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Global existence and uniqueness of mild solution for a fractional Keller-Segel system in Besov-Morrey spaces

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Abstract. This paper investigates the fractional Keller-Segel system involving both temporal and spatial variables. We examine fractional dissipation mechanisms for the physical variables, including chemotactic diffusion with fractional dissipation, and incorporate a time-fractional variation in the Caputo sense. Our analysis centers on the fractional heat semigroup, deriving time-decay and integral estimates for Mittag-Leffler operators in Besov-Morrey spaces. Furthermore, we establish a bilinear estimate arising from the nonlinearity of the Keller-Segel system, avoiding reliance on auxiliary norms. These results are then employed to demonstrate the existence and uniqueness of mild solution in Besov-Morrey spaces.

Keywords: existence and uniqueness of mild solution; Mittag-Leffler operators; Besov-Morrey spaces; Caputo sense; fractional system.

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1 Introduction

The Keller-Segel system (**KS**) is a set of partial differential equations (PDEs) that model chemotaxis, which refers to the directed movement of living organisms (such as bacteria or cells) in response to chemical gradients in their environment. This system is primarily used to describe the collective behavior of cellular populations, particularly in biological contexts such as the aggregation of amoebae or pattern formation in bacterial colonies (see [1, 3, 4, 9] and references therein).

The Keller-Segel system is given by:

$$\begin{cases} u_t - D_u \Delta u = -\varpi \nabla \cdot (u \nabla \rho), & \text{in } \mathbb{R}^d \times (0,\infty), \\ \rho_t - D_\rho \Delta \rho = -\xi \rho + \zeta u, & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^d, \\ \rho(x,0) = \rho_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here, the functions $u = u(x,t)$ and $\rho = \rho(x,t)$ represent the cell density and the concentration of an attractive chemical signal, respectively. The parameters D_u and D_ρ denote the corresponding diffusion coefficients for u and ρ. The constants ξ, ζ and ϖ are non-negative parameters representing decay rate, production rate, and chemotactic sensitivity, respectively.

The classical Keller-Segel model neglects non-local diffusion effects. Yet, in environments with scarce chemoattractants or food, organisms often follow Lévy flights instead of Brownian motion [5, 11], as seen in immune responses, ecological dynamics, and human mobility [6]. This justifies using fractional diffusion in (1.1). To model chemotactic flux more realistically, we also consider a less singular interaction kernel, aligning with studies on propagation of chaos in aggregation-diffusion systems. Furthermore, since biological systems often exhibit memory, we adopt a fractional time derivative to incorporate temporal non-locality [7].

Based on these observations, our focus is on the theoretical analysis of the Keller-Segel system in a fractional setting:

$$\begin{cases} {}^cD_t^\beta u + D_u(-\Delta)^{\alpha/2} u = -\varpi \nabla \cdot (uF(\rho)), & \text{in } \mathbb{R}^d \times (0,\infty), \\ {}^cD_t^\beta \rho - D_\rho(-\Delta)^{\alpha/2} \rho = -\xi \rho + \zeta u, & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^d, \\ \rho(x,0) = \rho_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{1.2}$$

Here, ${}^cD_t^\beta$ represents the Caputo fractional derivative of order $\beta \in (0,1)$ and $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian operator of order $\alpha/2$. Furthermore, $F(\rho)$ represents a nonlocal term given by:

$$F(\rho)(x) = \nabla((-\triangle)^{-\alpha_1/2}\rho)(x) \ , \ x \in \mathbb{R}^d.$$

Replacing the classical time differential operator with the time-fractional differential operator introduces a new parameter into the problem, thereby providing fresh insights into the overall dynamics of the system.

The remainder of the paper is organized as follows. In Section 2, we review some fundamental preliminaries related to the functional framework in which we will address our problem. In Section 3, we present important estimates concerning Mittag-Leffler operators. Section 4 focuses on proving the linear and nonlinear estimates, including the established product and bilinear estimates. Finally, in Section 5, we introduce and prove the main results, specifically the existence and uniqueness of mild solution to our problem.

2 Preliminaries

In the analysis of the system (1.2), we will employ Besov-Morrey space theory and Caputo derivative theory. We will also make use of the paraproduct formula, the Mainardi function, and the Mittag-Leffler operators, which are essential for determining the integral solution of our problem. At the end of this section, we will recall the mild solution of system (1.2).

2.1 Basic property of Besov-Morrey spaces

We start with some basic facts that will be used later. For more details, we refer to [8] and its references.
We begin by recalling some properties of the functional spaces $\mathcal{M}_a^\mu$.

DEFINITION 1. Let $1 \leq a \leq \infty$ and $0 \leq \mu < n$. The homogeneous Morrey space $\mathcal{M}_a^\mu$ is defined by

$$\mathcal{M}_a^\mu(\mathbb{R}^d) := \Big\{ \phi \in L^1_{loc}(\mathbb{R}^d), \|\phi\|_{\mathcal{M}_a^\mu} < \infty \Big\},$$

with the norm given by:

$$\|\phi\|_{\mathcal{M}_a^\mu} := \sup_{x_0 \in \mathbb{R}^d} \sup_{D>0} D^{-\frac{\mu}{a}} \|\phi\|_{L^a\left(B(x_0,D)\right)},$$

where $B(x_0, D)$ is the open ball in $\mathbb{R}^d$ centered at x_0 and with radius $D > 0$.

Let $\phi \in C_0^\infty$ with $supp(\phi) \subseteq \Big\{ \mathcal{K} \in \mathbb{R}^d \quad : \quad \frac{1}{2} < |\mathcal{K}| < 2 \Big\}$ such that:

$$\sum_{l=-\infty}^{\infty} \phi\Big(\frac{1}{2^l}\mathcal{K}\Big) = \sum_{l=-\infty}^{\infty} \hat{\phi}_l(\mathcal{K}) = 1, \quad \text{for all} \quad \mathcal{K} \neq 0,$$

where ϕ_l is defined in terms of the Fourier transform as:

$$\hat{\phi}_l(\mathcal{K}) = \phi\Big(\frac{1}{2^l}\mathcal{K}\Big) \quad \text{for all integer} \quad l.$$

For $f \in S'$, we define the norm:

$$\|f\|_{\mathcal{N}^r_{a,\mu,b}} := \begin{cases} \left(\sum_{l \in \mathbb{Z}} \Big(2^{lr} \|\phi_l * f\|_{\mathcal{M}_a^\mu} \Big)^b \right)^{\frac{1}{b}} & 1 \leq a \leq \infty, \ 1 \leq b < \infty, \ r \in \mathbb{R}, \\ \sup_{l \in \mathbb{Z}} \Big(2^{lr} \|\phi_l * f\|_{\mathcal{M}_a^\mu} \Big), & 1 \leq a \leq \infty, \ b = \infty, \ r \in \mathbb{R}. \end{cases}$$

It should be noted that the pair $(\mathcal{N}^r_{a,\mu,b}, \ \|\cdot\|_{\mathcal{N}^r_{a,\mu,b}})$ forms a Banach space.

The following Lemma can be found in [8].

Lemma 1. *Let* $r_1, r_2 \in \mathbb{R}, 1 \leq a_j, b_j \leq \infty$ *and* $0 \leq \mu_j < n$, *when* $j = 1, 2, 3$.
(i) If $1 \leq r_1 \leq r_2$ *and* $1 \leq b_2 \leq b_1 \leq \infty$, *then,*

$$\mathcal{N}^{r_2}_{a,\mu,b_2}\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{N}^{r_1}_{a,\mu,b_1}\left(\mathbb{R}^d\right).$$

(ii) If $a_1 > a_2, \quad r_1 - \frac{n-\mu_1}{a_1} = r_2 - \frac{n-\mu_2}{a_2}$*, then,*

$$\mathcal{N}^{r_2}_{a_2,\mu_2,b}\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{N}^{r_1}_{a_1,\mu_1,b}\left(\mathbb{R}^d\right) \quad \text{and}$$
$$\mathcal{N}^{0}_{a_1,\mu_1,1}\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{M}_{a_1,\mu_1}\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{N}^{0}_{a_1,\mu_1,\infty}\left(\mathbb{R}^d\right).$$

(iii) If $1 \le b_1 \le b_2 \le \infty$*, then,*

$$\mathcal{N}^{r}_{a,\mu,b_1}\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{N}^{r}_{a,\mu,b_2}\left(\mathbb{R}^d\right).$$

(iv) If $\frac{\mu_3}{a_3} = \frac{\mu_2}{a_2} + \frac{\mu_1}{a_1}$ *and* $g_i \in \mathcal{M}_{a_i,\mu_i}\left(\mathbb{R}^d\right)$ *when* $i = 1,\ 2$*, then,*

$$\|g_1 g_2\|_{\mathcal{M}^{\mu_3}_{a_3}} \le \|g_1\|_{\mathcal{M}^{\mu_1}_{a_1}} \|g_2\|_{\mathcal{M}^{\mu_2}_{a_2}}.$$

Lemma 2. *[13]For* $r_1 \neq r_2 \in \mathbb{R}$ *and* $\nu \in (0,1)$*, the following interpolation relations hold:*

$$\left(\mathcal{N}^{r_1}_{a,\mu,b_1}, \mathcal{N}^{r_2}_{a,\mu,b_2}\right)_{\nu,b} = \mathcal{N}^{r}_{a,\mu,b},$$

where $r = (1-\nu)r_1 + \nu r_2,\ b, b_1, b_2 \in [1,\infty]$*, and* $1 \le b \le a < \infty$*.*

We now present a few operators for frequency localization:

$$\dot{\Delta}_l g = \mathcal{F}^{-1}\phi_l g = 2^{d.l}\int_{\mathbb{R}^d} h\left(2^l y\right) g(\tilde{y}-y)dy,$$
$$\dot{S}_l g = \sum_{k\le l-1} \dot{\Delta}_k g = \mathcal{F}^{-1}\delta_l * g = 2^{dl}\int_{\mathbb{R}^d} w\left(2^l y\right) g(\tilde{y}-y)dy,$$

where $\phi_l(\mathcal{K}) = \phi\left(2^{-l}\mathcal{K}\right),\ \delta_l(\mathcal{K}) = \sum_{k\le l-1}\phi_k(\mathcal{K}),\ h(\tilde{y}) = \mathcal{F}^{-1}\phi(\tilde{y}),\ w(\tilde{y}) = \mathcal{F}^{-1}\delta(\tilde{y})$. The operator $\dot{\Delta}_l = \dot{S}_l - \dot{S}_{l-1}$ represents a frequency projection to the annulus $\left\{|\mathcal{K}| \sim 2^l\right\}$. From the definition of $\dot{S}_l$ and $\dot{\Delta}_l$, we observe the following properties:

$$\dot{\Delta}_l\dot{\Delta}_k g = 0, \quad \text{if } |l-k| \ge 2,$$
$$\dot{\Delta}_l\left(\dot{S}_{k-1}g\dot{\Delta}_k g\right) = 0, \quad \text{if } |l-k| \ge 5, \quad \widehat{\dot{\Delta}_l g} = \phi_l\widehat{g}.$$

We now recall the Bony decomposition [2], which allows us to decompose the product fg as follows:

$$\begin{aligned} fg &= \dot{P}_f g + \dot{P}_g f + Q(f,g) \\ &= \sum_{l\in\mathbb{Z}} \dot{S}_{l-1} f\dot{\Delta}_l g + \sum_{l\in\mathbb{Z}} \dot{S}_{l-1} g\dot{\Delta}_l f + \sum_{l\in\mathbb{Z}} \dot{\Delta}_l f\tilde{\Delta}_l g, \end{aligned}$$

where

$$\tilde{\Delta}_l g = \sum_{|l'-l|\le 1} \dot{\Delta}_{l'} g.$$

2.2 Mainardi function and Mittag-Leffler operators

To understand the concept of fractional derivatives, it is essential to review a few aspects of fractional calculus theory.

We define the function f_β for any positive value of β from $\mathbb{R}$ with value in $\mathbb{R}$ as follows:

$$f_\beta(t) := \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma(\beta)$ is the Euler's Gamma function.

Now, assume that $T > 0$. For a Banach space $\mathcal{Y}$ and a function $v \in L^1(0,T;\mathcal{Y})$, the Riemann-Liouville fractional integral of order β of ω is given by:

$$H_t^\beta \omega(t) := f_\beta * \omega(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(s) ds, \quad t \in [0,T].$$

This definition of H_t^β emphasizes the connection between convolution properties and the formulation of the fractional integral operator, providing a fundamental tool for analyzing fractional differential equations and their solutions.

Using the Riemann-Liouville fractional integral operator, we now define the Caputo fractional differential operator.

Definition 2. Let $\omega \in C([0,T];\mathcal{Y})$ satisfy $f_{1-\beta} * \omega \in W^{1,1}(0,T;\mathcal{Y})$, β in $(0,1)$. The Caputo fractional derivative of order β of the function ω is defined as:

$$^cD_t^\beta \omega(t) := \frac{d}{dt} \left\{ H_t^{1-\beta} [\omega(t) - \omega(0)] \right\} = \frac{d}{dt} \left\{ \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} [\omega(\tau) - \omega(0)] ds \right\}.$$

Let $\beta \in (0,1)$. The Mainardi function $\mathcal{U}_\beta$ is a mapping from $\mathbb{C}$ to $\mathbb{C}$ defined by: $\mathcal{U}_\beta(x) := \mathcal{G}_{-\beta,1-\beta}(-x)$, $x \in \mathbb{C}$, where $\mathcal{G}_{\theta,\eta}(x)$ denotes the Wright function, defined by the following convergent complex series in the complex plane:

$$\mathcal{G}_{\theta,\eta}(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!\Gamma(\theta n + \eta)} \quad \forall \theta > -1\,,\, \eta \in \mathbb{C}.$$

The function $\mathcal{U}_\beta$ was introduced by Mainardi in his book [12] as a specific instance of Wright's function.

The following proposition gives two important properties of the Mainardi function.

Proposition 1. *For $-1 < q < \infty$, the Mainardi function satisfies the following properties:*

- $\mathcal{U}_\beta(x) \geq 0$ *for all* $x \geq 0$.
- $\displaystyle\int_0^\infty x^q \mathcal{U}_\beta(x) dx = \frac{\Gamma(q+1)}{\Gamma(\beta q + 1)}$.

This subsection is devoted to introducing Mittag-Leffler operators. Let $\mathcal{Y}$ a Banach space and let $-\mathbb{A}$: $D(\mathbb{A}) \subset \mathcal{Y} \to \mathcal{Y}$ be an infinitesimal generator of a semigroup $\{\mathcal{T}(t) : t \geq 0\}$, where:

For any $\beta \in (0,1)$, we define the Mittag-Leffler families $\left\{\mathcal{Q}_\beta(-t^\beta \mathbb{A}) : t \geq 0\right\}$ and $\left\{\mathcal{Q}_{\beta,\beta}(-t^\beta \mathbb{A}) : t \geq 0\right\}$ as follows:

$$\mathcal{Q}_\beta(-t^\beta \mathbb{A}) = \int_0^\infty \mathcal{U}_\beta(\tau)\mathcal{T}(\tau, t^\beta)d\tau, \quad \mathcal{Q}_{\beta,\beta}(-t^\beta \mathbb{A}) = \int_0^\infty \beta\tau\mathcal{U}_\beta(\tau)\mathcal{T}(\tau t^\beta)d\tau.$$

It is noteworthy that the Mainardi functions bridge the gap between classical theories and abstract fractional theories. More details can be found in [10]. On the other hand, at the end of this section, we recall the mild formulation of system (1.2). Following Duhamel's principle, we express the system in the following mild form:

$$\begin{cases} u(t) = \mathcal{Q}_\beta\left(-t^\beta(-\Delta)^{\alpha/2}\right)u_0 \\ \qquad - \displaystyle\int_0^t (t-s)^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}\left(-(t-s)^\beta(-\Delta)^{\alpha/2}\right)(\varpi F(\rho))(s)\, ds, \\ \rho(t) = \mathcal{Q}_\beta\left(-t^\beta(-\Delta^{\alpha/2}) - \xi\right)\rho_0 \\ \qquad + \displaystyle\int_0^t (t-s)^{\beta-1}\mathcal{Q}_{\beta,\beta}\left(-(t-s)^\beta((-\Delta)^{\alpha/2} - \xi)\right)u(s)\, ds. \end{cases} \tag{2.1}$$

In the remainder of this paper, we will refer to the bilinear and linear operators appearing in (2.1) as:

$$\begin{aligned} \mathbb{B}(u,\rho)(t) &= -\int_0^t (t-s)^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}\left(-(t-s)^\beta(-\Delta)^{\alpha/2}\right)(\varpi F(\rho))(s)\, ds, \\ \Psi(u)(t) &= \int_0^t (t-s)^{\beta-1}\mathcal{Q}_{\beta,\beta}\left(-(t-s)^\beta((-\Delta)^{\alpha/2} - \xi)\right)u(s)\, ds. \end{aligned}$$

We can therefore reformulate system (2.1) as:

$$\begin{cases} u(t) = \mathcal{Q}_\beta\left(-t^\beta(-\Delta)^{\alpha/2}\right)u_0 + \mathbb{B}(u,\rho)(t), \\ \rho(t) = \mathcal{Q}_\beta\left(-t^\beta(-\Delta^{\alpha/2}) - \xi\right)\rho_0 + \Psi(u)(t). \end{cases} \tag{2.2}$$

3 A few estimates of the Mittag-Leffler operators

By conducting a thorough study of the Mittag-Leffler families, this section aims to establish a number of estimates and properties for both the nonlinear and linear parts of system (2.2). These results provide essential tools and utilities that will be used throughout this paper.

We employ arguments similar to those applied in the proof of [15, Lemma 2.2] to demonstrate the following lemma.

Lemma 3. *Let $\alpha > 0$, $0 \leq \mu < d$ and $\nu \geq 0$. Consider the fractional heat operator $T_\alpha(t)$ defined in Fourier variables as $\widehat{T_\alpha(t)g} = \mathrm{e}^{-t|\zeta|^\alpha}\hat{g}$.*
If $r_1 \leq r_2, 1 \leq a_1 \leq a_2 \leq \infty$ and $1 \leq b \leq \infty$, then the following inequality holds

$$\|(-\triangle)^{\nu/2}T_\alpha(t)g\|_{\mathcal{N}^{r_2}_{a_2,\mu,b}} \leq Ct^{-\frac{r_2-r_1+\nu}{\alpha}-\frac{1}{\alpha}(\frac{d-\mu}{a_1}-\frac{d-\mu}{a_2})}\|g\|_{\mathcal{N}^{r_1}_{a_1,\mu,b}}. \tag{3.1}$$

Lemma 4. *Let $\alpha > 0$ and $\nu \geq 0$.*
If $r_1 \leq r_2, 1 \leq a_1 \leq a_2 \leq \infty,\ 1 \leq b \leq \infty,\ 0 \leq \mu < d$ and

$$\frac{1}{\alpha}\Big(r_1 - r_2 + \nu + \frac{d-\mu}{a_1} - \frac{d-\mu}{a_2}\Big) < 1,$$

then the following inequality holds:

$$\Big\|(-\triangle)^{\nu/2}\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)g\Big\|_{\mathcal{N}^{r_2}_{a_2,\mu,b}} \leq Ct^{-\frac{\beta}{\alpha}(r_2-r_1+\nu)-\frac{\beta}{\alpha}(\frac{d-\mu}{a_1}-\frac{d-\mu}{a_2})}\|g\|_{\mathcal{N}^{r_1}_{a_1,\mu,b}} \tag{3.2}$$

Furthermore, if $r_1 \leq r_2$,$1 \leq a_1 \leq a_2 \leq \infty$, $1 \leq b \leq \infty$ and

$$\frac{1}{\alpha}\Big(r_1 - r_2 + \nu + \frac{d-\mu}{a_1} - \frac{d-\mu}{a_2}\Big) < 2,$$

then the following inequality holds

$$\Big\|(-\triangle)^{\nu/2}\mathcal{Q}_{\beta,\beta}\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)g\Big\|_{\mathcal{N}^{r_2}_{a_2,\mu,b}} \leq Ct^{-\frac{\beta}{\alpha}(r_2-r_1+\nu)-\frac{\beta}{\alpha}(\frac{d-\mu}{a_1}-\frac{d-\mu}{a_2})}\|g\|_{\mathcal{N}^{r_1}_{a_1,\mu,b}}. \tag{3.3}$$

Proof. Let $g \in \mathcal{N}^{r_1}_{a_1,\mu,b}$. By Proposition 1 and estimate (3.1), we have:

$$\begin{aligned}
&\Big\|(-\triangle)^{\nu/2}\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)g\Big\|_{\mathcal{N}^{r_2}_{a_2,\mu,b}} \leq \int_0^\infty \mathcal{U}_\beta(s)\|(-\triangle)^{\nu/2}T_\alpha(st^\beta)g\|_{\mathcal{N}^{r_2}_{a_2,\mu,b}}ds \\
&\leq C\Big[\int_0^\infty \mathcal{U}_\beta(s)s^{-\frac{1}{\alpha}(r_2-r_1+\nu)-\frac{1}{\alpha}(\frac{d-\mu}{a_1}-\frac{d-\mu}{a_2})}ds\Big]t^{-\frac{\beta}{\alpha}(r_2-r_1+\nu)-\frac{\beta}{\alpha}\Big(\frac{d-\mu}{a_1}-\frac{d-\mu}{a_2}\Big)} \\
&\quad \times \|g\|_{\mathcal{N}^{r_1}_{a_1,\mu,b}} \leq Ct^{-\frac{\beta}{\alpha}(r_2-r_1+\nu)-\frac{\beta}{\alpha}(\frac{d-\mu}{a_1}-\frac{d-\mu}{a_2})}\|g\|_{\mathcal{N}^{r_1}_{a_1,\mu,b}},
\end{aligned}$$

where $t > 0$. This completes the proof of inequality (3.2). The proof of (3.3) follows in a similar manner and is omitted for brevity. □

The following lemma presents the estimation of the integral term in (2.1).

Lemma 5. *Let $1 \leq a \leq \infty$, $\theta \geq 0$, $\beta > 0$ and $\alpha > \theta$. Suppose r_0, $r \in \mathbb{R}$ satisfy: $-r + \alpha - \theta = -r_0$. Then, for all $f \in \mathcal{N}^{-r}_{a,\mu,1}$, there exists a constant $C > 0$ such that:*

$$\int_0^\infty \Big\|s^{\beta-1}(-\triangle)^{\theta/2}\mathcal{Q}_{\beta,\beta}\Big(-s^\beta(-\triangle)^{\alpha/2}\Big)f\Big\|_{\mathcal{N}^{-r_0}_{a,\mu,1}}ds \leq C\|f\|_{\mathcal{N}^{-r}_{a,\mu,1}}.$$

Proof. Let $f \in \mathcal{N}^{-r}_{a,\mu,1}$ and define the function G_f by:

$$G_f(s) = \left\| s^{\beta-1}(-\triangle)^{\theta/2}\mathcal{Q}_{\beta,\beta}\Big(- s^{\beta}(-\triangle)^{\alpha/2}\Big) f \right\|_{\mathcal{N}^{-r_0}_{a,\mu,1}}.$$

By applying Lemma 4 for $-r_i \leq -r_0$ $(i = 1, 2)$, we have:

$$\begin{aligned} G_f(s) &= \left\| s^{\beta-1}(-\triangle)^{\theta/2}\mathcal{Q}_{\beta,\beta}\Big(- s^{\beta}(-\triangle)^{\alpha/2}\Big) f \right\|_{\mathcal{N}^{-r_0}_{a,\mu,1}} \\ &\leq C s^{-\frac{\beta}{\alpha}(-r_0+r_i+\theta+\frac{(1-\beta)}{\beta}\alpha)} \|f\|_{\mathcal{N}^{-r_i}_{a,\mu,1}}. \end{aligned}$$

Taking, for example, $-r_1 = -r - \varepsilon$ and $-r_2 = -r + \varepsilon$ for ε small enough, since $\alpha > \theta$, we have that $-r_i < -r_0$, and defining: $\frac{1}{x_i} = \frac{\beta}{\alpha}(-r_0 + r_i + \theta)$. For $i = 1$:

$$\begin{aligned} \frac{1}{x_1} &= \frac{\beta\Big(-r_0+r_1+\theta+\frac{(1-\beta)}{\beta}\alpha\Big)}{\alpha} = \frac{\beta\Big(-r_0+r+\varepsilon+\theta+\frac{(1-\beta)}{\beta}\alpha\Big)}{\alpha} \\ &= \frac{\beta}{\alpha}\Big(\alpha + \varepsilon + \frac{(1-\beta}{\beta}\alpha\Big) = 1 + \frac{\beta}{\alpha}\varepsilon > 1. \end{aligned}$$

For $i = 2$:

$$\begin{aligned} 0 < \frac{1}{x_2} &= \frac{\beta\Big(-r_0+r_2+\theta+\frac{(1-\beta)}{\beta}\alpha\Big)}{\alpha} = \frac{\beta\Big(-r_0+r-\varepsilon+\theta+\frac{(1-\beta)}{\beta}\alpha\Big)}{\alpha} \\ &= \frac{\beta}{\alpha}\Big(\alpha - \varepsilon + \frac{(1-\beta)}{\beta}\alpha\Big) = 1 - \frac{\beta}{\alpha}\varepsilon < 1. \end{aligned}$$

Hence, $0 < x_1 < 1 < x_2 < \infty$. Setting $\psi = 1/2$, we have:

$$1 = \frac{\psi}{x_1} + \frac{1-\psi}{x_2} \quad \text{and} \quad -r = (1-\psi)(-r_1) + \psi(-r_2).$$

Then, for $i = 1, 2$, it follows that: $G_f \in L^{x_i,\infty}(0,\ \infty)$ with the estimate:

$$\|G_f\|_{L^{x_i,\infty}(0,\infty)} \leq C\|f\|_{\mathcal{N}^{-r_i}_{a,\mu,1}}.$$

Using interpolation in Lorentz spaces, we obtain:

$$\|G_f\|_{L^1(0,\infty)} \leq C\|f\|_{\mathcal{N}^{-r}_{a,\mu,1}}.$$

This completes the proof. □

Remark 1. Under the same conditions as in Lemma 5, we can also derive the following estimate:

$$\int_0^\infty \left\| s^{\beta-1}(-\triangle)^{\theta/2}\mathcal{Q}_{\beta,\beta}\Big(- s^{\beta}((-\triangle)^{\alpha/2} - \xi)\Big) f \right\|_{\mathcal{N}^{-r_0}_{a,\mu,1}} ds \leq C\|f\|_{\mathcal{N}^{-r}_{a,\mu,1}}.$$

The following lemma provides an estimate for the operator $\mathcal{A}_1$, defined by:

$$\mathcal{A}_1\ (f) := \int_0^\infty s^{\beta-1}\nabla\cdot\mathcal{Q}_{\beta,\beta}\Big(- s^{\beta}(-\triangle)^{\alpha/2}\Big) f(\cdot,\ s)ds.$$

Lemma 6. *Let $1 < a \leq \infty,\ \beta > 0,\ \alpha > 1$, and $r_0,\ r \in \mathbb{R}$ satisfy: $-r+\alpha-1 = -r_0$. Then, there exists a constant $C > 0$ such that:*

$$\|\mathcal{A}_1\ (f)\|_{\mathcal{N}^{r}_{a,\mu,\infty}} \leq C \sup_{t>0} \|f(t)\|_{\mathcal{N}^{r_0}_{a,\mu,\infty}},$$

for all $f \in L^\infty\Big((0,\ \infty); \mathcal{N}^{r_0}_{a,\mu,\infty}\Big)$.

Proof. Using Lemma 5 and duality, we get:

$$\begin{aligned}
\|\mathcal{A}_1\ (f)\|_{\mathcal{N}^{r}_{a,\mu,\infty}} &= \sup_{\|g\|_{\mathcal{N}^{-r}_{a',\mu,\infty}}=1} \Big|\langle \mathcal{A}_1\ (f), g\rangle\Big| \\
\leq & C \sup_{\|g\|_{\mathcal{N}^{-r}_{a',\mu,1}}=1} \int_0^\infty \Big|\langle s^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}\Big(-s^{\beta}(-\triangle)^{\alpha/2}\Big)f(s), g\rangle\Big| ds \\
\leq & C \sup_{\|g\|_{\mathcal{N}^{-r}_{a',\mu,1}}=1} \int_0^\infty \Big|\langle f(s), s^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}\Big(-s^{\beta}(-\triangle)^{\alpha/2}\Big)g\rangle\Big| ds \\
\leq & C \sup_{\|g\|_{\mathcal{N}^{-r}_{a',\mu,1}}=1} \int_0^\infty \|f(s)\|_{\mathcal{N}^{r_0}_{a,\mu,\infty}} \|s^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}\Big(-s^{\beta}(-\triangle)^{\alpha/2}\Big)g\|_{\mathcal{N}^{-r_0}_{a',\mu,1}} ds \\
\leq & C \sup_{s>0} \|f(s)\|_{\mathcal{N}^{r_0}_{a,\mu,\infty}} \sup_{\|g\|_{\mathcal{N}^{-r}_{a',\mu,1}}=1} \int_0^\infty \|s^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}\Big(-s^{\beta}(-\triangle)^{\alpha/2}\Big)g\|_{\mathcal{N}^{-r_0}_{a',\mu,1}} ds \\
\leq & C \sup_{s>0} \|f(s)\|_{\mathcal{N}^{r_0}_{a,\mu,\infty}} \sup_{\|g\|_{\mathcal{N}^{-r}_{a',\mu,1}}=1} \|g\|_{\mathcal{N}^{-r_0}_{a',\mu,1}} \leq C \sup_{s>0} \|f(s)\|_{\mathcal{N}^{r_0}_{a,\mu,\infty}}.
\end{aligned}$$

This completes the proof. □

We now estimate the operator $\mathcal{A}_2$, which is defined by:

$$\mathcal{A}_2(u) := \int_0^\infty \tau^{\beta-1}\mathcal{Q}_{\beta,\beta}(-\tau^{\beta}((-\triangle)^{\alpha/2}-\xi))u(\tau)d\tau.$$

This operator plays a significant role in the analysis of the mild formulation and will be utilized in deriving key estimates.

Lemma 7. *Let $\beta > 0$, $1 < a \leq b$, $0 \leq \mu < d$ and $\alpha > 0$. Then, there exists a constant $c > 0$ such that:*

$$\|\mathcal{A}_2(u)\|_{\mathcal{N}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}_{b,\mu,\infty}} \leq C \sup_{t>0} \|u(t)\|_{\mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}_{a,\mu,\infty}},$$

for all $u \in L^\infty\Big((0,\ \infty); \mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}_{a,\mu,\infty}\Big)$.

Proof. Using Lemma 5 and duality, we have:

$$\begin{aligned}
\|\mathcal{A}_2(u)\|_{\mathcal{N}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}_{b,\mu,\infty}} &= \sup_{\|g\|_{\mathcal{N}^{-(2-\alpha-\alpha_1+\frac{d-\mu}{b})}_{b',\mu,\infty}}=1} \left|\langle \mathcal{A}_2\ (u), g\rangle\right| \\
&= \sup_{\|g\|_{\mathcal{N}^{-(2-\alpha-\alpha_1+\frac{d-\mu}{b})}_{b',\mu,\infty}}=1} \int_0^\infty |\langle s^{\beta-1}\nabla\cdot\mathcal{Q}_{\beta,\beta}\Big(-s^\beta(-\triangle)^{\alpha/2}\Big)u(s), g\rangle|ds \\
&= \sup_{\|g\|_{\mathcal{N}^{-(2-\alpha-\alpha_1+\frac{d-\mu}{b})}_{b',\mu,\infty}}=1} \int_0^\infty |\langle u(s), s^{\beta-1}\nabla\cdot\mathcal{Q}_{\beta,\beta}\Big(-s^\beta(-\triangle)^{\alpha/2}\Big)g\rangle|ds \\
&\le \sup_{s>0}\|u(s)\|_{\mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}_{a,\mu,\infty}} \sup_{\|g\|_{\mathcal{N}^{-(2-\alpha-\alpha_1+\frac{d-\mu}{b})}_{b',\mu,\infty}}=1} \|g\|_{\mathcal{N}^{-(2-\alpha-\alpha_1+\frac{d-\mu}{a})}_{a',\mu,\infty}} ds \\
&\le \sup_{s>0}\|u(s)\|_{\mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}_{a,\mu,\infty}}.
\end{aligned}$$

This completes the proof. □

4 Product and bilinear estimates

In order to present the main results, we introduce and demonstrate two fundamental lemmas, referred to as the product estimate and the bilinear estimate. These lemmas are essential tools in establishing the core results of this study.

Lemma 8. *(Product estimate). Let $d \ge 1$, $0 \le \mu < d$, $\alpha_1 \in [0,\ d)$, and let the parameters satisfy:*

$$\frac{6(d-\mu)}{5(d-\mu)+\alpha_1} < a \le b \le a',$$

$$\max\left\{1, 1-\frac{d-\mu}{2}-\frac{\alpha_1}{2}+\frac{d-\mu}{a}\right\} < \alpha < 1+\frac{(d-\mu)-\alpha_1}{3}.$$

Additionally, let $\gamma_1,\ \gamma_2 \ge 0$ be sufficiently small. Then, for $\phi \in \mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}+\gamma_1}_{a,\mu,\infty}$ and $\psi \in \mathcal{N}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}+\gamma_2}_{b,\mu,\infty}$, we have:

$$\phi F(\psi) \in \mathcal{N}^{3-3\alpha-\alpha_1+\frac{d-\mu}{a}+\gamma_1+\gamma_2}_{a,\mu,\infty}.$$

Furthermore, the following estimate holds:

$$\|\phi F(\psi)\|_{\mathcal{N}^{3-3\alpha-\alpha_1+\frac{d-\mu}{a}+\gamma_1+\gamma_2}_{a,\mu,\infty}} \le C\|\phi\|_{\mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}+\gamma_1}_{a,\mu,\infty}}\|\psi\|_{\mathcal{N}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}+\gamma_2}_{b,\mu,\infty}}. \tag{4.1}$$

Proof. To demonstrate this Lemma, let us denote

$$r_1 = 2-2\alpha-\alpha_1+\frac{d-\mu}{a},\quad r_0 = 3-3\alpha-\alpha_1+\frac{d-\mu}{a},\quad r_2 = 2-\alpha-\alpha_1+\frac{d-\mu}{b}.$$

By the Bony's decomposition, we write:

$$\begin{aligned}\triangle_i\Big(\phi F(\psi)\Big) = &\sum_{|j-i|\le 4}\triangle_i\Big(S_{j-2}\phi\triangle_i F(\psi)\Big) + \sum_{|j-i|\le 4}\triangle_i\Big(S_{j-2}F(\psi)\triangle_j f\Big)\\ &+ \sum_{j\ge i-2}\triangle_i\Big(\triangle_j\phi\widetilde{\triangle}_j F(\psi)\Big) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3. \end{aligned}\tag{4.2}$$

Here:

- $\mathcal{J}_1$ corresponds to the low-frequency interaction between ϕ and $F(\psi)$.
- $\mathcal{J}_2$ corresponds to the low-frequency interaction between $F(\psi)$ and ϕ.
- $\mathcal{J}_3$ corresponds to the high-frequency interaction between ϕ and $F(\psi)$.

In order to estimate $\mathcal{J}_1$, let a^* such that $\frac{1}{a} = \frac{1}{a^*} + \frac{1}{b}$, then,

$$\begin{aligned}\|\mathcal{J}_1\|_{\mathcal{M}^\mu_a} &\le C\sum_{|j-i|\le 4}\|S_{j-2}\phi\|_{\mathcal{M}^\mu_{a^*}}\|\triangle_j F(\psi)\|_{\mathcal{M}^\mu_b}\\ &\le C\sum_{|j-i|\le 4}\Bigg(\sum_{k\le j-2}\|\triangle_k\phi\|_{\mathcal{M}^\mu_{a^*}}\Bigg)\|F(\triangle_j\psi)\|_{\mathcal{M}^\mu_b}\\ &\le C\sum_{|j-i|\le 4}\Bigg(\sum_{k\le j-2}2^{k(\frac{d-\mu}{a}-\frac{d-\mu}{a^*})}\|\triangle_k\phi\|_{\mathcal{M}^\mu_a}\Bigg)2^{j(1-\alpha_1)}\|\triangle_j\psi\|_{\mathcal{M}^\mu_b}\\ &\le C\|\phi\|_{\mathcal{N}^{r_1+\gamma_1}_{a,\mu,\infty}}\|\psi\|_{\mathcal{N}^{r_2+\gamma_2}_{b,\mu,\infty}}\sum_{|j-i|\le 4}\Bigg(\sum_{k\le j-2}2^{k(-2+2\alpha+\alpha_1-\frac{d-\mu}{a^*}-\gamma_1)}\Bigg)\\ &\quad\times 2^{j(1-\alpha_1-2+\alpha+\alpha_1-\frac{d-\mu}{b}-\gamma_2)} \le C\|\phi\|_{\mathcal{N}^{r_1+\gamma_1}_{a,\mu,\infty}}\|\psi\|_{\mathcal{N}^{r_2+\gamma_2}_{b,\mu,\infty}}\sum_{|j-i|\le 4}\\ &\quad\times\Bigg(\sum_{k\le j-2}2^{k(-2+2\alpha+\alpha_1-\frac{d-\mu}{a^*}-\gamma_1)}\Bigg)2^{j(-1+\alpha-\frac{d-\mu}{b}-\gamma_2)}.\end{aligned}$$

Note that $-2+2\alpha+\alpha_1+\frac{d-\mu}{b}-\frac{d-\mu}{a}-\gamma_1 > 0$ holds for small enough $\gamma_1 > 0$. Since $\dfrac{d-\mu}{b} \ge \dfrac{d-\mu}{a^*}$, it suffices to verify: $-2+2\alpha+\alpha_1+\frac{d-\mu}{a^*}-\frac{d-\mu}{a} > 0$. which reduces to:

$$\alpha > 1 - \frac{d-\mu}{2} - \frac{\alpha_1}{2} + \frac{d-\mu}{a}.$$

Therefore,

$$\begin{aligned}\|\mathcal{J}_1\|_{\mathcal{M}^\mu_a} &\le C\|\phi\|_{\mathcal{N}^{r_1+\gamma_1}_{a,\mu,\infty}}\|\psi\|_{\mathcal{N}^{r_2+\gamma_2}_{b,\mu,\infty}}\sum_{|j-i|\le 4}2^{j(-2+2\alpha+\alpha_1+\frac{d-\mu}{b}-\frac{d-\mu}{a}-\gamma_1)}\\ &\quad\times 2^{j(-1+\alpha-\frac{d-\mu}{b}-\gamma_2)}\\ &\le C\|\phi\|_{\mathcal{N}^{r_1+\gamma_1}_{a,\mu,\infty}}\|\psi\|_{\mathcal{N}^{r_2+\gamma_2}_{b,\mu,\infty}}2^{i\left(-2+2\alpha+\alpha_1+\frac{d-\mu}{b}-\frac{d-\mu}{a}-\gamma_1-1+\alpha-\frac{d-\mu}{b}-\gamma_2\right)}\\ &\le C\|\phi\|_{\mathcal{N}^{r_1+\gamma_1}_{a,\mu,\infty}}\|\psi\|_{\mathcal{N}^{r_2+\gamma_2}_{b,\mu,\infty}}2^{i\left(-3+3\alpha+\alpha_1-\frac{d-\mu}{a}-\gamma_1-\gamma_2\right)}.\end{aligned}$$

Thus, we obtain:

$$\|\mathcal{J}_1\|_{\mathcal{M}_a^\mu} \leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} 2^{-i\left(r_0-\gamma_1+\gamma_2\right)}. \tag{4.3}$$

This completes the estimate for $\mathcal{J}_1$.

To estimate $\mathcal{J}_2$, we follow a similar approach:

$$\begin{aligned}
\|\mathcal{J}_2\|_{\mathcal{M}_a^\mu} &\leq C \sum_{|j-i|\leq 4} \|S_{j-2}F(\psi)\|_{\mathcal{M}_\infty^\mu} \|\triangle_j\phi\|_{\mathcal{M}_a^\mu} \\
&\leq C \sum_{|j-i|\leq 4} \left(\sum_{k\leq j-2} \|\triangle_k F(\psi)\|_{\mathcal{M}_\infty^\mu} \right) \|\triangle_j\phi\|_{\mathcal{M}_a^\mu} \\
&\leq C \sum_{|j-i|\leq 4} \left(\sum_{k\leq j-2} 2^{k\left(\frac{d-\mu}{b}\right)} \|\triangle_k F(\psi)\|_{\mathcal{M}_b^\mu} \right) \|\triangle_j\phi\|_{\mathcal{M}_a^\mu} \\
&\leq C \sum_{|j-i|\leq 4} \left(\sum_{k\leq j-2} 2^{k\left(\frac{d-\mu}{b}+1-\alpha_1\right)} \|\triangle_k \psi\|_{\mathcal{M}_b^\mu} \right) \|\triangle_j\phi\|_{\mathcal{M}_a^\mu} \\
&\leq C\|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} \sum_{|j-i|\leq 4} \left(\sum_{k\leq j-2} 2^{k\left(\frac{d-\mu}{b}+1-\alpha_1-2+\alpha+\alpha_1-\frac{d-\mu}{b}-\gamma_2\right)} \right) \|\triangle_j\phi\|_{\mathcal{M}_a^\mu} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} \sum_{|j-i|\leq 4} \left(\sum_{k\leq j-2} 2^{k(-1+\alpha-\gamma_2)} \right) 2^{j\left(-2+2\alpha+\alpha_1-\frac{d-\mu}{a}\right)} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} \sum_{|j-i|\leq 4} 2^{k(-1+\alpha-\gamma_2)} 2^{j\left(-2+2\alpha+\alpha_1-\frac{d-\mu}{a}\right)} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} 2^{i\left(-3+3\alpha+\alpha_1-\frac{d-\mu}{a}-\gamma_1-\gamma_2\right)}.
\end{aligned}$$

For $\alpha > 1$, we can take γ_2 to be small enough so that $-1+\alpha-\gamma_2 > 0$. This ensures the convergence of the inner sum. Therefore, we conclude:

$$\|\mathcal{J}_2\|_{\mathcal{M}_a^\mu} \leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} 2^{-i(r_0+\gamma_1+\gamma_2)}. \tag{4.4}$$

This completes the estimate for $\mathcal{J}_2$.

Under the given conditions, we know: $-3+3\alpha+\alpha_1-d+\mu < 0$, and consequently: $-3+3\alpha+\alpha_1-(d-\mu)-\gamma_1-\gamma_2 < 0$.

We now estimate $\mathcal{J}_3$:

$$\begin{aligned}
\|\mathcal{J}_3\|_{\mathcal{M}_1^\mu} &\leq C \sum_{j\geq i-2} \|\triangle_j\phi\widetilde{\triangle}_j F(\psi)\|_{\mathcal{M}_1^\mu} \leq C \sum_{j\geq i-2} \|\triangle_j\phi\|_{\mathcal{M}_a^\mu} \|\widetilde{\triangle}_j F(\psi)\|_{\mathcal{M}_{a'}^\mu} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \sum_{j\geq i-2} 2^{j\left(-2+2\alpha+\alpha_1-\frac{d-\mu}{a}-\gamma_1\right)} \|\widetilde{\triangle}_j F(\psi)\|_{\mathcal{M}_{a'}^\mu} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \sum_{j\geq i-2} 2^{j(-2+2\alpha+\alpha_1-\frac{d-\mu}{a}-\gamma_1)} 2^{j(1-\alpha_1)} \|\widetilde{\triangle}_j \psi\|_{\mathcal{M}_{a'}^\mu} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \sum_{j\geq i-2} 2^{j\left(-2+2\alpha+\alpha_1-\frac{d-\mu}{a}-\gamma_1\right)} 2^{j(1-\alpha_1)} 2^{j\left(\frac{d-\mu}{b}-\frac{d-\mu}{a'}\right)} \|\widetilde{\triangle}_j \psi\|_{\mathcal{M}_b^\mu} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} \sum_{j\geq i-2} 2^{j(-3+3\alpha+\alpha_1-(d-\mu)-\gamma_1-\gamma_2)} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}} \|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}} 2^{i(-3+3\alpha+\alpha_1-(d-\mu)-\gamma_1-\gamma_2)}.
\end{aligned}$$

Then,

$$
\begin{aligned}
\|\mathcal{J}_3\|_{\mathcal{M}_a^\mu} &\leq C2^{\left(\frac{d-\mu}{1}-\frac{d-\mu}{a}\right)}\|\mathcal{J}_3\|_{\mathcal{M}_1^\mu} \\
&\leq C2^{\left(\frac{d-\mu}{1}-\frac{d-\mu}{a}\right)}\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}}\|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}}2^{i(-3+3\alpha+\alpha_1-(d-\mu)-\gamma_1-\gamma_2)} \\
&\leq C\|\phi\|_{\mathcal{N}_{a,\mu,\infty}^{r_1+\gamma_1}}\|\psi\|_{\mathcal{N}_{b,\mu,\infty}^{r_2+\gamma_2}}2^{-i(r_0+\gamma_1+\gamma_2)}. \qquad (4.5)
\end{aligned}
$$

By computing the $\mathcal{M}_a^\mu$ norm in (4.2) and applying the estimates from (4.3), (4.4) and (4.5), we obtain the desired result. □

We now establish a crucial bilinear estimate for the operator $\mathbb{B}(u,\rho)$.

Lemma 9. *(Bilinear estimate). Let $d \geq 1$, $0 \leq \mu < d$, $0 < T \leq \infty$, $\alpha_1 \in [0,\ d)$ and $\beta > 0$. Assume the parameters satisfy:*

$$
\begin{aligned}
&\frac{6(d-\mu)}{5(d-\mu)+\alpha_1} < a \leq b \leq a', \\
&\max\left\{1, 1-\frac{d-\mu}{2}-\frac{\alpha_1}{2}+\frac{d-\mu}{a}\right\} < \alpha < 1+\frac{d-\mu-\alpha_1}{3}.
\end{aligned}
$$

Then, there exists a constant $\varepsilon > 0$ (independent of T) such that:

$$
\begin{aligned}
&\|\mathbb{B}(u,\rho)\|_{L^\infty\left((0,T),\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}\right)} \\
&\leq \varepsilon\|u\|_{L^\infty\left((0,T),\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}\right)}\|\rho\|_{L^\infty\left((0,T),\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}\right)},
\end{aligned} \qquad (4.6)
$$

for all $u \in L^\infty\left((0,T),\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}\right)$ and $\rho \in L^\infty\left((0,T),\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}\right)$.

Proof. Let $0 < T \leq \infty$ and $t \in (0,\ T)$. The bilinear term $\mathbb{B}(u,\rho)$ can be written as

$$
\mathbb{B}(u,\rho)(t) = -\int_0^t (t-s)^{\beta-1}\nabla\cdot\mathcal{Q}_{\beta,\beta}(-(t-s)^\beta(-\triangle)^{\alpha/2})(uF(\rho))ds \;=\; \mathcal{A}_1(h_t),
$$

where the function $h_t(\cdot,s)$ is defined as:

$$
\begin{aligned}
h_t(\cdot,\ s) &= (uF(\rho))(\cdot,\ t-s)\ ,\ \text{a.e.}\quad s \in (0,\ t)\ , \\
h_t(\cdot,\ s) &= 0, \text{ a.e.}\quad s \in (t,\ \infty)\ .
\end{aligned}
$$

Using Lemma 6 (with $r = 2-2\alpha-\alpha_1+\frac{d-\mu}{a}$ and $r_0 = 3-3\alpha-\alpha_1+\frac{d-\mu}{a}$), we have:

$$
\|\mathcal{A}_1\ (h_t)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}} \leq C\sup_{s>0}\|h_t(s)\|_{\mathcal{N}_{a,\mu,\infty}^{3-3\alpha-\alpha_1+\frac{d-\mu}{a}}}.
$$

Applying the Product Estimate from inequality (4.1) (with $\alpha_1 = \alpha_2 = 0$), we obtain:

$$
\begin{aligned}
\sup_{0<s<T}\|h_t(s)\|_{\mathcal{N}_{a,\mu,\infty}^{3-3\alpha-\alpha_1+\frac{d-\mu}{a}}} &= \sup_{0<s<t<T}\|(uF(\rho))(\cdot,t-s)\|_{\mathcal{N}_{a,\mu,\infty}^{3-3\alpha-\alpha_1+\frac{d-\mu}{a}}} \\
&\leq C\sup_{0<s<t<T}\|u(\cdot,t-s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}}\|\rho(\cdot,\ t-s)\|_{\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}} \\
&\leq C\sup_{0<s<t<T}\|u(\cdot,t-s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}}\sup_{0<s<t<T}\|\rho(\cdot,\ t-s)\|_{\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}} \\
&\leq C\sup_{0<s<T}\|u(\cdot,\ s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}}\sup_{0<s<T}\|\rho(\cdot,\ s)\|_{\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}}.
\end{aligned}
$$

Thus, we can conclude:

$$\sup_{0<s<T} \|\mathbb{B}(u,\rho)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}} \leq \varepsilon \sup_{0<s<T} \|u\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}} \sup_{0<s<T} \|\rho\|_{\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}}.$$

This completes the proof. □

We now proceed to derive an essential estimate for the operator $\Psi(u)$, which plays a key role in analyzing the linear part of our system and ensuring the desired regularity of solutions.

Lemma 10. *Let* $1 < a \leq b,\ 0 \leq \mu < d,\ \beta > 0$ *and* $\alpha > 0$. *Then, there exists a constant* $\varepsilon > 0$ *(independent of* T*) such that:*

$$\|\Psi(u)\|_{L^\infty\left((0,T);\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}\right)} \leq \varepsilon \|u(t)\|_{L^\infty\left((0,T);\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}\right)},$$

for all $u \in L^\infty\left((0,\ T);\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}\right)$.

Proof. Let $0 < T \leq \infty$ and $t \in (0,\ T)$. The operator $\Psi(u)$ can be written as

$$\Psi(u) = \int_0^t (t-s)^{\beta-1} \mathcal{Q}_{\beta,\beta}(-(t-s)^\beta((-\triangle)^{\alpha/2} - \xi))u(s)ds = -\mathcal{A}_2(h_t),$$

where the function $h_t(\cdot,\ s)$ is defined as:

$$\begin{aligned} h_t(\cdot,\ s) &= u(\cdot,\ t-s)\ ,\ \text{a.e.}\quad s \in (0,\ t)\ , \\ h_t(\cdot,\ s) &= 0,\ \text{a.e.}\quad s \in (t,\ \infty)\ . \end{aligned}$$

By Lemma 7, we get:

$$\|\mathcal{A}_2(h_t)\|_{\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}} \leq C \sup_{s>0} \|h_t(s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}}.$$

By substituting the definition of h_t, we find:

$$\begin{aligned} \sup_{0<s<T} \|h_t(s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}} &= \sup_{0<s<t<T} \|u(\cdot,t-s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}} \\ &\leq \sup_{0<s<T} \|u(\cdot,\ s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}}. \end{aligned}$$

Combining the two inequalities, we conclude:

$$\sup_{0<t<T} \|\Psi(u)\|_{\mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}} \leq \varepsilon \sup_{0<s<T} \|u(\cdot,\ s)\|_{\mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}}.$$

This completes the proof. □

5 Main results

In this section, we establish the existence and uniqueness of a mild solution to the system (1.2). We begin with the result concerning the existence of solutions.

Theorem 1. *Let $d \geq 1$, $0 \leq \mu < d$, $\alpha_1 \in [0,\ d)$, and $\beta > 0$. Assume that the parameters satisfy:*

$$\frac{6(d-\mu)}{5(d-\mu)+\alpha_1} < a \leq b \leq a',$$
$$\max\left\{1, 1 - \frac{(d-\mu)}{2} - \frac{\alpha_1}{2} + \frac{(d-\mu)}{a}\right\} < \alpha < 1 + \frac{(d-\mu)-\alpha_1}{3}.$$

There exist $\kappa > 0$ and $\lambda > 0$ such that, if:

$$\|u_0\|_{\mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}_{a,\mu,\infty}} < \kappa, \quad and \quad \|\rho_0\|_{\mathcal{N}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}_{b,\mu,\infty}} < \kappa,$$

then there exists a unique mild solution $[u,\ \rho]$ to the system (1.2)*, satisfying:*

$$\|u\|_{L^\infty\left((0,\infty);\mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}_{a,\mu,\infty}\right)} < \lambda, \quad and \quad \|\rho\|_{L^\infty\left((0,\infty);\mathcal{N}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}_{b,\mu,\infty}\right)} < \lambda.$$

Proof. For the sake of simplicity, we denote:

$$\mathcal{X} := L^\infty\Big((0,\infty);\mathcal{N}^{2-2\alpha-\alpha_1+\frac{d-\mu}{a}}_{a,\mu,\infty}\Big), \qquad \mathcal{Y} := L^\infty\Big((0,\infty);\mathcal{N}^{2-\alpha-\alpha_1+\frac{d-\mu}{b}}_{b,\mu,\infty}\Big).$$

To establish Theorem 1, we analyze the following iterative system:

$$u^1 := \mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0, \qquad \rho^1 := \mathcal{Q}_\beta\Big(-t^\beta((-\triangle)^{\alpha/2}-\xi)\Big)\rho_0,$$

and for $d \geq 1$:

$$u^{d+1} := u^1 + \mathbb{B}(u^d,\ \rho^d), \qquad \rho^{d+1} := \rho^1 + \Psi(u^{d+1})\ .$$

Using Lemma 4, we deduce:

$$\|u^1\|_{\mathcal{X}} = \|\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0\|_{\mathcal{X}} \leq C_1\|u_0\|_{\mathcal{X}},$$
$$\|\rho^1\|_{\mathcal{Y}} = \|\mathcal{Q}_\beta(-t^\beta\big((-\triangle)^{\alpha/2}\big)-\xi\big)\rho_0\|_{\mathcal{Y}} \leq C_2\|\rho_0\|_{\mathcal{Y}}.$$

Furthermore, by applying Lemmas 9 and 10, we obtain:

$$\begin{aligned}\|u^{d+1}\|_{\mathcal{X}} &= \|u^1 + \mathbb{B}(u^d,\ \rho^d)\|_{\mathcal{X}} \leq \|u^1\|_{\mathcal{X}} + \|\mathbb{B}(u^d,\ \rho^d)\|_{\mathcal{X}} \\ &\leq C_1\|u_0\|_{\mathcal{X}} + \varepsilon\|u^d\|_{\mathcal{X}}\|\rho^d\|_{\mathcal{Y}},\end{aligned}$$

and

$$\begin{aligned}\|\rho^{d+1}\|_{\mathcal{Y}} &= \|\rho^1 + \Psi(u^{d+1})\|_{\mathcal{Y}} \leq \|\rho^1\|_{\mathcal{Y}} + \|\Psi(u^{d+1})\|_{\mathcal{Y}} \\ &\leq C_2\|\rho_0\|_{\mathcal{Y}} + \varepsilon\|u^{d+1}\|_{\mathcal{X}}.\end{aligned}$$

Let $0<\kappa<\dfrac{1}{2\varepsilon}$, and suppose u_0 and ρ_0 satisfy the following conditions:

$$C_1\|u_0\|_{\mathcal{X}}\leq \frac{\kappa}{4C}<\frac{\kappa}{2C},\quad C_2\|\rho_0\|_{\mathcal{Y}}\leq \frac{\kappa}{2}<\kappa.$$

Then, for u^2 and ρ^2 we have:

$$\begin{aligned}\|u^2\|_{\mathcal{X}}&<\frac{\kappa}{4C}+\varepsilon\frac{\kappa}{2C}\kappa<\frac{\kappa}{4C}+\frac{\kappa}{4C}=\frac{\kappa}{2C},\\ \|\rho^2\|_{\mathcal{Y}}&<\frac{\kappa}{2}+C\frac{\kappa}{2C}=\kappa.\end{aligned}$$

By mathematical induction, we can demonstrate that:

$$\|u^{d+1}\|_{\mathcal{X}}<\kappa/(2C),\quad \|\rho^{d+1}\|_{\mathcal{Y}}<\kappa.$$

Next, we establish that the sequences (u^d) and (ρ^d) are Cauchy sequences in their respective spaces. In fact, we have:

$$\begin{aligned}u^{d+1}-u^d &=\mathbb{B}(u^d,\ \rho^d)-\mathbb{B}(u^{d-1},\ \rho^{d-1})\\ &=\mathbb{B}(u^d-u^{d-1},\ \rho^d)+\mathbb{B}(u^{d-1},\ \rho^d-\rho^{d-1}).\end{aligned}$$

Taking the norm in $\mathcal{X}$ and applying Lemma 9, we get:

$$\begin{aligned}\|u^{d+1}-u^d\|_{\mathcal{X}} &\leq\varepsilon\Big(\|u^d-u^{d-1}\|_{\mathcal{X}}\|\rho^d\|_{\mathcal{Y}}+\|u^{d-1}\|_{\mathcal{X}}\|\rho^d-\rho^{d-1}\|_{\mathcal{Y}}\Big)\\ &\leq\varepsilon\Big(\kappa\|u^d-u^{d-1}\|_{\mathcal{X}}+\tfrac{\kappa}{2C}\|\rho^d-\rho^{d-1}\|_{\mathcal{Y}}\Big).\end{aligned}\tag{5.1}$$

On the other hand, since

$$\rho^{d+1}-\rho^d=\Psi(u^{d+1}-u^d).$$

Taking the norm in $\mathcal{Y}$ and applying Lemma 10, we get:

$$\|\rho^{d+1}-\rho^d\|_{\mathcal{Y}}\leq C\|u^{d+1}-\eta^d\|_{\mathcal{X}}.\tag{5.2}$$

Substituting inequality (5.1) into (5.2), we have:

$$\begin{aligned}\|u^{d+1}-u^d\|_{\mathcal{X}} &\leq\varepsilon\left(\kappa\|u^d-u^{d-1}\|_{\mathcal{X}}+\tfrac{\kappa}{2C}C\|u^d-u^{d-1}\|_{\mathcal{X}}\right)\\ &\leq\tfrac{3\varepsilon\kappa}{2}\|u^d-u^{d-1}\|_{\mathcal{X}}\leq C(\kappa)\|u^d-u^{d-1}\|_{\mathcal{X}}.\end{aligned}\tag{5.3}$$

With an additional condition on κ (if necessary), we ensure that $C(\kappa)<1$, and from (5.3), it follows that (u^d) is Cauchy in $\mathcal{X}$. Similarly, from inequality (5.2), we conclude that the sequence (ρ^d) is Cauchy in $\mathcal{Y}$.
Let u and ρ denote the limits of these sequences, such that:

$$u^d\to u,\qquad \rho^d\to\rho.$$

We now verify that the limits u and ρ satisfy the mild formulation. For u, we have:

$$\begin{aligned}0 &\leq\|u-\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0-\mathbb{B}(u,\ \rho)\|_{\mathcal{X}}\\ &\leq\|u-\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0-\mathbb{B}(u,\ \rho)+u^d-u^d\|_{\mathcal{X}}\\ &\leq\|u-u^d\|_{\mathcal{X}}+\|u^d-\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0-\mathbb{B}(u,\ \rho)\|_{\mathcal{X}}\\ &\leq\|u-u^d\|_{\mathcal{X}}+\|-\mathbb{B}(u,\ \rho)+\mathbb{B}(u^{d-1},\ \rho^{d-1})\|_{\mathcal{X}}\\ &\leq\|u-u^d\|_{\mathcal{X}}+\|\mathbb{B}(u^{d-1}-u,\ \rho^{d-1})+\mathbb{B}(u,\ \rho^{d-1}-\rho)\|_{\mathcal{X}}\\ &\leq\|u-u^d\|_{\mathcal{X}}+\varepsilon\left(\kappa\|u^{d-1}-u\|_{\mathcal{X}}+\tfrac{\kappa}{2C}\|\rho^{d-1}-\rho\|_{\mathcal{Y}}\right)\to 0.\end{aligned}\tag{5.4}$$

For ρ, we similarly have:

$$\begin{aligned} 0 &\le \|\rho - \mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2} - \xi))\rho_0 - \Psi(u)\|_{\mathcal{Y}} \\ &\le \|\rho - \mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}) - \xi)\rho_0 - \Psi(u) + \rho^d - \rho^d\|_{\mathcal{Y}} \\ &\le \|\rho - \rho^d\|_{\mathcal{Y}} + \|\rho^d - \mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2} - \xi))\rho_0 - \Psi(u)\|_{\mathcal{Y}} \\ &\le \|\rho - \rho^d\|_{\mathcal{Y}} + \|\Psi(u^d - u)\|_{\mathcal{Y}} \\ &\le \|\rho - \rho^d\|_{\mathcal{Y}} + C\|u^d - u\|_{\mathcal{X}} \to 0. \end{aligned} \tag{5.5}$$

From the above estimates (5.4) and (5.5), we conclude that:

- u satisfies the mild formulation for u.
- ρ satisfies the mild formulation for ρ.

Thus, $[u, \rho]$ is indeed a mild solution of the system (1.2).

To prove uniqueness, assume there exist two mild solutions $[u,\ \rho]$ and $[\tilde{u}, \tilde{\rho}]$ corresponding to the same initial data under the conditions of Theorem 1. Using the same approach as in inequality (5.3), we have:

$$\|u - \tilde{u}\|_{\mathcal{X}} \le C(\kappa)\|u - \tilde{u}\|_{\mathcal{X}}.$$

Since $C(\kappa) < 1$, it follows that:

$$u - \tilde{u} = 0, \text{ which implies } u = \tilde{u}.$$

Similarly, using the formulation for $\rho =$, we conclude: $\rho = \tilde{\rho}$. Then, the existence and uniqueness of a mild solution $[u, \rho]$ to the system (1.2) have been established under the stated conditions. □

We now turn to the uniqueness result.

Theorem 2. *Let $d \ge 1$, $0 \le \mu < d$, $0 < T \le \infty$, $\alpha_1 \in [0,\ d)$, and $\beta > 0$. Suppose the parameters satisfy:*

$$\begin{aligned} &\frac{6(d-\mu)}{5(d-\mu)+\alpha_1} < a \le b \le a', \\ &\max\left\{1, 1 - \frac{d-\mu}{2} - \frac{\alpha_1}{2} + \frac{d-\mu}{a}\right\} < \alpha < 1 + \frac{d-\mu-\alpha_1}{3}. \end{aligned}$$

If $[u^1, \rho^1]$ and $[u^2, \rho^2]$ are two mild solutions of (1.2) in

$$C\left([0,\ T); \mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_{1+\frac{d-\mu}{a}}}\right) \times C\left([0,\ T); \mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_{1+\frac{d-\mu}{b}}}\right),$$

then

$$[u^1(t), \rho^1(t)] = [u^2(t), \rho^2(t)] \quad in \quad \mathcal{N}_{a,\mu,\infty}^{2-2\alpha-\alpha_{1+\frac{d-\mu}{a}}} \times \mathcal{N}_{b,\mu,\infty}^{2-\alpha-\alpha_{1+\frac{d-\mu}{b}}},$$

for all $t \in [0,\ T)$.

Proof. Using the bilinear estimate (4.6) alongside the product estimate (4.1), the uniqueness of the mild solution can be established by adapting an argument from Jhean [14]. For clarity, define the following parameters:

$$r_1 = 2-2\alpha-\alpha_1+\frac{d-\mu}{a}, \quad r_0 = 3-3\alpha-\alpha_1+\frac{d-\mu}{a}, \quad r_2 = 2-\alpha-\alpha_1+\frac{d-\mu}{b}.$$

Suppose $[u^1,\ \rho^1]$ and $[u^2,\ \rho^2]$ are two mild solutions in:

$$C\big([0,\ T);\mathcal{N}^{r_1}_{a,\mu,\infty}\big)\times C\big([0,\ T);\mathcal{N}^{r_2}_{b,\mu,\infty}\big).$$

The first step is to show that there exists $0<T_1<T$ such that $[u^1(t),\ \rho^1(t)] = [u^2(t),\ \rho^2(t)]$ in $\mathcal{N}^{r_1}_{a,\mu,\infty}\times\mathcal{N}^{r_2}_{b,\mu,\infty}$ for all $t\in[0,\ T_1)$. Define the following quantities:

$$\begin{aligned}
&\varUpsilon = u^1-u^2, \quad \Omega=\rho^1-\rho^2,\\
&\varUpsilon_1 = \mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}))u_0-u^1, \quad \varUpsilon_2 = \mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}))u_0-u^2,\\
&\Omega_1{=}\mathcal{Q}_\beta(-t^\beta\big((-\triangle)^{\alpha/2}-\xi\big))\rho_0-\rho^1,\ \Omega_2{=}\mathcal{Q}_\beta(-t^\beta\big((-\triangle)^{\alpha/2}-\xi\big))\rho_0-\rho^2.
\end{aligned}$$

From the mild formulation of the system, we know that:

$$\Omega = \varPsi(\varUpsilon). \tag{5.6}$$

Taking the norm in $\mathcal{N}^{r_2}_{b,\mu,\infty}$ and applying Lemma 10, we have:

$$\sup_{0<t<T_1}\|\Omega\|_{\mathcal{N}^{r_2}_{b,\mu,\infty}} \le \varepsilon \sup_{0<t<T_1}\|\varUpsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}}. \tag{5.7}$$

We consider the difference in the nonlinear interaction terms:

$$u^1F(\rho^1)-u^2F(\rho^2) = \varUpsilon F(\rho^1)+u^2F(\Omega),$$

and substituting the mild solution representations, we write:

$$\begin{aligned}
&u^1F(\rho^1)-u^2F(\rho^2) = \varUpsilon F\Big(\mathcal{Q}_\beta(-t^\beta\big((-\triangle)^{\alpha/2}-\xi\big))\rho_0-\Omega_1\Big)\\
&+\Big(\mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}))u_0{-}\varUpsilon_2\Big)F(\Omega){=}\varUpsilon F\Big(\mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}))\rho_0{-}\Omega_1\Big)\\
&+\Big(\mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}))u_0\Big)F(\Omega)-\varUpsilon F(\Omega_1)-\varUpsilon_2F(\Omega).
\end{aligned}$$

Substituting the difference into the bilinear operator, we obtain:

$$\begin{aligned}
&\Big\|\varUpsilon\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} = \Big\|\mathbb{B}(u^1,\rho^1)-\mathbb{B}(u^2,\rho^2)\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} = \Big\|\int_0^t(t-s)^{\beta-1}\\
&\times\nabla\cdot\mathcal{Q}_{\beta,\beta}(-(t-s)^\beta((-\triangle)^{\alpha/2}))\Big((u^1F(\rho^1))-(u^2F(\rho^2))\Big)(s)\,ds\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}}\\
\le&\Big\|\int_0^t(t-s)^{\beta-1}\nabla\cdot\mathcal{Q}_{\beta,\beta}(-(t-s)^\beta((-\triangle)^{\alpha/2}))\\
&\times\big(\varUpsilon F(\Omega_1)+\varUpsilon_2G(\Omega)\big)(s)\,\mathrm{d}s\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}}+\Big\|\int_0^t(t{-}s)^{\beta-1}\nabla\cdot\mathcal{Q}_{\beta,\beta}(-(t{-}s)^\beta\\
&\times((-\triangle)^{\alpha/2}))\beta\Big(\varUpsilon F\Big(\mathcal{Q}_\beta(-s^\beta((-\triangle)^{\alpha/2}-\gamma))\rho\Big)\\
&+\mathcal{Q}_\beta(-s^\beta((-\triangle)^{\theta/2}))u\Big)\,F(\Omega)\,ds\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} := \mathcal{I}_1(t)+\mathcal{I}_2(t).
\end{aligned}$$

For $\mathcal{I}_1(t)$, Using the product estimate (Lemma 8) and the previously derived inequality (5.7), we have:

$$\mathcal{I}_1(t) \le \varepsilon\Big(\sup_{0<t<T_1} \|\Upsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} \sup_{p,\infty 0<t<T_1} \|\Omega_1\|_{\mathcal{N}^{r_2}_{b,\mu,\infty}} + \sup_{0<t<T_1} \|\Upsilon_2\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} \sup_{0<t<T_1} \|\Omega\|_{\mathcal{N}^{r_2}_{b,\mu,\infty}}\Big).$$

Substituting inequality (5.7) into the above:

$$\mathcal{I}_1(t) \le C \sup_{0<t<T_1} \|\Upsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} \Big(\sup_{0<t<T_1} \|\Omega_1\|_{\mathcal{N}^{r_2}_{b,\mu,\infty}} + \sup_{0<t<T_1} \|\Upsilon_2\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}}\Big). \quad (5.8)$$

However, for $\mathcal{I}_2(t)$, it ensues that

$$\begin{aligned}
\mathcal{I}_2(t) &= \Big\| \int_0^t (t-s)^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}(-(t-s)^\beta((-\triangle)^{\alpha/2})) \\
&\quad \times \Big(\Upsilon F(\mathcal{Q}_\beta(-s^\beta((-\triangle)^{\alpha/2}-\xi))\rho_0)\Big) + \Big(\mathcal{Q}_\beta(-s^\beta(-\triangle)^{\alpha/2})u_0\Big)F(\Omega)\,ds\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} \\
&\le \int_0^t \Big\|(t-s)^{\beta-1}\nabla\cdot \mathcal{Q}_{\beta,\beta}(-(t-s)^\beta((-\triangle)^{\alpha/2}))\big(\Upsilon F(\mathcal{Q}_\beta(-s^\beta((-\triangle)^{\alpha/2} \\
&\quad -\xi))\rho_0)\big) + \Big(\mathcal{Q}_\beta(-s^\beta(-\triangle)^{\alpha/2})u_0\Big)F(\Omega)\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} ds \\
&\le C\int_0^t (t-s)^{\beta-1}(t-s)^{-\frac{\beta}{\alpha}(r-(r_0+\gamma)+1)}\Big\|\Upsilon F\Big(\mathcal{Q}_\beta(-s^\beta((-\triangle)^{\alpha/2}-\xi))\rho_0\Big) \\
&\quad + \mathcal{Q}_\beta(-s^\beta(-\triangle)^{\alpha/2})u_0\,F(\Omega)\Big\|_{\mathcal{N}^{r_0+\gamma}_{a,\mu,\infty}} ds \\
&\le C\int_0^t (t-s)^{\beta-1-\frac{\beta}{\alpha}(\alpha-\gamma)}\Big\|\Upsilon F(\mathcal{Q}_\beta(-s^\beta((-\triangle)^{\alpha/2}-\xi))\rho_0) \\
&\quad + \mathcal{Q}_\beta(-s^\beta(-\triangle)^{\alpha/2})u_0\,F(\Omega)\Big\|_{\mathcal{N}^{r_0+\gamma}_{a,\mu,\infty}} ds \le C\int_0^t (t-s)^{-1+\frac{\beta}{\alpha}\gamma} \\
&\quad \times\Big\|\Upsilon F(\mathcal{Q}_\beta(-s^\beta((-\triangle)^{\alpha/2}-\xi))\rho_0) + \mathcal{Q}_\beta(-s^\beta(-\triangle)^{\alpha/2})u_0\,F(\Omega)\Big\|_{\mathcal{N}^{r_0+\gamma}_{a,\mu,\infty}} ds \\
&\le C\int_0^t (t-s)^{-1+\frac{\beta}{\alpha}\gamma}s^{-\frac{\beta}{\alpha}\gamma}\Big\|\Upsilon\Big\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}}\Big\|\mathcal{Q}_\beta\big(-s^\beta((-\Delta)^{\alpha/2}-\xi)\big)\rho_0\Big\|_{\mathcal{N}^{r_2+\gamma}_{a,\mu,\infty}} ds \\
&\quad + C\int_0^t (t-s)^{-1+\frac{\beta}{\alpha}\gamma}s^{\frac{\beta}{\alpha}\gamma}\|\Upsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}}\Big\|\mathcal{Q}_\beta\big(-s^\beta(-\Delta)^{\alpha/2}\big)u_0\Big\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}} ds.
\end{aligned}$$

Using the uniform bounds of the Mittag-Leffler operator and simplifying:

$$\begin{aligned}
\mathcal{I}_2(t) &\le C\sup_{0<t<T_1}\Big\{t^{\frac{\beta}{\alpha}\gamma}\Big\|\mathcal{Q}_\beta\big(-t^\beta((-\Delta)^{\alpha/2}-\xi)\big)\rho_0\Big\|_{\mathcal{N}^{r_2+\gamma}_{a,\mu,\infty}}\Big\}\times\sup_{0<t<T_1}\|\Upsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} \\
&\quad + C\sup_{0<t<T_1}\Big\{t^{\frac{\beta}{\alpha}\gamma}\Big\|\mathcal{Q}_\beta\big(-t^\beta(-\Delta)^{\alpha/2}\big)u_0\Big\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}}\Big\}\times\sup_{0<t<T_1}\|\Upsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}}. \quad (5.9)
\end{aligned}$$

By appropriately applying Lemma 8, along with relation (5.7) and the fact that for sufficiently small $\gamma > 0$, the following integral is valid:

$$\int_0^t (t-s)^{-1+\frac{\beta}{\alpha}\gamma}s^{-\frac{\beta}{\alpha}\gamma}ds = \int_0^1 (1-\tau)^{-1+(1-\beta)+\frac{\beta}{\alpha}\gamma}\tau^{-\frac{\beta}{\alpha}\gamma}d\tau = C.$$

Therefore, using the estimates (5.8) and (5.9), we obtain:

$$\sup_{0<t<T_1} \|\Upsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} \leq C\mathcal{R}(T_1) \sup_{0<t<T_1} \|\Upsilon\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}},$$

where:

$$\begin{aligned}\mathcal{R}(T_1) = &\sup_{0<t<T_1} \|\Omega_1\|_{\mathcal{N}^{r_2}_{b,\mu,\infty}} + \sup_{0<t<T_1} t^{\frac{\beta}{\alpha}\gamma}\|\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}}\\ &+ \sup_{0<t<T_1} \|\Upsilon_2\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} + \sup_{0<t<T_1} t^{\frac{\beta}{\alpha}\gamma}\|\mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}-\xi))\rho_0\|_{\mathcal{N}^{r_2+\gamma}_{b,\mu,\infty}}.\end{aligned}$$

From the properties of the Mittag-Leffler operators and the assumptions on the initial data:

1. $\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0 \to u_0$ in $\mathcal{N}^{r_1}_{a,\mu,\infty}$ as $t \to 0^+$.
2. $\mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}-\xi))\rho_0 \to \rho_0$ as $t \to 0^+$.

From the above and Lemma 4:

$$\lim_{t\to 0^+} \|\Upsilon_2\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} = \lim_{t\to 0^+} \|\rho_1\|_{\mathcal{N}^{r_2}_{b,\mu,\infty}} = 0. \tag{5.10}$$

We aim now to show:

$$\lim_{t\to 0^+} \sup t^{\frac{\beta}{\alpha}\gamma}\|\mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}))\eta_0\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}} = 0.$$

Indeed, define an approximation sequence:

$$u_{0,j} = \mathcal{Q}_\beta(-(\frac{t}{j})^\beta(-\triangle)^{\alpha/2})u_0 \quad \text{for all} \quad j \in \mathbb{N}.$$

From Lemma 3.1, it follows that $u_{0,j} \in \mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}$. Furthermore, based on the hypothesis for u_0, we have $u_{0,j} \to u_0$ in $\mathcal{N}^{r_1}_{a,\mu,\infty}$ as $j \to \infty$. Then,

$$\begin{aligned}&\lim_{t\to 0^+} t^{\frac{\beta}{\alpha}\gamma}\|\mathcal{Q}_\beta\Big(-t^\beta(-\triangle)^{\alpha/2}\Big)u_0\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}}\\ \leq &\lim_{t\to 0^+} \sup\Big\{t^{\frac{\beta}{\alpha}\gamma}\|\mathcal{Q}_\beta\big(-t^\beta(-\triangle)^{\alpha/2}\big)(u_0-u_{0,j})\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}}\\ &+t^{\frac{\beta}{\alpha}\gamma}\|\mathcal{Q}_\beta\big(-t^\beta(-\triangle)^{\alpha/2}\big)u_{0,j}\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}}\Big\}\\ \leq &C\|u_0-u_{0,j}\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} + C\|u_{0,j}\|_{\mathcal{N}^{r_1+\gamma}_{a,\mu,\infty}} \lim_{t\to 0^+} \sup t^{\frac{\beta}{\alpha}\gamma}\\ \leq &C\|u_0-u_{0,j}\|_{\mathcal{N}^{r_1}_{a,\mu,\infty}} \to 0, \quad \text{as} \quad j\to\infty.\end{aligned}$$

A similar reasoning can be applied to demonstrate that:

$$\lim_{t\to\ 0^+} \sup t^{\frac{\beta}{\alpha}\gamma}\|\mathcal{Q}_\beta(-t^\beta((-\triangle)^{\alpha/2}-\xi))\rho_0\|_{\mathcal{N}^{r_2+\gamma}_{a,\mu,\infty}} = 0. \tag{5.11}$$

By combining the estimates (5.10)–(5.11), we deduce that there exists a small time $T_1 > 0$ such that:

$$= C\mathcal{R}(T_1) < 1.$$

This inequality implies that:

$$\Upsilon(t) = 0 \quad \text{for all} \quad t \in [0,\ T_1).$$

From relation (5.6), it immediately follows that:

$$\Omega(t) = 0 \quad \text{for all} \quad t \in [0,\ T_1).$$

To complete the proof, we will demonstrate that T_1 can be extended arbitrarily to cover the entire interval $(0,\ T]$. Define:

$$T_* = \sup\left\{\tilde{T}; 0 < \tilde{T} < T,\ u^1(t) = u^2(t) \quad \text{in} \quad \mathcal{N}^{r_1}_{a,\mu,\infty}, \text{for all} \quad t \in [0, \tilde{T})\right\}.$$

If $T_* = T$, the proof is complete since the uniqueness extends to the entire interval. Otherwise, we have $u^1(t) = u^2(t)$ for $t \in [0,\ T_*)$, and by time continuity of u^1 and u^2, it follows that:

$$u^1(T_*) = u^2(T_*).$$

Similarly, for ρ^1 nd ρ^2, we have: $\rho^1(t) = \rho^2(t)$ for $t \in [0,\ T_*)$, and by continuity:

$$\rho^1(T_*) = \rho^2(T_*).$$

Now, restarting the argument from the initial time T_*, the same reasoning shows that there exists a small enough $\lambda > 0$ such that:

$$u^1(t) = u^2(t) \quad \text{for} \quad t \in [T_*,\ T_* + \lambda).$$

This implies:

$$u^1(t) = u^2(t) \quad \text{for} \quad t \in [0,\ T_* + \lambda).$$

However, this contradicts the definition of T_* as the supremum of such intervals. Therefore, we must have: $T_* = T$. This completes the proof. □

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