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Constitutive memory equations for auxetic materials

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Abstract. In this note we suggest a set of constitutive equations for anelastic materials whose internal structure can present anomalous variations resulting from external effects of tension/compression, as in auxetic media. For these problems we require of a new function related with the variation of internal structure, which allows us to define a threshold separating the internal structure media from the classic ones. The equations are formulated in the case of static problems but also when the material has memory or plastic properties in addition to auxetic ones. In order to limit the complexity of the formulae, the discussion is limited to the case where the perturbation is one dimensional, which however does not limit the significance of the results.

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1 Introduction

The special properties of auxetic materials are described in many notes; we would like to cite that of [9], mostly concentrated in depth but also with details, on the Poisson ratio and that of [10] which describe with great detail the large variety of properties of these materials which are now attracting an increasing number of possible users and are applied in many fields such as medicine, aerospace, and textiles. Among the important properties of auxetic materials we note that they dilate when subjected to traction and contract when subjected to pressure such that the Poisson ratio is negative. Among the important problems concerning these materials are the mathematical models of these phenomena, specifically the set of constitutive equations adequately modeling their mechanical properties. Various attempts have been made for the solution of these problems, among them we cite that very successful of [1] who presented an elegant theory for auxetic materials giving "the explicit relation between the local auxetic strain and the local rotation and expansion of the elementary building blocks of auxetic materials" which stimulates the present note. We quote also the model of [5], where, with the use of a Landau model [7] and memory represented by first-order derivatives of the stress strain relations, the authors reproduce the properties of auxetic media and the seminal paper [3]. Concerning the mathematical modeling of auxetic materials in this note we present 3-dimensional stress strain relations based on 3 elastic parameters, on a memory formalism represented by a fractional derivative and on a term depending on a dimensionless parameter introduced in order to model the properties of auxetic media which, among other, could be the temperature [6]. The presence of 3 elastic parameters instead of the classic 2 represents the possibility to have negative or positive Poisson ratio. The dimensionless parameter represents the state of the material which could depend on temperature or other physical conditions. The note is structured as follows. First, we introduce the memory dependent stress strain relation, then we apply to the medium a one directional stress and compute directly the Poisson ratio in the case of the absence of memory and assuming that it is given by the classic formula. Then we compute the strains caused by this stress and compute from them the Poisson ratio cases where the memory with $t-z$ (z real) kernel is present showing that, asymptotically, the two values of the Poisson ratio coincide. We show also that, formally, one obtains the same results by using a memory based on any other fractional derivative equivalent to the Caputo one [4]. Finally, we show that formally one may obtain the negative Poisson ratio using also memory-based stress strain relations specific for plastic materials, such that when they are subject to a stress decreasing to zero, they relax and asymptotically remain strained also when the stress is eventually zero.

2 The stress strain relation for auxetic materials

The stress strain relations with memory used in this note are:

$$\tau_{ij} = \delta_{ij}\Big(n - m/g\Big)\epsilon_{rr} + 2\Big(f/g\Big)\epsilon_{ij} + 2\Big(f/g\Big)\eta D^{\nu}\Big(\epsilon_{ij} - \delta_{ij}\epsilon_{rr}/3\Big), \qquad (2.1)$$

where δ_{ij} is the Kronecker delta, τ_{ij} and ϵ_{ij} are stress and strain components respectively dependent on spatial coordinates and time, n, m, f, are positive elastic parameters, ϵ_{rr} is the dilatation, and $g(y)$ dimensionless, monotonically increasing function representing a phase field that describes the supposed evolution of the medium's internal structure; y is a generic physical parameter such as temperature or other. η has dimensions $kg \cdot m^{-1} \cdot s^{-2} \cdot s^{z}$, and D^{ν} represents the [2] fractional derivative or any equivalent fractional derivative, y is an internal parameter denoting a phase field able to describe the supposed evolution of the internal structure of the medium. We note that in this note the effective elastic parameters are 3: ng, m/g and f/g, since $g(y)$ is a parameter introduced to fit the model to auxetic materials. It is worth noting that since $g(y)$ is assumed to increase with increasing y, the rigidity properties vanish when g is sufficiently large. Since we will operate in the Laplace Transform

(LT) domain with LT variable p, the LT of Equations (2.1), assuming zero values of stress and strain at $t = 0$, gives

$$T_{ij} = \delta_{ij}\Big(n - m/g\Big)E_{rr} + 2\Big(f/g\Big)E_{ij} + 2\Big(f/g\Big)\eta D^{\nu}\Big(E_{ij} - \delta_{ij}E_{rr}/3\Big),$$

where $D^z = LT(D^\nu)$, $T_{ij} = LT(\tau_{ij})$. In order to simplify the equations in the following sections we also set:

$$\begin{aligned} &r = n - m/g(y), \quad s = f/g(y),\\ &n > 0,\ m > 0,\ f > 0. \end{aligned} \tag{2.2}$$

3 Studying the constitutive equations of auxetic media: the effect of a stress

In the following we use the Caputo derivative [8] defined as:

$$\frac{\partial^u p}{\partial t^u} = \frac{1}{\Gamma(u-1)}\int_0^t \frac{1}{(t-x)^u}\frac{\partial p}{\partial x}dx,$$

with $0 < u < 1$, and $\Gamma(x)$ the Euler Gamma function, for the somewhat formally simpler notation of its LT. The 3D constitutive equations are then:

$$T_{ij} = r\delta_{ij}E_{rr} + 2sE_{ij} + 2s\eta p^z\Big(E_{ij} - \delta_{ij}E_{rr}/3\Big),$$

or, referring to the normal components of the stress, which are of interest here,

$$\begin{aligned} T_{11} &= rE_{rr} + 2sE_{11} + 2s\eta p^z\Big(E_{11} - E_{rr}/3\Big),\\ T_{22} &= rE_{rr} + 2sE_{22} + 2s\eta p^z\Big(E_{22} - E_{rr}/3\Big),\\ T_{33} &= rE_{rr} + 2sE_{33} + 2s\eta p^z\Big(E_{33} - E_{rr}/3\Big), \end{aligned} \tag{3.1}$$

from which one may express the algebraic expressions of the components E_{ij}. In the case of auxetic media, it is necessary to operate in 3 dimensions, and in order to simplify the computation, without losing generality in the study of these media, we assume nil the normal stress along the x_2 and x_1 axes and different from zero only τ_{11} consequently yielding $E_{22} = E_{33}$ and obtaining the following system in the LT of the 2 unknowns principal strains where T_{11} is assigned

$$\begin{aligned} &T_{11} = r(E_{11} + E_{22} + E_{33}) + 2sE_{11} + 2s\eta p^z(E_{11} - (E_{11} + E_{22} + E_{33})/3),\\ &E_{11}(r - 2s\eta p^z/3) + E_{22}(r - 2s + 4s\eta p^z/3) + E_{33}(r - 2s\eta p^z/3) = 0,\\ &E_{11}(r - 2s\eta p^z/3) + E_{22}(r - 2s\eta p^z/3) + E_{33}(s + 2\mu + 4s\eta p^z/3) = 0. \end{aligned}$$

Adding the 3 Equations (3.1) gives

$$T_{11} + T_{22} + T_{33} = (3r + 2s)(E_{11} + E_{22} + E_{33}),$$

which, in the case of our hypothesis

$$E_{22} = E_{33}, \quad T_{22} \quad = T_{33} = 0, \tag{3.2}$$

is

$$T_{11} = (3r + 2s)(E_{11} + 2E_{22}). \tag{3.3}$$

4 The computation of the deformations and of the Poisson ratio

We first proceed with the computation of the deformation in the LT domain, using the first of Equations (3.1) and Equation (3.3) which we recall below for convenience:

$$T_{11} = r(E_{11} + E_{22} + E_{33}) + 2sE_{11} + 2s\eta p^z(E_{11} - (E_{11} + E_{22} + E_{33})/3),$$
$$T_{11} = (3r + 2s)(E_{11} + 2E_{22}).$$

Then, setting

$$A = (3r + 2s), \quad B \quad = \eta s p^\nu/3, \tag{4.1}$$

we find:

$$\begin{aligned} E_{22} =&(T_{11}/A)(-r + 2B)/(2s + 6B), \\ E_{11} =&(T_{11}/A)(2r + 2s + 2B)/(2s + 6B), \\ E_{22}/E_{11} =&(-r + 2B)/(2r + 2s + 2B), \end{aligned} \tag{4.2}$$

finally, using the Extreme Value Theorem (EVT) on the expressions (4.2) of the strains, we obtain at $t = \infty$

$$E_{22}/E_{11} = -r/[2(r + s)].$$

Provided that the initial conditions concerning Equation (3.2) are satisfied, the expression of the ratio of the LT of the strains in terms of the new parameters gives:

$$E_{22}/E_{11} = -(1/2)(ng - m - 2gB)/(ng - m + f + gB),$$

which asymptotically converges to:

$$\epsilon_{22}/\epsilon_{11} = -(1/2)[ng - m)/(ng - m + f)].$$

The LT of the dilatation is also of interest: $E_{rr} = T_{11}/A$, which, in the case of a 1-D stress parallel to the x_2 or the x_3 axes, is independent of the rheology and is valid for any set of stresses normal to the reference axes.

In the case of absence of memory ($\eta = 0$) the Poisson ratio

$$P = -0.5(ng - m)/(ng - m + f) \tag{4.3}$$

is positive if

$$((m - f)/n) < g < (m/n), \tag{4.4}$$

and it is negative if:

$$((m-f)/n) > g \quad \text{or} \quad g > (m/n).$$

With the exploring assumption that g is monotonically increasing and in order to study the variation of the Poisson ratio, we set: $w = ng - m$, the (4.3) is:

$$P = -w/2(w+f),$$

which is positive (auxetic) when $-f < w < 0$, but when $0 < w$ or $w < -f$, P is negative. Moreover, considering the derivative of P

$$dP/dw = -f/(w+f)^2 > 0,$$

we see that dP/dw, as well as P, has a singular point at $w = -f$. P is decreasing from -0.5 to ∞ when $g < (m-f)/n$; then, in the interval where $P > 0$, P is decreasing from ∞ to 0 at $g = m/n(w = 0)$ and decreasing from 0 at $g = m/n(w = 0)$. When $w > 0$, P is decreasing from ∞ at $g = m/n(w = 0)$ to the asymptotic value 0.5.

The velocity v_p and v_s of the P and S waves respectively are

$$v_p = [(3n - (3m - 2f)/g)/d]^{0.5}, \quad v_s = [(f/g)/d]^{0.5},$$

where d is the density of the medium.
Considering that a real velocity is required, it must be:

$$3ng - 3m - 2f > 0 \quad \text{or} \quad g > m/n - 2f/3n.$$

Considering this conditions together with Equation (4.2), which defines the auxetic media, the Poisson ratio is in the limits $0 < P < 2/3$. However, taking into account that we require that the velocity of the P waves be positive, then the condition for the parameters is $w > 2f/3$ or $g > m/n - 2f/3n$, which is more restrictive. When $x > 0$, the Poisson ratio in negative.

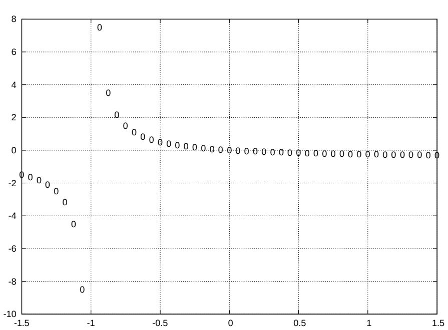

Figure 1. Plot of $P(x) = -0.5/(1 + 1/x)$ with $x = (ng - m)/f$.

Where we note (Figure 1), that assuming m, n, f and g are positive with $x = (ng - m)/f$ the material would be auxetic, when $-4/7 < x < 0$ or $-4f/7 < ng - m < 0$, or $0 < g < m - 4f/7$ and the maximum possible value of P of the model would be $2/3$ at $x = -4/7$.
Some critical values are of interest and are shown in Table 1.

Table 1. Values of P for different values of the parameters w, and x.

w	g	x	P
$-2f$	$(m-2f)/n$	-2	-2
$-3f/2$	$(m-3f/2)/n$	$-3/2$	$-3/2$
$-2f/3$	$(m-2f/3)/n$	$-2/3$	$+1$
$-4f/7$	$(m-4f/7)/n$	$-4/7$	$2/3$
$-f$	$(m-f)/n$	-1	∞
$-f/2$	$(m-f/2)/n$	$-1/2$	$+1/2$
0	$m-n$	0	0
$f/2$	$(m+f/2)/n$	$1/2$	$-1/6$
$2f/3$	$(m+2f/3)/n$	$2/3$	$-2/5$
∞	∞	∞	$-1/2$

5 The time domain expression of the deformations

Now, we proceed to express the deformations in terms of the parameters of the constitutive equations, that is in terms of n, m, f, and $g(y)$. In order to satisfy the nil initial condition of stress and strain, we assume that τ_{11} is a step function $H(\sigma, t)$ we assume $H(\sigma, t)$ a step function at $t = \sigma$,with σ as small as desired, whose LT is

$$T_{11} = \exp(-\sigma p)/p. \tag{5.1}$$

Remembering that:

$$\begin{aligned} E_{22} &= (T_{11}/A)(-r + 2B)/(2s + 6B), \\ E_{11} &= (T_{11}/A)2(2r + 2s + 2B)/2(2s + 6B), \end{aligned} \tag{5.2}$$

we then find:

$$\begin{aligned} E_{22} &= (T_{11}/A)[m - ng + 2Bg]/[2f + 6Bg], \\ E_{11} &= (T_{11}/A)[2ng - 2m + 2f + 2Bg]/[2f + 6Bg], \end{aligned}$$

where T_{11} is given by Equation (5.1). Reformulating Equations (5.2) we obtain:

$$\begin{aligned} E_{11} &= (T_{11}/3A)[(6 + 4s + 2s + 6B)/(2s + 6B)], \\ E_{22} &= (T_{11}/3A)[(6r + 4s)/(2s + 6B) + 1]. \end{aligned}$$

The LT^{-1} of both deformations are readily expressed in terms of the Mittag-Leffler function, taking into account Equations (4.1) and (2.2):

$$\begin{aligned} &A = (3r + 2s), \quad s = f/g, \quad r = n - m/g, \\ &B = \eta s p^\nu/3, \quad B = f\eta p^\nu/3g, \end{aligned}$$

we find:

$$E_{11} = (T_{11}/3A)[1 + (6r + 4s)/[2f\eta/g]/[\eta^{-1} + p^{\nu}]],$$
$$E_{22} = (T_{11}/3A)[1 + (-3r - 2s)/[2f\eta/g]/[\eta^{-1} + p^{\nu}]].$$

And finally:

$$\begin{aligned} E_{11} &= (T_{11}/3A)[1 + (\alpha/\beta)(\eta^{-1} + p^{\nu}], \\ E_{22} &= (T_{11}/3A)[1 + (\gamma/\beta)(\eta^{-1} + p^{\nu}], \end{aligned} \tag{5.3}$$

where: $\alpha = (6r + 4s)$, $\beta = (2f\eta/g)$, $\gamma = (-3r - 2s)$. We note that the asymptotic values of the strain are:

$$\epsilon_{22}(\infty) = (\tau_{11}/3A)/(1 + \gamma\eta/\beta),$$
$$\epsilon_{11}(\infty) = (\tau_{11}/3A)/(1 + \alpha\eta/\beta).$$

The time domain expression of the deformations are readily found in terms of Mittag-Leffler function:

$$\epsilon_{11}(t) = (\tau_{11}/3A)H(\sigma,t) + \tau_{11}/3A)(\alpha/\eta\beta)[\sin(\pi v)/(\pi v)]$$
$$\times \left\{ H(\sigma,t) * \int_0^\infty \exp(-(\eta^{-1}u)^{1/v}t)du/(u^2 + 2u\cos(\pi v) + 1) \right\},$$
$$\epsilon_{11}(t) = (\tau_{11}/3A)H(\sigma,t) + \tau_{11}/3A)(\alpha/\eta\beta)[\sin(\pi v)/(\pi v)]$$
$$\times \int_0^\infty [\exp(-(\eta^{-1}u)^{1/v}\sigma) - \exp(-(\eta^{-1}u)^{1/v}t)]du/(u^2 + 2u\cos(\pi v) + 1), \tag{5.4}$$
$$\epsilon_{22}(t) = (\tau_{11}/3A)H(\sigma,t) + \tau_{11}/3A)(\gamma/\eta\beta)[\sin(\pi v)/(\pi v)]$$
$$\times \left\{ H(\sigma,t) * \int_0^\infty \exp(-(\eta^{-1}u)^{1/v}t)du/(u^2 + 2u\cos(\pi v) + 1) \right\},$$
$$\epsilon_{22}(t) = (\tau_{11}/3A)H(\sigma,t) + \tau_{11}/3A)(\gamma/\eta\beta)[\sin(\pi v)/(\pi v)]$$
$$\times \int_0^\infty [\exp(-(\eta^{-1}u)^{1/v}\sigma) - \exp(-(\eta^{-1}u)^{1/v}t)]du/(u^2 + 2u\cos(\pi v) + 1), \tag{5.5}$$

which are valid for $t \geq \sigma$, as consequence of the convolution; the integrals are monotonically increasing to a finite value. From which the Green functions are readily obtained substituting $H(\sigma,t)$ with $\delta(t - \sigma)$. The Figure 2 shows the values of the integral appearing in both deformations for different values of v and with the time t in abscissa in units of $1/\eta^{1/v}$.

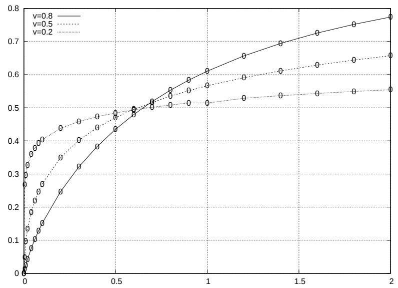

Figure 2. Values of the integrals present in formulae (5.4) and (5.5) calculated for different values of v .

Figure 2 gives the values of:

$$[\sin(\pi v)/(\pi v)]\int_0^\infty[\exp(-(\eta^{-1}u)^{1/v}\sigma)-\exp(-(\eta^{-1}u)^{1/v}t)]du/(u^2+2u\cos(\pi v)+1),$$

which appear in formulae (5.4) and (5.5). These values should be completed with the terms appearing in formulae (5.4) and (5.5), namely the factor $(\tau_{11}/3A)(\alpha/\eta\beta)$ concerning ϵ_{11} and the factor $(\tau_{11}/3A)(\gamma/\eta\beta)$ concerning ϵ_{22}.

It is verified that with values of g in the interval:

$$m/n > g > m/n - 2f/3n,$$

the corresponding values of α and γ in formulae (5.4) and (5.5), with $\tau_{11} > 0$ ($\tau_{11} < 0$), give positive (negative) values of deformations, that is, give $P > 1$. We note that the effect of the memory on the deformations, represented by the integrals in Equations (5.4) and (5.5), depends on the 2 parameters ν and η. With reference to the Figure 2 , where the time is measured in units of $\eta\ 1/\nu$, for a given ν, which we assume here $0 < \nu < 1$, an increase of η implies that a shorter time is required to reach a given value of the integral, that is the time scale is dilated and vice versa.

6 The auxetic media with memory based on the exponential kernel

If we were to use a memory based on the fractional derivative with exponential kernel [4], then the term B of Equation (4.4) is:

$$B_1 = \mu\eta p/(p(1-v)+v),$$

where η is dimensionless and the ratio of the strains is:

$$\begin{aligned}E_{22}/E_{11} =& -(1/2)[((ng^2-m)(1-v)-2\mu\eta)p\\ &+(ng^2-m)]/(p(1-v)+v))(ng^2+(f-m)),\end{aligned}$$

which asymptotically gives:

$$\epsilon_{22}/\epsilon_{11} = -(1/2)[((ng^2 - m)/v(ng^2 + (f - m))],$$

which is positive in the range defined in Equation (4.3).
In a similar manner, we find the strains using Equation (5.3).

$$\begin{aligned} E_{11} &= -(T_{11}/A)(r - A)[p(1 - v) + v]/(2s(1 - v) + 4s\eta)p + 2sv, \\ E_{22} &= -(T_{11}/A)(2r - 4s\eta p)/(2r + 4s\eta p). \end{aligned} \tag{6.1}$$

We note that in Equations (6.1), the right-hand members are formed by the ratio of linear forms of the variable p; then the LT^{-1} is readily expressed in terms of exponentials.

Using the EVT one finds that the asymptotic values of the strains are:

$$\epsilon_{22} = -(\tau_{11}/3A)(1 + \gamma\eta/\beta), \quad \epsilon_{11} = -(\tau_{11}/3A)(1 + \alpha\eta/\beta),$$

which, as expected, are equal to the asymptotic values obtained using the fractional derivative with kernel t^{-v}; only the travel times to these values are different.

7 Conclusions

The constitutive equations proposed in this note to discuss some the properties of materials with internal structures include 3 parameters instead of the classic 2 of elastic media, but there is also a generic dimensionless function $g(y)$ which defines the thresholds for possible phase changes to the different properties of the material. The presence of 3 parameters instead of the classic 2 of perfectly elastic media allows the comparison with the elastic media and with the plastic ones. The properties of auxetic materials modeled concerning the change of sign and values of their Poisson ratio are valid also when the material has a memory.

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