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One-signed rotationally symmetric solutions of singular Dirichlet problems with the prescribed higher mean curvature operator in Minkowski spacetime

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Abstract. We investigate the existence, uniqueness and multiplicity of one-signed rotationally symmetric solutions of singular Dirichlet problems with the prescribed higher mean curvature operator in Minkowski spacetime. The main tools are the Schauder fixed point theorem along with cut-off technique and the Leggett-Williams fixed point theorem. In addition, we give some practical models to illustrate the effectiveness of our results.

Keywords: higher mean curvature operator; Minkowski spacetime; singular Dirichlet problem; one-signed rotationally symmetric solutions.

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1 Introduction

Throughout this paper, $\mathbb{L}^{N+1}$ denotes the $(N+1)$-dimensional Minkowski spacetime endowed with the standard Lorentzian metric $\langle\ ,\ \rangle = -(\mathrm{d}t)^2 + \sum_{i=1}^{N}(\mathrm{d}x_i)^2$. For a spacelike hypersurface Σ in $\mathbb{L}^{N+1}$, the k-th mean curvature S_k is a geometric invariant encoding the geometry of Σ. From the perspective of physics, the k-th mean curvature S_k plays an important role in General Relativity. Each S_k intuitively measures the time evolution towards the future or the past of the spatial universe. From an algebraic perspective, each one of these functions corresponds to a coefficient of the characteristic polynomial of the shape operator corresponding to a unit timelike vector field pointing to future. In fact, each k-th mean curvature S_k is described as a certain type of average measure of the principal curvatures of the hypersurface Σ [11]. Specifically, S_1 is opposite of the usual mean curvature of the principal curvature, S_2 is equal to the scalar curvature up to a constant factor, and S_N is $(-1)^{N+1}$ times the Gauss-Kronecker curvature.

For a given prescription function H_k, the prescribed k-th mean curvature problem is to find a spacelike hypersurface Σ in $\mathbb{L}^{N+1}$ satisfying

$$S_k(p) = H_k(p), \quad \forall p \in \Sigma, \tag{1.1}$$

where $S_k(1 \leq k \leq N)$ is the k-th mean curvature of the hypersurface. We make a hyperplane Π that passes through $\gamma(0)$ and is orthogonal to the line γ in $\mathbb{L}^{N+1}$, where γ is a timelike parameter line pointing towards the future. The hypersurface we are looking for can be denoted as $\Sigma = \{(v(x), x) : x \in \Pi\} \subset \mathbb{R} \times \mathbb{R}^N$. On the one hand, if we suppose that the prescription function H_k is rotationally symmetric concerning γ, we can naturally assume that v has the same symmetric, namely, $v(x) = v(r)$ where $r = r(x)$ is the distance from x to $\gamma(0)$ in Π. On the other hand, it follows from [11, 21] that the differential operator $S_k(1 \leq k \leq N)$, associated with the k-curvature of rotationally symmetric graphs in $\mathbb{L}^{N+1}$, can be written as follows:

$$\mathcal{M}_k : \{v \in C^2(\mathbb{R}^+) : v'(0) = 0, |v'| < 1\} \to \mathbb{R},$$
$$(\mathcal{M}_k v)(r) = \begin{cases} \dfrac{1}{Nr^{N-1}} \left(r^{N-k}\phi^k(v')\right)', & r \in (0, +\infty); \\ 0, & r = 0, \end{cases}$$

where $\phi(s) := s/\sqrt{1-s^2}$ and $1 \leq k \leq N$. Therefore, to obtain rotationally symmetric solutions of Equation (1.1) with the Dirichlet boundary condition, it is only sufficient to consider the following one-dimensional boundary value problem

$$(\mathcal{M}_k v)(r) = H_k(v(r), r), \quad r \in (0, R) \tag{1.2}$$

with the mixed boundary condition

$$v'(0) = 0, \quad v(R) = 0. \tag{1.3}$$

Over the past few decades, the study of the prescribed mean curvature spacelike equation in Minkowski spacetime $\mathbb{L}^{N+1}$ has received widespread attention. When $k = 1$, the existence, uniqueness, multiplicity and bifurcation of rotationally symmetric solutions for Dirichlet problems with prescribed mean curvature equation in $\mathbb{L}^{N+1}$ have been widely studies, for instance, see [4, 5, 6, 7, 8, 9, 10, 15, 18, 24, 25, 26, 32, 33, 35] and references therein. Also, we refer for examples to [3, 22, 23, 34] for other types boundary value problems with the prescribed mean curvature equation in $\mathbb{L}^{N+1}$. When $k = 2$, we refer the reader to [1, 2, 29] for Dirichlet problems with the prescribed scalar curvature equation, and on the case of more general ambient spacetimes, see [14]. When $k = N$, we specifically mention [13, 20] for Dirichlet problems with the Gauss-Kronecker curvature. However, for Dirichlet problems involving the prescribed k-th mean curvature operator when $3 \leq k < n$, the existing relevant research remains scarce. The study of Dirichlet problems with a prescribed k-th $(1 \leq k \leq N)$ mean curvature operator originated from Ivochkina [17], in which the main tools utilized are the implicit function theorem and Leray-Schauder principle. And then, the problems have been considered by some authors, for instance, [16, 27, 28, 30]. In recent years, using the Schauder fixed point theorem and the standard prolongability theorem of ordinary differential equations, de la Fuente, Romero and Torres [11] derived the existence and multiplicity of rotationally symmetric solutions to Eq. (1.1) with the Dirichlet boundary condition. Ma and Xu [21] offered a geometric interpretation of the results of [11] and obtained the existence of rotationally symmetric entire solutions via the global bifurcation theory. More recently, Xu [31] considered the existence of rotationally symmetric entire solutions for (1.2)–(1.3) by using topological degree methods.

We note that most of the literature mentioned above is non-singular problems, the research on singular problems is rarely seen [5, 6, 10, 24, 26], and such problems correspond to the case of $k = 1$, while the results are extremely rare [31] when $k > 1$.

Motivated and inspired by the above works, the purpose of this article is to establish the existence, uniqueness and multiplicity of the one-signed rotationally symmetric solutions of the singular Equation (1.1) with the Dirichlet boundary condition. From the above discussion, we are led to consider the singular problem (1.2)–(1.3), where the prescription function H_k may be singular at $r = 0$, $r = R$, $v = 0$, and $v = \pm R$. Our main models of the prescription function are

$$H_k(v,r) = (-1)^k \left(\frac{a}{r^{\alpha_1}(R-r)^{\beta_1} v^p} + \frac{b v^q}{r^{\alpha_2}(R-r)^{\beta_2}} + \frac{c}{r^{\alpha_3}(R-r)^{\beta_3}} \right),$$
$$H_k(v,r) = (-1)^k \lambda \mu(r) v^\rho / (R-v)^\sigma,$$

where $1 \le k \le N$, $a > 0$, $b, c \ge 0$, $p, q > 0$, $\alpha_i, \beta_i \in [0, 1)$, $i = 1, 2, 3$ and $\rho > k$, $\mu(\cdot) \in C([0, R], (0, +\infty))$, $\lambda > 0$, $0 < \sigma < k/2$. We note that the above models of prescription functions take those in [25, 26] and [24] as special cases, respectively.

It is worth emphasizing that our results are not only new, but also generalize and improve the corresponding results in [24, 25, 26]. In addition, we allow that the prescription function H_k may be strongly singular at $v = 0$, and we do not assume that H_k satisfies the monotonicity condition as in [25, 26] when obtaining the existence of solutions. Meanwhile, H_k may be strongly singular at $v = \pm R$ provided $k > 2$, when we establish the multiplicity of solutions to problem (1.2)–(1.3).

This paper is organized as follows: In Section 2, we introduce some notations and two fixed point theorems. In Section 3, we separate k into odd and even cases and establish the existence of positive(or negative) solutions to the singular problem (1.2)–(1.3) by using the Schauder fixed point theorem. And then, by imposing the monotonicity condition on H_k, the uniqueness of positive and negative solutions of the singular problem (1.2)–(1.3) is obtained. In Section 4, we also divide k into odd and even cases and present the multiplicity of positive(or negative) solutions to the singular problem (1.2)–(1.3) by using the Leggett-Williams fixed point theorem. In the last section, some model examples are given as applications of our main results.

2 Preliminaries

In this section, we introduce some concepts and fixed point theorems which will be used later.

Lemma 1. (Schauder Fixed Point Theorem [12]) *Let Ω be a bounded closed convex subset of Banach space E. Then the continuous compact mapping $T: \Omega \to \Omega$ has a fixed point.*

Let P be a cone in the real Banach space E. A map α is a nonnegative concave functional on the cone P if it satisfies

(i) $\alpha : P \to [0, +\infty)$ is continuous;

(ii) $\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$ for all $x, y \in P$ and $0 \leq t \leq 1$.

In addition, let $P_c := \{x \in P : \|x\| \leq c\}$ and $P(\alpha, a, b) := \{x \in P : a \leq \alpha(x), \|x\|) \leq b\}$.

Lemma 2. (Leggett-Williams Fixed Point Theorem [19]) *Let P be a cone in the real Banach space E, $A : \overline{P}_c \to \overline{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P with $\alpha(x) \leq \|x\|$ for all $x \in \overline{P}_c$. Suppose there exist $0 < a < b < d \leq c$ such that the following conditions hold:*

(i) *$\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Ax) > b$ for all $x \in P(\alpha, b, d)$;*

(ii) *$\|Ax\|) > b$ for $x \in P(\alpha, b, c)$ with $\|Ax\|) > d$.*

Then, the operator A has at least three fixed points $x_1, x_2, x_3 \in \overline{P}_c$ satisfying

$$\|x_1\| < a, \quad \alpha(x_2) > b, \quad a < \|x_3\| \text{ with } \alpha(x_3) < b.$$

We introduce some symbols that will be used in this article. The continuous function space $C[0, R]$ equipped with the maximum norm $\|\cdot\|$ and B_R is an open ball of center 0 and radius R in $C[0, R]$. Let

$$P = \{v \in C[0, R] : v(r) \text{ is nonnegative and nonincreasing on } [0, R]\}.$$

Evidently, P is a cone in $C[0, R]$.

3 Existence and uniqueness

We now consider the one-dimensional singular problem (1.2)–(1.3), that is,

$$\begin{cases} \left(r^{N-k}\phi^k(v')\right)' = Nr^{N-1}H_k(v, r), \quad r \in (0, R), \\ v'(0) = 0, \quad v(R) = 0, \end{cases} \tag{3.1}$$

where $\phi(s) := s/\sqrt{1-s^2}$, $1 \leq k \leq N$, $H_k \in C(((-\infty, 0) \cup (0, +\infty)) \times (0, R))$ may be singular at $r = 0$, $r = R$ and $v = 0$ satisfying $(-1)^k H_k(v, r) > 0$ for all $(v, r) \in ((-\infty, 0) \cup (0, +\infty)) \times (0, R)$.

Next, we divide k into odd and even cases to discuss the existence and uniqueness of the solution of the singular problem (3.1).

3.1 k is odd

We first utilize the Schauder fixed point theorem to establish the existence of positive solutions to problem (3.1).

Theorem 1. *Let k be an odd number. Assume that*

(C$_1$) $\int_0^R \max_{\delta \le v \le R} |H_k(v,r)| \mathrm{d}r < +\infty$ *for all* $\delta \in (0,R)$.

Then, problem (3.1) *has at least one positive solution.*

Proof. Firstly, we define the modification $H_k^*(v,r)$ of $H_k(v,r)$ by

$$H_k^*(v,r) = \begin{cases} H_k(-R,r), & v < -R; \\ H_k(v,r), & 0 < |v| \le R; \\ H_k(R,r), & v > R. \end{cases} \tag{3.2}$$

Then, $H_k^* \in C(((-\infty,0) \cup (0,+\infty)) \times (0,R))$ satisfies

$$\int_0^R \max_{\delta \le v \le R+1} |H_k^*(v,r)| \mathrm{d}r < +\infty, \quad \forall \delta \in (0,R+1). \tag{3.3}$$

We consider the modified singular problem

$$\begin{cases} \left(r^{N-k}\phi^k(v')\right)' = Nr^{N-1}H_k^*(v,r), \quad r \in (0,R), \\ v'(0) = 0, \quad v(R) = 0. \end{cases} \tag{3.1*}$$

For each $n \in \mathbb{N}$, we first define the set Ω_n as follows:

$$\Omega_n = \Big\{ v \in C[0,R] : v(r) \text{ is nonincreasing and } \frac{1}{n} \le v(r) \le R+1 \text{ on } [0,R] \Big\}.$$

Obviously, Ω_n is a nonempty bounded closed convex subset in $C[0,R]$. We also define the operator $T_n : \Omega_n \to C[0,R]$ by

$$(T_n v)(r) = \frac{1}{n} - \int_r^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}} \int_0^t \tau^{N-1} H_k^*(v(\tau),\tau) \mathrm{d}\tau \right)^{1/k} \right) \mathrm{d}t. \tag{3.4}$$

By the standard proof, we know that T_n is well defined.

Next, we divide the proof into the following three steps.

Step 1. We show that there exists $v_n \in \Omega_n$ such that $T_n v_n = v_n$ for each $n \in \mathbb{N}$. Assume that $n \in \mathbb{N}$ is given. Note that $H_k^*(v,r) < 0$ for all $(v,r) \in ((-\infty,0) \cup (0,+\infty)) \times (0,R)$ and

$$0 < \phi^{-1}(s) = \frac{s}{\sqrt{1+s^2}} < 1, \quad \forall s > 0. \tag{3.5}$$

Then, for all $v \in \Omega_n$, we have

$$\frac{1}{n} \le (T_n v)(r) \le R + \frac{1}{n} \le R+1, \quad r \in [0,R],$$

and thus $T_n(\Omega_n) \subset C[0,R]$ is bounded. On the other hand, from (3.4) and (3.5), for all $r_1, r_2 \in [0,R]$ and $v \in \Omega_n$, we can obtain

$$\begin{aligned}|(T_n v)(r_1) - (T_n v)(r_2)| \le & \left| \int_{r_2}^{r_1} \phi^{-1}\left(\left(\frac{N}{t^{N-k}} \int_0^t \tau^{N-1} H_k^*(v(\tau),\tau)\mathrm{d}\tau \right)^{1/k} \right) \mathrm{d}t \right| \\ \le & |r_1 - r_2|.\end{aligned}$$

This implies $T_n(\Omega_n)$ is equicontinuous. Then, we know that $T_n(\Omega_n)$ is compact in $C[0,R]$ by the Arzelà-Ascoli theorem. Taking into account that Ω_n is bounded and closed, there is a sequence $\{v_m\} \subset \Omega_n$ satisfying $\lim_{m\to\infty} v_m = \hat{v} \in \Omega_n$. Let

$$q_n(r) = \max_{\frac{1}{n} \le v \le R+1} |H_k^*(v,r)|, \quad r \in (0,R).$$

It follows from (3.3) that $q_n \in L^1(0,R)$. Since $1/n \le v_m(r) \le R+1$ for $r \in [0,R]$, $m \in \mathbb{N}$, we have

$$\tau^{N-1}|H_k^*(v_m(\tau),\tau)| \le \tau^{N-1} q_n(\tau) < R^{N-1} q_n(\tau), \quad \forall \tau \in (0,R).$$

This together with Lebesgue dominated convergence theorem implies that

$$\lim_{m\to\infty} (T_n v_m)(r) = (T_n \hat{v})(r)$$

uniformly for $r \in [0,R]$. Hence T_n is continuous in Ω_n. In addition, it is clear that $(T_n v)(r)$ is nonincreasing on $[0,R]$ for each $v \in \Omega_n$, and then $T_n(\Omega_n) \subset \Omega_n$. Thus by Lemma 1, there exists $v_n \in \Omega_n$ such that $T_n v_n = v_n$.

Step 2. We prove that there is $z \in C[0,R]$ satisfying

$$z(r) > 0, \quad \forall r \in [0,R), \tag{3.6}$$

such that

$$v_n(r) \ge 1/n + z(r), \quad \forall r \in (0,R).$$

Indeed, we let

$$p(r) = \min_{1/n \le v \le R+1} |H_k^*(v,r)|, \quad r \in (0,R). \tag{3.7}$$

Clearly, $p \in L^1(0,R)$ by (3.3). Since $H_k^*(v,r)$ is singular at $v=0$, it follows from (3.7) that for all $(v,r) \in [1/n, R+1] \times (0,R)$,

$$|H_k^*(v,r)| \ge \min_{\frac{1}{n} \le v \le R+1} |H_k^*(v,r)| = p(r) > 0. \tag{3.8}$$

Notice that $1/n \le v_n(r) \le R+1$, $r \in [0,R]$ and (3.8), we obtain

$$|H_k^*(v_n(r),r)| \ge p(r) > 0, \quad r \in (0,R),\ n \in \mathbb{N}. \tag{3.9}$$

Due to $T_n v_n = v_n$, it follows from (3.5) and (3.9) that

$$\begin{aligned}v_n(r) =& \frac{1}{n} - \int_r^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}} \int_0^t \tau^{N-1} H_k^*(v_n(\tau),\tau)\mathrm{d}\tau \right)^{1/k} \right) \mathrm{d}t \\ \ge& \frac{1}{n} + \int_r^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}} \int_0^t \tau^{N-1} p(\tau)\mathrm{d}\tau \right)^{1/k} \right) \mathrm{d}t \\ :=& 1/n + z(r), \quad \forall r \in (0,R).\end{aligned}$$

We note that $z \in C[0, R]$ satisfies (3.6) because of $p \in L^1(0, R)$.

Step 3 We assert that $\{v_n\}$ has a convergent subsequence. Because $1/n \leq \|v_n\|_\infty \leq R + 1$, $n \in \mathbb{N}$, we know that $\{v_n\}$ is uniformly bounded. Notice that for all $n \in \mathbb{N}$,

$$|v_n'(r)| < 1, \quad \forall r \in [0, R],$$

it follows that for all $r_1, r_2 \in [0, R]$ and $n \in \mathbb{N}$,

$$|v_n(r_1) - v_n(r_2)| \leq |r_1 - r_2|.$$

Thus, $\{v_n\}$ is equicontinuous in $[0, R]$. From the Arzelà-Ascoli theorem, $\{v_n\}$ has convergent subsequences. Without loss of generality, we can assume that $v_n \to \check{v} \in \Omega_n (n \to \infty)$. Obviously, $\check{v}(R) = 0$. Notice that $T_n v_n = v_n$, we have

$$v_n'(r) = \phi^{-1}\left(\left(\frac{N}{r^{N-k}} \int_0^r \tau^{N-1} H_k^*(v_n(\tau), \tau) \mathrm{d}\tau\right)^{1/k}\right), \quad r \in (0, R). \tag{3.10}$$

Taking $\eta \in (0, R)$, from (3.10), we get for all $r \in [0, \eta]$,

$$v_n(r) = v_n(\eta) - \int_r^\eta \phi^{-1}\left(\left(\frac{N}{t^{N-k}} \int_0^t \tau^{N-1} H_k^*(v_n(\tau), \tau) \mathrm{d}\tau\right)^{1/k}\right) \mathrm{d}t. \tag{3.11}$$

Let $\sigma = \min_{s \in [0,\eta]} z(s) > 0$, where $z(s)$ is given by Step 2. For each $n \in \mathbb{N}$, we can obtain

$$\sigma \leq \frac{1}{n} + z(r) \leq v_n(r) \leq R + 1, \quad \forall r \in [0, \eta]. \tag{3.12}$$

Furthermore, we also let $w(r) = \max_{\sigma \leq v \leq R+1} |H_k^*(v, r)|, r \in (0, R)$. Then, $w \in L^1(0, R)$ by (3.3). Hence, from (3.12), we have

$$\tau^{N-1} |H_k^*(v_n(\tau), \tau)| \leq \tau^{N-1} w(\tau) < \eta^{N-1} w(\tau), \quad \tau \in (0, \eta). \tag{3.13}$$

It follows from (3.11), (3.13) and Lebesgue dominated convergence theorem that for $r \in (0, \eta)$,

$$\check{v}(r) = \check{v}(\eta) - \int_r^\eta \phi^{-1}\left(\left(\frac{N}{t^{N-k}} \int_0^t \tau^{N-1} H_k^*(\check{v}(\tau), \tau) \mathrm{d}\tau\right)^{1/k}\right) \mathrm{d}t. \tag{3.14}$$

Let $\eta \to R^-$ in (3.14) and from the absolute continuity of Lebesgue integral, we have

$$\check{v}(r) = -\int_r^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}} \int_0^t \tau^{N-1} H_k^*(\check{v}(\tau), \tau) \mathrm{d}\tau\right)^{1/k}\right) \mathrm{d}t, \quad r \in (0, R),$$

and thus,

$$\check{v}'(r) = \phi^{-1}\left(\left(\frac{N}{r^{N-k}} \int_0^r \tau^{N-1} H_k^*(\check{v}(\tau), \tau) \mathrm{d}\tau\right)^{1/k}\right) \mathrm{d}t, \quad r \in (0, R).$$

This implies $\check{v}'(0) = 0$ and

$$\left(r^{N-k} \phi^k \left(\check{v}'(r)\right)\right)' = N r^{N-1} H_k^*(\check{v}(r), r), \quad r \in (0, R).$$

At the same time, from (3.11), $\check{v}(r) > 0$, $r \in [0, R)$. Therefore, $\check{v}$ is a positive solution of modified problem (3.1*). Notice that $0 < \check{v}(r) < R$ on $(0, R)$, it follows that $\check{v}$ is a positive solution of problem (3.1). This completes the proof of the theorem. □

Remark 1. Theorem 1 generalizes and improves the corresponding existence results in [25].

By imposing a monotonicity condition for the prescription function H_k, we can establish the uniqueness of the positive solution to the singular problem (3.1).

Theorem 2. *Let k be an odd number. Assume that* (C_1) *hold. Suppose also that*

(C_2) *for each fixed $r \in (0, R)$, $H_k(v, r)$ is nondecreasing in $v \in (0, R)$.*

Then, problem (3.1) *has a unique positive solution.*

Proof. The existence of positive solutions to problem (3.1) follows from Theorem 1 immediately.

Uniqueness. By contradiction, suppose that $v_1(r)$ and $v_2(r)$ are two positive solutions of problem (3.1) satisfying $v_1 \neq v_2$. Then, there exists $r_0 \in [0, R]$ such that $v_1(r_0) \neq v_2(r_0)$. Without loss of generality, we assume that $v_1(r_0) > v_2(r_0)$. Hence, there is a closed interval $[a, b] \subset [0, R]$ such that $v_1(a) - v_2(a)$ is the maximum value of $v_1(r) - v_2(r)$ on $[a, b]$, $v_1(r) > v_2(r)$, $r \in [a, b)$, $v_1'(a) = v_2'(a)$ and $v_1(b) = v_2(b)$. Note that

$$\left(r^{N-k}\phi^k\left(v_i'(r)\right)\right)' = Nr^{N-1}H_k(v_i(r), r), \quad r \in (0, R),\ i = 1, 2.$$

Integrating the above equation on $[a, t] \subset [a, b] (t \in (a, b])$, we get for $i = 1, 2$,

$$v_i'(t) = \phi^{-1}\left(\left((a/t)^{N-k}\phi^k\left(v_i'(a)\right) + \frac{N}{t^{N-k}}\int_a^t \tau^{N-1}H_k(v_i(\tau), \tau)\mathrm{d}\tau\right)^{1/k}\right).$$

It follows that for $i = 1, 2$,

$$\begin{aligned} &v_i(b) - v_i(a) \\ &= \int_a^b \phi^{-1}\left(\left((a/t)^{N-k}\phi^k\left(v_i'(a)\right) + \frac{N}{t^{N-k}}\int_a^t \tau^{N-1}H_k(v_i(\tau), \tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t. \end{aligned}$$

This together with (C_2) implies that $v_1(a) - v_2(a) \leq 0$, which contradicts $v_1(a) > v_2(a)$. This completes the proof of the theorem. □

3.2 *k* is even

Similar to the case that k is an odd number, we now establish the existence and uniqueness of positive and negative solutions to the singular problem (3.1), when k is an even number.

Theorem 3. *Let k be an even number. Assume that*

$(\overline{\mathrm{C}}_1)$ $\int_0^R \max_{\delta\le|v|\le R} H_k(v,r)\mathrm{d}r < +\infty$ *for all* $\delta \in (0,R)$.

Then, problem (3.1) *has at least two solutions, one is positive and the other is negative.*

Proof. Firstly, we prove the existence of positive solutions. To do this, for each $n \in \mathbb{N}$, we define a bounded closed subset Ω_n^+ in $C[0,R]$ as follows:

$$\Omega_n^+ = \Big\{v \in C[0,R] : v(r) \text{ is nonincreasing and } \frac{1}{n} \le v(r) \le R+1 \text{ on } [0,R]\Big\}.$$

We also define the operator T_n^+ on Ω_n^+ by

$$(T_n^+ v)(r) = \frac{1}{n} + \int_r^R \phi^{-1}\bigg(\Big(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}H_k^*(v(\tau),\tau)\mathrm{d}\tau\Big)^{1/k}\bigg)\mathrm{d}t,$$

where H_k^* is defined in (3.2). Then T_n^+ is well defined. Similar to the Step 1 of Theorem 1, we have that there exists $v_n \in \Omega_n^+$ such that $T_n^+ v_n = v_n$ for each $n \in \mathbb{N}$. Furthermore, using the Arzelà-Ascoli theorem, we know that $\{v_n\}$ has a convergent subsequence whose limit v_0^+ is a positive solution of modified problem (3.1*). Notice that $0 < v_0^+(r) < R$ for $r \in (0,R)$, then v_0^+ is a positive solution of problem (3.1). Next, we show the existence of negative solutions. For each $n \in \mathbb{N}$, we define a bounded closed subset Ω_n^- as follows:

$$\Omega_n^- = \Big\{v\in C[0,R] : v(r) \text{ is nondecreasing and } -R-1\le v(r) \le -\frac{1}{n} \text{ on } [0,R]\Big\}.$$

We also define the operator T_n^- on Ω_n^- by

$$(T_n^- v)(r) = -\frac{1}{n} - \int_r^R \phi^{-1}\bigg(\Big(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}H_k^*(v(\tau),\tau)\mathrm{d}\tau\Big)^{1/k}\bigg)\mathrm{d}t.$$

It is easy to know that T_n^- is well defined. Similar to the proof of Theorem 1, using the Schauder fixed point theorem and the Arzelà-Ascoli theorem, we can deduce that the modified problem (3.1*) has a negative solution v_0^-, which is a negative solution of problem (3.1). The details of the process are omitted. This completes the proof of the theorem. □

Theorem 4. *Let k be an even number. Assume that* $(\overline{\mathrm{C}}_1)$ *hold. Suppose also that*

$(\overline{\mathrm{C}}_2)$ *for each fixed* $r \in (0,R)$, $H_k(v,r)$ *is nondecreasing in* $v \in (0,R)$ *and nonincreasing in* $v \in (-R,0)$.

Then, problem (3.1) *has only two solutions, one is positive and the other is negative.*

Proof. The existence of positive and negative solutions to problem (3.1) follows from Theorem 3, immediately.

Uniqueness. The proof of the uniqueness of the positive and negative solution for problem (3.1) is completely analogous to one of Theorem 2, and is omitted. This completes the proof of the theorem. □

4 Multiplicity

In this section, we consider the multiplicity of one-signed solutions to the following singular problem

$$\begin{cases} \left(r^{N-k}\phi^k(v')\right)' = Nr^{N-1}H_k(v,r), & r \in (0,R), \\ v'(0) = 0, \quad v(R) = 0, \end{cases} \tag{4.1}$$

where $\phi(s) := s/\sqrt{1-s^2}$, $1 \le k \le N$, $H_k \in C((-R,R)\times[0,R])$ may be singular at $v = \pm R$ satisfying $(-1)^k H_k(v,r) > 0$ for all $(v,r) \in (-R,R) \times [0,R]$.

4.1 *k* is odd

Now we use the Leggett-Williams fixed point theorem to establish the multiplicity of positive solutions for problem (4.1).

Theorem 5. *Let k be an odd number. Assume that there exist constants a, b, c, d and η with $0 < a < b < \frac{\eta}{R}d < d \le c < R$ such that*

(C$_3$) $|H_k(v,r)| \le \frac{1}{R^k}\phi^k\left(\frac{a}{R}\right)$ *for all* $(v,r) \in [0,a] \times [0,R]$;

(C$_4$) $|H_k(v,r)| \le \frac{1}{R^k}\phi^k\left(\frac{c}{R}\right)$ *for all* $(v,r) \in [0,c] \times [0,R]$;

(C$_5$) $|H_k(v,r)| \ge \frac{R^{N-k}}{(R-\eta)^N}\phi^k\left(b/\eta\right)$ *for all* $(v,r) \in [b,d] \times [0,R-\eta]$;

(C$_6$) $(R-\eta)^N \min_{[b,c]\times[0,R-\eta]} |H_k(v,r)| \ge R^N\phi^k\left(\frac{Rb}{\eta d}\right)\max_{[0,c]\times[0,R]} |H_k(v,r)|$.

Then, problem (4.1) *has at least three positive solutions v_1, v_2, v_3 satisfying*

$$\|v_1\| < a, \quad b < \min_{[0,R-\eta]} v_2(r), \quad \|v_3\| > a \text{ with } \min_{[0,R-\eta]} v_3(r) < b. \tag{4.2}$$

Proof. Firstly, we define a nonlinear operator A on $P \cap B_R$ by

$$(Av)(r) = -\int_r^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t. \tag{4.3}$$

Then, A is well defined. Evidently, $(Av)(r) \ge 0$ and $(Av)'(r) \ge 0$ for all $r \in [0,R]$, which imply $A(P \cap B_R) \subset P$. Furthermore, $(Av)(R) = 0$ and $(Av)'(0) = 0$. By the standard argument and using the Arzelà-Ascoli theorem, we find that A is compact on $P \cap \overline{B}_\rho(\rho \in (0,R))$. It is easy to verify that v is a positive solution of problem (4.1) if and only if $v \in P \cap B_R$ is a fixed point of A.

Let $\alpha : P \to [0,+\infty)$ be a nonnegative continuous concave functional defined by

$$\alpha(v) = \min_{[0,R-\eta]} v(r), \quad \forall v \in P.$$

Then, $\alpha(v) \le \|v\|$, $v \in P$ and

$$\alpha(v) = v(R-\eta), \quad \alpha(Av) = (Av)(R-\eta), \quad \forall v \in P. \tag{4.4}$$

Next, we will distinguish the proof into three steps.

Step 1. Since ϕ^{-1} is odd, (C_4) and (4.3), for all $v \in \overline{P}_c$, we have

$$\|Av\| = \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}|H_k(v(\tau),\tau)|\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t$$
$$\leq \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}\frac{1}{R^k}\phi^k\left(\frac{c}{R}\right)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t < c.$$

Hence, $A(\overline{P}_c) \subset \overline{P}_c$. Similarly, we can get $\|Av\| < a$ for all $v \in \overline{P}_a$ by (C_3), which implies that the condition (ii) of Lemma 2 holds.

Step 2. Let $v = (b+d)/2$, then $v \in P$, $\|v\| < d$ and $\alpha(v) = (b+d)/2 > b$, which show that $\{v \in P(\alpha, b, d) : \alpha(v) > b\} \neq \emptyset$. We also let $v \in P(\alpha, b, d)$, then $\alpha(v) \geq b$ and $\|v\| \leq d$, i.e., $b \leq v(r) \leq d$ for all $r \in [0, R-\eta]$. It follows from the oddness of ϕ^{-1}, (4.4) and (C_5) that

$$\alpha(Av) = -\int_{R-\eta}^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t$$
$$\geq \int_{R-\eta}^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^{R-\eta} \tau^{N-1}|H_k(v(\tau),\tau)|\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t$$
$$\geq \int_{R-\eta}^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^{R-\eta} \tau^{N-1}\frac{R^{N-k}}{(R-\eta)^N}\phi^k\left(\frac{b}{\eta}\right)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t$$
$$> \eta\phi^{-1}\Big(\phi\left(b/\eta\right)\Big) = b.$$

Hence, the condition (i) of Lemma 2 is satisfied.

Step 3. We check that the condition (iii) of Lemma 2 holds. Notice that for all $v \in P(\alpha, b, c)$, we have $b \leq v(r) \leq c$ for $r \in [0, R-\eta]$, and so

$$\begin{aligned}\|Av\| &= \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}|H_k(v(\tau),\tau)|\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\ &\leq \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}\max_{[0,c]\times[0,R]}|H_k(v,\tau)|\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\ &\leq R\phi^{-1}\Big(R\big(\max_{[0,c]\times[0,R]}|H_k(v,\tau)|\big)^{1/k}\Big).\end{aligned} \tag{4.5}$$

Since

$$\phi^{-1}(s_1 s_2) \geq \phi^{-1}(s_1)\phi^{-1}(s_2), \quad \forall s_1, s_2 \in [0, +\infty), \tag{4.6}$$

it follows from (4.5) and (C_6) that for all $v \in P(\alpha, b, c)$ with $\|Av\| > d$,

$$\alpha(Av) = -\int_{R-\eta}^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t$$
$$\geq \int_{R-\eta}^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^{R-\eta} \tau^{N-1}\min_{[b,c]\times[0,R-\eta]}|H_k(v,\tau)|\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t$$
$$\geq \int_{R-\eta}^R \phi^{-1}\left(\left(\frac{(R-\eta)^N}{R^{N-k}}\min_{[b,c]\times[0,R-\eta]}|H_k(v,\tau)|\right)^{1/k}\right)\mathrm{d}t$$

$$\geq \eta\phi^{-1}\left(\left(R^k\phi^k\left(\frac{Rb}{\eta d}\right)\max_{[0,c]\times[0,R]}\left|H_k(v,\tau)\right|\right)^{1/k}\right)$$

$$\geq \eta\frac{Rb}{\eta d}\phi^{-1}\left(R\Big(\max_{[0,c]\times[0,R]}|H_k(v,\tau)|\Big)^{1/k}\right) \geq \frac{b}{d}\|Av\| > b.$$

In conclusion, all the conditions of Lemma 2 are satisfied, and thus the singular problem (4.1) has at least three positive solutions v_1, v_2, v_3 satisfying (4.2). This completes the proof of the theorem. □

Remark 2. Let k be an odd number. Assume that there exist constants b, d and η with $0 < b < \frac{\eta}{R}d < d < R$ such that all the conditions in Theorem 5 hold, except that (C_3) and (C_4) are replaced by

(C_3') $\overline{\lim}_{v\to 0^+}\max_{r\in[0,R]}\frac{|H_k(v,r)|}{\phi^k(v/R)} < \frac{1}{R^k}$;

(C_4') $\overline{\lim}_{v\to R^-}\max_{r\in[0,R]}\frac{|H_k(v,r)|}{\phi^k(v/R)} < \frac{1}{R^k}$.

Then, the conclusion of Theorem 5 is still true. In fact, we point out that (C_3) is to show that (ii) of Lemma 2, and (C_4) is used to verify $A(\overline{P}_c) \subset \overline{P}_c$. It is easy to see that the condition (C_3') implies that (C_3) holds, then the condition (ii) of Lemma 2 holds. Hence, we only need to show that $A(\overline{P}_c) \subset \overline{P}_c$ when (C_4') holds. It follows from (C_4') that there exist $\rho \in (0, 1/R^k)$ and $\delta \in (0, R)$ such that

$$|H_k(v,r)| \leq M + \rho\phi^k\left(v/R\right), \quad \forall (v,r) \in [0,R) \times [0,R],$$

where $M = \max\{|H_k(v,r)| : (v,r) \in [0,\delta]\times[0,R]\}$. We now choose the constant c such that

$$\max\left\{d, R\phi^{-1}\left(R\Big(\frac{M}{1-\rho R^k}\Big)^{\frac{1}{k}}\right)\right\} \leq c < R.$$

Then, for all $v \in \overline{P}_c$, we have

$$\begin{aligned}\|Av\| &= \int_0^R \phi^{-1}\left(\Big(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}\big|H_k(v(\tau),\tau)\big|\mathrm{d}\tau\Big)^{1/k}\right)\mathrm{d}t\\ &\leq \int_0^R \phi^{-1}\left(\Big(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}\Big(M + \rho\phi^k\left(c/R\right)\Big)\mathrm{d}\tau\Big)^{1/k}\right)\mathrm{d}t\\ &< R\phi^{-1}\left(R\Big(M + \rho\phi^k\left(cR\right)\Big)^{1/k}\right) \leq c.\end{aligned}$$

This implies $A(\overline{P}_c) \subset \overline{P}_c$.

Remark 3. In Theorem 5, the hypothesis condition (C_6) is unnecessary provided $d = c$.

Remark 4. If $(-1)^k H_k(v,r) \geq 0$ for all $(v,r) \in (-R,R) \times [0,R]$ in problem (4.1), the three positive solutions of Theorem 5 are nonnegative.

4.2 k is even

Similar to the case that k is an odd number, we now present the multiplicity of positive and negative solutions of problem (4.1) when k is an even number.

Theorem 6. *Let k be an even number. Assume that there exist constants a, b, c, d and η with $0 < a < b < \frac{\eta}{R}d < d \leq c < R$ such that*

($\overline{\mathrm{C}}_3$) $H_k(v,r) \leq \frac{1}{R^k}\phi^k\left(\frac{a}{R}\right)$ *for all* $(v,r) \in [-a,a] \times [0,R]$;

($\overline{\mathrm{C}}_4$) $H_k(v,r) \leq \frac{1}{R^k}\phi^k\left(\frac{c}{R}\right)$ *for all* $(v,r) \in [-c,c] \times [0,R]$;

($\overline{\mathrm{C}}_5$) $H_k(v,r) \geq \frac{R^{N-k}}{(R-\eta)^N}\phi^k\left(b/\eta\right)$ *for all* $(v,r) \in ([-d,-b] \cup [b,d]) \times [0,R-\eta]$;

($\overline{\mathrm{C}}_6$) $(R-\eta)^N \min_{(v,r)\in D} H_k(v,r) \geq R^N \phi^k\left(\frac{Rb}{\eta d}\right) \max_{[-c,c]\times[0,R]} H_k(v,r)$, *where* $D = ([-c,-b] \cup [b,c]) \times [0,R-\eta]$.

Then, there exist at least six solutions to problem (4.1)*, three of which are positive solutions v_1, v_2, v_3 satisfying*

$$\|v_1\| < a, \quad b < \min_{[0,R-\eta]} v_2(r), \quad \|v_3\| > a \text{ with } \min_{[0,R-\eta]} v_3(r) < b, \tag{4.7}$$

and the other three are negative solutions v_4, v_5, v_6 satisfying

$$\|v_4\| < a, \quad -b > \max_{[0,R-\eta]} v_5(r), \quad \|v_6\| > a \text{ with } \max_{[0,R-\eta]} v_6(r) > -b. \tag{4.8}$$

Furthermore, if $H_k(v,r) = H_k(-v,r)$ for all $(v,r) \in (-R,R) \times [0,R]$,

$$v_1(r) = -v_4(r), \quad v_2(r) = -v_5(r), \quad v_3(r) = -v_6(r), \quad \forall r \in [0,R].$$

Proof. We first consider the multiplicity of positive solutions. Let

$$P^+ = \left\{v \in C[0,R] : v(r) \text{ is nonnegative and nonincreasing on } [0,R]\right\}.$$

Then, P^+ is a cone in $C[0,R]$. We define the nonlinear operator A^+ on $P^+ \cap B_R$ by

$$(A^+v)(r) = \int_r^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t,$$

then, A^+ is well defined. Exactly analogous to the proof of Theorem 5, we can easily prove that there exist at least three positive solutions v_1, v_2, v_3 to problem (4.1) satisfying (4.7).

Next, we consider the multiplicity of negative solutions. Let

$$P^- = \left\{v \in C[0,R] : v(r) \text{ is nonpositive and nondecreasing on } [0,R]\right\},$$

then, P^- is a cone in $C[0,R]$. We define the operator A^- on $P^- \cap B_R$ by

$$(A^-v)(r) = -\int_r^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t. \tag{4.9}$$

It is easy to see that A^- is well defined. Clearly, for each $v \in P^- \cap B_R$, $(A^-v)(r) \le 0$ and $(A^-v)'(r) \ge 0$ when $r \in [0,R]$, which imply $A^-(P^- \cap B_R) \subset P^-$. Moreover, $(A^-v)(R) = 0$ and $(A^-v)'(0) = 0$. Using the Arzelà-Ascoli theorem, we can show that A^- is compact on $P^- \cap \overline{B}_\rho (\rho \in (0,R))$. Meanwhile, it is easy to know that v is a negative solution to problem (4.1) if and only if v is a fixed point of A^-. Let $\beta : P^- \to [0,+\infty)$ be a nonnegative continuous concave functional defined by

$$\beta(v) = -\max_{r\in[0,R-\eta]} v(r), \quad \forall v \in P^-.$$

It follows that $\beta(v) \le \|v\|$, $v \in P^-$ and

$$\beta(v) = -v(R-\eta), \quad \beta(A^-v) = -(A^-v)(R-\eta), \quad \forall v \in P^-. \tag{4.10}$$

Next, we will split the proof into three steps.

Step 1. We prove $\|A^-v\| < a$ for all $v \in \overline{P_a^-}$. From $(\overline{\mathrm{C}}_3)$ and (4.9), for all $v \in \overline{P_a^-}$, we have

$$\begin{aligned}\|A^-v\| &= \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1} H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\ &\le \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}\frac{1}{R^k}\phi^k\left(a/R\right)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t < a,\end{aligned}$$

which implies that the condition (ii) of Lemma 2 holds. Similarly, we can deduce $A^-(\overline{P_c^-}) \subset \overline{P_c^-}$ by $(\overline{\mathrm{C}}_4)$.

Step 2. Let $v = -(b+d)/2$, then $v \in P^-$, $\|v\| < d$ and $\beta(v) = (b+d)/2 > b$, which show that $\{v \in P^-(\beta,b,d) : \beta(v) > b\} \ne \emptyset$. We also let $v \in P^-(\beta,b,d)$, then $\beta(v) \ge b$ and $\|v\| \le d$, i.e., $-d \le v(r) \le -b$ for all $r \in [0,R-\eta]$. It follows from (4.10) and $(\overline{\mathrm{C}}_5)$ that for all $v \in P^-(\beta,b,d)$,

$$\begin{aligned}\beta(A^-v) &= \int_{R-\eta}^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1} H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\ &\ge \int_{R-\eta}^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^{R-\eta} \tau^{N-1}\frac{R^{N-k}}{(R-\eta)^N}\phi^k\left(b/\eta\right)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\ &> \int_{R-\eta}^R \phi^{-1}\Big(\phi\left(b/\eta\right)\Big)\mathrm{d}t = b.\end{aligned}$$

Hence, the condition (i) of Lemma 2 is satisfied.

Step 3. We show that the condition (iii) of Lemma 2 holds. Notice that for all $v \in P^-(\beta,b,c)$, we have $-c \le v(r) \le -b$ for $r \in [0,R-\eta]$, and so

$$\begin{aligned}\|A^-v\| &= \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1} H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\ &\le \int_0^R \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1}\max_{[-c,0]\times[0,R]} H_k(v,\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\ &\le R\phi^{-1}\left(R\Big(\max_{[-c,0]\times[0,R]} H_k(v,\tau)\Big)^{1/k}\right).\end{aligned}$$

It follows from (4.6) and ($\overline{\mathrm{C}}_6$), when $v \in P^-(\beta, b, c)$ with $\|A^- v\| > d$, we can obtain

$$\begin{aligned}
\beta(A^- v) &= \int_{R-\eta}^{R} \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^t \tau^{N-1} H_k(v(\tau),\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\
&\geq \int_{R-\eta}^{R} \phi^{-1}\left(\left(\frac{N}{t^{N-k}}\int_0^{R-\eta} \tau^{N-1} \min_{[-c,-b]\times[0,R-\eta]} H_k(v,\tau)\mathrm{d}\tau\right)^{1/k}\right)\mathrm{d}t \\
&\geq \int_{R-\eta}^{R} \phi^{-1}\left(\left(\frac{(R-\eta)^N}{R^{N-k}} \min_{[-c,-b]\times[0,R-\eta]} H_k(v,\tau)\right)^{1/k}\right)\mathrm{d}t \\
&\geq \eta\phi^{-1}\left(\left(R^k\phi^k\left(\frac{Rb}{\eta d}\right) \max_{[-c,0]\times[0,R]} H_k(v,\tau)\right)^{1/k}\right) \\
&\geq \eta\frac{Rb}{\eta d}\phi^{-1}\left(R\left(\max_{[-c,0]\times[0,R]} H_k(v,\tau)\right)^{1/k}\right) \geq \frac{b}{d}\|A^- v\| > b.
\end{aligned}$$

In summary, all the conditions of Lemma 2 are satified. Thus, the problem (4.1) has at least three negative solutions v_4, v_5, v_6 satisfying (4.8). This completes the proof of the theorem. □

Remark 5. Let k be an even number. Assume that there exist constants b, d and η with $0 < b < \frac{\eta}{R}d < d < R$ such that all the conditions in Theorem 6 hold, except that ($\overline{\mathrm{C}}_3$) and ($\overline{\mathrm{C}}_4$) are replaced by

($\overline{\mathrm{C}}_3'$) $\overline{\lim}_{v\to 0} \max_{r\in[0,R]} \frac{H_k(v,r)}{\phi^k(v/R)} < \frac{1}{R^k}$;

($\overline{\mathrm{C}}_4'$) $\overline{\lim}_{|v|\to R^-} \max_{r\in[0,R]} \frac{H_k(v,r)}{\phi^k(v/R)} < \frac{1}{R^k}$.

Then, the conclusion of Theorem 6 is still true.

Remark 6. In Theorem 6, the hypothesis condition ($\overline{\mathrm{C}}_6$) is unnecessary provided $d = c$.

Remark 7. If $(-1)^k H_k(v,r) \geq 0$ for all $(v,r) \in (-R,R)\times[0,R]$ in problem (4.1), the three positive solutions of Theorem 6 are nonnegative and the three negative solutions of Theorem 6 are nonpositive.

5 Some examples

In this section, we demonstrate the importance of our results through some illustrative examples.

Example 1. Consider the nonlinear singular problem with the prescribed k-th mean curvature operator in Minkowski space

$$\begin{cases} (\mathcal{M}_k v)(r) = (-1)^k\left(\frac{a}{r^{\alpha_1}(R-r)^{\beta_1}v^p} + \frac{bv^q}{r^{\alpha_2}(R-r)^{\beta_2}} + \frac{c}{r^{\alpha_3}(R-r)^{\beta_3}}\right),\ r\in(0,R), \\ v'(0)=0,\quad v(R)=0, \end{cases} \tag{5.1}$$

where $1 \leq k \leq N$, $a > 0$, $b, c \geq 0$, $p, q > 0$ and $\alpha_i, \beta_i \in [0, 1)$, $i = 1, 2, 3$.

The following is divided into two cases to discuss.

Case 1. k is an odd number. We define

$$H_k(v, r) = -\frac{a}{r^{\alpha_1}(R-r)^{\beta_1}|v|^p} - \frac{b|v|^q}{r^{\alpha_2}(R-r)^{\beta_2}} - \frac{c}{r^{\alpha_3}(R-r)^{\beta_3}}$$

for $(v, r) \in ((-\infty, 0) \cup (0, +\infty)) \times (0, R)$. We consider the following modified singular problem

$$\begin{cases} \frac{1}{Nr^{N-1}}\left(r^{N-k}\phi^k(v')\right)' = H_k(v, r), \quad r \in (0, R), \\ v'(0) = 0, \quad v(R) = 0. \end{cases} \tag{5.1*}$$

It's not difficult to verify that for all $\delta \in (0, R)$,

$$\int_0^R \max_{\delta \leq v \leq R} |H_k(v, r)| \mathrm{d}r < +\infty.$$

Thus, from Theorem 1, the modified problem (5.1*) has at least one positive solution. This implies that the problem (5.1) has at least one positive solution.

Note that the model (5.1) take the model examples in [25, 26] as special cases when $k = 1$.

Case 2. k is an even number and p, q are even. Let

$$H_k(v, r) = \frac{a}{r^{\alpha_1}(R-r)^{\beta_1}v^p} + \frac{bv^q}{r^{\alpha_2}(R-r)^{\beta_2}} + \frac{c}{r^{\alpha_3}(R-r)^{\beta_3}}$$

for $(v, r) \in ((-\infty, 0) \cup (0, +\infty)) \times (0, R)$. Similar to Case 1, we can show that for all $\delta \in (0, R)$,

$$\int_0^R \max_{\delta \leq |v| \leq R} H_k(v, r) \mathrm{d}r < +\infty.$$

Therefore, from Theorem 3, there exist at least two solutions to problem (5.1), one of which is positive and the other is negative.

Example 2. Consider the nonlinear singular problem with the prescribed k-th mean curvature operator in Minkowski space

$$\begin{cases} (\mathcal{M}_k v)(r) = -\frac{a}{r^{\alpha_1}(R-r)^{\beta_1}v^p} - \frac{b}{r^{\alpha_2}(R-r)^{\beta_2}}, \quad r \in (0, R), \\ v'(0) = 0, \quad v(R) = 0, \end{cases} \tag{5.2}$$

where $1 \leq k \leq N$ is an odd number, $a > 0$, $b \geq 0$, $p > 0$ and $\alpha_i, \beta_i \in [0, 1)$, $i = 1, 2$.

Let

$$H_k(v, r) = -\frac{a}{r^{\alpha_1}(R-r)^{\beta_1}|v|^p} - \frac{b}{r^{\alpha_2}(R-r)^{\beta_2}}$$

for $(v, r) \in ((-\infty, 0) \cup (0, +\infty)) \times (0, R)$. Considering the following modified singular problem

$$\begin{cases} \frac{1}{Nr^{N-1}}\left(r^{N-k}\phi^k(v')\right)' = H_k(v, r), \quad r \in (0, R), \\ v'(0) = 0, \quad v(R) = 0. \end{cases} \tag{5.2*}$$

It is easy to verify that for all $\delta \in (0, R)$, we have

$$\int_0^R \max_{\delta \le v \le R} |H_k(v, r)| \mathrm{d}r < +\infty.$$

Furthermore, it is easy to see that for fixed $r \in (0, R)$, $H_k(v, r)$ is nondecreasing with respect to $v \in (0, R)$, and thus by Theorem 2, there exists a unique positive solution to modified problem (5.2*). Therefore, problem (5.2) has a unique positive solution.

Example 3. Consider the nonlinear singular problem with the prescribed k-th mean curvature operator in Minkowski space

$$\begin{cases} (\mathcal{M}_k v)(r) = \frac{a\lambda^{-\frac{1}{v^p}}}{r^{\alpha_1}(R-r)^{\beta_1}} + \frac{bv^q}{r^{\alpha_2}(R-r)^{\beta_2}} + \frac{c}{r^{\alpha_3}(R-r)^{\beta_3}}, & r \in (0, R), \\ v'(0) = 0, \quad v(R) = 0, \end{cases} \tag{5.3}$$

where $1 \le k \le N$ is an even number, $\lambda > 1$, $a > 0$, $b, c \ge 0$, p, q are two even numbers and $\alpha_i, \beta_i \in [0, 1)$, $i = 1, 2, 3$.

Let

$$H_k(v, r) = \frac{a\lambda^{-\frac{1}{v^p}}}{r^{\alpha_1}(R-r)^{\beta_1}} + \frac{bv^q}{r^{\alpha_2}(R-r)^{\beta_2}} + \frac{c}{r^{\alpha_3}(R-r)^{\beta_3}}$$

for $(v, r) \in ((-\infty, 0) \cup (0, +\infty)) \times (0, R)$. Then for all $\delta \in (0, R)$, we have

$$\int_0^R \max_{\delta \le |v| \le R} H_k(v, r) \mathrm{d}r < +\infty.$$

Meanwhile, for each fixed $r \in (0, R)$, $H_k(v, r)$ is nondecreasing with respect to $v \in (0, R)$ and nonincreasing with respect to $v \in (-R, 0)$. Thus by Theorem 4, the problem (5.3) has a unique positive solution and a unique negative solution.

Example 4. Consider the nonlinear singular problem with the prescribed k-th mean curvature operator in Minkowski space

$$\begin{cases} (\mathcal{M}_k v)(r) = (-1)^k \frac{\lambda\mu(r)v^p}{(R-v)^q}, & r \in (0, R), \\ v'(0) = 0, \quad v(R) = 0, \end{cases} \tag{5.4}$$

where $1 \le k \le N$, $p > k$, $\mu(\cdot) \in C([0, R], (0, +\infty))$, $\lambda > 0$ and $0 < q < k/2$.

The following is divided into two cases to discuss.

Case 1. k is an odd number. We select $b, \eta \in (0, R)$ with $b < \eta$. Let

$$H_k(v, r) = -\frac{\lambda\mu(r)|v|^p}{(R-v)^q}, \quad (v, r) \in (-R, R) \times [0, R].$$

A simple computation leads that

$$\overline{\lim_{v \to 0^+}} \max_{r \in [0,R]} \frac{|H_k(v, r)|}{\phi^k\left(v/R\right)} = 0 \quad \text{and} \quad \overline{\lim_{v \to R^-}} \max_{r \in [0,R]} \frac{|H_k(v, r)|}{\phi^k\left(v/R\right)} = 0.$$

So we can choose $a \in (0,b)$ and $c = d \in (b,R)$ with $Rb < \eta c$ such that (C_3) and (C_4) hold. Let $\Lambda = \frac{R^{N-k}(R-b)^q \phi^k(b/\eta)}{Mb^p(R-\eta)^N}$, where $M = \max_{r\in[0,R]} \mu(r)$. Then for $\lambda \geq \Lambda$, we have

$$\begin{aligned} |H_k(v,r)| \geq & \frac{R^{N-k}(R-b)^q \phi^k(b/\eta)}{Mb^p(R-\eta)^N} \cdot \frac{Mv^p}{(R-v)^q} \\ \geq & \frac{R^{N-k}\phi^k(b/\eta)}{(R-\eta)^N}, \quad \forall (v,r) \in [b,d] \times [0, R-\eta]. \end{aligned}$$

This implies (C_5) holds. Hence by Remark 2–4, when $\lambda \geq \Lambda$, the modified problem

$$\begin{cases} \frac{1}{Nr^{N-1}} \left(r^{N-k}\phi^k(v')\right)' = H_k(v,r), & r \in (0,R), \\ v'(0) = 0, \quad v(R) = 0 \end{cases}$$

has at least three nonnegative solutions v_1, v_2, v_3. Therefore, the problem (5.4) has three nonnegative solutions v_1, v_2, v_3 provided λ is sufficiently large.

Note that the problem (5.4) take the model example in [24] as special case when $k = 1$.

Case 2. k is an even number and p is even. We select $b, \eta \in (0,R)$ with $b < \eta$. Let

$$H_k(v,r) = \frac{\lambda\mu(r)v^p}{(R-v)^q}, \quad (v,r) \in (-R,R) \times [0,R].$$

Then we can deduce that

$$\overline{\lim_{v\to 0}} \max_{r\in[0,R]} \frac{H_k(v,r)}{\phi^k\left(v/R\right)} = 0 \quad \text{and} \quad \overline{\lim_{|v|\to R^-}} \max_{r\in[0,R]} \frac{H_k(v,r)}{\phi^k\left(v/R\right)} = 0.$$

Thus, there exist $a \in (0,b)$ and $c = d \in (b,R)$ with $Rb < \eta c$ such that ($\overline{C}_3$) and ($\overline{C}_4$) hold. Let $\Lambda = \frac{R^{N-k}(R+d)^q \phi^k(b/\eta)}{Mb^p(R-\eta)^N}$. Then, for $\lambda \geq \Lambda$, we have

$$H_k(v,r) \geq \frac{R^{N-k}\phi^k(b/\eta)}{(R-\eta)^N}, \quad \forall (v,r) \in ([-d,-b] \cup [b,d]) \times [0, R-\eta].$$

This shows that ($\overline{C}_5$) holds. It follows from Remarks 5–7 that the problem (5.4) has at least three nonnegative solutions and three nonpositive solutions provided λ is sufficiently large.

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