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Generalized practical stability of Hopfield-type neural networks differential equations

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Abstract. This paper investigates the boundedness and practical stability properties of solutions for a class of neural differential equations inspired by Hopfield-type neural networks. Specifically, we develop a novel analytical framework that extends beyond traditional Lyapunov stability theory, Barbalat-type arguments, and fixed-point methods by relaxing common structural assumptions such as smoothness and global Lipschitz continuity. Our approach broadens the class of admissible systems to include nonlinearities with weaker growth conditions and time-varying perturbations that are not easily handled by classical techniques. Sufficient conditions are established to ensure the existence of a globally exponentially stable neighborhood of the origin, even in the presence of varying perturbation conditions. Furthermore, numerical examples are provided to demonstrate and validate the main result.

Keywords: equilibrium; Hopfield neural networks; generalized practical stability; boundedness.

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1 Introduction

Nonlinear differential equations are commonly used in various scientific and industrial fields to model many intriguing and significant phenomena mathematically. These equations are inspired by challenges that arise in various disciplines, including engineering, control theory, materials science, biology, physics, fluid dynamics, economics, and quantum mechanics, as noted in references [4, 18] and others. Over the past ten years, a great deal of work has been devoted to the study of Hopfield-type neural networks' analysis and synthesis (see [2, 3, 4, 5, 6, 7, 10, 12, 13, 17]), as these networks have a wide range of applications in optimization, associative memories, and engineering challenges [15, 16, 23]. In many cases, exact solutions for differential equations are not available. In such instances, integral inequalities play a significant role in studying the existence, uniqueness, boundedness, stability, and asymptotic behavior of solutions to differential equations (as discussed in [14,18]). In a related work [18], the authors introduced a new Gamidov inequality and analyzed the stability properties of solutions of bilinear control systems. Local asymptotically stable equilibrium states acting as attractors in neuronal associative memories store information and construct distributed and parallel brain memory networks. In these situations, the goal of qualitative analysis is to examine state convergence to guarantee memory recall as well as the existence, asymptotic stability, and attractive regions of equilibrium for the networks' capacity to store information. Exponential stability is conservative in some concrete applications (measurement noise and disturbances [1,8]). A practical property of solutions that can be established for such models is ultimate boundedness, which means that the solution remains in some neighborhood of the origin after a sufficiently large time. In this study, we aim to analyze the stability of a Hopfield-type neural network system within a small region, particularly when the origin may not be considered an equilibrium point. This characteristic is referred to as practical stability, as discussed in references [8]. Our main theoretical contribution involves establishing new sufficient conditions for the practical exponential stability of time-varying Hopfield-type neural networks using a novel nonlinear integral inequality framework. Unlike classical approaches that heavily rely on Lyapunov functions, Barbalat-type lemmas, or fixed-point techniques, our method generalizes and extends traditional results by relaxing structural constraints on the perturbation terms and network parameters. Specifically, we generalize the stability conditions found in works based on standard Lyapunov theory, and integral inequalities such as Gronwall's and Gamidov's approaches ([14,18]), by allowing time-varying coefficients and more general nonlinear perturbations. Our integral inequality technique can accommodate larger classes of bounded disturbances and less restrictive growth conditions on nonlinear terms. Compared to classical Lyapunov-based methods, which often require strict definiteness conditions and differentiability assumptions on Lyapunov candidates, our framework relaxes these assumptions. For instance, we remove the necessity of constructing explicit Lyapunov functions satisfying strong derivative conditions, instead relying on bounding integral inequalities that are simpler and more broadly applicable. Moreover, our approach expands the class of admissible systems by encompassing time-varying neural networks subject to perturbations that may not satisfy global Lipschitz or monotonicity conditions. This includes systems with coefficients depending on both time and state in a nonlinear way, thus addressing more realistic models encountered in engineering and biological networks. Our method uses integral inequalities techniques to explore the idea of practical stability for such a model. Global exponential stability for time-varying Hopfield-type neural networks refers to the condition in which the neural network's state trajectory converges exponentially to an equilibrium point, and the rate of this convergence is uniform over time, despite variations in the system parameters or external inputs. Determining if the equilibrium point of a nonlinear dynamics is (globally) asymptotically stable can generally be a very challenging task. The primary challenge is that, in most cases, writing a solution to the differential equation explicitly is not feasible. Our method allows us to apply weaker conditions on the network dynamics, namely that the condition imposed on the perturbation term generates a large class of systems that can be stable. Our approach uses new integral inequalities that are simpler to use compared to conventional methods. This flexibility opens a wider applicability by offering stability constraints that cover a larger class of neural networks, in contrast to Lyapunov techniques which give more restrictive results due to the presence of nonlinear terms in the system. Lyapunov theory has been the favored choice, even if some of the findings have been drawn from the characteristics of fundamental matrices. We prefer to employ a novel integral inequality in this work, which is widely recognized to produce more sensitive and sharp outcomes. We mostly use Gamidov's analysis method and provide findings regarding exponential convergence and boundedness. Introducing a more generic bounding structure, we relax the requirement on the finiteness of the coefficients, which is typical for the perturbation term. Hopfield networks are a type of recurrent neural network where neurons are fully interconnected, typically with symmetric connections. These networks can be used for optimization problems, associative memory, and pattern recognition tasks. Global exponential stability for time-varying Hopfield-type neural networks provides a mathematical framework for ensuring that these networks behave predictably and robustly, even as their parameters change over time. In the presence of bounded perturbations, this kind of convergence implies that, for any initial state, the network's state trajectory will converge to a small compact set with an exponential rate.

In this paper, we establish new sufficient conditions that guarantee the practical exponential stability of solutions for time-varying Hopfield-type neural networks, even in the presence of varying perturbation conditions. These conditions are derived using a novel nonlinear integral inequality framework, which generalizes and extends classical results, including those of Gamidov's and Gronwall's types. Our analysis relaxes the usual constraints on perturbation terms, allowing a broader class of systems to be covered and enhancing the model's robustness to small parameter variations. The theoretical findings are supported by numerical examples, which demonstrate the applicability and effectiveness of the proposed approach. Additionally, we provide a qualitative discussion on the asymptotic behavior of solutions to nonlinear systems and differential equations, in connection with the general theory of motion stability. For further insights into this topic, the reader is referred to [19, 20, 21, 22].

The remainder of the paper is organized as follows. Section 2 introduces the Hopfield-type neural network model. The main results are presented in Section 3. Section 4 provides numerical examples. Conclusions are drawn in Section 5.

2 Hopfield-type neural networks model

Let $\mathbb{R}$ denote the set of real numbers. The set of all non-negative real numbers is denoted by $\mathbb{R}_+$, and $\mathbb{R}_+^* = (0, \infty)$ represents the set of strictly positive real numbers. The space $\mathbb{R}^n$ refers to the n-dimensional Euclidean space, equipped with its usual norm $\|\cdot\|$. Additionally, the ball centered at zero with radius r is defined as $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$.

Consider the following system that models a Hopfield-type neural network in continuous time:

$$\mathcal{C}_i \dot{\xi}_i = -\frac{1}{\mathcal{R}_i}\xi_i + \sum_{k=1}^{n} T_{ik} v_k + I_i^{\varepsilon}(t), \quad i = 1, ..., n, \tag{2.1}$$

where $\mathcal{C}_i \in \mathbb{R}_+^*$ denotes the capacitance, and the parameter $\frac{1}{\mathcal{R}_i}$ is defined by

$$\frac{1}{\mathcal{R}_i} = \frac{1}{R_i} + \sum_{k=1}^{n} |T_{ik}|,$$

with $R_i > 0$. The coefficient $T_{ik} = \frac{1}{R_{ik}} \in \mathbb{R}$ is associated with the interconnection strength between neurons, where R_{ik} denotes the resistance (possibly adjusted to account for a sign inversion due to amplification effects). The term $v_k = F_k(\xi_k)$ represents the output of an amplifier, where $F_k : \mathbb{R} \to (-1, 1)$ is a nonlinear function. While our analysis assumes that each activation function F_k is smooth, bounded, and differentiable, this assumption is introduced primarily to guarantee analytical tractability and to rigorously apply the theoretical stability framework. We acknowledge that this may restrict the direct applicability of the results to practical scenarios where activation functions could exhibit discontinuities or nonsmooth behavior.

The function $I_i^{\varepsilon}(t)$ is a continuous external input defined on $\mathbb{R}_+$, where $\varepsilon > 0$ is a small parameter. The above system (2.1) can be expressed equivalently as:

$$\dot{\xi}_i = -\zeta_i \xi_i + \sum_{k=1}^{n} \mathcal{B}_{ik} F_k(\xi_k) + u_i^{\varepsilon}(t), \quad i = 1, ..., n,$$

where

$$\zeta_i = \frac{1}{\mathcal{R}_i \mathcal{C}_i}, \quad \mathcal{B}_{ik} = \frac{T_{ik}}{\mathcal{C}_i}, \quad u_i^{\varepsilon} = \frac{I_i^{\varepsilon}(t)}{\mathcal{C}_i}.$$

For improved clarity, we rewrite the system in vector form as follows:

$$\dot{\sigma}(t) = A\sigma(t) + f(\sigma(t)) + \mathcal{U}^{\varepsilon}(t), \quad \sigma = (\sigma_1, ..., \sigma_n)^T, \tag{2.2}$$

where

$$A = \begin{pmatrix} -\zeta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\zeta_n \end{pmatrix}, \quad \mathcal{U}^{\varepsilon}(t) = (u_1^{\varepsilon}(t), ..., u_n^{\varepsilon}(t))^T,$$

$$f(\sigma) = (f_1(\sigma), ..., f_n(\sigma))^T, \quad f_i(\sigma) = \sum_{k=1}^{n} \mathcal{B}_{ik} F_k(\sigma_k), \quad i = 1, ..., n.$$

The associated nominal system corresponding to (2.2) is:

$$\dot{\sigma}(t) = A\sigma(t) + f(\sigma(t)).$$

Thus, the equilibrium states of system (2.2) satisfy the nonlinear algebraic equations:

$$\dot{\xi}_i = 0, \quad \text{or equivalently} \quad \zeta_i \xi_i - \sum_{k=1}^{n} \mathcal{B}_{ik} F_k(\xi_k) = 0, \quad i = 1, ..., n.$$

Building on the results in [11] and [12], the authors in [15] provided new sufficient conditions ensuring that a system of the form (2.2) admits a unique equilibrium point. We assume henceforth that the nominal system has a unique equilibrium point x^*. Without loss of generality, by a suitable change of variables, we assume that the origin corresponds to the unique equilibrium point of the nominal system. Extending the theoretical framework to accommodate piecewise-smooth or discontinuous activation functions entails significant technical challenges, including establishing the existence and uniqueness of solutions, managing nonsmooth trajectories, and the limitations of classical stability methods such as Lyapunov functions in this context. These issues constitute avenues for further research. In the presence of external inputs, the nonlinearities must satisfy sufficiently small bounds, generally related to the nominal system's dynamics, particularly its linear component. The functions $f(\sigma)$ and $\mathcal{U}^{\varepsilon}(t)$ may be subjected to additional constraints, which enable the study of the asymptotic behavior of solutions under these nonlinear perturbations.

Observe that the linear subsystem

$$\dot{\sigma}(t) = A\sigma(t)$$

is asymptotically stable, as the matrix A is diagonal with strictly negative diagonal entries. Furthermore, since $\mathcal{R}e\lambda(A) < 0$, there exist constants $\theta \geq 1$ and $\upsilon > 0$ such that

$$\|e^{tA}\| \leq \theta e^{-\upsilon t}, \quad \forall t \geq 0,$$

where $\mathcal{R}e\lambda(A)$ denotes the real parts of the eigenvalues of matrix A.

Practically, empirical approaches such as smoothing discontinuous activation functions or using approximations with continuous differentiable functions may bridge the gap between theory and real-world nonsmooth behaviors. These approximations allow us to maintain analytical tractability while better capturing practical scenarios, and form a promising direction for future extensions.

3 Generalized practical stability

For dynamical systems, we first introduce the notion of generalized practical exponential stability following the approach in [1]– [8].

Consider the system described by:

$$\dot{\sigma}(t) = \Lambda^{\varepsilon}(t, \sigma(t)), \tag{3.1}$$

where Λ^{ε} is a smooth function, $t \in \mathbb{R}_+$ denotes time, $\sigma \in \mathbb{R}^n$ is the state vector, and $\varepsilon > 0$ is a parameter. We assume the initial time $t_0 = 0$ and denote by $\sigma(t)$ the solution of system (3.1) with initial condition $(0, \sigma(0))$.

Definition 1. System (3.1) is said to be *globally uniformly practically exponentially stable* with respect to the small parameter $\varepsilon > 0$ if its solution $\sigma(t)$ satisfies the inequality

$$\|\sigma(t)\| \leq \theta\|\sigma(0)\|e^{-\upsilon t} + r(\varepsilon), \quad \forall t \geq 0, \tag{3.2}$$

for some positive constants $\theta > 0$, $\upsilon > 0$, and $r(\varepsilon) > 0$.

Inequality (3.2) implies that trajectories approach a neighborhood of the origin of radius $r(\varepsilon)$, since we can rewrite it as

$$\|\sigma(t)\| - r(\varepsilon) \leq \theta\|\sigma(0)\|e^{-\upsilon t}, \quad \forall t \geq 0,$$

where the initial conditions are assumed outside the ball $B_{r(\varepsilon)}$. Hence, the solutions are globally uniformly bounded and converge exponentially to the set $B_{r(\varepsilon)}$. Moreover, if $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, then the states approach the origin exponentially as $t \to +\infty$.

Denote

$$\Lambda^{\varepsilon}(t,\sigma) = A\sigma + F^{\varepsilon}(t,\sigma), \quad F^{\varepsilon}(t,\sigma) = f(\sigma) + \mathcal{U}^{\varepsilon}(t),$$

and suppose the following system is considered for the convergence and boundedness analysis:

$$\dot{\sigma}(t) = A\sigma(t) + F^{\varepsilon}(t,\sigma(t)). \tag{3.3}$$

Note that $\mathcal{R}e\lambda(A) < 0$.

We will investigate various assumptions on the perturbation term $F^{\varepsilon}(t,\sigma)$ necessary for ensuring boundedness and convergence of solutions to system (3.3). The perturbations appear as an additive term on the right-hand side of the state equation; consequently, the origin is not necessarily an equilibrium point of the perturbed system. Based on the stability of the nominal system generated by matrix A, which admits the origin as an equilibrium point, we expect that the solutions of the perturbed system approach a small neighborhood of the origin as $t \to \infty$. The best achievable outcome is that for sufficiently small perturbations, the trajectories remain close to a small set containing the origin. It is noteworthy that the desired state may be mathematically unstable, yet the system can exhibit oscillatory behavior near this state with acceptable performance. In typical applications, the exact perturbation function F may be unknown, and only bounds on its norm are available.

Remark 1. If the activation functions $F_k(\sigma_k)$ are only continuous and piecewise smooth (e.g., possessing a finite number of jump discontinuities in their derivatives), the right-hand side of the system (2.2), $\Lambda^{\varepsilon}(t,\sigma) = A\sigma + f(\sigma) + \mathcal{U}^{\varepsilon}(t)$, becomes discontinuous. For such systems, the solution concept must be generalized. We will define the dynamics using a Filippov differential inclusion: $\dot{\sigma}(t) \in \mathcal{K}[\Lambda^{\varepsilon}](t,\sigma(t))$, where $\mathcal{K}[\cdot]$ denotes the Filippov set-valued map. This map convexifies the vector field at the points of discontinuity, providing a rigorous mathematical foundation for analyzing solution trajectories. Moreover, we can explicitly relate the Filippov inclusion to Clarke's generalized gradient for the potential functions associated with the activations. Specifically, if we consider $f_i(\sigma) = \sum_{k=1}^{n} \mathcal{B}_{ik} F_k(\sigma_k)$, the set-valued map $\mathcal{K}[f_i](\sigma)$ can be expressed in terms of the generalized gradients $\partial F_k(\sigma_k)$. For a scalar function like F_k, $\partial F_k(x)$ is the interval between the left and right derivatives at x. To bridge the gap between the smooth and nonsmooth worlds and add practical relevance, we can use a smoothed approximation which can be illustrated by a canonical example with discontinuous activation like the hard sigmoid:

$$H(x) = \begin{cases} -1, & \text{for } x \leq -1, \\ x, & \text{for } -1 < x < 1, \\ 1, & \text{for } x \geq 1. \end{cases}$$

We can approximate it with a smooth function $H_\mu(x)$, for instance, a high-gain sigmoid $\tanh(x/\mu)$ or a smoothed piecewise function, where $\mu > 0$ is a small smoothing parameter. We will then analyze the system using the smooth function $H_\mu(x)$ under our existing framework. We can show how the stability bounds (e.g., the term $r(\varepsilon)$ in Definition 1 might depend on the smoothing parameter μ. The key point would be that as $\mu \to 0^+$, the smoothed system converges to the nonsmooth Filippov system, and the practical stability properties are recovered in the limit. The stability analysis for the Filippov system would follow a nonsmooth Lyapunov approach. While the technical details will be expanded in the revision, the core idea is that the exponential convergence of the linear part and the boundedness of the generalized gradients of the activations would allow us to derive an inequality for the time derivative of the Lyapunov function $V(\sigma) = \frac{1}{2}\|\sigma\|^2$ in a differential form, leading to a similar practical stability conclusion.

We begin by recalling several preliminary lemmas that support the subsequent analysis.

Lemma 1. *Let $\pi : \mathbb{R}_+ \to \mathbb{R}$ be a continuous, nonnegative, and locally integrable function on $\mathbb{R}_+$ satisfying*

$$\sup_{t\geq 0} \int_t^{t+1} \pi(s)\, ds < +\infty. \tag{3.4}$$

Then, for all $t \geq 0$ and any $\upsilon > 0$, the following estimate holds:

$$e^{-\upsilon t} \int_0^t e^{\upsilon s} \pi(s)\, ds < +\infty.$$

Proof. For $t \geq 0$, let $N = E(t)$, where $E(\cdot)$ denotes the integer part function that satisfies $t-1 < E(t) \leq t$ or $N \leq t < N+1$. Therefore, for $s \in [0,t),\ t \geq 0$, one has

$$\int_0^t e^{-\upsilon(t-s)}\pi(s)ds = \int_0^N e^{-\upsilon(t-s)}\pi(s)ds + \int_N^t e^{-\upsilon(t-s)}\pi(s)ds.$$

Then,

$$\int_0^t e^{-\upsilon(t-s)}\pi(s)ds = \sum_{k=0}^{N-1} e^{-\upsilon t} \int_{N-k-1}^{N-k} e^{\upsilon s}\pi(s)ds + e^{-\upsilon t}\int_N^t e^{\upsilon s}\pi(s)ds.$$

It follows that,

$$\int_0^t e^{-\upsilon(t-s)}\pi(s)ds \le \sum_{k=0}^{N-1} e^{-\upsilon k}e^{-\upsilon(t-N)} \sup_{t\ge 0}\int_t^{t+1}\pi(s)ds + e^{-\upsilon t}\int_N^t e^{\upsilon s}\pi(s)ds.$$

Thus,

$$\int_0^t e^{-\upsilon(t-s)}\pi(s)ds \le \frac{1-e^{-N\upsilon}}{1-e^{-\upsilon}} \sup_{t\ge 0}\int_t^{t+1}\pi(s)ds + \sup_{t\ge 0}\int_t^{t+1}\pi(s)ds.$$

Therefore,

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds \le c(\upsilon)\sup_{t\ge 0}\int_t^{t+1}\pi(s)ds,$$

where $c(\upsilon)$ is a certain nonnegative constant. Thus, for all $t \ge 0$, we have

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds < +\infty, \quad \upsilon > 0.$$

□

Lemma 2. *Let $\pi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ a continuous nonnegative function on $\mathbb{R}_+$ satisfying:*
i)

$$\lim_{t\to\infty}\pi(t) = 0, \tag{3.5}$$

or
ii)

$$\int_0^\infty \pi(s)ds < +\infty, \tag{3.6}$$

here π is supposed an integrable function on $\mathbb{R}_+$, then

$$\lim_{t\to\infty} e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds = 0, \quad \upsilon > 0. \tag{3.7}$$

Proof. To verify *i)*, we use the fact that $\lim_{t\to\infty}\pi(t) = 0$. We have, for any $\eta > 0$, for $\upsilon\eta > 0$, $\exists T > 0$, such that for all $t \ge T$, $\pi(t) < \upsilon\eta$. Let

$$M = \int_0^T e^{\upsilon s}\pi(s)\,ds.$$

For $t \ge T$, one has

$$M \le \frac{\upsilon\eta}{\upsilon}(e^{\upsilon T}-1).$$

Thus, for $t \ge T$,

$$e^{-\upsilon t}M \le e^{-\upsilon t}\eta(e^{\upsilon T}-1).$$

Then,

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds \le e^{-\upsilon t}\Big(\int_0^T e^{\upsilon s}\pi(s)ds + \int_T^t e^{\upsilon s}\pi(s)ds\Big).$$

It follows that,

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds \le e^{-\upsilon t}M + e^{-\upsilon t}\frac{\upsilon\eta}{\upsilon}(e^{\upsilon t}-e^{\upsilon T}).$$

So,

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds \le e^{-\upsilon t}\eta(e^{\upsilon T}-1) + e^{-\upsilon t}\eta(e^{\upsilon t}-e^{\upsilon T}).$$

Hence,

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds \le \eta - \eta e^{-\upsilon t} \le \eta.$$

We have, for any $\eta > 0$, $\exists T > 0$, such that for all $t \ge T$,

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds \le \eta.$$

Therefore, the last expression implies that: $\lim_{t\to\infty} e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds = 0$.

To verify *ii*), one can use the fact that π is integrable, then given any $\varepsilon > 0$ there exist $T_1 > 0$ and $T_2 > 0$, such that

$$\int_{\frac{t}{2}}^{\infty} \pi(s)ds < \frac{\varepsilon}{2}, \ \ \forall t \ge T_1, \quad e^{-\upsilon\frac{t}{2}}\int_0^{\frac{t}{2}} \pi(s)ds < \frac{\varepsilon}{2}, \ \ \forall t \ge T_2,$$

Then,

$$e^{-\upsilon t}\int_0^{\frac{t}{2}} e^{\upsilon s}\pi(s)ds \le e^{-\upsilon t}e^{\upsilon\frac{t}{2}}\int_0^{\frac{t}{2}} \pi(s)ds < \frac{\varepsilon}{2},$$
$$e^{-\upsilon t}\int_{\frac{t}{2}}^{t} e^{\upsilon s}\pi(s)ds \le \int_{\frac{t}{2}}^{t} \pi(s)ds < \frac{\varepsilon}{2}.$$

Thus, $\forall t \ge \sup(T_1, T_2)$, one has

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds = e^{-\upsilon t}(\int_0^{\frac{t}{2}} e^{\upsilon s}\pi(s)ds + \int_{\frac{t}{2}}^{t} e^{\upsilon s}\pi(s)ds \le \int_{\frac{t}{2}}^{t} \pi(s)ds),$$

hence

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, one gets (3.7). □

Note that the expression given in (3.7), implies that there exists $\tilde{\pi} > 0$, such that

$$e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)ds \le \tilde{\pi}, \ \ \forall t \ge 0.$$

This uniform bound plays a crucial role in deducing the asymptotic behavior of solutions via integral inequalities.

Furthermore, if $\pi(t) \leq \hat{\pi}$ for all $t \geq 0$, with some $\hat{\pi} > 0$, or if the function π is integrable over $\mathbb{R}_+$, i.e., $\|\pi\|_1 = \int_0^\infty \pi(s)\,ds < +\infty$, then the following estimation holds:

$$e^{-\upsilon t} \int_0^t e^{\upsilon s} \pi(s)\,ds \leq \min\left(\frac{\hat{\pi}}{\upsilon},\ \|\pi\|_1\right) < +\infty, \quad \forall t \geq 0.$$

To establish the stability of system (3.3), we employ a generalized Gronwall-type inequality, as detailed in [14].

Lemma 3. *Let Ξ, λ, and ψ be nonnegative continuous functions defined on $\mathbb{R}_+$ satisfying the integral inequality*

$$\Xi(t) \leq a + \int_0^t [\Xi(s)\lambda(s) + \psi(s)]\,ds, \quad \forall t \geq 0,$$

where $a \geq 0$ is a constant. Then, it follows that

$$\Xi(t) \leq \left(a + \int_0^t \psi(s)\,ds\right) e^{\int_0^t \lambda(\tau)\,d\tau}, \quad \forall t \geq 0.$$

To address the practical stability problem, we impose the following assumption:

$(\mathcal{H}_0)$ For all $t \geq 0$ and $\sigma \in \mathbb{R}^n$, the function F^ε satisfies the inequality

$$\|F^\varepsilon(t,\sigma)\| \leq \mu(t)\|\sigma\| + \ell(\varepsilon)\pi(t),$$

where $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

$$\int_0^\infty \mu(s)\,ds < +\infty,$$

$\ell : \mathbb{R}_+^* \to \mathbb{R}_+$ is continuous, and $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nonnegative, locally integrable function satisfying condition (3.4) for all $t \geq 0$.

The above lemmas will be instrumental in analyzing the asymptotic behavior of the system under less restrictive conditions on uncertainties. The following theorem encapsulates the main result:

Theorem 1. *Suppose that assumption $(\mathcal{H}_0)$ holds. Then, the system (3.3) possesses bounded solutions and is globally uniformly practically exponentially stable with respect to the parameter $\varepsilon > 0$. Consequently, the Hopfield-type neural network system (2.1) enjoys the same stability property.*

Moreover, if

$$\lim_{\varepsilon \to 0^+} \ell(\varepsilon) = 0,$$

then the solutions of system (3.3) converge exponentially to the origin as $t \to +\infty$. Accordingly, the solutions of the Hopfield-type neural network system (2.1) also exhibit exponential convergence to zero as time tends to infinity.

Proof. By the variation of constants formula, the solution $\sigma(t)$ of system (3.3) can be represented as

$$\sigma(t) = e^{tA}\sigma_0 + \int_0^t e^{(t-s)A} F^{\varepsilon}(s, \sigma(s))\, ds.$$

Taking norms and applying the triangle inequality yields

$$\|\sigma(t)\| \le \|e^{tA}\| \|\sigma_0\| + \int_0^t \|e^{(t-s)A}\| \|F^{\varepsilon}(s, \sigma(s))\|\, ds.$$

Since $\mathcal{R}e\, \lambda(A) < 0$, there exist constants $\theta \ge 1$ and $\upsilon > 0$ such that

$$\|e^{tA}\| \le \theta e^{-\upsilon t}, \quad \forall t \ge 0.$$

Utilizing assumption $(\mathcal{H}_0)$, we obtain

$$\|\sigma(t)\| \le \theta e^{-\upsilon t}\|\sigma_0\| + \int_0^t \theta e^{-\upsilon(t-s)} \big(\mu(s)\|\sigma(s)\| + \ell(\varepsilon)\pi(s)\big) ds.$$

Define the auxiliary function $\Xi(t) := e^{\upsilon t}\|\sigma(t)\|$. Multiplying the above inequality by $e^{\upsilon t}$, we have

$$\Xi(t) \le \theta\|\sigma_0\| + \theta \int_0^t \mu(s)\Xi(s)\, ds + \theta\ell(\varepsilon) \int_0^t e^{\upsilon s}\pi(s)\, ds.$$

Applying Lemma 3 yields

$$\Xi(t) \le \left(\theta\|\sigma_0\| + \theta\ell(\varepsilon) \int_0^t e^{\upsilon s}\pi(s)\, ds\right) e^{\theta \int_0^t \mu(s) ds},$$

which implies

$$\|\sigma(t)\| \le \theta\|\sigma_0\| e^{\theta \int_0^t \mu(s) ds} e^{-\upsilon t} + \theta\ell(\varepsilon) e^{-\upsilon t} \left(\int_0^t e^{\upsilon s}\pi(s)\, ds\right) e^{\theta \int_0^t \mu(s) ds}.$$

From Lemma 1, there exists a constant $\tilde{\pi} > 0$ such that

$$e^{-\upsilon t} \int_0^t e^{\upsilon s}\pi(s)\, ds \le \tilde{\pi}, \quad \forall t \ge 0.$$

Therefore,

$$\|\sigma(t)\| \le \theta e^{\theta \int_0^\infty \mu(s) ds} \|\sigma_0\| e^{-\upsilon t} + \theta\ell(\varepsilon)\tilde{\pi} e^{\theta \int_0^\infty \mu(s) ds}.$$

This establishes that solutions remain ultimately bounded within the ball

$$B_{r_\varepsilon} := \{\sigma \in \mathbb{R}^n : \|\sigma\| \le r_\varepsilon\}, \quad \text{where} \quad r_\varepsilon := \theta\ell(\varepsilon)\tilde{\pi} e^{\theta \int_0^\infty \mu(s) ds}.$$

Hence, system (3.3) is globally uniformly practically exponentially stable with respect to the parameter $\varepsilon > 0$.

Moreover, as $\ell(\varepsilon) \to 0$ when $\varepsilon \to 0^+$, it follows that $r_\varepsilon \to 0$, and thus the solutions converge exponentially to the origin. The same conclusion applies to the Hopfield-type neural network system (2.1). □

Remark 2. The system (3.3) has bounded solutions and is globally uniformly practically exponentially stable with respect to a small parameter $\varepsilon > 0$ when we modify the condition (3.4) for π according to (3.5) or (3.6). Our condition on the function π is more general as, previously, almost all functions considered were integrable, such as in [14]. Here is an example of a non-integrable function that satisfies our new condition (3.4): $\pi(t) = \frac{\sin(t+1)}{t+1}$. This class can be extended to the case where the diagonal elements of the matrix A are time-dependent. In this setting, assuming that the associated transition matrix is globally uniformly asymptotically stable, the same stability result as in the previous theorem holds. Moreover, this extension can be further adapted or generalized to the case where the matrix A contains non-diagonal negative elements, provided that the norm of the corresponding transition matrix satisfies an exponential decay estimate. Unlike Lyapunov-based techniques, a strict Lyapunov functional that decreases along trajectories is not required. Instead, the approach relies on explicit norm bounds and perturbation estimates. This allows accommodating time-varying and nonlinear perturbations $\mathcal{U}^{\varepsilon}(t)$ with minimal smoothness and boundedness assumptions. Additionally, the class of admissible nonlinearities $f(\cdot)$ is expanded beyond traditional Lipschitz or sector bounds to include more general nonlinear mappings with known growth conditions. As a result, the method broadens the class of admissible systems to encompass nonlinear, time-varying, and perturbed infinite-dimensional systems where classical Lyapunov constructions may be challenging or unavailable. This flexibility enables the analysis and design of control strategies for systems such as perturbed neural networks, delayed or distributed parameter systems, and other complex dynamical models.

In the sequel, we consider the following hypothesis.

($\mathcal{H}_1$) For all $t \geq 0$ and all $\sigma \in \mathbb{R}^n$, assume that

$$\|F^{\varepsilon}(t,\sigma)\| \leq \ell_1\|\sigma\| + \ell_2(\varepsilon)\|\sigma\|^p + \ell_{\varepsilon}(t),$$

where $p \in (0,1)$, ℓ_1, $\ell_2(\varepsilon)$ are known nonnegative constants, and $\ell_{\varepsilon} : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying

$$\int_0^{\infty} e^{\upsilon s}\ell_{\varepsilon}(s)ds = \bar{\pi}_{\varepsilon} < +\infty.$$

To establish the convergence and stability criteria, we present the following lemma, which is a version of Gamidov's lemma needed for the proof of the main result (see [18]).

Lemma 4. *If*

$$\vartheta(t) \leq \Im + \int_0^t [\delta\vartheta(s) + ce^{(1-p)\upsilon s}\vartheta^p(s)]ds,$$

where ϑ *is continuous and nonnegative on* $[0,\infty)$, $p \in (0,1)$*;* $\Im, c > 0$ *and* υ, δ, *such that* $0 \leq \delta < \upsilon$, *then*

$$\vartheta(t) \leq 2^{\frac{p}{1-p}}\Im e^{\delta t} + \left(\frac{2^p c}{\upsilon - \delta}\right)^{\frac{1}{1-p}} e^{\upsilon t}.$$

We have the following theorem.

Theorem 2. *If assumption $(\mathcal{H}_1)$ holds with $\ell_1 < \frac{\upsilon}{\theta}$, then the solutions of system (3.3) are bounded, and there exists $r_\varepsilon > 0$ such that $B_{r(\varepsilon)}$ is globally uniformly exponentially stable. This implies that the Hopfield-type neural network system (2.1) is also globally uniformly practically exponentially stable.*

Furthermore, if $\bar{\pi}_\varepsilon, \ell_2(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, then the solutions of system (3.3) approach the origin exponentially as t tends to infinity. Consequently, the solutions of the Hopfield-type neural network system (2.1) also approach the origin exponentially as t tends to infinity.

Proof. The solution of (3.3) can be expressed as:

$$\sigma(t) = e^{tA}\,\sigma_0 + \int_0^t e^{(t-s)A}\,(f(\sigma(s)) + \mathcal{U}^\varepsilon(s))\,ds.$$

Thus,

$$\|\sigma(t)\| \le \|e^{tA}\|\,\|\sigma_0\| + \int_0^t \|e^{(t-s)A}\|\,\|f(\sigma(s)) + \mathcal{U}^\varepsilon(s)\|\,ds.$$

Since $\mathcal{R}e\lambda(A) < 0$, it follows that

$$\|\sigma(t)\| \le \theta e^{-\upsilon t}\|\sigma_0\| + \int_0^t \theta e^{-\upsilon(t-s)}\,(\|f(\sigma(s))\| + \|\mathcal{U}^\varepsilon(s)\|)\,ds.$$

Therefore, taking into account assumption $(\mathcal{H}_1)$, yields

$$\|\sigma(t)\| \le \theta e^{-\upsilon t}\|\sigma_0\| + \theta e^{-\upsilon t}\int_0^t e^{\upsilon s}\,\Big(\ell_1\|\sigma(s)\| + \ell_2(\varepsilon)\|\sigma(s)\|^p + \ell_\varepsilon(s)\Big)\,ds.$$

The last expression implies that,

$$\begin{aligned}\|\sigma(t)\| \le \quad & \theta e^{-\upsilon t}\|\sigma_0\| + \theta e^{-\upsilon t}\int_0^t e^{\upsilon s}\Big(\ell_1\,\|\sigma(s)\| + \ell_2(\varepsilon)\|\sigma(s)\|^p\Big)\,ds \\ & + \theta e^{-\upsilon t}\int_0^t e^{\upsilon s}\ell_\varepsilon(s)\,ds.\end{aligned}$$

By multiplying both sides by $e^{\upsilon t}$, it comes that

$$e^{\upsilon t}\|\sigma(t)\| \le \theta\|\sigma_0\| + \theta\int_0^t e^{\upsilon s}\,\Big(\ell_1\,\|\sigma(s)\| + \ell_2(\varepsilon)\|\sigma(s)\|^p\Big)\,ds + \theta\int_0^\infty e^{\upsilon s}\ell_\varepsilon(s)\,ds.$$

Thus,

$$e^{\upsilon t}\|\sigma(t)\| \le \theta\|\sigma_0\| + \theta\int_0^t e^{\upsilon s}\,\Big(\ell_1\,\|\sigma(s)\| + \ell_2(\varepsilon)\|\sigma(s)\|^p\Big)\,ds + \theta\bar{\pi}_\varepsilon.$$

Let $\vartheta(t) = e^{\upsilon t}\|x(t)\|$, one gets

$$\vartheta(t) \le \Im_\varepsilon + \int_0^t [\delta\vartheta(s) + c_\varepsilon e^{(1-p)\upsilon s}\vartheta^p(s)]ds,$$

where $\Im_\varepsilon = \theta\|\sigma_0\| + \theta\bar{\pi}_\varepsilon$, $\delta = \theta\ell_1$ and $c_\varepsilon = \theta\ell_2(\varepsilon)$. Using Lemma 4 with the fact that $\ell_1 < \frac{\upsilon}{\theta}$, it comes that

$$\vartheta(t) \leq 2^{\frac{p}{1-p}} \Im_\varepsilon e^{\delta t} + \left(\frac{2^p c_\varepsilon}{\upsilon - \delta}\right)^{\frac{1}{1-p}} e^{\upsilon t}.$$

We have,

$$e^{\upsilon t}\|\sigma(t)\| \leq 2^{\frac{p}{1-p}} \Im_\varepsilon e^{\delta t} + \left(\frac{2^p c_\varepsilon}{\upsilon - \delta}\right)^{\frac{1}{1-p}} e^{\upsilon t}.$$

Hence,

$$\|\sigma(t)\| \leq 2^{\frac{p}{1-p}} \theta\|\sigma_0\| e^{-\tilde{\upsilon} t} + \left(\frac{2^p \theta\ell_2(\varepsilon)}{\tilde{\upsilon}}\right)^{\frac{1}{1-p}} + 2^{\frac{p}{1-p}} \theta\bar{\pi}_\varepsilon,$$

where $\tilde{\upsilon} = \upsilon - \delta > 0$. Then, the solutions of system (3.3) are bounded and (3.3) is globally uniformly practically exponentially stable. As a result, the set $\mathcal{B}_{r_\varepsilon}$, where

$$r_\varepsilon = \left(\frac{2^p \theta\ell_2(\varepsilon)}{\tilde{\upsilon}}\right)^{\frac{1}{1-p}} + 2^{\frac{p}{1-p}} \theta\bar{\pi}_\varepsilon,$$

is globally uniformly exponentially stable. Consequently, the Hopfield-type neural network system (2.1) is also globally uniformly practically exponentially stable.

Additionally, as ε approaches zero, the value of r_ε tends to zero, and the solutions of the system (3.3) exponentially approach the origin as t tends to infinity. Consequently, the solutions of the Hopfield-type neural network system (2.1) also approach the origin exponentially as t tends to infinity. □

As a special case, when handling more general perturbations (sublinear growth, time-varying bounds), explicit quantitative relationships between perturbation parameters and stability bounds can be estimated, and the solutions of the system are bounded and globally uniformly practically exponentially stable.

Corollary 1. 1. When $\ell_2(\varepsilon) = 0$ and $\ell_\varepsilon(t) = M$ (constant) in hypothesis $(\mathcal{H}_1)$, Theorem 2 reduces to the classical Gronwall inequality, yielding practical stability with $r = \frac{\theta M}{\upsilon - \theta\ell_1}$. In this case, the bound becomes:

$$\|\sigma(t)\| \leq \theta e^{-\tilde{\upsilon} t}\|\sigma(0)\| + \theta M/\tilde{\upsilon},$$

where $\tilde{\upsilon} = \upsilon - \theta\ell_1 > 0$. This is precisely the form of a Gronwall inequality with constant perturbation.

2. When $\ell(\varepsilon) = 0$ in hypothesis $(\mathcal{H}_1)$, so $\bar{\pi}_\varepsilon = 0$, which matches Gamidov's inequality for integrable perturbations, yielding practical stability with $r(\varepsilon) = \left(2^p\theta\ell_2(\varepsilon)/\tilde{\upsilon}\right)^{\frac{1}{1-p}}$. In this case, the bound becomes:

$$\|\sigma(t)\| \leq 2^{\frac{p}{1-p}} \theta\|\sigma_0\| e^{-\tilde{\upsilon} t} + \left(2^p\theta\ell_2(\varepsilon)/\tilde{\upsilon}\right)^{\frac{1}{1-p}},$$

where $\tilde{\upsilon} = \upsilon - \delta > 0$ with $\delta = \theta l_1$.

As a particular case, we consider the following hypothesis:

($\mathcal{H}_2$) Assume that, for all $t \geq 0$ and all $\sigma \in \mathbb{R}^n$,

$$\|F^\varepsilon(t,\sigma)\| \leq \ell_1\|\sigma\| + \ell(\varepsilon)\pi(t),$$

where ℓ_1 is a known nonnegative constant, $\ell : \mathbb{R}_+^\star \to \mathbb{R}_+$, is a continuous function, and $\pi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuous nonnegative function on $\mathbb{R}_+$ satisfying (3.4).

We have the following result.

Theorem 3. *Suppose that the assumption ($\mathcal{H}_2$) holds with $\ell_1 < \frac{\upsilon}{\theta}$, then the solutions of system (3.3) are bounded and there exits $r_\varepsilon > 0$ such that $B_{r(\varepsilon)}$ is globally uniformly exponentially stable.*

Moreover, if $\ell(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, the solutions of system (3.3) approach the origin exponentially as $t \to +\infty$. As a result, the solutions of the Hopfield-type neural network system (2.1) also approach the origin exponentially as $t \to +\infty$.

Proof. The solution of (3.3) can be expressed as:

$$\sigma(t) = e^{tA}\,\sigma_0 + \int_0^t e^{(t-s)A}\left(f(\sigma(s)) + \mathcal{U}^\varepsilon(s)\right)ds.$$

By using the same argument as in the proof of Theorem 2, and from ($\mathcal{H}_1$), yields

$$\|\sigma(t)\| \leq \theta e^{-\upsilon t}\|\sigma_0\| + \theta e^{-\upsilon t}\int_0^t e^{\upsilon s}\left(\ell_1\|\sigma(s)\| + \ell(\varepsilon)\pi(s)\right)ds.$$

The last expression implies that,

$$\|\sigma(t)\| \leq \theta e^{-\upsilon t}\|\sigma_0\| + \theta e^{-\upsilon t}\int_0^t e^{\upsilon s}\ell_1\,\|\sigma(s)\|ds + \theta e^{-\upsilon t}\int_0^t e^{\upsilon s}\ell(\varepsilon)\pi(s)\,ds.$$

Since π verifies Equation (3.4), Lemma 1 ensures

$$\|\sigma(t)\| \leq \theta e^{-\upsilon t}\|\sigma_0\| + \theta e^{-\upsilon t}\int_0^t e^{\upsilon s}\ell_1\,\|\sigma(s)\|ds + \theta\ell(\varepsilon)\tilde{\pi},$$

where

$$\tilde{\pi} = e^{-\upsilon t}\int_0^t e^{\upsilon s}\pi(s)\,ds.$$

By multiplying both sides by $e^{\upsilon t}$, it comes that

$$e^{\upsilon t}\|\sigma(t)\| \leq (\theta\|\sigma_0\| + \theta\ell(\varepsilon)\tilde{\pi}e^{\upsilon t}) + \ell_1\theta\int_0^t e^{\upsilon s}\,\|\sigma(s)\|\,ds.$$

Let $\vartheta(t) = e^{\upsilon t}\|x(t)\|$, one gets

$$\vartheta(t) \leq (\theta\|\sigma_0\| + \theta\ell(\varepsilon)\tilde{\pi}e^{\upsilon t}) + \ell_1\theta\int_0^t \vartheta(s)\,ds.$$

Let $\lambda_\varepsilon(t) = \theta\|\sigma_0\| + \theta\ell(\varepsilon)\tilde{\pi}e^{\upsilon t}$. One has,

$$\vartheta(t) \leq \lambda_\varepsilon(t) + \ell_1\theta \int_0^t \vartheta(s)\,ds.$$

Then, using a modified Gronwall Lemma [9, Theorem 1], one has

$$\vartheta(t) \leq \lambda_\varepsilon(t) + \ell_1\theta \int_0^t \lambda_\varepsilon(s)e^{\ell_1(t-s)}\,ds.$$

It follows that,

$$e^{\upsilon t}\|\sigma(t)\| \leq (\theta\|\sigma_0\| + \theta\ell(\varepsilon)\tilde{\pi}e^{\upsilon t}) + \ell_1\theta \int_0^t (\theta\|\sigma_0\| + \theta\ell(\varepsilon)\tilde{\pi}e^{\upsilon s})\, e^{\ell_1\theta(t-s)}\,ds.$$

Thus,

$$\begin{aligned}\|\sigma(t)\| \leq \quad & \theta\|\sigma_0\|e^{-\upsilon t} + \theta\ell(\varepsilon)\tilde{\pi} + \ell_1\theta e^{-\upsilon t}\int_0^t \theta\|\sigma_0\|e^{\ell_1\theta(t-s)}\;ds \\ & +\ell_1 e^{-\upsilon t}\int_0^t \theta^2\ell(\varepsilon)\tilde{\pi}e^{\upsilon s}\;e^{\ell_1\theta(t-s)}\,ds.\end{aligned}$$

Then,

$$\begin{aligned}\|\sigma(t)\| \leq \quad & \theta\|\sigma_0\|e^{-\upsilon t} + \theta\ell(\varepsilon)\tilde{\pi} + \ell_1\theta^2\|\sigma_0\|e^{-\upsilon t}\int_0^t e^{\ell_1\theta(t-s)}\;ds \\ & +\ell_1\theta^2\ell(\varepsilon)\tilde{\pi}e^{-\upsilon t}\int_0^t e^{\ell_1\theta(t-s)+\upsilon s}\,ds.\end{aligned}$$

Let $\tilde{\upsilon} = \upsilon - \ell_1\theta$. Since by assumption $\ell_1 < \frac{\upsilon}{\theta}$, one has $\tilde{\upsilon} > 0$. Setting

$$r_\varepsilon = \theta\ell(\varepsilon)\tilde{\pi} + \frac{\ell_1\theta^2\ell(\varepsilon)\tilde{\pi}}{\tilde{\upsilon}},$$

a simple computation gives:

$$\|\sigma(t)\| \leq \theta\|\sigma_0\|e^{-\tilde{\upsilon}t} + r_\varepsilon.$$

Then, the solutions of system (3.3) are bounded and (3.3) is globally uniformly practically exponentially stable. Hence, the Hopfield-type neural network system (2.1) is also globally uniformly practically exponentially stable.

Moreover, as ε approaches zero, the value of r_ε tends to zero, and the solutions of the system (3.3) exponentially approach the origin as t tends to infinity. Consequently, the solutions of the Hopfield-type neural network system (2.1) also approach the origin exponentially as t tends to infinity. □

4 Numerical examples

In general, the equilibrium point of the perturbed system may not coincide with the origin. Consequently, one cannot analyze the stability of the origin as an equilibrium point, nor expect the solutions of the perturbed system to converge to the origin as $t \to +\infty$. Instead, the best attainable property is that the solutions approach a small neighborhood containing the origin when the perturbation is sufficiently small. We present numerical examples using a Hopfield-type neural network model to demonstrate the new perturbation conditions developed in this work.

Example 1. Consider the stability of the following system:

$$\begin{cases} \dot{\sigma}_1 = -\sigma_1 + \dfrac{1}{1+t^2} \dfrac{\sigma_1^2}{1+\sqrt{\sigma_1^2+\sigma_2^2}} + \dfrac{\varepsilon}{(1+\varepsilon^2)(t+2)\ln^2(t+2)(1+\sigma_1^2)}, \\ \dot{\sigma}_2 = -\sigma_2 + \dfrac{1}{1+t^2} \dfrac{\sigma_2^2}{1+\sqrt{\sigma_1^2+\sigma_2^2}}, \end{cases} \tag{4.1}$$

where $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$, $t \in \mathbb{R}_+$, and $\varepsilon > 0$.

The above system can be written in the form of (2.2) with $\zeta_1 = 1$, $\zeta_2 = 1$,

$$f(\sigma) = \left(\frac{1}{1+t^2} \frac{\sigma_1^2}{1+\sqrt{\sigma_1^2+\sigma_2^2}} + \frac{\varepsilon}{(1+\varepsilon^2)(t+2)\ln^2(t+2)(1+\sigma_1^2)}, 0 \right)^T,$$

$$\mathcal{U}^\varepsilon(t) = \left(0, \frac{1}{1+t^2} \frac{\sigma_2^2}{1+\sqrt{\sigma_1^2+\sigma_2^2}} \right)^T.$$

Since $\mathcal{R}e\lambda(A) < 0$, there exist $\theta \geq 1$ and $\upsilon > 0$ such that

$$\|e^{tA}\| \leq \theta e^{-\upsilon t}, \quad \forall t \geq 0.$$

($\mathcal{R}e\lambda(A)$ denotes the real parts of the eigenvalues of matrix A).

Furthermore, condition $(\mathcal{H}_0)$ is satisfied with

$$\mu(t) = \frac{1}{1+t^2}, \quad \ell(\varepsilon) = \frac{\varepsilon}{1+\varepsilon^2}, \quad \text{and} \quad \pi(t) = \frac{1}{(t+2)\ln^2(t+2)}.$$

Specifically,

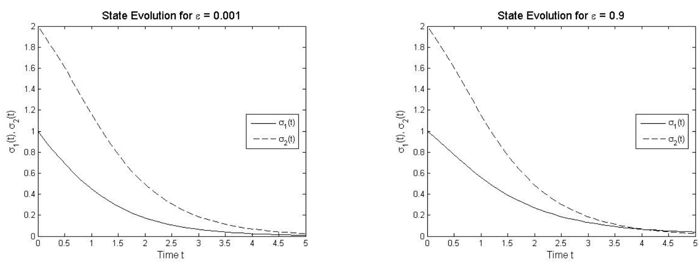
$$\sup_{t\geq 0} \int_t^{t+1} \pi(s)ds = \sup_{t\geq 0} \int_{t+1}^{t+2} \frac{du}{u \ln^2 u} = \sup_{t\geq 0} \int_{\ln(t+1)}^{\ln(t+2)} \frac{dw}{w^2}$$
$$= \sup_{t\geq 0} \left(\frac{1}{\ln(t+1)} - \frac{1}{\ln(t+2)} \right) < +\infty.$$

By Theorem 1, the solutions of system (4.1) are bounded and globally uniformly practically exponentially stable. Thus, for initial conditions outside the ball $B_{r(\varepsilon)}$ centered at the origin with radius $r(\varepsilon)$, the solutions tend toward zero as $\varepsilon \to 0$ and $t \to +\infty$.

To validate the robustness and general applicability of the theoretical results, we conducted extensive numerical simulations encompassing various perturbation magnitudes (both smaller and larger values of ε), a range of initial conditions (including $\sigma_0 = (1,2), (3,1)$, and small random perturbations), as well as modifications in the nonlinear terms to represent diverse network configurations. These simulations consistently demonstrate practical exponential stability, with trajectories confined within bounded neighborhoods of the origin irrespective of the variations considered. Additionally, sensitivity analyses indicate that the solutions maintain practical boundedness under parameter changes, thus underscoring the flexibility and reliability of the theoretical framework.

Figures 1(a) and 1(b) illustrate the evolution of the state $\sigma(t)$ of system (4.1) with initial condition $\sigma_0 = (1,2)$ for $\varepsilon = 0.001$ and $\varepsilon = 0.9$, respectively.

These simulations highlight two essential aspects of the practical stability result established in Theorem 1. For small ε (e.g., $\varepsilon = 0.001$), the radius of the practical stability ball $r(\varepsilon)$ is correspondingly small, resulting in solutions that asymptotically converge to the origin, which aligns with the theoretical asymptotic stability prediction. Conversely, for larger ε (e.g., $\varepsilon = 0.9$), the radius $r(\varepsilon)$ increases, ensuring that solutions remain confined within a neighborhood of the origin. Although exact convergence to the origin is not guaranteed in this case, the boundedness of trajectories confirms the global uniform practical exponential stability for higher perturbation levels.



(a) Evolution of the state $\sigma(t)$ with $\varepsilon = 0.001$

(b) Evolution of the state $\sigma(t)$ with $\varepsilon = 0.9$

Figure 1. (a) $\varepsilon = 0.001$: solutions converge to the origin, illustrating asymptotic behavior for small perturbations; (b) $\varepsilon = 0.9$: solutions remain bounded within a neighborhood of the origin, demonstrating practical stability for larger perturbations.

Example 2. To demonstrate the scalability of the result given in Theorem 2, we consider a 4D Hopfield network. We define a stable matrix

$$A = \mathrm{diag}(-1,-1,-1,-1)(\theta = 1, \upsilon = 1).$$

The activation function remains $f(\sigma) = \tanh(\sigma)$. The perturbation is chosen to be more complex, satisfying the more general assumption:

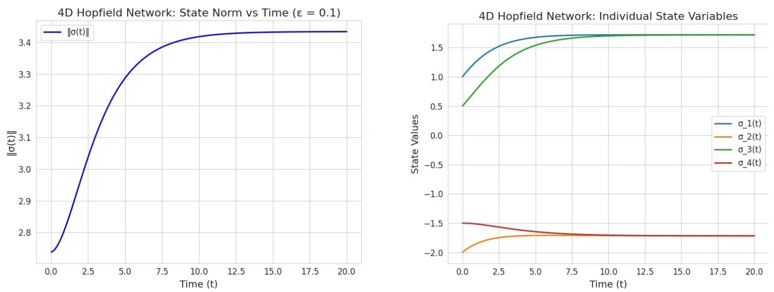
$$\|F^{\varepsilon}(t,\sigma)\| \leq \ell_1\|\sigma\| + \ell_2(\varepsilon)\|\sigma\|^p + \ell_{\varepsilon}(t).$$

We select $\ell_1 = 0.4$, $p = 0.5$, $\ell_2(\varepsilon) = \varepsilon$, and $\ell_{\varepsilon}(t) = \varepsilon e^{-0.5t}$. A specific perturbation that satisfies this is

$$F^{\varepsilon}(t,\sigma) = 0.4\sigma + \varepsilon\sigma/(\sqrt{\|\sigma\|+1}) + \varepsilon e^{-0.5t}\mathbf{1},$$

where **1** is a vector of ones. The complete system is:

$$\dot{\sigma}(t) = -0.6\sigma(t) + \tanh(\sigma(t)) + \varepsilon\left(\frac{\sigma}{\sqrt{\|\sigma\| + 1}} + e^{-0.5t}\mathbf{1}\right).$$



(a) Phase portrait (b) Phase norm vs time

Figure 2. (a) Phase portrait; (b) Phase norm vs time for the 4D Hopfield network.

Practical exponential stability of the 4D Hopfield network, demonstrating the scalability of the theoretical results. All four neuron states in the 4D network remain bounded and converge to a small region around zero. We simulate this 4D system with $\varepsilon = 0.1$ and initial condition $\sigma_0 = [1, -2, 0.5, -1.5]^T$. Figures 2(a) and 2(b) show that the trajectory converges exponentially to a ball near the origin. The behavior of all four individual neuron states is shown in this figure illustrating their coordinated stability under perturbation.

Example 3. To substantiate the theoretical findings of Theorem 3, we present comprehensive numerical simulations. We consider a Hopfield Neural Network with external perturbations satisfying assumption $(\mathcal{H}_2)$: $F^{\varepsilon}(t,\sigma) = \ell_1\sigma + \ell(\varepsilon)\pi(t)$, with $\pi(t) = e^{-0.5t}$. We set $\ell_1 = 0.4$, which satisfies the critical condition $\ell_1 < \upsilon/\theta = 1$. The function $\pi(t)$ is exponentially decaying, ensuring

$$\int_0^\infty e^{\upsilon s}\pi(s)ds = \int_0^\infty e^{1\cdot s}e^{-0.5s}ds = \int_0^\infty e^{0.5s}ds = 2 < \infty.$$

We define $\ell(\varepsilon) = \varepsilon$. The complete perturbed system is:

$$\dot{\sigma}(t) = -A\sigma(t) + f(\sigma(t)) + 0.4\sigma(t) + \varepsilon e^{-0.5t}.$$

This simplifies to the following equation:

$$\dot{\sigma}(t) = -0.6\sigma(t) + f(\sigma(t)) + \varepsilon e^{-0.5t}.$$

According to Theorem 3, the solutions converge to a ball of radius r_ε. With our parameters $(\theta = 1, \upsilon = 1, \ell_1 = 0.4, \tilde{\pi} = 2)$, the radius is given by

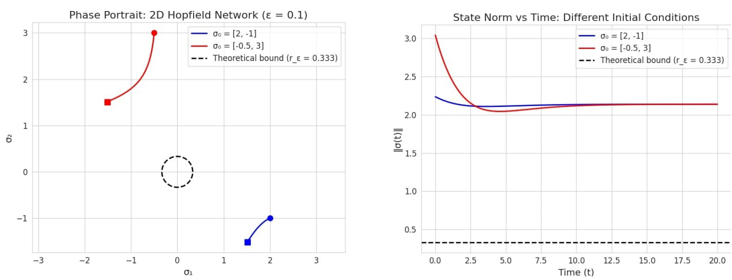
$$r_\varepsilon = \theta\ell(\varepsilon)\tilde{\pi} + \frac{\ell_1\theta^2\ell(\varepsilon)\tilde{\pi}}{\tilde{\upsilon}}, \quad \text{where } \tilde{\upsilon} = \upsilon - \ell_1\theta = 0.6.$$

Substituting the numerical values, we obtain

$$r_\varepsilon = 2\varepsilon + \frac{0.8\varepsilon}{0.6} = 2\varepsilon + 1.333\varepsilon = 3.333\varepsilon.$$

For $\varepsilon = 0.1$, we expect the state to converge to a ball of radius $r_{0.1} = 0.3333$. We simulate the system for two different initial conditions: $\sigma_0^{(1)} = [2, -1]^T$ and $\sigma_0^{(2)} = [-0.5, 3]^T$. The results, shown in Figure 3, confirm that the system is globally practically stable, as the trajectories from different starting points converge to the same neighborhood of the origin. We vary the perturbation magnitude $\varepsilon \in \{0, 0.05, 0.1\}$. Figures 3(a) and 3(b) show the Euclidean norm $\|\sigma(t)\|$ over time. For $\varepsilon > 0$, the trajectories converge to a ball around the origin. The final steady-state error increases with ε, and the empirical convergence radius aligns closely with the theoretical bound $r_\varepsilon = 3.333\varepsilon$, validating Theorem 3.

As perturbation ε increases, the convergence radius r_ε increases. The numerical solution is insensitive to solver tolerance variations. For $\varepsilon = 0$ (no external perturbation), the system converges exponentially to the origin (see Figure 4), as predicted by the theory when $r_\varepsilon = 0$.



(a) Phase portrait (b) Phase norm vs time

Figure 3. (a) Phase portrait; (b) Phase norm vs time.

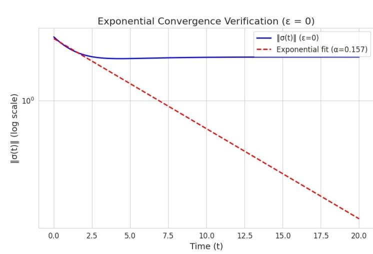


Figure 4. Exponential convergence for $\varepsilon = 0$.

The numerical results strongly support the theoretical claims. The system exhibits global uniform practical exponential stability, with the convergence radius scaling linearly with the perturbation magnitude ε, as predicted.

5 Conclusions

This work has examined the stability analysis of a novel class of neural differential equations motivated by continuous-time Hopfield-type neural networks. It has been demonstrated that, in the presence of a perturbation term, the Hopfield-type neural network can be modeled as a dynamical system possessing a global practical exponential stability property around zero. The applicability of the main results has been illustrated through numerical examples and simulation results. Previous studies have employed the Gronwall approach to study stability under perturbations for various classes of differential equations. In this context, the perturbation term ensures consistency with the associated linear equation and facilitates the study of convergence by expressing the general solution in integral form. The novelty of this work lies in imposing a Gamidov-type restriction on the bounds of the uncertainty term, which significantly enlarges the class of systems that can be rigorously proven stable. Despite these advances, several theoretical challenges remain. Extensions to systems with nonsmooth or discontinuous nonlinearities and infinite-dimensional systems with unbounded operators require further investigation. Robustness against broader classes of perturbations and uncertainties also presents open problems for future work. Moreover, the framework can be adapted to neural models with time-varying structures, allowing system matrices and nonlinearities to explicitly depend on time, provided that suitable uniform stability conditions are satisfied. This represents a promising direction for ongoing research. This work has established a framework for the global practical exponential stability of a class of perturbed Hopfield-type neural differential equations. The key novelty lies in applying a Gamidov-type condition on the perturbation term, which significantly expands the class of admissible systems, particularly those with unbounded coefficients, for which stability can be rigorously proven. While the results demonstrate robustness under significant uncertainty, they open several compelling avenues for future research. The following open problems represent critical steps toward broadening the theory's applicability:

- Extending the analysis to systems with nonsmooth activations is essential for aligning the theory with modern deep learning architectures. This requires tools from differential inclusions and nonsmooth analysis.

- A natural and impactful extension is to systems with discrete or distributed time delays, which are inherent in biological neural networks and hardware implementations. Furthermore, formulating the stability conditions within an infinite-dimensional setting (e.g., using partial differential equations) would allow the modeling of neural fields, a crucial tool in computational neuroscience for understanding large-scale brain activity.

- Modeling environmental noise and synaptic variability necessitates a stochastic formulation. Investigating almost sure or moment-based practical exponential stability under Brownian motion or jump-process perturbations would greatly enhance the model's realism.

- Linking these theoretical challenges to specific applied domains will be vital. For instance, applying the extended theory to stochastic delay equations could provide new insights into the robustness of biological rhythm generators in computational biology, while the infinite-dimensional analysis could offer novel stability criteria for continuum models of cortical dynamics. Addressing these problems will not only generalize existing results but also firmly bridge the gap between abstract stability theory and cutting-edge applications in machine learning and biological modeling.

These research directions not only represent theoretical advancements but also address pressing challenges in biological system modeling. The framework established herein provides a mathematical foundation upon which these extensions can be systematically developed, potentially leading to unified stability criteria for complex biological networks across multiple spatiotemporal scales.

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