta$. An analogous argument proves that a local minimum should be greater than α. Thus, any L-periodic solution holds the uniform bounds

$$\alpha \leq u(\eta) \leq \beta,$$

for all $\eta \in [0, L]$. As a matter of fact, α, β is a couple of ordered lower and upper solutions of (4.1), and a classical theorem (see for instance Theorem 1.1 in [2, Section 1.1]) provides the existence of at least one L-periodic solution in between. This closes the remaining gap in Theorem 1.

5 Final remarks

In conclusion, we have proved rigorously the existence of a uniparametric family of periodic travelling waves for the Cahn-Hilliard equation with an external forcing in the form of a travelling wave. The proof is independent of the specific form of the external forcing term, as is pointed out at the end of Section 2.

Moreover, the quantitative estimates derived in Sections 2 and 5 are easily interpretable from the point of view of the physical model of phase separation of binary mixtures, providing additional information. For example, when $c_0 = 0$, they can be used to understand possible strategies for homogenization. In view of (2.2), if $c_0 = 0$ we observe that

$$\lim_{v \to +\infty} C_1 = 0,$$

so from (2.5) also C_2 tends to 0 when $v \to +\infty$. This fact can be interpreted as that the external forcing acts as a mixer of the binary mixture, in such a way that increasing the travelling velocity enhances the homogenization. The same effect occurs when k or γ tends to infinity. In contrast with the analytical findings of [6], our results do not rely on perturbation theory.

An interesting open question concerns the uniqueness of the travelling wave once all the parameters are fixed, including the proper parameter of the found uniparametric family, which of course is not a proper parameter of the original Cahn-Hilliard equation. This deep question was proposed by one of the referees and requires further study.

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