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Analytic-Numerical Solution of Random Parabolic Models: A Mean Square Fourier Transform Approach

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Abstract. This paper deals with the construction of mean square analytic-numerical solution of parabolic partial differential problems where both initial condition and coefficients are stochastic processes. By using a random Fourier transform, an infinite integral form of the solution stochastic process is firstly obtained. Afterwards, explicit expressions for the expectation and standard deviation of the solution are obtained. Since these expressions depend upon random improper integrals, which are not computable in an exact manner, random Gauss-Hermite quadrature formulae are introduced throughout an illustrative example.

Keywords: mean square random calculus, random parabolic models, analytic-numerical solution, random mean square quadrature formulae, random Fourier transform.

AMS Subject Classification: 35R60, 60H15; 60H35; 68U20.

1 Introduction

Random partial differential initial value problems (IVP) was considered an emergent mathematical subject since the celebrated surveys [3] edited by Albert T. Bharucha-Reid. Diffusion models with uncertainties are frequent due to material impurities apart from the appearance of error measurements. The consideration of pollutants in presence of impurities is another situation where uncertainty is relevant in diffusion problems. In the evaluation of microwave heating processes, the time dependent model is more appropriate to avoid misleading results due to the complexity of the field distribution within the oven and the variation of dielectric properties of the material with temperature, moisture content, density and other parameters [15, 20, 27].

Random heat transfer models have been studied in [9] using a random perturbation method, in [24] using finite methods, and in [11] by applying finite difference methods. Random linear advection equation has been treated in [16] and the statistical moments of the solution of the random Burgers-Riemann problem and of the transport random differential equation are studied in [12] and [21, 22], respectively. Stochastic heat transfer problems modeled in a different way to the one considered here, in fact based upon Browniann motion and Itô calculus, may be found in [29]. As indicated in [2], numerous problems like continuum mechanics systems can be modeled by partial differential equations with random coefficients or random operators and stochastic initial and/or boundary conditions. One of the main difficulties in dealing with random partial differential equations is the fact the search for solutions and the analysis has to be carried out for every realization of the random parameters of the model equation. In this respect one usually faces arduous problems when trying to apply the usual and well known numerical techniques of the deterministic case. Consequently, it appears very interesting to look for approximated analytic solution. Also in [26], authors say that in many complex models understanding the behavior of the system requires obtaining many realizations of the state equations which necessitates performing simulations over a range of model parameter values. Because performing many simulations for complex partial differential equations (PDEs) is typically computationally expensive, methods have been developed to reduce the work. In [26], authors propose interesting stabilizations methods to overcome these drawbacks for the advection-diffusion-reaction equation. The aim of this paper is just to progress in this direction and we propose the construction of analytic-numerical solutions for random parabolic-type models. To achieve this goal, we use a mean square Fourier transform approach.

Random constant coefficient parabolic models have been recently treated using a mean square (m.s.) approach based on an integral transform technique in [7] using the random Laplace transform, and in [5, 6], using several random Fourier transforms (trigonometric and exponential). In all these cases, the constant coefficient model allows to obtain the exact m.s. solution of the random transformed differential problem as well as of the inverse integral transform captures the solution stochastic process (s.p.) of the original problem. For the random time dependent case, the capture of the solution s.p. of the original problem involves, throughout the inverse integral transform, unbounded random integrals that makes advisable the numerical evaluation of random complicated integrals. This is a major contribution introduced here, where random numerical quadrature formulae are applied to approximate the solution s.p. to random parabolic problems obtained after using the random Fourier transform.

In this paper we solve the time dependent random parabolic problem

$$\begin{aligned} u_t(x,t) &= a_2(t)\, u_{xx}(x,t) + a_1(t)\, u_x(x,t) + a_3(t)\, u(x,t)\,, \qquad (1.1)\\ &\qquad\qquad -\infty < x < +\infty\,,\ t > 0\,,\\ u(x,0) &= f(x)\,,\ \ -\infty < x < +\infty\,, \qquad (1.2)\end{aligned}$$

where $a_i(t) \equiv a_i(t;\omega) :]0,+\infty[\times\Omega \longrightarrow \mathbb{R}$, $1 \le i \le 3$ and $f(x) \equiv f(x;\omega) : \mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ are s.p.'s, defined in a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, that satisfy certain hypotheses that will be specified later. To achieve this goal, firstly we will establish some new results related to the so-called L^p-random calculus. Afterwards, we will extend some classical quadrature formulae to the random context in order to compute reliable approximations of the mean and the variance of the solution s.p., $u(x,t)$, also termed random field, to the IVP given in (1.1)–(1.2). All our theoretical findings will be illustrated by means of several examples. The model (1.1)–(1.2) for the deterministic case, using the Fourier transform was studied in [23].

The paper is organized as follows. Section 2 begins with some notational and adapted results that are introduced for the sake of clarity in the presentation. Some new auxiliary results that will be required throughout the paper are also established. In Section 2.1, the numerical method of random Gauss-Hermite for the evaluation of random improper integrals is introduced and it is applied to an example strategically placed that will be used later in Section 3, where problem (1.1)–(1.2) is firstly analytically solved using the random Fourier transform. Then, using the random Gauss-Hermite quadrature introduced in Section 2.1, the solution of problem (1.1)–(1.2) is numerically approximated. Numerical examples illustrating the theoretical results are included in Section 3.

2 Preliminaries

This section is addressed to introduce some preliminaries, definitions and results that will be required throughout this paper. Further details about these preliminaries can be checked [1, 28]. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space, a complex random variable (r.v.), $x : \Omega \longrightarrow \mathbb{C}$, is said to be of order $p \ge 1$, if $\mathbb{E}[|x|^p] < +\infty$, being $\mathbb{E}[\cdot]$ the expectation operator. It can be shown that the set of all r.v.'s of order p,

$$\mathrm{L}_p^{\mathrm{RV}}(\Omega) = \{x : \Omega \longrightarrow \mathbb{C} \,/\, \mathbb{E}[|x|^p] < +\infty\}, \quad 1 \le p < +\infty, \tag{2.1}$$

endowed with the norm

$$\|x\|_{p,\mathrm{RV}} = \left(\mathbb{E}[|x|^p]\right)^{1/p} < +\infty, \tag{2.2}$$

is a Banach space, [1, p.9]. The convergence inferred by the $\|\cdot\|_{p,\mathrm{RV}}$-norm is usually referred to as the p-th mean convergence. More precisely, a sequence of r.v.'s $\{x_n : n \ge 0\}$ in $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$ is p-th mean convergent to the r.v. $x \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$, and it is denoted as $x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{p,\mathrm{RV}}} x$, if and only if, $\|x_n - x\|_{p,\mathrm{RV}} = \left(\mathbb{E}[|x_n - x|^p]\right)^{1/p} \xrightarrow[n\to+\infty]{} 0$. The cases $p = 2$ and $p = 4$, corresponding to the so called mean square and mean fourth convergence, respectively, play a major role in the study of random differential equations [10, 30]. This key role will also be manifested throughout this paper as well.

Below, we state some inequalities for r.v.'s, belonging to the space $(\mathrm{L}_p^{\mathrm{RV}}(\Omega), \|\cdot\|_{p,\mathrm{RV}})$, that will be required subsequently. In accordance with the Liapunov's inequality

$$\left(\mathbb{E}[|x|^p]\right)^{1/p} \le \left(\mathbb{E}[|x|^q]\right)^{1/q}, \quad 1 \le p < q < +\infty, \tag{2.3}$$

one gets

$$\mathrm{L}_q^{\mathrm{RV}}(\Omega) \subset \mathrm{L}_p^{\mathrm{RV}}(\Omega), \quad 1 \le p < q < +\infty. \tag{2.4}$$

In dealing with random differential equations, a primary goal is try to formulate general results in the space $(\mathrm{L}_2^{\mathrm{RV}}(\Omega), \|\cdot\|_{2,\mathrm{RV}})$. Although the biggest space corresponding to $p = 1$ has its own mathematical interest, when dealing with random differential equations the reference space is $\mathrm{L}_2^{\mathrm{RV}}(\Omega)$. It is because in practice most of the r.v.'s have finite variance. However, the legitimation of some mean square operational rules often requires to assume hypotheses involving information related to $\mathrm{L}_4^{\mathrm{RV}}(\Omega)$. For example, it can be seen that [4]

$$\left. x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{4,\mathrm{RV}}} x, \quad y \in \mathrm{L}_4^{\mathrm{RV}}(\Omega), \right\} \Longrightarrow x_n y \xrightarrow[n\to+\infty]{\|\cdot\|_{2,\mathrm{RV}}} xy.$$

Nevertheless, in general, this property does not hold if either $x_n \xrightarrow[n\to+\infty]{\|\cdot\|_{2,\mathrm{RV}}} x$ or $y \in \mathrm{L}_2^{\mathrm{RV}}(\Omega)$. This matter is a consequence of the fact that the $\|\cdot\|_{p,\mathrm{RV}}$-norm is not submultiplicative, i.e., for $x, y \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$, $\|xy\|_{p,\mathrm{RV}} \not\le \|x\|_{p,\mathrm{RV}} \|y\|_{p,\mathrm{RV}}$, in general (see [30]). In the particular case that $x, y \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$ are independent r.v.'s, the above relationship is just an identity, i.e.,

$$\text{if } x, y \in \mathrm{L}_p^{\mathrm{RV}}(\Omega) \text{ are independent r.v.'s} \Rightarrow \|xy\|_{p,\mathrm{RV}} = \|x\|_{p,\mathrm{RV}} \|y\|_{p,\mathrm{RV}}\,. \tag{2.5}$$

This result is a consequence of the following Proposition 1 together with the definition of the $\|\cdot\|_{p,\mathrm{RV}}$-norm in terms of the operator expectation (see (2.2)).

Proposition 1. *[19, p.92] Let $f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be measurable transformations and $x_1, x_2 : \Omega \longrightarrow \mathbb{R}$ be independent r.v.'s. Then, $f_1(x_1)$ and $f_2(x_2)$ are independent r.v.'s and*

$$\mathbb{E}\left[f_1(x_1) f_2(x_2)\right] = \mathbb{E}\left[f_1(x_1)\right] \mathbb{E}\left[f_2(x_2)\right],$$

provided the above expectations exist.

A set $\{x(v) : v \in \mathcal{V} \subset \mathbb{R}\}$ of r.v.'s in $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$ indexed by the index v, is said to be a s.p. of order p. As usual, the definitions of continuity, differentiability and integrability of a s.p. of order p can be established in terms of the $\|\cdot\|_{p,\mathrm{RV}}$-norm. For instance, a s.p. of order p, $x(v)$, is said to be continuous at $v \in \mathcal{V}$ ($x(v)$ is $\|\cdot\|_{p,\mathrm{RV}}$-continuous at $v \in \mathcal{V}$, for short) if

$$\|x(v+h) - x(v)\|_{p,\mathrm{RV}} \xrightarrow[h\to 0]{} 0, \quad v, v+h \in \mathcal{V}.$$

As a direct consequence of the Liapunov's inequality (2.3), one deduces that if $x(v)$ is $\|\cdot\|_{q,\mathrm{RV}}$-continuous (differentiable or integrable) s.p., then $x(v)$ is $\|\cdot\|_{p,\mathrm{RV}}$-continuous (differentiable or integrable) s.p., being $q \ge p \ge 1$. However, the reverse is not true, in general.

Additionally to the definition of $\|\cdot\|_{p,\mathrm{RV}}$-integrability of a s.p. $x(v)$ defined in the space $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$, we will use the concept of $\|\cdot\|_{p,\mathrm{RV}}$-absolutely integrable of a s.p. Namely, a s.p. $x(v) \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$ is said to be $\|\cdot\|_{p,\mathrm{RV}}$-absolutely integrable s.p. if the following deterministic integral

$$\int_{-\infty}^{+\infty} \|x(v)\|_{p,\mathrm{RV}} \, \mathrm{d}v \tag{2.6}$$

exists and is finite. If $x(v) \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$ is $\|\cdot\|_{p,\mathrm{RV}}$-absolutely integrable s.p., then its random exponential $\|\cdot\|_{p,\mathrm{RV}}$-Fourier transform is defined by

$$X(\xi) := \mathfrak{F}[x(v)](\xi) = \int_{-\infty}^{+\infty} x(v) \exp(-\mathrm{i}\xi v) \, \mathrm{d}v, \quad \xi \in \mathbb{R}, \quad \mathrm{i} = +\sqrt{-1},$$

where this random integral defines a s.p. $\{X(\xi) : \xi \in \mathbb{R}\}$ in the Banach space $(\mathrm{L}_p^{\mathrm{RV}}(\Omega), \|\cdot\|_{p,\mathrm{RV}})$. If $x(v)$ is $\|\cdot\|_{p,\mathrm{RV}}$-absolutely integrable s.p., it is clear that it admits a random $\|\cdot\|_{p,\mathrm{RV}}$-Fourier transform since

$$\begin{aligned} \int_{-\infty}^{+\infty} \|x(v) \exp(-\mathrm{i}\xi v)\|_{p,\mathrm{RV}} \, \mathrm{d}v &\leq \int_{-\infty}^{+\infty} \|x(v)\|_{p,\mathrm{RV}} \, |\exp(-\mathrm{i}\xi v)| \, \mathrm{d}v \\ &= \int_{-\infty}^{+\infty} \|x(v)\|_{p,\mathrm{RV}} \, \mathrm{d}v < +\infty, \end{aligned}$$

where we have used that, $|\exp(\mathrm{i}\xi v)| = 1$ and that $x(v)$ is $\|\cdot\|_{p,\mathrm{RV}}$-absolutely integrable s.p., hence by (2.6) the last integral is finite. In [5, p.5926], it is proved the extension of the following well-known properties of the Fourier transform

$$\mathfrak{F}[x'(v)](\xi) = \mathrm{i}\xi \mathfrak{F}[x(v)](\xi), \quad \mathfrak{F}[x''(v)](\xi) = -\xi^2 \mathfrak{F}[x(v)](\xi), \tag{2.7}$$

to the random framework provided that the involved random $\|\cdot\|_{p,\mathrm{RV}}$-derivatives exist and $x(v)$, $x'(v)$ and $x''(v)$ are $\|\cdot\|_{p,\mathrm{RV}}$-absolutely integrable s.p.'s. These properties will be used later.

In order to formalize our study, besides the above Banach space of complex r.v.'s having absolute moments of order p, $(\mathrm{L}_p^{\mathrm{RV}}(\Omega), \|\cdot\|_{p,\mathrm{RV}})$, we will also need the following Banach space, $(\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R} \times \Omega), \|\cdot\|_{p,\mathrm{SP}})$ where

$$\begin{aligned} \mathrm{L}_p^{\mathrm{SP}}(\mathbb{R} \times \Omega) &= \left\{ f : \mathbb{R} \times \Omega \to \mathbb{C} \,/ \int_{-\infty}^{+\infty} \left(\mathbb{E}\left[|f(v)|^p\right]\right)^{1/p} \mathrm{d}v < +\infty \right\} \\ &= \left\{ f : \mathbb{R} \times \Omega \to \mathbb{C} \,/ \int_{-\infty}^{+\infty} \|f(v)\|_{p,\mathrm{RV}} \, \mathrm{d}v < +\infty \right\}, \end{aligned} \tag{2.8}$$

and

$$\|f\|_{p,\mathrm{SP}} = +\left(\int_{-\infty}^{+\infty} \|f(v)\|_{p,\mathrm{RV}} \, \mathrm{d}v \right)^{1/p}, \quad 1 \leq p < +\infty.$$

Notice that the elements of $\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R} \times \Omega)$ are $\|\cdot\|_{p,\mathrm{RV}}$-absolutely integrable s.p.'s (see (2.6)). Observe that if $f \in \mathrm{L}_p^{\mathrm{SP}}(\mathbb{R} \times \Omega)$, then the expectation $\mathbb{E}\left[|f(v)|^p\right]$ exists and is finite for every $v \in \mathbb{R}$ fixed (otherwise would not make sense the definition of the space $\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R}\times\Omega)$ given in (2.8)). Hence, for every $v\in\mathbb{R}$ fixed, $f(v)$ is a r.v. of the space $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$.

As usual, in dealing with s.p.'s for convenience sometimes the sample parameter $\omega\in\Omega$ will be hidden depending on the context. Hence, an element f of $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$ or $\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R}\times\Omega)$ will be denoted by $f(v)$ or $f(v;\omega)$ interchangeably throughout this paper.

Now, we recall an important class of r.v.'s that will be considered later. This class has been used in previous works where random differential equations are studied [5, 7].

Definition 1. A real r.v. $a:\Omega\longrightarrow\mathbb{R}$ defined on a probability space $(\Omega,\mathfrak{F},\mathbb{P})$ is said to be of class $\mathcal{C}$ if

$$\exists\, M,H>0:\quad \mathbb{E}\left[\,|a|^m\right]\le MH^m<+\infty,\quad \forall m\ge 0. \tag{2.9}$$

Remark 1. Condition (2.9) can be written in terms of the Landau's symbol as

$$\mathbb{E}\left[|a|^m\right]=\mathcal{O}((H)^m)\,.$$

As it has been demonstrated in [5], an important class of r.v.'s that belong to the class $\mathcal{C}$ are bounded r.v.'s. Thus, binomial, uniform, beta, λ-distributed r.v.'s, etc. satisfy condition (2.9). Unbounded r.v.'s can be approximated using the truncation method [25, ch.V] instead of checking the condition (2.9). This is particularly convenient because there exist families of r.v.'s for which a closed expression for their absolute statistical moments is not available.

Below, we establish an auxiliary result that will be used later. This result involves a class of s.p.'s that satisfy a natural generalization of condition (2.9).

Lemma 1. *Let $h(\xi)$ be a complex deterministic function and let $\hat{a}(t)$ be a real s.p. such that*

$$\exists\, M_{\hat{a}}>0,\ H_{t,\hat{a}}>0:\quad \mathbb{E}\left[|\hat{a}(t)|^m\right]\le M_{\hat{a}}\left(H_{t,\hat{a}}\right)^m<+\infty\,,\quad \forall m\ge 0\,, \tag{2.10}$$

for every $t>0$ fixed. Then,

$$\|\exp(h(\xi)\,\hat{a}(t))\|_{2,RV}\le\sqrt{M_{\hat{a}}}\,\exp(\mathrm{Re}\,(h(\xi))\,H_{t,\hat{a}}), \tag{2.11}$$

where $\mathrm{Re}\,(\cdot)$ *denotes the real part of a complex number.*

Proof. On the one hand, it is important to point out that following an analogous reasoning to the one exhibited in Section 3 of [8] and under condition (2.10), the exponential s.p. $\exp(h(\xi)\,\hat{a}(t))$ is well-defined for every $t>0$ and $h(\xi)$ given. On the other hand, using the definition of the p-norm for $p=2$ (see (2.2)), one gets

$$\begin{aligned}\left(\|\exp(h(\xi)\,\hat{a}(t))\|_{2,\mathrm{RV}}\right)^2&=\mathbb{E}\left[|\exp(h(\xi)\,\hat{a}(t))|^2\right]=\mathbb{E}\left[\exp(2\,\mathrm{Re}\,(h(\xi))\,\hat{a}(t))\right]\\ &=\mathbb{E}\left[\sum_{m\ge 0}\frac{(2\,\mathrm{Re}\,(h(\xi))\,\hat{a}(t))^m}{m!}\right]=\sum_{m\ge 0}\frac{(2\,\mathrm{Re}\,(h(\xi)))^m\,\mathbb{E}\left[(\hat{a}(t))^m\right]}{m!}\\ &\le M_{\hat{a}}\sum_{m\ge 0}\frac{(2\,\mathrm{Re}\,(h(\xi))\,H_{t,\hat{a}})^m}{m!}=M_{\hat{a}}\,\exp(2\,\mathrm{Re}\,(h(\xi))\,H_{t,\hat{a}})\,,\end{aligned}$$

where we have used that $|\exp(z)| = \exp(\mathrm{Re}(z))$, for every complex number z. This proves the result. $\square$

Now, we apply the previous Lemma 1 to a particular case that will required later in the Example 2.

Remark 2. Let $\hat{a}(t) = \{a\, t : \ t > 0\}$ be a s.p. such that a is a r.v. of class $\mathcal{C}$, i.e., satisfying condition (2.9).

Let $\xi > 0$ and let us observe that applying Lemma 1 to $h(\xi) = -\xi^2$, one gets

$$\left\|\exp\left(-\xi^2\, t\, a\right)\right\|_{2,\mathrm{RV}} \leq \sqrt{M} \exp\left(-\xi^2\, H\, t\right). \tag{2.12}$$

Observe that the constant $H_{t,\hat{a}}$ that appears in (2.11), now is just $H\, t$, for every $t > 0$ fixed.

2.1 Approximation of random improper integrals by the random Gauss-Hermite quadrature

We begin this section by extending to the random framework the practical Gauss-Hermite quadrature formulae for the evaluation of improper random integrals that appear in a natural way when using random integral transform methods.

For $f \in \mathrm{L}_2^{\mathrm{SP}}(\mathbb{R} \times \Omega)$, let us consider the following integral

$$I = I[f] = \int_{-\infty}^{+\infty} f(\xi) \exp(-\xi^2)\, \mathrm{d}\xi, \tag{2.13}$$

which is a r.v. Since $0 < \exp(-\xi^2) \leq 1$ for all $\xi \in \mathbb{R}$ and $f \in \mathrm{L}_2^{\mathrm{SP}}(\mathbb{R} \times \Omega)$ (see (2.8) with $p = 2$), one gets

$$\begin{aligned} \|I\|_{2,\mathrm{RV}} &= \left\|\int_{-\infty}^{+\infty} f(\xi)\, \exp(-\xi^2)\, \mathrm{d}\xi\right\|_{2,\mathrm{RV}} \leq \int_{-\infty}^{+\infty} \|f(\xi)\, \exp(-\xi^2)\|_{2,\mathrm{RV}}\, \mathrm{d}\xi \\ &\leq \int_{-\infty}^{+\infty} \|f(\xi)\|_{2,\mathrm{RV}}\, \mathrm{d}\xi < +\infty. \end{aligned}$$

Then, $I[f]$ is well-defined. If we further assume that $f \in \mathrm{L}_2^{\mathrm{SP}}(\mathbb{R} \times \Omega)$ has continuous sample trajectories, i.e. $f(x)(\omega)$ is continuous with respect to $x \in \mathbb{R}$ for all $\omega \in \Omega$, then the r.v. (2.13) coincides, with probability 1, with the (deterministic) sample integrals

$$I(\omega) = I[f](\omega) = \int_{-\infty}^{+\infty} f(\xi; \omega) \exp(-\xi^2)\, \mathrm{d}\xi, \quad \omega \in \Omega,$$

which are well-defined and thus they are convergent for all $\omega \in \Omega$ [28, Appendix I]. Then, taking advantage of the Gauss-Hermite quadrature formula of degree N, [13, 14], we can consider the following numerical approximation

$$I_N^{\text{G-H}}[f](\omega) \approx \sum_{j=1}^{N} \rho_j\, f(\xi_{j,\mathrm{H}}; \omega), \quad \rho_j = \frac{2^{N+1} N! \sqrt{\pi}}{\left(H_N'(\xi_{j,\mathrm{H}})\right)^2}, \quad 1 \leq j \leq N,\ \omega \in \Omega, \tag{2.14}$$

where $\xi_{j,\mathrm{H}}$ are the roots of the deterministic Hermite polynomial, H_N, of degree N.

Example 1. Let us consider the following random integral whose interest will be apparent later.

$$I(x,t) = \int_{-\infty}^{+\infty} \exp\left(-\xi^2\left(ta_2 + \frac{1}{4}\right)\right) \cos(\xi\,(ta_1 + x))\,\mathrm{d}\xi\,, \tag{2.15}$$

for fixed $(x,t) \in \mathbb{R} \times (0,+\infty)$ and given real r.v.'s $a_1, a_2 : \Omega \longrightarrow \mathbb{R}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying certain properties to be specified later.

Note that the integrand of (2.15) can be transformed into the form of the integrand (2.13), multiplying them by $\exp(\xi^2)$ to obtain the s.p. $f(\xi)(x,t)$, that is,

$$\exp\left(-\xi^2\left(ta_2 + \frac{1}{4}\right)\right) \cos(\xi\,(ta_1 + x))\,\exp(\xi^2) = f(\xi)(x,t)\exp(-\xi^2)\exp(\xi^2).$$

Then

$$\begin{aligned} I(x,t) = I(x,t)[f] &= \int_{-\infty}^{+\infty} f(\xi)(x,t)\,\exp\left(-\xi^2\right)\mathrm{d}\xi\,, \qquad (2.16)\\ f(\xi)(x,t) &= \exp\Big(-\xi^2\Big(ta_2 - \frac{3}{4}\Big)\Big)\cos\big(\xi\,(ta_1 + x)\big). \end{aligned}$$

Assuming that a_1 and a_2 are so that the s.p. $f(x,t) \in \mathrm{L}_2^{\mathrm{SP}}(\mathbb{R} \times \Omega)$ and its sample trajectories $f(x,t;\omega)$ are continuous, then according to (2.14) we can consider the following numerical approximations of (2.16)

$$\left.\begin{aligned} I_N^{\text{G-H}}(x,t)[f](\omega) &\approx \textstyle\sum_{j=1}^N \rho_j\,\exp\left(-\xi_j^2\left(ta_2 - \frac{3}{4}\right)\right)\,\cos(\xi_j\,(ta_1 + x))\,,\\ \rho_j &= \frac{2^{N+1}N!\sqrt{\pi}}{\left(H_N'(\xi_j)\right)^2}\,, \quad 1 \le j \le N\,, \qquad \omega \in \Omega. \end{aligned}\right\} \tag{2.17}$$

3 Solving random parabolic problems

In this section we consider the initial value problem (1.1)–(1.2) where the time s.p.'s, $a_i(t)$, $i = 1, 2, 3$, and the spatial s.p., $f(x)$, are assumed to satisfy the following conditions

$$\left.\begin{aligned} a_i(t), f(x) \;\text{ are independent r.v.'s},\quad \forall i : 1 \le i \le 3\,,\\ \forall (x,t)\,, -\infty < x < +\infty\,,\quad t > 0,\;\text{ both fixed,} \end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned} &f(x)\;\text{ is a } \|\cdot\|_{4,\mathrm{RV}} - \text{absolutely integrable s.p. such that its}\\ &\text{random Fourier transform } F(\xi) \in \mathrm{L}_4^{\mathrm{SP}}(\mathbb{R} \times \Omega)\,, \end{aligned}\right\} \tag{3.2}$$

$$a_i(t) \text{ are } \|\cdot\|_{4,\mathrm{RV}} - \text{continuous s.p.'s},\quad \forall i : 1 \le i \le 3 \tag{3.3}$$

and

$$\hat{a}_i(t) = \int_0^t a_i(s)\,\mathrm{d}s, \quad 1 \le i \le 3, \tag{3.4}$$

satisfy condition (2.10), i.e.,

$$\exists\, M_{\hat{a}_i} > 0,\ H_{t,\hat{a}_i} > 0 :\ \mathbb{E}\left[|\hat{a}_i(t)|^m\right] \le M_{\hat{a}_i}\left(H_{t,\hat{a}_i}\right)^m < +\infty\,, \qquad \forall m \ge 0\,,\ \forall i:\ 1 \le i \le 3. \tag{3.5}$$

Notice that $\{a_i(t) : t \ge 0\}$, $1 \le i \le 3$ are $\|\cdot\|_{4,\mathrm{RV}}$-continuous s.p.'s (see hypothesis (3.3)), then by Lemma 3.16 of [30] the integral s.p.'s $\hat{a}_i(t)$ given in (3.4) are well-defined in $(\mathrm{L}_4^{\mathrm{RV}}(\Omega), \|\cdot\|_{4,\mathrm{RV}})$ (and hence also in $(\mathrm{L}_2^{\mathrm{RV}}(\Omega), \|\cdot\|_{2,RV})$, see (2.4)).

In the following, we will apply the random Fourier transform approach, introduced in [5], by assuming for the time being, that problem (1.1)–(1.2) admits a solution s.p. $u(x,t)$ such that itself and its two first derivatives with respect to x, $u_x(x,t)$ and $u_{xx}(x,t)$, regarded as s.p.'s of the spatial variable x, all are $\|\cdot\|_{2,\mathrm{RV}}$-random Fourier transformable. Let $\mathfrak{F}[u(\cdot,t)](\xi) = U(t)(\xi)$ be the random Fourier transform of the solution s.p. $u(x,t)$ considering x as the active variable and t fixed. Applying the random Fourier transform to both sides of equation (1.1) and to the initial condition (1.2), and using its linearity, one gets

$$\mathfrak{F}[u_t(\cdot,t)](\xi) = a_2(t)\mathfrak{F}[u_{xx}(\cdot,t)](\xi) + a_1(t)\mathfrak{F}[u_x(\cdot,t)](\xi) + a_3(t)\mathfrak{F}[u(\cdot,t)](\xi), \tag{3.6}$$

$$\mathfrak{F}[u(\cdot,0)](\xi) = \mathfrak{F}[f(x)](\xi) = F(\xi)\,. \tag{3.7}$$

By the properties of the random Fourier transform of a s.p. stated in (2.7), one gets

$$\mathfrak{F}[u_{xx}(\cdot,t)](\xi) = -\xi^2\mathfrak{F}[u(\cdot,t)](\xi) = -\xi^2 U(t)(\xi)\,,$$

$$\mathfrak{F}[u_x(\cdot,t)](\xi) = \mathrm{i}\,\xi\,\mathfrak{F}[u(\cdot,t)](\xi) = \mathrm{i}\xi U(t)(\xi)\,.$$

Assuming that the solution s.p. $u(x,t)$ is such that $u_t(\cdot,t)$ is Fourier transformable and that hypotheses of Lemma 2 of [5] hold, then one gets

$$\mathfrak{F}\left[u_t(\cdot,t)\right](\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{F}\left[u(\cdot,t)\right]\right)(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}(U(t))(\xi)\,. \tag{3.8}$$

Therefore, from (3.6)–(3.8) one deduces that, for each $\xi \in \mathbb{R}$ fixed, $U(t)(\xi)$ satisfies the random IVP

$$\left.\begin{array}{lcl} \dfrac{\mathrm{d}}{\mathrm{d}t}(U(t))(\xi) & = & \left(-\xi^2 a_2(t) + \mathrm{i}\xi a_1(t) + a_3(t)\right) U(t)(\xi), \quad t > 0, \\ U(0)(\xi) & = & F(\xi). \end{array}\right\} \tag{3.9}$$

On the one hand, let us denote by

$$a(t) := a(t,\xi) = -\xi^2\, a_2(t) + \mathrm{i}\xi\, a_1(t) + a_3(t)\,, \quad \xi \in \mathbb{R} \text{ fixed}, \tag{3.10}$$

and assume that

$$\exists\,\delta>0\,,\ r>2p:\sup_{s,s^*\in[-\delta,\delta]}\mathbb{E}\Big[\exp\Big(r\int_{x+s}^{t+s^*}a(v)\,\mathrm{d}v\Big)\Big]<+\infty. \tag{3.11}$$

On the other hand, observe that by hypotheses (3.2) and (3.3), $a_i(t)$ and $f(x)$ are in $\mathrm{L}_4^{\mathrm{RV}}(\Omega)$ for each $t>0$ and $x\in\mathbb{R}$, respectively, then it is guaranteed that $a(t)$, defined by (3.10), and $F(\xi)$, defined by (3.7), also belong to $\mathrm{L}_4^{\mathrm{RV}}(\Omega)$ for $\xi\in\mathbb{R}$ fixed. Moreover, due to the hypothesis of independence among $a_i(t)$ and $f(x)$ assumed in (3.1), $a(t)$ and $F(\xi)$ are also independent. Finally, by hypothesis (3.3), it is clear that the s.p. $a(t)$ is $\|\cdot\|_{4,\mathrm{RV}}$-continuous. Then, Theorem 8 of [10] allows us to guarantee that the mean square solution s.p. of the IVP (3.9) is given by

$$U(t)(\xi)=\exp\left(\int_0^t a(s,\xi)\,\mathrm{d}s\right)\,F(\xi)\,,\quad t>0\,,\quad \text{being } \xi\in\mathbb{R} \text{ fixed}\,.$$

Now using formally the random inverse Fourier transform, the candidate solution s.p. of problem (1.1)–(1.2) is given by

$$\begin{aligned} u(x,t) &= \mathfrak{F}^{-1}\left[U(t)(\xi)\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}U(t)(\xi)\,\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(\mathrm{i}\xi x+\int_0^t a(s,\xi)\,\mathrm{d}s\right)\,F(\xi)\,\mathrm{d}\xi\,. \end{aligned} \tag{3.12}$$

For every $(x,t)\in\mathbb{R}\times[0,+\infty[$ fixed, it remains to justify the latter random integral given in (3.12) is convergent in the space $(\mathrm{L}_2^{\mathrm{SP}}(\mathbb{R}\times\Omega),\|\cdot\|_{2,\mathrm{SP}})$ defined in (2.8) with $p=2$. As a consequence, the s.p. $u(x,t)$ given in (3.12) is well-defined in the mean square sense, that is in the Banach space $(\mathrm{L}_2^{\mathrm{RV}}(\Omega),\|\cdot\|_{2,\mathrm{RV}})$ defined in (2.1). With this goal, let us observe that

$$\begin{aligned} &\int_{-\infty}^{+\infty}\Big\|\exp\Big(\mathrm{i}\xi x+\int_0^t a(s,\xi)\mathrm{d}s\Big)F(\xi)\Big\|_{2,\mathrm{RV}}\mathrm{d}\xi \\ &\qquad=\int_{-\infty}^{+\infty}|\exp\left(\mathrm{i}\xi x\right)|\,\Big\|\exp\Big(\int_0^t a(s,\xi)\mathrm{d}s\Big)\Big\|_{2,\mathrm{RV}}\|F(\xi)\|_{2,\mathrm{RV}}\;\mathrm{d}\xi \\ &\qquad=\int_{-\infty}^{+\infty}\Big\|\exp\Big(\int_0^t a(s,\xi)\,\mathrm{d}s\Big)\Big\|_{2,\mathrm{RV}}\|F(\xi)\|_{2,\mathrm{RV}}\;\mathrm{d}\xi, \end{aligned} \tag{3.13}$$

where in the last step we have applied the relationship (2.5), since by hypothesis (3.1), $\int_0^t a(s,\xi)\,\mathrm{d}s$ and $F(\xi)$ are independent r.v.'s for every $t>0$ and $\xi\in\mathbb{R}$. Moreover, according to hypotheses (3.1), (3.4)–(3.5), (3.10) and Lemma 1, one gets

$$\Big\|\exp\Big(\int_0^t a(s,\xi)\,\mathrm{d}s\Big)\Big\|_{2,\mathrm{RV}}=\Big\|\exp\Big(\int_0^t -\xi^2\,a_2(s)+\mathrm{i}\xi a_1(s)+a_3(s)\,\mathrm{d}s\Big)\Big\|_{2,\mathrm{RV}}$$

$$
\begin{aligned}
&= \left\| \exp\Big(\int_0^t -\xi^2\, a_2(s)\,\mathrm{d}s \Big) \exp\Big(\int_0^t \mathrm{i}\xi\, a_1(s)\,\mathrm{d}s \Big) \exp\Big(\int_0^t a_3(s)\,\mathrm{d}s \Big) \right\|_{2,\mathrm{RV}} \\
&= \left\| \exp\Big(-\xi^2 \int_0^t a_2(s)\mathrm{d}s \Big) \right\|_{2,\mathrm{RV}} \left\| \exp\Big(\mathrm{i}\xi \int_0^t a_1(s)\,\mathrm{d}s \Big) \right\|_{2,\mathrm{RV}} \\
&\qquad \times \left\| \exp\Big(\int_0^t a_3(s)\,\mathrm{d}s \Big) \right\|_{2,\mathrm{RV}} \\
&= \left\| \exp\left(-\xi^2\, \hat{a}_2(t) \right) \right\|_{2,\mathrm{RV}} \left\| \exp\left(\mathrm{i}\xi\, \hat{a}_1(t) \right) \right\|_{2,\mathrm{RV}} \left\| \exp\left(\hat{a}_3(t) \right) \right\|_{2,\mathrm{RV}} \\
&\le \Big(\sqrt{M_{\hat{a}_1}} \exp(\underbrace{\mathrm{Re}(\mathrm{i}\xi)}_{0}\, H_{t,\hat{a}_1}) \Big) \Big(\sqrt{M_{\hat{a}_2}} \exp(\underbrace{\mathrm{Re}(-\xi^2)}_{-\xi^2}\, H_{t,\hat{a}_2}) \Big) \sqrt{M_{\hat{a}_3}} \\
&\quad \times \exp\big(\mathrm{Re}(1) H_{t,\hat{a}_3} \big) \le \sqrt{M_{\hat{a}_1} M_{\hat{a}_2} M_{\hat{a}_3}} \exp\big(-(\xi^2 H_{t,\hat{a}_2} - H_{t,\hat{a}_3}) \big).
\end{aligned} \tag{3.14}
$$

Taking into account the bound (3.14) in (3.13), one gets

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \left\| \exp\Big(\mathrm{i}\xi x + \int_0^t a(s,\xi)\,\mathrm{d}s \Big)\, F(\xi) \right\|_{2,\mathrm{RV}} \mathrm{d}\xi \\
&\quad \le \sqrt{M_{\hat{a}_1}\, M_{\hat{a}_2}\, M_{\hat{a}_3}} \int_{-\infty}^{+\infty} \exp(-(\xi^2 H_{t,\hat{a}_2} - H_{t,\hat{a}_3}))\, \|F(\xi)\|_{2,\mathrm{RV}}\, \mathrm{d}\xi \\
&\quad = \sqrt{M_{\hat{a}_1}\, M_{\hat{a}_2}\, M_{\hat{a}_3}} \exp(H_{t,\hat{a}_3}) \int_{-\infty}^{+\infty} \exp(-\xi^2 H_{t,\hat{a}_2})\, \|F(\xi)\|_{2,\mathrm{RV}}\, \mathrm{d}\xi \\
&\qquad\qquad \le \sqrt{M_{\hat{a}_1}\, M_{\hat{a}_2}\, M_{\hat{a}_3}} \exp(H_{t,\hat{a}_3}) \int_{-\infty}^{+\infty} \|F(\xi)\|_{2,\mathrm{RV}}\, \mathrm{d}\xi < +\infty,
\end{aligned}
$$

where the finiteness of the last integral follows because by hypothesis (3.2), $F(\xi) \in \mathrm{L}_2^{\mathrm{SP}}(\mathbb{R} \times \mathit{\Omega})$.

Summarizing the following result has been established.

Theorem 1. *Let us consider the random IVP* (1.1)–(1.2) *and assume that the coefficients $a_i(t)$, $1 \le i \le 3$ and the initial condition $f(x)$ satisfy conditions* (3.1)–(3.5) *and* (3.10)–(3.11)*. Then, the mean square solution s.p. of* (1.1)–(1.2) *is given by*

$$
u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\Big(\mathrm{i}\xi x + \int_0^t a(s,\xi)\,\mathrm{d}s \Big)\, F(\xi)\,\mathrm{d}\xi\,, \tag{3.15}
$$

being $F(\xi)$ the random Fourier transform of the stochastic process $f(x)$.

Taking into account that $\int_0^t a(s,\xi)\,\mathrm{d}s$ and $F(\xi)$ in (3.15), are independent r.v.'s for every $t > 0$ and $\xi \in \mathbb{R}$ due to condition (3.1), we can obtain the following explicit expressions for the expectation and the standard deviation of the solution s.p. (3.15) of the random IVP (1.1)–(1.2)

$$
\mathbb{E}[u(x,t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(i\xi x)\, \mathbb{E}\Big[\exp\Big(\int_0^t a(s,\xi)\,\mathrm{d}s \Big) \Big] \mathbb{E}\left[F(\xi) \right]\, \mathrm{d}\xi,
$$

$$\mathbb{E}[(u(x,t))^2] = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(i(\xi_1+\xi_2)x\right)$$
$$\times\, \mathbb{E}\Big[\exp\Big(\int_0^t \big(a(s,\xi_1)+a(s,\xi_2)\big)\,\mathrm{d}s\Big)\Big]\,\mathbb{E}\big[F(\xi_1)F(\xi_2)\big]\,\mathrm{d}\xi_1\mathrm{d}\xi_2,$$
$$\sqrt{\mathrm{Var}[u(x,t)]} = +\sqrt{\mathbb{E}[(u(x,t))^2]-\left(\mathbb{E}[u(x,t)]\right)^2}.$$

Example 2. Let us consider the following particular case of the random IVP (1.1)–(1.2)

$$u_t(x,t) = a_2u_{xx}(x,t) + a_1u_x(x,t) + a_3u(x,t), \quad |x|<\infty,\ t>0 \qquad (3.16)$$
$$u(x,0) = \exp(-x^2), \quad -\infty < x < +\infty. \qquad (3.17)$$

Observe that the initial condition is deterministic and admits a deterministic Fourier transform, $F(\xi) = \mathfrak{F}[f(x)](\xi) = \frac{1}{\sqrt{2}}\exp\left(\frac{-\xi^2}{4}\right)$ (see Example 1 in [5], for instance). We will assume that coefficients a_i, $1\le i\le 3$, in (3.16), are independent r.v.'s satisfying condition (2.9), i.e.,

$$\mathbb{E}\left[|a_i|^m\right] \le M_i(H_i)^m < +\infty\,, \quad \forall m\ge 0\,, \quad \forall i:\ 1\le i\le 3. \qquad (3.18)$$

Thereby,

$$\hat{a}_i(t) = \int_0^t a_i\,\mathrm{d}s = a_it,$$
$$\mathbb{E}\left[|\hat{a}_i(t)|^m\right] = \mathbb{E}\left[|a_i|^m\right]t^m \le M_i\left(H_it\right)^m < +\infty, \quad \forall m\ge 0, \quad \forall i:\ 1\le i\le 3,$$

hence it is straightforwardly to check that all the hypotheses of Theorem 1 hold. Using expression (3.15) for each (x,t) fixed, one gets

$$u(x,t) = \frac{\exp(t\,a_3)}{2\pi\sqrt{2}}\int_{-\infty}^{+\infty}\exp\left(-\xi^2\left(ta_2+1/4\right)+\mathrm{i}\xi(ta_1+x)\right)\,\mathrm{d}\xi\,. \qquad (3.19)$$

Note that imaginary part of integral (3.19) vanishes because the s.p.

$$y(\xi) := \sin\left(\xi(ta_1+x)\right)\exp\left(-\xi^2\left(ta_2+1/4\right)\right)$$

is odd, i.e., $y(-\xi)(\omega) = -y(\xi)(\omega)$ for all $\omega\in\Omega$. Thus,

$$\int_{-\infty}^{+\infty}\sin\left(\xi(ta_1+x)\right)\exp\left(-\xi^2\left(ta_2+1/4\right)\right)\,\mathrm{d}\xi = 0\,,$$

and one gets

$$u(x,t) = \frac{\exp(t\,a_3)}{2\pi\sqrt{2}}\int_{-\infty}^{+\infty}\exp\left(-\xi^2\left(ta_2+1/4\right)\right)\cos\left(\xi(ta_1+x)\right)\,\mathrm{d}\xi\,. \qquad (3.20)$$

Using that $\exp\left(-\xi^2\left(ta_2+1/4\right)\right)\cos\left(\xi(ta_1+x)\right)$ is an even s.p. of the variable ξ, hence the solution s.p. of problem (3.16)–(3.17), given by (3.20), takes the form

$$u(x,t) = \frac{\exp(t\,a_3)}{\pi\sqrt{2}}\int_0^{+\infty}\exp\left(-\xi^2\left(ta_2+1/4\right)\right)\cos\left(\xi(ta_1+x)\right)\,\mathrm{d}\xi\,. \qquad (3.21)$$

The knowledge of deterministic integrals involving conditions of exponentials and trigonometric functions, and in particular (see [18, p.515])

$$\int_0^{\infty} \exp(-\beta\,\xi^2)\,\cos(b\xi)\,\mathrm{d}\xi = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{-b^2}{4\beta}\right), \quad \mathrm{Re}(\beta) > 0 \tag{3.22}$$

suggests the possibility of finding a closed-form expression of the integral given in (3.21). Following this strategy, previously developed in [6], let us consider the random improper integral

$$J(x,t) = \int_0^{\infty} \exp\left(-\xi^2\left(ta_2 + \frac{1}{4}\right)\right)\cos\left(\xi(ta_1 + x)\right)\,\mathrm{d}\xi, \quad x \in \mathbb{R},\ t > 0. \tag{3.23}$$

Note that as for each $\xi \in [0, +\infty[$, $t > 0$ and $x \in \mathbb{R}$ fixed, it is verified

$$\|\cos\left(\xi(ta_1 + x)\right)\|_{2,\mathrm{RV}} \le 1\,. \tag{3.24}$$

Taking into account the hypothesis of independence of a_1 and a_2 and, then applying Proposition 1 together with (3.24) and (2.12) (see Remark 2), one gets

$$\begin{aligned}
&\left\|\exp\left(-\xi^2\left(ta_2 + 1/4\right)\right)\cos\left(\xi(ta_1 + x)\right)\right\|_{2,\mathrm{RV}} \\
&\quad = \left\|\exp\left(-\xi^2\left(ta_2 + 1/4\right)\right)\right\|_{2,\mathrm{RV}} \|\cos(\xi(ta_1 + x))\|_{2,\mathrm{RV}} \\
&\quad = \left\|\exp\left(-\xi^2\left(ta_2 + 1/4\right)\right)\right\|_{2,\mathrm{RV}} = \exp\left(-\xi^2/4\right)\left\|\exp\left(-\xi^2 ta_2\right)\right\|_{2,\mathrm{RV}} \\
&\quad \le \sqrt{M_2}\exp\left(-\left(H_2\,t + 1/4\right)\,\xi^2\right),
\end{aligned}$$

being $M_2 > 0$ and $H_2 > 0$ the constants involved in (3.18) for $i = 2$. Observe that M_2 and H_2 are independent of x. Applying (3.22) with $\beta = H_2\,t + 1/4 > 0$ and $b = 0$ yields

$$\int_0^{+\infty} \exp\left(-\left(H_2 t + 1/4\right)\,\xi^2\right)\,\mathrm{d}\xi = \frac{1}{2}\sqrt{\frac{\pi}{H_2\,t + 1/4}} < +\infty,$$

hence, the integral $J(x,t)$, given by (3.23), is absolutely convergent in $\mathrm{L}_2^{\mathrm{RV}}(\Omega)$, for each (x,t) fixed. This guarantees the mean square random integral $J(x,t)$ defined in (3.23) and its sample representation,

$$J(x,t)(\omega) = \int_0^{+\infty} \exp\left(-\xi^2\left(ta_2(\omega) + 1/4\right)\right)\,\cos\left(\xi(ta_1(\omega) + x)\right)\,\mathrm{d}\xi\,,$$

both coincide (see [28, Appendix I]). Then, applying (3.22) with $\beta = ta_2(\omega) + \frac{1}{4}$, (provided $\mathrm{Re}\left(a_2(\omega)\right) > -\frac{1}{4\,t}$, for $t > 0$ fixed) and $b = ta_1(\omega) + x$, (with t and x fixed), one gets

$$J(x,t)(\omega) = \frac{\sqrt{\pi}}{\sqrt{4ta_2(\omega) + 1}}\exp\left(-\frac{(ta_1(\omega) + x)^2}{4ta_2(\omega) + 1}\right), \quad \forall\,\omega \in \Omega\,.$$

Thereby

$$J(x,t) = \frac{\sqrt{\pi}}{\sqrt{4ta_2 + 1}}\exp\left(-\frac{(ta_1 + x)^2}{4ta_2 + 1}\right),$$

and substituting the latter expression in (3.21), one finally obtains:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \frac{\exp(ta_3)}{\sqrt{4ta_2+1}} \exp\left(-\frac{(ta_1+x)^2}{4ta_2+1}\right), \qquad x \in \mathbb{R},\ t > 0. \tag{3.25}$$

Now, using Example 1, note that for fixed (x,t), $u(x,t)$ given by (3.20) can be approximated using random Gauss-Hermite quadrature formula (2.17)

$$u_N^{\text{G-H}}(x,t) = \frac{\exp(ta_3)}{2\pi\sqrt{2}} \sum_{j=1}^{N} \rho_j \exp\left(-\xi_j^2\left(ta_2 - \frac{3}{4}\right)\right) \cos\left(\xi_j(ta_1+x)\right), \tag{3.26}$$

$$\rho_j = \frac{2^{N+1} N! \sqrt{\pi}}{\left(H'_N(\xi_j)\right)^2}, \quad 1 \le j \le N.$$

By mean of the independence of r.v.'s a_i, $1 \le i \le 3$, one gets the expressions of the expectation and the standard deviation for the exact solution s.p. (3.25) and their numerical approximations by the random Gauss-Hermite quadrature (3.26), respectively,

$$\mathbb{E}[u(x,t)] = \frac{1}{\sqrt{2\pi}} \mathbb{E}\big[\exp(ta_3)\big]\mathbb{E}\Big[\frac{1}{\sqrt{4ta_2+1}}\Big]\,\mathbb{E}\Big[\exp\Big(-\frac{(ta_1+x)^2}{4ta_2+1}\Big)\Big], \tag{3.27}$$

$$\mathbb{E}[(u(x,t))^2] = \frac{1}{2\pi} \mathbb{E}\big[\exp(2ta_3)\big]\mathbb{E}\Big[\frac{1}{4ta_2+1}\Big]\mathbb{E}\Big[\exp\Big(-\frac{2(ta_1+x)^2}{4ta_2+1}\Big)\Big], \tag{3.28}$$

$$\mathbb{E}\left[u_N^{\text{G-H}}(x,t)\right] = \frac{1}{2\pi\sqrt{2}}\mathbb{E}\big[\exp(ta_3)\big] \sum_{j=1}^{N} \rho_j\, \mathbb{E}\Big[\exp\Big(-\xi^2\Big(ta_2 - \frac{3}{4}\Big)\Big)\Big] \times \mathbb{E}\big[\cos\left(\xi_j(ta_1+x)\right)\big], \tag{3.29}$$

$$\mathbb{E}\left[\left(u_N^{\text{G-H}}(x,t)\right)^2\right] = \frac{1}{8\,\pi^2}\mathbb{E}\big[\exp(2ta_3)\big] \sum_{j=1}^{N}\sum_{k=1}^{N} \rho_j\rho_k \mathbb{E}\Big[\exp\Big(-(\xi_j^2+\xi_k^2) \times \Big(ta_2 - 3/4\Big)\Big)\Big]\mathbb{E}\big[\cos\left(\xi_j(ta_1+x)\right)\cos\left(\xi_k(ta_1+x)\right)\big], \tag{3.30}$$

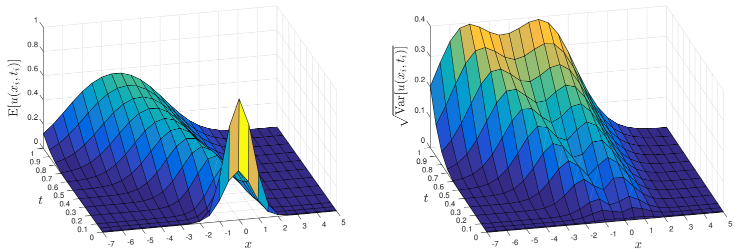
$$\sqrt{\text{Var}[u(x,t)]} = +\sqrt{\mathbb{E}[(u(x,t))^2] - \left(\mathbb{E}[u(x,t)]\right)^2}, \tag{3.31}$$

$$\sqrt{\text{Var}[u_N^{\text{G-H}}(x,t)]} = +\sqrt{\mathbb{E}\left[\left(u_N^{\text{G-H}}(x,t)\right)^2\right] - \left(\mathbb{E}[u_N^{\text{G-H}}(x,t)]\right)^2}. \tag{3.32}$$

In order to compute the values of (3.27), (3.28) and (3.31) for the random IVP (3.16)–(3.17) and compare these values with those numerical approximations obtained by the random Gauss-Hermite quadrature, (3.29), (3.30) and (3.32), respectively, we will assume that the input r.v.'s of IVP (3.16)–(3.17) follow some particular probabilistic distributions. Concretely, r.v. a_1 has a gamma distribution of parameters $(2;3)$ truncated on the interval $[0,6]$, $a_1 \sim \text{Gamma}_{[0,6]}(2;3)$; a_2 has a beta distribution of parameters $(2;1)$, $a_2 \sim \text{Beta}(2;1)$; and finally a_3 has an exponential distribution of parameter $\lambda = 1$ truncated on the interval $[1,2]$, $a_3 \sim \text{Exp}_{[1,2]}(1)$. For this choice of the r.v.'s a_i, $1 \le i \le 3$, it is guaranteed condition (3.18) because all of them are bounded r.v.'s and, in addition, for r.v. a_2 that $\text{Re}\,(a_2(\omega)) = a_2(\omega) > -\frac{1}{4\,t}$ with $t > 0$

fixed, since $a_2(\omega) \in (0,1)$, $\omega \in \Omega$. Observe that, the rest of the hypotheses of Theorem 1 clearly also hold in the context of this example.

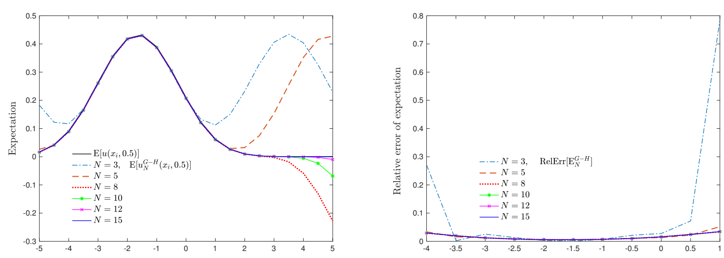
Figure 1 shows the evolution on the time domain $0 \le t \le 1$ of the expectation $\mathbb{E}[u(x,t)]$, computed by (3.27), and the standard deviation $\sqrt{\mathrm{Var}[u(x,t)]}$, computed by (3.27), (3.28) and (3.31), of exact solution s.p. (3.25) to the random IVP (3.16)–(3.17) on the spatial domain $-7 \le x \le 5$. Outside this spatial range, both the expectation and the standard deviation tends to zero.



(a) (b)

Figure 1. Plot (a): surface of the expectation $\mathbb{E}[u(x,t)]$, plot (b): surface of the standard deviation $\sqrt{\mathrm{Var}[u(x,t)]}$.

In Figure 2a and Figure 3a, we compare the expectation $\mathbb{E}[u(x_i, 0.5)]$, (3.27) and the standard deviation $\sqrt{\mathrm{Var}[u(x_i, 0.5)]}$, (3.31), respectively, vs. their numerical approximations, $\mathbb{E}\left[u_N^{\text{G-H}}(x_i, 0.5)\right]$, computed by means of (3.29), and $\sqrt{\mathrm{Var}[u_N^{\text{G-H}}(x_i, 0.5)]}$, computed by means of (3.29), (3.30) and (3.32), for some Hermite's polynomials of degree N, at the time instant $t = 0.5$ and on the spatial domain $-5 \le x \le 5$.



(a) (b)

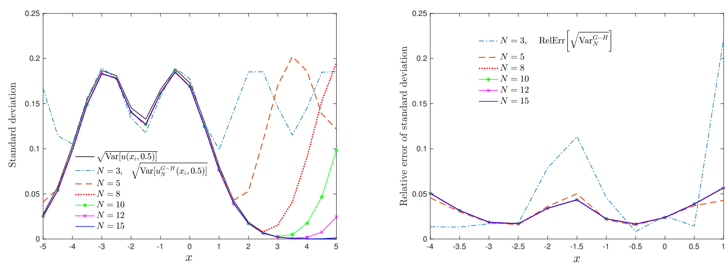
Figure 2. Plot (a): $\mathbb{E}[u(x_i, 0.5)]$ vs. $\mathbb{E}\left[u_N^{\text{G-H}}(x_i, 0.5)\right]$ by random Gauss-Hermite quadrature using Hermite's polynomials of degree $N \in \{3, 5, 8, 10, 12, 15\}$. Plot (b): $\mathrm{RelErr}\left[\mathbb{E}_N^{G-H}\right]$. In order to represent properly the relative error, the domain $-5 \le x \le 5$ has been shorten since the expectation is almost zero outside the interval $-4 \le x \le 1$.

It can be seen in Figure 2b and Figure 3b the numerical values of the relative errors for the approximate expectation, $\mathrm{RelErr}\left[\mathbb{E}_N^{G-H}\right]$, and the approximate standard deviation, $\mathrm{RelErr}\left[\sqrt{\mathrm{Var}_N^{G-H}}\right]$, respectively, computed using the following expressions

$$\mathrm{RelErr}\left[\mathbb{E}_N^{G-H}\right] = \left|\left(\mathbb{E}[u(x,t)] - \mathbb{E}\left[u_N^{\text{G-H}}(x,t)\right]\right)/\mathbb{E}[u(x,t)]\right|, \tag{3.33}$$

$$\mathrm{RelErr}\left[\sqrt{\mathrm{Var}_N^{G-H}}\right] = \left|\frac{\sqrt{\mathrm{Var}[u(x,t)]} - \sqrt{\mathrm{Var}[u_N^{\text{G-H}}(x,t)]}}{\sqrt{\mathrm{Var}[u(x,t)]}}\right|. \tag{3.34}$$

The spatial domain considered, $-4 \le x \le 1$, has been properly chosen to compute these relative errors since outside this piece the expectation, $\mathbb{E}[u(x_i, 0.5)]$, and the standard deviation, $\sqrt{\mathrm{Var}[u(x_i, 0.5)]}$, have very small values. The computed relative errors show us that, for time $t = 0.5$, it is sufficient to consider a Hermite's polynomial of degree $N = 8$, in the random Gauss-Hermite quadrature, in order to obtain good approximations of the exact expectation $\mathbb{E}[u(x_i, 0.5)]$ on the spatial domain $-4 \le x \le 1$.



(a) (b)

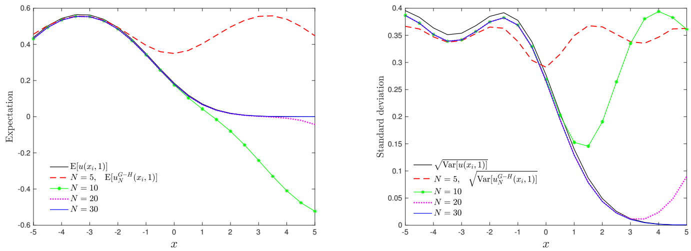
Figure 3. Plot (a): $\sqrt{\mathrm{Var}[u(x_i, 0.5)]}$ vs. $\sqrt{\mathrm{Var}[u_N^{\text{G-H}}(x_i, 0.5)]}$ by random Gauss-Hermite quadrature using Hermite's polynomials of degree $N \in \{3, 5, 8, 10, 12, 15\}$. Plot (b): $\mathrm{RelErr}\left[\sqrt{\mathrm{Var}_N^{\text{G-H}}}\right]$. To be consistent with Plot (b) in Figure 2, where the relative error has been represented on the spatial interval $-4 \le x \le 1$, here we keep the same interval to plot the relative error.

Figure 4 illustrates that, in a longer time $t = 1$, the approximations of the expectation, $\mathbb{E}\left[u_N^{\text{G-H}}(x_i, 1)\right]$, and the standard deviation, $\sqrt{\mathrm{Var}[u_N^{\text{G-H}}(x_i, 1)]}$, are being improved as the degree N of the Hermite's polynomials increases.

In Tables 1 and 2 we collect the numerical values for the approximations shown in Figure 4 as well as the relative errors on the spatial domain $-4 \le x \le 1$. We observe that increasing $N = 5$ up to $N = 30$, the proposed method provides a reasonable approximation to the exact expectation $\mathbb{E}[u(x_i, 1)]$, while for obtaining good approximations to the standard deviation $\sqrt{\mathrm{Var}[u(x_i, 1)]}$ it will be enough to consider $N = 10$.

Example 3. In this example, we shall illustrate the theoretical results previously established by a random parabolic problem where both initial condition and coefficients are s.p.'s. Let us consider random IVP (1.1)–(1.2) with

$$a_1(t) = a_1 \cos(t)\,, \quad a_2(t) = a_2\, t\,, \; a_3(t) = a_3\,, \; f(x) = \exp(-b\, x^2)\,,$$



(a) (b)

Figure 4. Plot (a): $\mathbb{E}[u(x_i,1)]$ vs. $\mathbb{E}\left[u_N^{\text{G-H}}(x_i,1)\right]$ for the degrees $N \in \{5, 10, 20, 30\}$. Plot (b): $\sqrt{\mathrm{Var}[u(x_i,1)]}$ vs. $\sqrt{\mathrm{Var}[u_N^{\text{G-H}}(x_i,1)]}$.

where the r.v.'s a_i, $1 \le i \le 3$, and b are given following the distributions

$$a_1 \sim \mathrm{Beta}(2;3)\,, \quad a_2 \sim \mathrm{N}_{[2,4]}(3;0.1)\,, a_3 \sim \mathrm{Exp}_{[0.5,1.5]}(1)\,, b \sim \mathrm{Un}([1,2])\,.$$

Following an analogous reasoning as the one exhibited in Example 2, it is straightforwardly to check that hypotheses of Theorem 1 hold. Notice that r.v. b has a positive lower bound $\ell_1 > 0$, i.e., $b(\omega) \ge 1 > 0$, $\forall \omega \in \Omega$. Also expressions of exact solution s.p. (3.15) and numerical approximation by the random Gauss-Hermite quadrature (2.14), now take the form, respectively,

$$u(x,t) = \frac{1}{\sqrt{2\pi\, b}} \frac{\exp(ta_3)}{\sqrt{2a_2t^2 + \frac{1}{b}}} \exp\left(-\frac{(a_1\,\sin(t)+x)^2}{2a_2t^2 + \frac{1}{b}}\right), \quad x \in \mathbb{R},\; t > 0, \tag{3.35}$$

$$u_N^{\text{G-H}}(x,t) = \frac{\exp(ta_3)}{2\pi\sqrt{2\,b}} \sum_{j=1}^{N} \rho_j \exp(\xi_j^2) \exp\left(-\xi_j^2\left(a_2\frac{t^2}{2} + \frac{1}{4\,b}\right)\right) \times \cos\left(\xi_j(a_1 \sin(t) + x)\right)\,. \tag{3.36}$$

Using statistical independence of r.v.'s a_i, $1 \le i \le 3$, and b, one obtains the expectation and standard deviation (using (3.31)–(3.32)) of $u(x,t)$ given by (3.35) and the expectation and standard deviation of their numerical approximations by the random Gauss-Hermite quadrature $u_N^{G-H}(x,t)$ given by (3.36), as well

$$\mathbb{E}[u(x,t)] = \frac{1}{\sqrt{2\pi}}\,\mathbb{E}\left[\frac{1}{\sqrt{b}}\right]\,\mathbb{E}\left[\exp(ta_3)\right]\mathbb{E}\left[\frac{1}{\sqrt{2a_2t^2 + 1/b}}\right] \times \mathbb{E}\left[\exp\left(-\frac{(a_1\sin(t)+x)^2}{2a_2t^2 + 1/b}\right)\right], \tag{3.37}$$

Table 1. Exact values of $\mathbb{E}[u(x_i,t)]$ and $\sqrt{\mathrm{Var}[u(x_i,t)]}$, at some spatial points $x_i \in [-4,-2]$ at the end time $t=1$. Approximations of $\mathbb{E}\left[u_N^{\text{G-H}}(x_i,t)\right]$, and $\sqrt{\mathrm{Var}[u_N^{\text{G-H}}(x_i,t)]}$, obtained by random Gauss-Hermite quadrature using Hermite's polynomials of degree $N \in \{5,10,20,30\}$. A comparison of these approximations with respect to the exact values is reported by means of the calculation of relative errors in the spatial points $x_i \in [-4,-2]$ considered and for each Hermite's polynomial of degree N.

$t=1$		x_i				
	N	-4.0	-3.5	-3	-2.5	-2.0
$\mathbb{E}[u(x_i,t)]$		0.5437	0.5644	0.5626	0.5387	0.4931
$\mathbb{E}[u_N^{G-H}(x_i,t)]$	5	0.5383	0.5559	0.5543	0.5331	0.4941
	10	0.5343	0.5543	0.5528	0.5295	0.4847
	20	0.5343	0.5543	0.5528	0.5295	0.4847
	30	0.5343	0.5543	0.5528	0.5295	0.4847
$\mathrm{RelErr}\left[\mathbb{E}_N^{G-H}\right]$	5	9.9863e-03	1.5080e-02	1.4671e-02	1.0383e-02	2.1077e-03
	10	1.7238e-02	1.7901e-02	1.7429e-02	1.7039e-02	1.7035e-02
	20	5.5588e-01	3.3138e-01	5.5945e-02	2.4677e-01	5.4834e-01
	30	1.7228e-02	1.7900e-02	1.7424e-02	1.7035e-02	1.6992e-02
$\sqrt{\mathrm{Var}[u(x_i,t)]}$		0.3636	0.3511	0.3539	0.3677	0.3849
$\sqrt{\mathrm{Var}[u_N^{G-H}(x_i,t)]}$	5	0.3471	0.3374	0.3402	0.3530	0.3652
	10	0.3515	0.3391	0.3417	0.3568	0.3747
	20	0.3515	0.3391	0.3417	0.3568	0.3746
	30	0.3515	0.3391	0.3417	0.3568	0.3746
$\mathrm{RelErr}\left[\sqrt{\mathrm{Var}_N^{G-H}}\right]$	5	4.5441e-02	3.8990e-02	3.8725e-02	4.0012e-02	5.1189e-02
	10	3.3189e-02	3.4337e-02	3.4367e-02	2.9479e-02	2.6660e-02
	20	3.3208e-02	3.4339e-02	3.4377e-02	2.9485e-02	2.6724e-02
	30	3.3208e-02	3.4339e-02	3.4377e-02	2.9485e-02	2.6724e-02

$$
\begin{aligned}
\mathbb{E}[(u(x,t))^2] &= \frac{1}{2\pi}\,\mathbb{E}\left[\frac{1}{b}\right]\,\mathbb{E}\left[\exp(2ta_3)\right] \\
&\quad\times \mathbb{E}\left[\frac{1}{2a_2t^2+1/b}\right]\mathbb{E}\left[\exp\left(-\frac{2(a_1\sin(t)+x)^2}{2a_2t^2+1/b}\right)\right], \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[u_N^{\text{G-H}}(x,t)\right] &= \frac{1}{2\pi\sqrt{2}}\,\mathbb{E}\left[\frac{1}{\sqrt{b}}\right]\,\mathbb{E}\left[\exp(ta_3)\right] \\
&\quad\times \sum_{j=1}^{N}\rho_j\mathbb{E}\left[\exp\left(\xi_j^2\left(1-\frac{2ba_2t^2+1}{4b}\right)\right)\right]\mathbb{E}\left[\cos\left(\xi_j(a_1\sin(t)+x)\right)\right], \qquad (3.39)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\left(u_N^{\text{G-H}}(x,t)\right)^2\right] &= \frac{1}{8\,\pi^2}\,\mathbb{E}\left[1/b\right]\,\mathbb{E}\left[\exp(2ta_3)\right] \\
&\quad\times \sum_{j=1}^{N}\sum_{k=1}^{N}\rho_j\;\rho_k\;\mathbb{E}\left[\exp\left((\xi_j^2+\xi_k^2)\left(1-\frac{2b\,a_2\,t^2+1}{4b}\right)\right)\right] \\
&\quad\times \mathbb{E}\left[\cos\left(\xi_j(a_1\sin(t)+x)\right)\cos\left(\xi_k(a_1\sin(t)+x)\right)\right]. \qquad (3.40)
\end{aligned}
$$

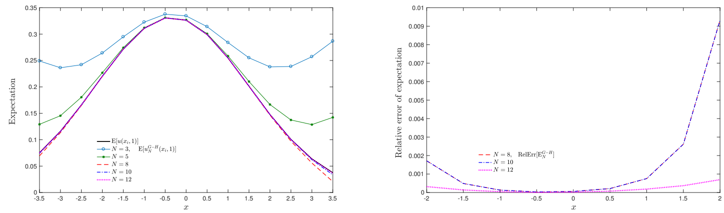
Table 2. The same random functions of Table 1 at some spatial points $x_i \in [-1, 1]$ at the end time instant $t = 1$.

$t = 1$		x_i				
	N	-1.0	-0.5	0	0.5	1
$\mathbb{E}[u(x_i, t)]$		0.3492	0.2651	0.1859	0.1197	0.0706
$\mathbb{E}[u_N^{G-H}(x_i, t)]$	5	0.3946	0.3596	0.3495	0.3663	0.4036
	10	0.3418	0.2572	0.1758	0.1043	0.0426
	20	0.3421	0.2585	0.1797	0.1145	0.0667
	30	0.3421	0.2585	0.1797	0.1145	0.0667
$\mathrm{RelErr}\left[\mathbb{E}_N^{G-H}\right]$	5	1.2978e-01	3.5644e-01	8.8040e-01	2.0606	4.7138
	10	2.1372e-02	3.0038e-02	5.3999e-02	1.2882e-01	3.9720e-01
	20	1.0635	1.2569	1.4052	1.5137	1.5902
	30	2.0301e-02	2.5157e-02	3.3004e-02	4.3317e-02	5.5300e-02
$\sqrt{\mathrm{Var}[u(x_i, t)]}$		0.3776	0.3379	0.2769	0.2066	0.1402
$\sqrt{\mathrm{Var}[u_N^{G-H}(x_i, t)]}$	5	0.3387	0.3038	0.2911	0.3153	0.3495
	10	0.3690	0.3292	0.2685	0.2023	0.1526
	20	0.3687	0.3282	0.2656	0.1944	0.1292
	30	0.3687	0.3282	0.2656	0.1944	0.1292
$\mathrm{RelErr}\left[\sqrt{\mathrm{Var}_N^{G-H}}\right]$	5	1.0294e-01	1.0073e-01	5.1349e-02	5.2626e-01	1.4926
	10	2.2619e-02	2.5642e-02	3.0474e-02	2.0711e-02	8.8231e-02
	20	2.3517e-02	2.8699e-02	4.0889e-02	5.8873e-02	7.8277e-02
	30	2.3517e-02	2.8699e-02	4.0890e-02	5.8875e-02	7.8294e-02

In Figure 5a and Figure 6a, we compare the expectation $\mathbb{E}[u(x_i, 1)]$, using (3.37), and standard deviation $\sqrt{\mathrm{Var}[u(x_i, 1)]}$, (3.31) and (3.37)–(3.38), respectively, vs. their $\mathbb{E}\left[u_N^{G-H}(x_i, 1)\right]$, (3.39), and $\sqrt{\mathrm{Var}[u_N^{G-H}(x_i, 1)]}$, (3.32) and (3.39)–(3.40), for $N \in \{3, 5, 8, 10, 12\}$ at the time instant $t = 1$ on the spatial domain $-3.5 \leq x \leq 3.5$. Notice that approximations improve as N increases. This behaviour can be observed in Figure 5b and Figure 6b, where relative errors, computed by (3.33)–(3.34), are shown. For the sake of clarity, since the order of magnitud of relative errors associated to $N \in \{3, 5, 8, 10, 12\}$ are very different, the latter two figures (Figure 5b and Figure 6b) have been plotted on a shorted spatial domain, $-2 \leq x \leq 2$, for $N \in \{8, 10, 12\}$.

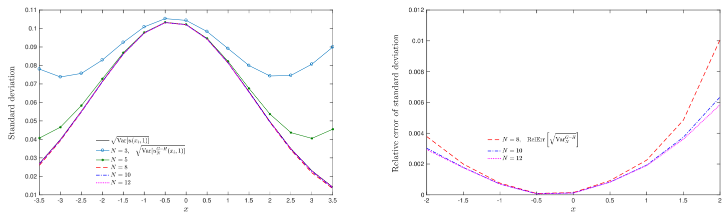
4 Conclusions

In this paper we have considered the construction of exact and approximate solution of random time dependent parabolic partial differential initial value problems where the uncertainty is treated in the mean square sense. We have shown that a random Fourier transform method can be applied so efficiently as it has been proved to be in the solution of deterministic problems [17]. However, in the random case, not only the solution stochastic process is important, but also its expectation and standard deviation. This has motivated the con-



(a) (b)

Figure 5. Plot (a): $\mathbb{E}[u(x_i,1)]$ vs $\mathbb{E}\left[u_N^{\text{G-H}}(x_i,1)\right]$ for the degrees $N \in \{3,5,8,10,12\}$. Plot (b): RelErr $\left[\mathbb{E}_N^{G-H}\right]$. In order to represent properly the relative error, the domain $-3.5 \leq x \leq 3.5$ has been shorten.



(a) (b)

Figure 6. Plot (a): $\sqrt{\text{Var}[u(x_i,0.5)]}$ vs $\sqrt{\text{Var}[u_N^{\text{G-H}}(x_i,0.5)]}$ for the degrees $N \in \{3,5,8,10,12\}$. Plot (b): RelErr $\left[\sqrt{\text{Var}_N^{\text{G-H}}}\right]$. To be consistent with Plot (b) in Figure 5, where the relative error has been represented on the spatial interval $-2 \leq x \leq 2$, here we keep the same interval to plot the relative error.

sideration of random integral numerical methods to approximate infinite mean square convergent integrals. In fact, random Gauss-Hermite quadrature formulae are proposed to approximate the solution stochastic process in a more computable way. Results have been illustrated with an example.

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