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Spectral algorithm for fractional BVPs via novel modified Chebyshev polynomials fractional derivatives

Mohamed Abdelhakem[a,d], Dina Abdelhamied[b,d] ✉, Mamdouh El-kady[a,d] and Youssri Hassan Youssri[c]

[a] *Mathematics Department, Faculty of Science, Helwan University, 11795 Cairo, Egypt*
[b] *Department of Basic Science, Faculty of Engineering, May University in Cairo (MUC), Cairo, Egypt*
[c] *Department of Mathematics, Faculty of Science, Cairo University, 12613 Giza, Egypt*
[d] *Helwan School of Numerical Analysis in Egypt (HSNAE), Helwan, Cairo, Egypt*

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Abstract. This paper introduces a spectral algorithm tailored for solving fractional boundary value problems (BVPs) using the fractional derivatives of modified Chebyshev polynomials. Specifically, it addresses linear and non-linear BVPs and Bratu equations in one dimension via spectral methods. The approach employs basis functions derived from first-kind shifted polynomials that satisfy the homogeneous boundary conditions. The fractional derivatives are formulated to facilitate the solution process. The convergence analysis is studied for the suggested basis expansion; some numerical results are exhibited to verify the applicability and accuracy of the method.

Keywords: modified shifted Chebyshev polynomials; collocation method; Galerkin method; fractional BVPs.

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✉ Corresponding author. E-mail: dina.abdelhamied90@gmail.com

1 Introduction

Fractional boundary value problems (F-BVPs) have gained considerable significance in recent decades due to their extensive applications across various disciplines, including physics, chemistry, and engineering [11, 16, 29]. Many applied research fields rely on these equations to model various phenomena [19, 21]. Since exact solutions are often unattainable for most of these equations, numerical methods are necessary [1, 18]. Several numerical algorithms are employed to derive these solutions. The authors in [22] used Mittag-Leffler to solve F-BVPs. The fractional-order Fibonacci wavelet technique was applied to solve the fractional Bagley-Torvik equation [30]. Also, the authors [20] solved the hyper-Bessel fractional differential operator and bi-ordinal Hilfer fractional derivative.

One of the most important methods is the spectral method. Spectral methods are highly effective tools in computational mathematics, and they are known for their exceptional accuracy and efficiency in solving various problems. The spectral methods involve three famous methods. Firstly, the collocation method (pseudospectral) depends on constructing a differentiation or integration matrix [13, 26]. The Galerkine method is the second method, with a condition to apply: the base function satisfies the initial and boundary conditions [17, 25]. The third method is the Tau method [7, 14].

The spectral method assumes the approximate solution as an assumption of unknown constant and suitable orthogonal polynomials [2]. The author of [8] used the shifted ultraspherical as a base function. At the same time, shifted Legendre is used in [5]. The authors in [32] chose shifted Jacobi polynomials. The Morgan-Voyce polynomials are applied in [27].

The Chebyshev polynomials (CH-Ps), including their different kinds, are the most used in the research papers. In addition, some of the authors modify these polynomials. The authors of [31] modified CH-Ps of the first kind to satisfy the homogeneity of initial and boundary conditions. The homogeneity of initial and boundary conditions (IBCs) was confident in [6]. Moreover, the CH-Ps of the second kind satisfied the homogeneity in [9], which achieved excellent results in solving problems and real applications [15]. Furthermore, generalized Chebyshev polynomials are used via the spectral Galerkin in [3, 4].

The basis function, modified shifted Chebyshev polynomials (MSCH-Ps), is employed throughout this paper, which satisfies the condition of the Galerkin method. This means we start with an expansion-approximated solution that satisfies the initial and boundary conditions. In addition, the number of equations of the obtained system will be reduced. The fractional derivative of MSCH-Ps will be determined and applied to solve F-BVPs. We utilize the Galerkin and the Tau spectral methods for solving F-BVPs.

This paper consists of six sections. Section 2 presents all the definitions and relations needed in our work. The fraction derivative of MSCH-Ps is shown in Section 3. The techniques of solving F-BVPs are presented in Section 4 using the base function MSCH-Ps and its derivatives via the two spectral methods, the Galerkin and the Tau method. The error analysis concepts are discussed in Section 5. Finally, in Section 6, some F-BVPs and real-life problems are solved and compared with other methods.

2 Preliminaries

Caputo fractional derivative of order v is defined as:

$$D_a^v Z(\xi) = \frac{1}{\Gamma(n-v)} \int_a^{\xi} \frac{Z^{(n)}(x)}{(\xi - x)^{(v-n+1)}} dx, \quad \text{for all } n-1 < v < n,\, n \in \mathbb{N},$$

where $Z^{(n)}(x) = \frac{d^n Z}{dx^n}$.

It is clear that

$$D^v \xi^b = \frac{\Gamma(b+1)}{\Gamma(b+1-v)} \xi^{b-v}. \tag{2.1}$$

The shifted Chebyshev polynomials of first kind SCH-Ps ($T_l^*(\xi); l = 0, 1, \ldots; \xi \in [\alpha, \beta]$) of degree l is described as [23]:

$$T_l^*(\xi) = T_l\left(\frac{2\xi - \beta - \alpha}{\beta - \alpha}\right), \quad l = 0, 1, 2, \ldots.$$

Its recurrence relation can be formed as:

$$T_l^*(\xi) = 2\left(\frac{2\xi - \beta - \alpha}{\beta - \alpha}\right) T_{l-1}^*(\xi) - T_{l-2}^*(\xi) \quad l = 2, 3, \ldots,$$

with initials $T_0^*(\xi) = 1$ and $T_1^*(\xi) = \frac{2\xi - \beta - \alpha}{\beta - \alpha}$. Its series form is defined as:

$$T_l^*(\xi) = l \sum_{j=0}^{l} \frac{(-1)^{l-j} \Gamma(l+j) 2^{2j}}{\Gamma(l-j+1)\Gamma(2j+1)} \xi^j, \quad l > 0. \tag{2.2}$$

The MSCH-Ps $\left(\psi_{m,l}(\xi);\, l, m = 0, 1, 2, \ldots; \quad \xi \in [\alpha, \beta]\right)$ of degree $(l + 2m)$ formed as [6]:

$$\psi_{m,l}(\xi) = (\beta - \xi)^m (\xi - \alpha)^m T_l^*(\xi), \quad l = 0, 1, 2, \ldots. \tag{2.3}$$

Consequently, the initial terms of MSCH-Ps are as follows:

$$\psi_{m,0}(\xi) = (\beta - \xi)^m (\xi - \alpha)^m, \tag{2.4}$$

$$\psi_{m,1}(\xi) = (\beta - \xi)^m (\xi - \alpha)^m \left(\frac{2\xi - \beta - \alpha}{\beta - \alpha}\right), \tag{2.5}$$

$$\psi_{m,2}(\xi) = (\beta - \xi)^m (\xi - \alpha)^m \left(\frac{8\xi^2 - 8(\alpha + \beta)\xi + (\alpha + \beta)^2 + 4\alpha\beta}{(\beta - \alpha)^2}\right).$$

The recurrence relation of MSCH-Ps is formed as follows:

$$\psi_{m,l+2}(\xi) = 2\left(\frac{2\xi - \beta - \alpha}{\beta - \alpha}\right) \psi_{m,l+1}(\xi) - \psi_{m,l}(\xi) \quad l = 0, 1, 2, \ldots,$$

while its initial Equations (2.4) and (2.5).

Furthermore, the initials and boundaries are:

$$\psi_{m,l}(\alpha) = \psi_{m,l}(\beta) = 0, \quad m > 0, \quad \psi'_{m,l}(\alpha) = \psi'_{m,l}(\beta) = 0, \quad m > 1.$$

The MSCH-Ps satisfy that:

$$|\psi_{m,l}(\xi)| \leq (\beta - \alpha)^{2m}, \quad |\psi'_{m,l}(\xi)| \leq (\beta - \alpha)^{2m} l^2. \tag{2.6}$$

The following equation expresses the orthogonality relation of polynomials $\{\psi_{m,l}(\xi)\}_{l,m \geq 0}$ with the weight function $\hat{w}(\xi) = \frac{1}{(\beta-\xi)^{2m}(\xi-\alpha)^{2m}\sqrt{(\xi-\alpha)(\beta-\xi)}}$ as:

$$\int_\alpha^\beta \hat{w}(\xi)\, \psi_{m,l}(\xi) \psi_{m,i}(\xi) d\xi = \begin{cases} 0, & i \neq l, \\ \pi, & i = l = 0, \\ \pi/2, & i = l > 0. \end{cases} \tag{2.7}$$

The first derivative of $\psi_{m,l}(\xi)$ defined as:

$$\frac{d}{d\xi}\psi_{m,l}(\xi) = \frac{2}{\beta-\alpha}\sum_{i=0}^{l-1}\frac{2\lambda_{l+i}}{\gamma_i}[i+(2m+1)(l-i)]\psi_{m,i}(\xi) + \Delta_l(\xi),$$

where

$$\lambda_i = \begin{cases} 0, & i \text{ even}, \\ 1, & i \text{ odd}, \end{cases} \qquad \gamma_i = \begin{cases} 2, & i=0, \\ 1, & i \neq 0, \end{cases}$$

and

$$\Delta_i = \begin{cases} -m((\beta-\xi)(\xi-\alpha))^{m-1}(2\xi-\alpha-\beta), & i \text{ even}, \\ -m((\beta-\xi)(\xi-\alpha))^{m-1}(\beta-\alpha), & i \text{ odd}. \end{cases}$$

Also, the first derivative of $\psi(\xi)$ formed as: $\psi'(\xi) = G\psi(\xi) + \Omega(\xi)$, where $\psi'(\xi) = [\psi'_{m,0}(\xi), \psi'_{m,1}(\xi), ..., \psi'_{m,N}(\xi)]^T$, $\Omega(\xi) = [\Delta_0(\xi), \Delta_1(\xi), ..., \Delta_N(\xi)]^T$, and $G = (g_{lk})_{l,k=0}^{N}$ is $(N+1)\times(N+1)$ matrix such that:

$$g_{lk} = \frac{4\lambda_{l+k}}{(\beta-\alpha)\gamma_k}[k+(2m+1)(l-k)], \qquad k,l = 0,\dots,N.$$

For example, at $m=2$ and $N=4$:

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{10}{\beta-\alpha} & 0 & 0 & 0 & 0 \\ 0 & \frac{24}{\beta-\alpha} & 0 & 0 & 0 \\ \frac{30}{b-a} & 0 & \frac{28}{\beta-\alpha} & 0 & 0 \\ 0 & \frac{64}{\beta-\alpha} & 0 & \frac{32}{\beta-\alpha} & 0 \end{bmatrix}.$$

Furthermore, the v^{th} integer order derivative of $\psi(\xi)$ defined as:

$$\psi^{(v)}(\xi) = \begin{cases} G^v\psi(\xi) + \sum_{i=0}^{v-1} G^{v-i-1}\Omega^{(i)}(\xi), & v = 1,\dots,N, \\ \sum_{i=0}^{v-1} G^{v-i-1}\Omega^{(i)}(\xi), & v > N, \end{cases}$$

where G^0 is the identity matrix.

3 The MSCH-Ps fractional order derivative

This section aims to calculate the fractional derivative of the MSCH-Ps. From this point forward $\xi \in [0,1]$.

Theorem 1. *The fractional derivative of order α for MSCH-Ps is formed as:*

$$\psi_{m,l}^{(\alpha)}(\xi) = l\sum_{j=0}^{l}\sum_{r=\min(0,\lceil\alpha\rceil-m-j)}^{m}\frac{C_{j,r}\Gamma(j+m+r+1)}{\Gamma(j+m+r-\alpha+1)}\xi^{j+m+r-\alpha}, \quad l>0,$$

$$\psi_{m,0}^{(\alpha)}(\xi) = \sum_{r=\min(0,\lceil\alpha\rceil-m)}^{m} \frac{(-1)^r\, \Gamma(m+1)\Gamma(m+r+1)}{\Gamma(r+1)\Gamma(m-r+1)\Gamma(m+r-\alpha+1)} \xi^{m+r-\alpha},$$

where

$$C_{j,r} = \frac{(-1)^{r+l-j} 2^{2j} \Gamma(l+j)\Gamma(m+1)}{\Gamma(l-j+1)\Gamma(2j+1)\Gamma(r+1)\Gamma(m-r+1)},$$

such that $m+l \geq \alpha$, *and* $\psi_{m,l}^{(\alpha)}(\xi) = 0$, *where* $m+l < \alpha$.

Proof. For $l = 0$: $\psi_{m,0}(\xi) = \xi^m(1-\xi)^m$. By using the binomial theorem to convert $(1-\xi)^m$ into summation as:

$$(1-\xi)^m = \sum_{r=0}^{m} \frac{(-1)^r \Gamma(m+1)}{\Gamma(r+1)\Gamma(m-r+1)} \xi^r, \tag{3.1}$$

then,

$$\psi_{m,0}(\xi) = \sum_{r=0}^{m} \frac{(-1)^r\, \Gamma(m+1)}{\Gamma(r+1)\Gamma(m-r+1)} \xi^{m+r}.$$

As the definition of Caputo fractional derivative of order α, Equation (2.1):

$$\psi_{m,0}^{(\alpha)}(\xi) = \sum_{r=\min(0,\lceil\alpha\rceil-m)}^{m} \frac{(-1)^r\, \Gamma(m+1)\Gamma(m+r+1)}{\Gamma(r+1)\Gamma(m-r+1)\Gamma(m+r-\alpha+1)} \xi^{m+r-\alpha},$$

where $m \geq \alpha$. While $\psi_{m,0}^{(\alpha)}(\xi) = 0, \quad m < \alpha$. For $l > 0$:

Let us define the MSCH-Ps as a summation in the interval $[0,1]$ by using Equations (2.2) and (2.3):

$$\psi_{m,l}(\xi) = l \sum_{j=0}^{l} \frac{(-1)^{l-j} 2^{2j} \Gamma(l+j)}{\Gamma(l-j+1)\Gamma(2j+1)} \xi^j \xi^m (1-\xi)^m, \quad l > 0.$$

From Equation (3.1), $\psi_{m,l}(\xi)$ can be formed as:

$$\psi_{m,l}(\xi) = l \sum_{j=0}^{l} \sum_{r=0}^{m} \frac{(-1)^{r+l-j} 2^{2j} \Gamma(l+j)\Gamma(m+1)}{\Gamma(l-j+1)\Gamma(2j+1)\Gamma(r+1)\Gamma(m-r+1)} \xi^{j+m+r},$$

where $l > 0$.

Apply Caputo fractional derivative of order α:

$$\psi_{m,l}^{(\alpha)}(\xi) = l \sum_{j=0}^{l} \sum_{r=\min(0,\lceil\alpha\rceil-m-j)}^{m} \frac{C_{j,r}\Gamma(j+m+r+1)}{\Gamma(j+m+r-\alpha+1)} \xi^{j+m+r-\alpha},$$

such that $m+l \geq \alpha$. While $\psi_{m,l}^{(\alpha)}(\xi) = 0, \quad m+l < \alpha$. □

In the following section, we provide a detailed methodology for solving F-BVPs. The two spectral methods, Galerkin and Tau, are used to solve F-BVPs.

4 Problem formulation and the presented method for solving F-BVPS MSCH-Ps

This section presents the techniques for solving fractional boundary value problems (F-BVPs) using the Galerkin and Tau methods. The emphasis is on exploring and explaining the methodologies in effectively tackling fractional BVPs using these specific numerical approaches.

Consider the F-BVPs as:

$$Z^{(v)}(\xi) = \mathbb{F}(\xi, Z(\xi), Z^{(v_1)}(\xi), \dots, Z^{(v_k)}), \quad \xi \in [0,1], \tag{4.1}$$

The initial and boundary conditions are homogeneous as:

$$Z(0) = Z(1) = 0,$$

where v is a real number, assumed that the approximated spectral solution of Equation (4.1) is as:

$$Z(\xi) \simeq \sum_{l=0}^{N} c_l \psi_{m,l}(\xi). \tag{4.2}$$

Evaluating the residual of Equation (4.1) is gained from Theorem 1 as:

$$\mathcal{R}(\xi)=\sum_{l=0}^{N} c_l \psi_{m,l}^{(v)}(\xi)-\mathbb{F}\left(\xi,\sum_{l=0}^{N} c_l \psi_{m,l}(\xi),\sum_{l=0}^{N} c_l \psi_{m,l}^{(v_1)}(\xi),\dots,\sum_{l=0}^{N} c_l \psi_{m,l}^{(v_k)}(\xi)\right). \tag{4.3}$$

4.1 Collocation spectral method for fraction order BVP via MSCH-Ps (MSCH-Coll)

We utilize the collocation points $\xi_j \in [0,1]$; $j = 0, 1, \dots, N$. By collocating equation (4.3), we obtain $N+1$ algebraic system equations for the unknowns c_l; $j = 0, 1, \dots, N$, represented as:

$$\mathcal{R}(\xi_j) = 0, \quad j = 0, 1, \dots, N. \tag{4.4}$$

Solving this algebraic system, as given in Equation (4.4), allows us to determine the values of the constants c_l. Subsequently, these calculated constants are substituted into the approximated solution presented in Equation (4.2).

4.2 Galerkin spectral method for fraction order BVP via MSCH-Ps (MSCH-Galerkin)

According to the introduced function in Equation (2.3) specification, the initial and boundary of the function and their derivatives are equal to zero for specified values of m. Consequently, this assumption is consistent with the homogeneity of BVP's initial and boundary conditions for Galerkin. The suitable weight function that is used:

$$\bar{w}(\xi) = \frac{1}{\xi^m (1-\xi)^m \sqrt{\xi(1-\xi)}}.$$

And the trial function will be $\psi_{m,l}(\xi)$. Then the Tau integration will be:

$$\int_0^1 \mathcal{R}(\xi)\psi_{m,l}(\xi)\bar{w}(\xi)d\xi = 0, \quad l = 0, 1, \ldots, N. \tag{4.5}$$

Equation (4.5) yields a system of algebraic $N+1$ equations in $N+1$ unknowns. By solving this system, we can determine the spectral coefficients for the approximated solution (4.2).

Remark 1. In many practical applications, ensuring homogeneous initial and boundary conditions is often only achievable sometimes. Therefore, converting these conditions to be homogeneous becomes necessary. The transformation is achieved in the following form. Let:

$$z(\xi) = Z(\xi) + \sum_{i=0}^{l-1} M_i \xi^i, \tag{4.6}$$

such that

$$\begin{aligned} z(0) &= z'(0) = z''(0) = \cdots = z^{(\frac{l}{2}-1)}(0) = 0, \\ z(1) &= z'(1) = z''(1) = \cdots = z^{(\frac{l}{2}-1)}(1) = 0, \end{aligned} \tag{4.7}$$

M_i are constants can be determined by solving Equations (4.6), (4.7). Thus, the F-BVP (4.1), (4.7) can be solved to get the unknown function $z(\xi)$.

5 Error analysis

In this section, some concepts related to error analysis and convergence will be introduced.

Definition 1. [31] Consider the Sobolev space $\mathcal{H}^n_\theta(\alpha,\beta)$ as the following:

$$\mathcal{H}^n_\theta(\alpha,\beta) = \{\omega \in \mathcal{L}^2_\theta(\alpha,\beta) : \qquad \omega^{(r)} \in \mathcal{L}^2_\theta(\alpha,\beta),\ 0 \le r \le n\},$$

Consider the subspace $\mathcal{H}^n_{0,\theta}(0,1)$ of the space $\mathcal{H}^n_\theta(\alpha,\beta)$ such that:

$$\mathcal{H}^n_{0,\theta}(0,1) = \{\omega \in \mathcal{H}^n_\theta(\alpha,\beta) : \qquad \omega^{(r)}(0) = \omega^{(r)}(1) = 0,\ 0 \le r \le n\}.$$

Theorem 2. *Let $Z(\xi)$ be a function that can be defined as $Z(\xi) = \xi^m\ (1-\xi)^m\, \bar{Z}(\xi) \in \mathcal{H}^n_{0,\theta}(0,1)$ and expanded in terms of the basis function $\psi_{m,l}(\xi)$, Equation (2.3). Then,*

$$|c_l| \lesssim \frac{\mathcal{M}_n}{l^n}, \qquad \forall l > 1,$$

where $|\bar{Z}^{(n)}(\xi)| < \mathcal{M}_n$, and $\xi \in [0,1]$.

Proof. Let $Z(\xi)$ be a function that satisfies the conditions stated in the theorem. Then,

$$\xi^m(1-\xi)^m\bar{Z}(\xi)=\sum_{l=0}^{\infty}c_l\psi_{m,l}(\xi).$$

According to the definition of the basis function in Equation (2.3) and the orthogonality relation Equation (2.7), we get:

$$c_l=h_l\int_0^1\frac{\bar{Z}(\xi)T_l^*(\xi)}{\sqrt{\xi-\xi^2}}d\xi,$$

where $h_0=\frac{1}{\pi}$ and $h_l=\frac{2}{\pi}\ \forall l>0$.

Let $\cos\gamma=2\xi-1$ and use the integration by parts to get:

$$c_l=\frac{h_l}{2l}\int_0^{\pi}\bar{Z}'(\xi)\sin\gamma\sin l\gamma\,d\gamma. \tag{5.1}$$

Thus, $|c_l|\lesssim\frac{M_l}{l}$.

Applying the integration by parts to the integral (5.1) to get:

$$c_l=\frac{h_l}{2l(l^2-1)}\int_0^{\pi}\bar{Z}''(\xi)\sin\gamma[\sin l\gamma\cos\gamma-\cos l\gamma\sin\gamma]d\gamma.$$

Thus, $|c_l|\lesssim\frac{M_2}{l^2}$.

Applying the same procedures $n-2$ to complete the proof. □

Lemma 1. *[28] Let $Z(\xi)$ be function with $Z(l)=c_l$. If the following assumptions are verified:*

i. $Z(\xi)$ is positive, decreasing, and continuous for $\xi\geq\mu$.

ii. $\sum c_\mu$ is convergent, and $\mathcal{R}_\mu=\sum\limits_{l=\mu+1}^{\infty}a_l$.

Then,

$$\mathcal{R}_\mu\leq\int_\mu^\infty Z(\xi)\,d\xi.$$

Theorem 3. *Let $Z(\xi)$ be a function that satisfies Lemma 1 and Theorem 2. Then,*

$$|Z(\xi)-Z_N(\xi)|\lesssim\mathcal{O}\big(N^{1-n}\big).$$

Proof. It is clear that

$$|Z(\xi)-Z_N(\xi)|=\left|\sum_{l=N+1}^{\infty}c_l\psi_{m,l}(\xi)\right|.$$

Applying Lemma 1 with the aid of the property (2.6), for $\xi\in[0,1]$, and Theorem 2 completes the proof. □

Corollary 1. Let $Z(\xi)$ be a function that satisfies Theorem 3. Then,

$$|DZ(\xi) - DZ_N(\xi)| \lesssim \mathcal{O}\left(N^{3-n}\right).$$

Theorem 4. *Let $Z(\xi)$ be a function that satisfies Lemma 1 and Theorem 2, and $Z(\xi)$ satisfy the following differential equation*

$$Z'(\xi) = \chi(\xi, Z),$$

and χ satisfy Lipchitz condition in the variable Z, then we have the following residual error bound.

Proof.

$$\begin{aligned}|D\,Z_N(\xi) - \chi(\xi, Z_N)| &\leq |D\,Z_N(\xi) - D\,Z(\xi)| + |\chi(\xi, Z_N) - \chi(\xi, Z)| \\ &\leq |D\,Z_N(\xi) - D\,Z(\xi)| + L\,|Z_N(\xi) - Z(\xi)|.\end{aligned}$$

Now, by Theorem 3, and Corollary 1, we get

$$|D\,Z_N(\xi) - \chi(\xi, Z_N)| \leq c_1\,N^{1-n} + c_2\,N^{3-n} \lesssim \mathcal{O}\left(N^{3-n}\right),$$

which completes the proof. □

For the fractional case,

$$Z^{(\alpha)}(\xi) = \chi(\xi, Z).$$

Theorem 4 will work with the aid of the following corollary.

Corollary 2.

$$|Z^{(\alpha)}(\xi)| \leq \frac{1}{\Gamma(1-\alpha)} \int_0^{\xi} \frac{|Z'(\eta)|}{(\xi-\eta)^{1-\alpha}}\, d\eta.$$

The next section will apply all the previous techniques to some types of F-BVPs.

6 Solving fractional boundary value problems

Some F-BVPs are solved using two spectral methods: Galerkin and Tau. A comparison between these methods and other techniques is presented. Two types of errors are displayed:

1. Absolute Error (AE): $AE_j = |Z_N(\xi_j) - Z(\xi_j)|$,
2. Maximum Absolute Error (MAE): $MAE = \max_j(AE_j)$,

where $\xi_j \in [0,1]$, $Z_N(\xi)$ is the approximate solution, and $Z(\xi)$ is the exact solution.

Example 1. Consider the following nonlinear fractional ordinary differential equation [24]:

$$Z^{\alpha}(\xi) = \frac{\Gamma(5+\alpha)}{24}\xi^4 + \xi^{8+2\alpha} - Z^2(\xi), \quad 0<\xi<X, \quad Z(0)=0,$$

its exact solution is $Z(\xi) = \xi^{4+\alpha}$. As $Z(1) \neq 0$, we apply the homogeneity converting. The constants in Equation (4.6) were calculated as $M_0 = 0$ and $M_1 = -1$, as $m = 1$. Tables 1, 2 and 3 show the MAE of Example 1 at different values of α. Figure 1 presents the log error at various N.

Table 1. The MAE at $\alpha = 0.2$ and $m = 1$ for Example 1.

N	MSCH-Coll	MSCH-Galerkin	[24]
4	$1.1 \cdot 10^{-05}$	$3.7 \cdot 10^{-06}$	-
6	$8.5 \cdot 10^{-07}$	$2.8 \cdot 10^{-07}$	-
10	$1.6 \cdot 10^{-08}$	$6.6 \cdot 10^{-09}$	$1.2 \cdot 10^{-07}$
14	$1.9 \cdot 10^{-09}$	$5.9 \cdot 10^{-10}$	$5.7 \cdot 10^{-09}$

Table 2. The MAE at $\alpha = 0.5$ and $m = 1$ for Example 1.

N	MSCH-Coll	MSCH-Galerkin	[24]
4	$4.2 \cdot 10^{-05}$	$6.5 \cdot 10^{-06}$	-
6	$3.4 \cdot 10^{-06}$	$2.9 \cdot 10^{-07}$	-
10	$1.2 \cdot 10^{-07}$	$6.6 \cdot 10^{-09}$	$1.1 \cdot 10^{-07}$
14	$1.4 \cdot 10^{-09}$	$9.1 \cdot 10^{-10}$	$4.1 \cdot 10^{-09}$

Table 3. The MAE at $\alpha = 1$ and $m = 1$ for Example 1.

N	MSCH-Coll	MSCH-Galerkin	[24]
4	$2.8 \cdot 10^{-16}$	$1.5 \cdot 10^{-15}$	-
6	$2.2 \cdot 10^{-16}$	$3.8 \cdot 10^{-16}$	-
10	$1.4 \cdot 10^{-15}$	$2.2 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$
14	$4.4 \cdot 10^{-14}$	$1.0 \cdot 10^{-16}$	$2.2 \cdot 10^{-16}$

Example 2. Consider the fractional ordinary differential equation [33]:

$$Z^{(\alpha)}(\xi) = \frac{40320}{\Gamma(9-\alpha)}\xi^{8-\alpha} - 3\frac{\Gamma(5+\frac{\alpha}{2})}{\Gamma(5-\frac{\alpha}{2})}\xi^{4-\frac{\alpha}{2}} + \frac{9}{4}\Gamma(\alpha+1), \quad Z(0)=0,$$

its exact solution $Z(\xi) = \xi^8 - 3\xi^{4+\frac{\alpha}{2}} + \frac{9}{4}\xi^{\alpha}$.

At $m = 1$, the constants in Equation (4.6) were calculated as $M_0 = 0$ and $M_1 = -\frac{1}{4}$. Table 4 shows the AE at $N = 10$ and $\alpha = 0.1$.

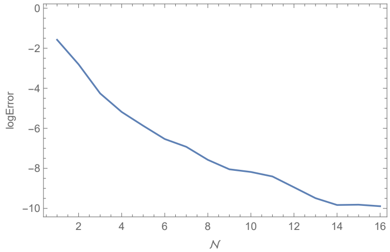

Figure 1. Log error for Example 1 at $\alpha = 0.5$ using MSCH-Galerkin.

Table 4. The AE at $N = 10$, $\alpha = 0.1$, and $m = 1$ for Example 2.

ξ	MSCH-Coll	MSCH-Galerkin	[33]
0.1	$3.5 \cdot 10^{-02}$	$6.1 \cdot 10^{-03}$	$2.1 \cdot 10^{-02}$
0.2	$2.2 \cdot 10^{-02}$	$4.7 \cdot 10^{-02}$	$3.4 \cdot 10^{-01}$
0.3	$1.3 \cdot 10^{-02}$	$2.0 \cdot 10^{-01}$	$2.2 \cdot 10^{-02}$
0.4	$1.0 \cdot 10^{-02}$	$4.7 \cdot 10^{-01}$	$8.3 \cdot 10^{-01}$
0.5	$8.7 \cdot 10^{-03}$	$5.0 \cdot 10^{-01}$	$2.4 \cdot 10^{-01}$
0.6	$7.3 \cdot 10^{-03}$	$2.7 \cdot 10^{-02}$	$5.9 \cdot 10^{-02}$
0.7	$6.5 \cdot 10^{-03}$	$2.7 \cdot 10^{-02}$	$1.2 \cdot 10^{-02}$
0.8	$5.3 \cdot 10^{-03}$	$1.7 \cdot 10^{-01}$	$2.1 \cdot 10^{-01}$
0.9	$6.2 \cdot 10^{-03}$	$7.5 \cdot 10^{-02}$	$1.4 \cdot 10^{-02}$
1.0	0	0	$1.4 \cdot 10^{-02}$

Example 3. An application problem of fractional Riccati differential equations is given by [33]:

$$Z^{(\alpha)}(\xi) = -Z^2(\xi) + 1, \qquad Z(0) = 0,$$

its exact solution at $\alpha = 1$ is $Z(\xi) = \tanh \xi$. By converting the conditions to

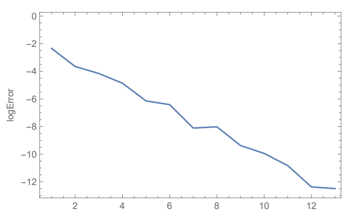

Figure 2. Log Error for Example 3 at $\alpha = 1$ using MSCH-Galerkin.

be homogeneous at $m = 1$, the constants in Equation (4.6) were calculated as $M_0 = 0$ and $M_1 = -\frac{e^2-1}{1+e^2}$. Table 5 shows the AE at $\alpha = 1$ and $N = 10$. Also, Figure 2 shows the log error at several N using the MSCH-Galerkin method, proving the stability of these techniques.

Table 5. The AE at $N = 10$, $\alpha = 1$, and $m = 1$ for Example 3.

ξ	MSCH-Coll	MSCH-Galerkin	[33]
0.1	$1.9 \cdot 10^{-08}$	$6.8 \cdot 10^{-11}$	1
0.2	$1.8 \cdot 10^{-08}$	$1.1 \cdot 10^{-10}$	$9.5 \cdot 10^{-05}$
0.3	$1.7 \cdot 10^{-08}$	$5.9 \cdot 10^{-11}$	$1.8 \cdot 10^{-04}$
0.4	$1.6 \cdot 10^{-08}$	$2.6 \cdot 10^{-12}$	$2.6 \cdot 10^{-04}$
0.5	$1.5 \cdot 10^{-08}$	$9.2 \cdot 10^{-11}$	$3.3 \cdot 10^{-04}$
0.6	$1.3 \cdot 10^{-08}$	$6.9 \cdot 10^{-11}$	$4.0 \cdot 10^{-04}$
0.7	$1.2 \cdot 10^{-08}$	$2.9 \cdot 10^{-11}$	$4.5 \cdot 10^{-04}$
0.8	$1.1 \cdot 10^{-08}$	$2.9 \cdot 10^{-11}$	$5.0 \cdot 10^{-04}$
0.9	$9.5 \cdot 10^{-09}$	$1.5 \cdot 10^{-11}$	$5.5 \cdot 10^{-04}$
1.0	0	0	$5.9 \cdot 10^{-04}$

Example 4. Consider the Bratu equation:

$$Z^{(\alpha)}(\xi) + \lambda e^{Z(\xi)} = 0, \qquad Z(0) = Z(1) = 0,$$

its exact solution at $\alpha = 2$ is $Z(\xi) = -2\ln\left[\frac{\cosh(\frac{\eta}{4}(2\xi-1))}{\cosh(\frac{\eta}{4})}\right]$, where η is the solution of the equation $\eta = \sqrt{2\lambda}\cosh\eta$, for the three corresponding values of η for $\lambda = 1, 2$, and 3.51 are 1.51716, 2.35755, and 4.66781 , respectively. The conditions of this problem are homogeneous, so MSCH-Coll is applied directly for $m = 1$.

Table 6. The AE at $m = 1$ and $\lambda = 1$ for Example 4.

ξ	MSCH-Coll		[10]	[12]
	$N = 10$	$N = 20$	$N = 10$	$N = 20$
0	0	0	-	-
0.1	$1.76 \cdot 10^{-11}$	$8.88 \cdot 10^{-16}$	$1.32 \cdot 10^{-06}$	$2.01 \cdot 10^{-04}$
0.2	$1.82 \cdot 10^{-11}$	$4.72 \cdot 10^{-16}$	$6.33 \cdot 10^{-07}$	$2.83 \cdot 10^{-04}$
0.3	$1.88 \cdot 10^{-11}$	$7.63 \cdot 10^{-16}$	$3.29 \cdot 10^{-06}$	$2.95 \cdot 10^{-04}$
0.4	$1.91 \cdot 10^{-11}$	$7.49 \cdot 10^{-16}$	$2.81 \cdot 10^{-06}$	$2.56 \cdot 10^{-04}$
0.5	$1.92 \cdot 10^{-11}$	$5.27 \cdot 10^{-16}$	$4.75 \cdot 10^{07}$	$1.96 \cdot 10^{-04}$
0.6	$1.91 \cdot 10^{-11}$	$6.39 \cdot 10^{-16}$	$1.48 \cdot 10^{-06}$	$1.37 \cdot 10^{-04}$
0.7	$1.87 \cdot 10^{-11}$	$5.72 \cdot 10^{-16}$	$2.31 \cdot 10^{-06}$	$9.16 \cdot 10^{-05}$
0.8	$1.82 \cdot 10^{-11}$	$8.32 \cdot 10^{-16}$	$2.26 \cdot 10^{-06}$	$5.71 \cdot 10^{-05}$
0.9	$1.76 \cdot 10^{-11}$	$4.09 \cdot 10^{-16}$	$8.62 \cdot 10^{-07}$	$2.83 \cdot 10^{-05}$
1.0	0	0	-	-

Tables 6 and 7 present the AE for $\lambda = 1$, and 2. While the best MAE of several methods is presented in Table 8. The stability of the proposed technique is proved in Figure 3.

Table 7. The AE at $m = 1$ and $\lambda = 2$ for Example 4.

ξ	MSCH-Coll		[10]	[12]
	$N = 10$	$N = 20$	$N = 10$	$N = 20$
0	0	0	-	-
0.1	$7.42 \cdot 10^{-09}$	$1.25 \cdot 10^{-14}$	$5.69 \cdot 10^{-06}$	$9.14 \cdot 10^{-04}$
0.2	$8.18 \cdot 10^{-09}$	$1.46 \cdot 10^{-14}$	$6.10 \cdot 10^{-06}$	$1.34 \cdot 10^{-03}$
0.3	$8.78 \cdot 10^{-09}$	$1.58 \cdot 10^{-14}$	$1.13 \cdot 10^{-05}$	$1.21 \cdot 10^{-03}$
0.4	$9.16 \cdot 10^{-09}$	$1.64 \cdot 10^{-14}$	$1.28 \cdot 10^{-05}$	$7.07 \cdot 10^{-04}$
0.5	$9.29 \cdot 10^{-09}$	$1.69 \cdot 10^{-14}$	$9.52 \cdot 10^{-06}$	$1.32 \cdot 10^{-04}$
0.6	$9.16 \cdot 10^{-09}$	$1.68 \cdot 10^{-14}$	$8.49 \cdot 10^{-06}$	$2.57 \cdot 10^{-04}$
0.7	$8.78 \cdot 10^{-09}$	$1.64 \cdot 10^{-14}$	$9.66 \cdot 10^{-06}$	$3.88 \cdot 10^{-04}$
0.8	$8.18 \cdot 10^{-09}$	$1.56 \cdot 10^{-14}$	$6.80 \cdot 10^{-06}$	$3.26 \cdot 10^{-04}$
0.9	$7.42 \cdot 10^{-09}$	$1.40 \cdot 10^{-14}$	$3.60 \cdot 10^{-06}$	$1.74 \cdot 10^{-04}$
1.0	0	0	-	-

Table 8. The MAE for Example 4.

Method	$\lambda = 1$	$\lambda = 2$	$\lambda = 3.51$
MSCH-Coll	$8.88 \cdot 10^{-16}$	$1.68 \cdot 10^{-14}$	$5.80 \cdot 10^{-07}$
[10]	$8.62 \cdot 10^{-06}$	$1.28 \cdot 10^{-05}$	$5.46 \cdot 10^{-04}$
[12]	$2.95 \cdot 10^{-04}$	$1.32 \cdot 10^{-03}$	-

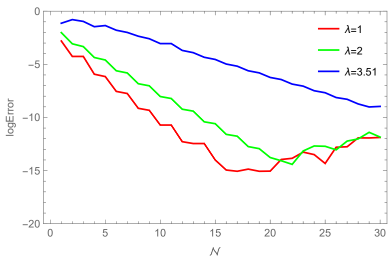

Figure 3. Log error for Example 4 at $\lambda = 1, 2, 3.51$ using MSCH-Coll.

7 Conclusions

This paper develops a spectral algorithm based on the fractional derivatives of modified Chebyshev polynomials for solving fractional boundary value problems (BVPs). The method is applied to linear and non-linear BVPs, including Bratu equations. The algorithm utilizes modified shifted Chebyshev polynomials as basis functions, which naturally satisfy the homogeneous boundary conditions, simplifying the solution process. By introducing the fractional derivatives of the presented polynomials, the approach effectively addresses these problems. The convergence analysis supports the method's validity, and the numerical results confirm its effectiveness and precision.

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References

[1] W.M. Abd-Elhameed, O.M. Alqubori, N.M.A. Alsafri, A.K. Amin and A.G. Atta. A matrix approach by convolved Fermat polynomials for solving the fractional Burgers' equation. *Mathematics*, **13**(7):1135, 2025. https://doi.org/10.3390/math13071135.

[2] W.M. Abd-Elhameed, O.M. Alqubori and A.G. Atta. A collocation procedure for the numerical treatment of FitzHugh–Nagumo equation using a kind of Chebyshev polynomials. *AIMS Mathematics*, **10**(1):1201–1223, 2025. https://doi.org/10.3934/math.2025057.

[3] W.M. Abd-Elhameed and M.M. Alsuyuti. Numerical treatment of multi-term fractional differential equations via new kind of generalized Chebyshev polynomials. *Fractal and Fractional*, **7**(1):74, 2023. https://doi.org/10.3390/fractalfract7010074.

[4] W.M. Abd-Elhameed and M.M. Alsuyuti. New spectral algorithm for fractional delay pantograph equation using certain orthogonal generalized Chebyshev polynomials. *Communications in Nonlinear Science and Numerical Simulation*, **141**:108479, 2025. https://doi.org/10.1016/j.cnsns.2024.108479.

[5] M. Abdelhakem, D. Abdelhamied, M.G. Alshehri and M. El-Kady. Shifted Legendre fractional pseudospectral differentiation matrices for solving fractional differential problems. *Fractals*, **30**(1):2240038, 2022. https://doi.org/10.1142/S0218348X22400382.

[6] M. Abdelhakem, D. Abdelhamied, M. El-Kady and Y.H. Youssri. Enhanced shifted Tchebyshev operational matrix of derivatives: two spectral algorithms for solving even-order BVPs. *Journal of Applied Mathematics and Computing*, **69**(5):3893–3909, 2023. https://doi.org/10.1007/s12190-023-01905-4.

[7] M. Abdelhakem, M. Fawzy, M. El-Kady and H. Moussa. Legendre polynomials' second derivative tau method for solving Lane-emden and Ricatti equations. *Applied Mathematics & Information Sciences*, **17**(3):437–445, 2023. https://doi.org/10.18576/amis/170305.

[8] M. Abdelhakem, D. Mahmoud, D. Baleanu and M. El-Kady. Shifted ultraspherical pseudo-Galerkin method for approximating the solutions of some types of ordinary fractional problems. *Advances in Difference Equations*, **2021**(1):110, 2021. https://doi.org/10.1186/s13662-021-03247-6.

[9] D. Abdelhamied, M. Abdelhakem, M. El-Kady and Y.H. Youssri. Adapted shifted ChebyshevU operational matrix of derivatives: Two algorithms for solving even-order BVPs. *Applied Mathematics & Information Sciences*, **17**(3):575–581, 2023. https://doi.org/10.18576/amis/170318.

[10] A. Ahmad, M. Sulaiman, A.J. Aljohani, A. Alhindi and H. Alrabaiah. Design of an efficient algorithm for solution of Bratu differential equations. *Ain Shams Engineering Journal*, **12**(2):2211–2225, 2021. https://doi.org/10.1016/j.asej.2020.11.007.

[11] I. Ahmad, H. Ahmad, P. Thounthong, Y.M. Chu and C. Cesarano. Solution of multi-term time-fractional PDE models arising in mathematical biology and physics by local meshless method. *Symmetry*, **12**(7):1195, 2020. https://doi.org/10.3390/sym12071195.

[12] M. Al-Smadi, S. Momani, N. Djeddi, A. El-Ajou and Z. Al-Zhour. Adaptation of reproducing kernel method in solving Atangana–Baleanu fractional Bratu model. *International Journal of Dynamics and Control*, **11**(1):136–148, 2023. https://doi.org/10.1007/s40435-022-00961-1.

[13] I. Ali and S.U. Khan. A dynamic competition analysis of stochastic fractional differential equation arising in finance via pseudospectral method. *Mathematics*, **11**(6):1328, 2023. https://doi.org/10.3390/math11061328.

[14] A.G. Atta, W.M. Abd-Elhameed, G.M. Moatimid and Y.H. Youssri. Advanced shifted sixth-kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem. *Mathematical Sciences*, **17**(4):415–429, 2023. https://doi.org/10.1007/s40096-022-00460-6.

[15] A.G. Atta and Y.H. Youssri. Shifted second-kind Chebyshev spectral collocation-based technique for time-fractional KdV-Burgers' equation. *Iranian Journal of Mathematical Chemistry*, **14**(4):207–224, 2023. https://doi.org/10.22052/ijmc.2023.252824.1710.

[16] U. Farooq, H. Khan, F. Tchier, E. Hincal, D. Baleanu and H.B. Jebreen. New approximate analytical technique for the solution of time fractional fluid flow models. *Advances in Difference Equations*, **2021**(1):81, 2021. https://doi.org/10.1186/s13662-021-03240-z.

[17] M. Gamal, M. El-Kady and M. Abdelhakem. Solving real-life BVPs via the second derivative Chebyshev pseudo-Galerkin method. *International Journal of Modern Physics C*, **35**(7):2450089, 2024. https://doi.org/10.1142/S012918312450089X.

[18] R.M. Hafez, H.M. Ahmed, O.M. Alqubori, A.K. Amin and W.M. Abd-Elhameed. Efficient spectral Galerkin and collocation approaches using telephone polynomials for solving some models of differential equations with convergence analysis. *Mathematics*, **13**(6):918, 2025. https://doi.org/10.3390/math13060918.

[19] R. Hajaj and Z. Odibat. Numerical solutions of fractional epidemic models with generalized Caputo-type derivatives. *Physica Scripta*, **98**(4):045206, 2023. https://doi.org/10.1088/1402-4896/acbfef.

[20] E. Karimov, M. Ruzhansky and B. Toshtemirov. Solvability of the boundary-value problem for a mixed equation involving hyper-Bessel fractional differential operator and bi-ordinal Hilfer fractional derivative. *Mathematical Methods in the Applied Sciences*, **46**(1):54–70, 2023. https://doi.org/10.1002/mma.8491.

[21] H. Khan, S. Ahmed, J. Alzabut and A.T. Azar. A generalized coupled system of fractional differential equations with application to finite time sliding mode control for Leukemia therapy. *Chaos, Solitons & Fractals*, **174**:113901, 2023. https://doi.org/10.1016/j.chaos.2023.113901.

[22] X.Y. Li and X.Y. Liu. A hybrid kernel functions collocation approach for boundary value problems with Caputo fractional derivative. *Applied Mathematics Letters*, **142**:108636, 2023. https://doi.org/10.1016/j.aml.2023.108636.

[23] J.C. Mason and D.C. Handscomb. *Chebyshev polynomials*. CRC Press, New York, 2002. https://doi.org/10.1201/9781420036114.

[24] S.A. Mohamed. A fractional differential quadrature method for fractional differential equations and fractional eigenvalue problems. *Mathematical Methods in the Applied Sciences*, **47**(13):10995–11018, 2024. https://doi.org/10.1002/mma.6753.

[25] L. Sadek, A.S. Bataineh, H. Talibi Alaoui and I. Hashim. The novel Mittag-Leffler–Galerkin method: Application to a Riccati differential equation of fractional order. *Fractal and Fractional*, **7**(4):302, 2023. https://doi.org/10.3390/fractalfract7040302.

[26] M. Shahriari, B.N. Saray, B. Mohammadalipour and S. Saeidian. Pseudospectral method for solving the fractional one-dimensional dirac operator using Chebyshev cardinal functions. *Physica Scripta*, **98**(5):055205, 2023. https://doi.org/10.1088/1402-4896/acc7d3.

[27] H.M. Srivastava, W. Adel, M. Izadi and A.A. El-Sayed. Solving some physics problems involving fractional-order differential equations with the Morgan-Voyce polynomials. *Fractal and Fractional*, **7**(4):301, 2023. https://doi.org/10.3390/fractalfract7040301.

[28] J. Stewart. *Single Variable Essential Calculus: Early Transcendentals*. Cengage Learning, Boston, 2012.

[29] D. Vignesh and S. Banerjee. Reversible chemical reactions model with fractional difference operator: Dynamical analysis and synchronization. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **33**(3):033126, 2023. https://doi.org/10.1063/5.0139967.

[30] P. Yadav, S. Jahan and K.S. Nisar. Solving fractional Bagley-Torvik equation by fractional order Fibonacci wavelet arising in fluid mechanics. *Ain Shams Engineering Journal*, **15**(1):102299, 2024. https://doi.org/10.1016/j.asej.2023.102299.

[31] Y.H. Youssri, W.M. Abd-Elhameed and M. Abdelhakem. A robust spectral treatment of a class of initial value problems using modified Chebyshev polynomials. *Mathematical Methods in the Applied Sciences*, **44**(11):9224–9236, 2021. https://doi.org/10.1002/mma.7347.

[32] Y.H. Youssri and A.G. Atta. Spectral collocation approach via normalized shifted Jacobi polynomials for the nonlinear Lane-Emden equation with fractal-fractional derivative. *Fractal and Fractional*, **7**(2):133, 2023. https://doi.org/10.3390/fractalfract7020133.

[33] N.A. Zabidi, Z.A. Majid, A. Kilicman and Z.B. Ibrahim. Numerical solution of fractional differential equations with Caputo derivative by using numerical fractional predict–correct technique. *Advances in Continuous and Discrete Models*, **2022**(1):26, 2022. https://doi.org/10.1186/s13662-022-03697-6.