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Re-iterated approximation methods for nonlinear Volterra integral equations

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Abstract. In this article, the Newton-iteration scheme based upon iterated Galerkin operator is applied for solving non-linear Volterra Urysohn integral equations of the second kind for smooth and weakly singular kernels. A one step of improvement by iteration to the Galerkin method, named as iterated Galerkin method is a well discussed method and it gives improved convergence rates than Galerkin method. But if we iterate them one more time, then there is no guarantee that we get any improved convergence rates. The proposed Newton-iteration scheme based upon iterated Galerkin operator ensures improved convergence rates at every step of iteration. Specifically, we establish that the convergence rate in iterated Galerkin method increases by $\mathcal{O}(h^r)$ for smooth kernel, and $\mathcal{O}(h^{1-\alpha})$ for weakly singular kernel, in each step of reiteration, where h is the norm of the partition. Numerical examples are provided to justify the reliability and efficiency of the proposed technique.

Keywords: nonlinear Volterra integral equations; Newton method; smooth kernel; weakly singular kernel; iterated Galerkin method.

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1 Introduction

Consider the non-linear second kind Volterra integral equations of the form:

$$v(\xi) - \int_0^{\xi} \ell(\xi, s)k(\xi, s, v(s))\, ds = f(\xi), \tag{1.1}$$

where

$$\ell(\xi, s) = \begin{cases} 1, & \text{for smooth kernel, } \alpha = 0, \\ |\xi - s|^{-\alpha}, & \text{for weakly singular kernel, } 0 < \alpha < 1. \end{cases} \tag{1.2}$$

Here k is a given sufficiently smooth function and f is a given function. We have to find the approximation of the unknown function v.

Non-linear Volterra Urysohn integral equations arise in many physical as well as mathematical problems such as heat conduction, heat transfer, population dynamics, epidemic diffusion, gas absorption, crystal growth, elasticity, reaction–diffusion in small cells, diffusion in a semi-infinite region, boundary layer problems, etc. (see [16, 23, 25]). As explicitly solving these kind of integral equations is difficult, we require to apply some numerical approximation methods. Several approximation methods such as Nystrom, collocation, Galerkin, Petrov-Galerkin, multi-Galerkin, multi-collocation, are there (see [4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 21, 26, 27, 30]) to solve (1.1). In [10], Blom et al. used collocation method to solve non-linear Volterra Urysohn integral equations, whereas in [12] discussed those results for weakly singular kernels. In [30], Zhang et al. discussed the numerical solutions for second-kind Volterra integral equation using Galerkin method based on piecewise polynomials. In [11,12], Brunner applied collocation and iterated collocation methods with graded mesh for non-linear Volterra integral equation using the piecewise polynomials of degree $\leq r - 1$ as basis functions and found the convergence orders $\mathcal{O}(h^r)$ and $\mathcal{O}(h^{r+1-\alpha})$, respectively for weakly singular kernel. Newton's method is one of the most important method among the fixed point iteration methods (see [29]) to solve non-linear operator equations. In [6], Newton iteration method is discussed for non-linear Fredholm integral operator equations. In [3], Argyros considered Newton and Newton-like methods to obtain the convergence results for non-linear operator equations. In [20], Kelly and Sachs have applied Broyden's method as an approximation of Newton's method to approximate non-linear integral equations. For more extensive discussions and applications of Newton method and its discretizations to solve non-linear operator equations, (see [2, 5, 24]). To the best of our knowledge, in the study of Newton iteration method for iterated projection operators, there are no convergence rates available for non-linear Volterra integral operator equations. In [28], Wan et al. used spectral Galerkin method to solve Volterra-Urysohn integral equations of the second kind and proved that the error in approximation decay exponentially, provided that the source as well as kernel functions are sufficiently smooth. A one step of improvement by iteration to the Galerkin method, named as iterated Galerkin method is well discussed by many authors (see [19, 21]) and it gives improved convergence rates than Galerkin method. But if we iterate them one more time, then there is no guarantee that we get any improved convergence rates. In this paper, we apply Newton-iteration scheme based upon iterated Galerkin operator to derive the improved convergence rates for every step of iteration for solving Volterra Urysohn integral equations for both smooth and weakly singular kernels.

In this paper, Newton-iteration scheme based upon iterated Galerkin operator is applied for solving the nonlinear Volterra Urysohn integral equations for both smooth and weakly singular kernels. We prove that in j-th Newton iteration scheme, the iterated Galerkin approximation converges with $\mathcal{O}(h^{(j+2)r})$ and $\mathcal{O}(h^{r+(j+1)(1-\alpha)})$, $0<\alpha<1$, respectively, for smooth and weakly singular kernels. Thus, in Newton iteration scheme, convergence rates are improving in each step for iterated Galerkin method for Volterra Urysohn integral operator equations.

We organize the article in the following way. In Section 2, Newton-iteration scheme for solving (1.1) using iterated-Galerkin operator is discussed and corresponding convergence results are obtained in Section 3 for this Newton iteration scheme. Numerical results are provided in Section 4. We assume C as a generic constant throughout the article which may vary time to time.

2 Newton iteration method to solve Volterra Urysohn integral equations

Let $\mathbb{X} = L^\infty[0,1]$. We consider the non-linear Volterra Urysohn integral equations of the form (1.1)–(1.2). Now to get superconvergence results, we eventually apply a conversion $s : [0,1]\times[0,1] \to [0,1]$ by $s = \xi\eta$. Then the integral equations (1.1)–(1.2) become

$$v(\xi) - \int_0^1 \ell(\xi, s(\xi,\eta))k(\xi, s(\xi,\eta), v(s(\xi,\eta)))\, d\eta = f(\xi),$$

where

$$\ell(\xi, s(\xi,\eta)) = \begin{cases} \xi, & \text{for smooth kernel, } \alpha = 0, \\ |1-\eta|^{-\alpha}\xi^{1-\alpha}, & \text{for weakly singular kernel, } 0<\alpha<1. \end{cases} \tag{2.1}$$

Considering the non-linear operator $\mathcal{K}$ from $\mathbb{X}$ to $\mathbb{X}$ by

$$\mathcal{K}v(\xi) = \int_0^1 \ell(\xi, s(\xi,\eta))k(\xi, s(\xi,\eta), v(s(\xi,\eta)))\, d\eta, \tag{2.2}$$

we write (1.1) as

$$v(\xi) - \mathcal{K}(v)(\xi) = f(\xi), \quad \xi \in [0,1]. \tag{2.3}$$

Then the Fréchet derivative of $\mathcal{K}$ at any point $v \in \mathbb{X}$ is given by

$$\mathcal{K}'(v)y(\xi) = \int_0^1 \ell(\xi, s(\xi,\eta))\frac{\partial k}{\partial v}(\xi, s(\xi,\eta), v(s(\xi,\eta)))y(s(\xi,\eta))\, d\eta, \quad y \in \mathbb{X}.$$

Throughout this paper, the following assumptions are made on f and $k(\cdot,\cdot,\cdot)$:

- A.1: $f \in \mathcal{C}^r[0,1]$ for $\alpha = 0$ and $\mathcal{C}^{r,\alpha}[0,1]$ for $0<\alpha<1$.
- A.2: The non-linear function $k(\cdot,\cdot,v(\cdot))$ and its partial derivative $\frac{\partial k}{\partial v}(\cdot,\cdot,v(\cdot))$ are bounded and Lipschitz continuous w.r.t. v, i.e., $\exists\, c_1, c_2 > 0$ such that for any $y_1, y_2 \in \mathbb{R}$,

$$|k(\xi,s,y_1) - k(\xi,s,y_2)| \le c_1|y_1-y_2|,$$
$$\left|\frac{\partial k}{\partial v}(\xi,s,y_1) - \frac{\partial k}{\partial v}(\xi,s,y_2)\right| \le c_2|y_1-y_2|.$$

A.3 The function $\frac{\partial k}{\partial v}(\xi, s, v(\cdot))$ are also Lipschitz continuous w.r.t. all variables ξ, s and v, i.e., $\exists\, c_3, c_4, c_5 > 0$ such that for any $\xi_1, \xi_2, s_1, s_2 \in [0,1]$, $y_1, y_2 \in \mathbb{R}$,

$$\left|\frac{\partial k}{\partial v}(\xi_1, s_1, y_1) - \frac{\partial k}{\partial v}(\xi_2, s_2, y_2)\right| \leq c_3|\xi_1 - \xi_2| + c_4|s_1 - s_2| + c_5|y_1 - y_2|.$$

Also for any $v \in \mathcal{C}^r[0,1]$, we denote $\|v\|_{r,\infty} = \max_{1\leq i\leq r}\{\|v^{(i)}\|_\infty\}$. Now, for the kernel (2.1), we have that

$$c_\ell := \sup_{0\leq\xi\leq 1}\int_0^1 |\ell(\xi, s(\xi,\eta))|\, d\eta < \infty.$$

Then, considering $\mathcal{T}(v) = \mathcal{K}(v) + f$, (2.2) can be written as $v = \mathcal{T}(v)$. Then, using A.2, we have

$$\|\mathcal{T}(v_1) - \mathcal{T}(v_2\|_\infty = \|\mathcal{K}(v_1) - \mathcal{K}(v_2)\|_\infty \leq c_1 c_\ell \|v_1 - v_2\|_\infty.$$

Then, by Banach contraction principle, if $c_1 c_\ell < 1$, $\mathcal{T}$ posses a fixed point v in $\mathbb{X}$.

Now, to define Newton iteration scheme for Volterra Urysohn integral equations based upon iterated-Galerkin operator, first we discuss iterated-Galerkin method. We consider the partition $\Pi_n : 0 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$ of $[0,1]$ such that

$$h = \max_j\{|\xi_{j-1} - \xi_j| : 1 \leq j \leq n\} \to 0, \quad as \;\; n \to \infty.$$

Then, we let $\mathbb{X}_n = S^{\nu}_{r,n}(\Pi_n)$, the space of piecewise polynomials subspace of degree $\leq r-1$ as the approximating subspace having ν $(-1 \leq \nu \leq r-2)$ continuous derivatives at the breakpoints $\xi_1, \xi_2, \ldots, \xi_{n-1}$.

The orthogonal projection operator $\pi_n : \mathbb{X} \to \mathbb{X}_n$ is defined as

$$\langle \pi_n v, z\rangle = \langle v, z\rangle, \quad v \in \mathbb{X},\, z \in \mathbb{X}_n,$$

where $\langle v, z\rangle = \int_0^1 v(\xi)z(\xi)\, d\xi$. Then, according to Chatelin [17], π_n satisfies

i) There exists a constant $c_6 > 0$, independent of n such that $\|\pi_n\|_\infty \leq c_6 < \infty$.

ii) For any $v \in \mathcal{C}^r[0,1]$, there exists a constant $c_7 > 0$, independent of n such that

$$\|v - \pi_n v\|_\infty \leq c_7 h^r \|v^{(r)}\|_\infty. \tag{2.4}$$

Then, the Galerkin method to find the numerical solution of (2.2) is to find $u_n \in \mathbb{X}_n$ such that

$$v_n - \pi_n\mathcal{K}(v_n) = \pi_n f.$$

The iterated Galerkin approximation is defined as

$$\tilde{v}_n = \mathcal{K}(v_n) + f. \tag{2.5}$$

From (2.6) and (2.5), it can be seen that $v_n = \pi_n \tilde{v}_n$. Using which we rewrite (2.5) in operator form by

$$\tilde{v}_n - \mathcal{K}(\pi_n \tilde{v}_n) = f. \tag{2.6}$$

Then, from [19], [21], $\tilde{v}_n$ converges to the exact solution v with

$$\|v - \tilde{v}_n\|_\infty = \begin{cases} \mathcal{O}(h^{2r}), & \text{for } \alpha = 0, \\ \mathcal{O}(h^{r+1-\alpha}), & \text{for } 0 < \alpha < 1. \end{cases} \tag{2.7}$$

Now we propose the Newton iteration scheme for the operator equation (2.6) to obtain the numerical solution of Volterra Urysohn integral equations (1.1) of the second kind. Set $\tilde{v}_n^{(0)} = \tilde{v}_n$, and for $j = 0, 1, 2, \ldots$ define

$$\tilde{v}_n^{(j+1)} = \tilde{v}_n^{(j)} + \Big(I - \mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n\Big)^{-1} (-\tilde{v}_n^{(j)} + \mathcal{K}(\tilde{v}_n^{(j)}) + f). \tag{2.8}$$

This scheme can be equivalently written as
Step-I: Calculate $r_n^{(j)} = -\tilde{v}_n^{(j)} + \mathcal{K}(\tilde{v}_n^{(j)}) + f$. If $r_n^{(j)} = 0$, then $\tilde{v}_n^{(j)}$ gives exact solution of (2.2).
Step-II: If $r_n^{(j)} \neq 0$, solve for $y_n^{(j)}$ such that

$$(I - \mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n) y_n^{(j)} = r_n^{(j)}.$$

Step-III: Set $\tilde{v}_n^{(j+1)} = \tilde{v}_n^{(j)} + y_n^{(j)}$. Go to Step-I and repeat the process.
The above scheme is Newton iteration scheme for iterated-Galerkin operator.

3 Convergence analysis

Here, convergence results are discussed for Newton-iteration scheme defined by (2.8) for the iterated-Galerkin method. For this, for any $y \in \mathbb{X}$, we define

$$\mathcal{A}_n(y) = y + \Big(I - \mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n\Big)^{-1} (-y + \mathcal{K}(y) + f).$$

Using this, we write iterated scheme (2.8) by

$$\tilde{v}_n^{(j+1)} = \mathcal{A}_n(\tilde{v}_n^{(j)}), \quad j = 0, 1, 2, \ldots, \tag{3.1}$$

where $\tilde{v}_n^{(0)} = \tilde{v}_n$.

Now before discussing the convergence rates first we prove the following lemma for the norm convergence of $\mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n$.

Lemma 1. *Let v be the exact solution of (2.2) and $\tilde{v}_n$ be the corresponding iterated-Galerkin approximation defined as in (2.6). Then, $\mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n$ is norm convergent to $\mathcal{K}'(v)$.*

Proof. Using A.2 and $\tilde{v}_n^{(0)} = \tilde{v}_n$, we have

$$\begin{aligned}
&\|\mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n - \mathcal{K}'(v)\|_\infty \le \|\mathcal{K}'(\pi_n \tilde{v}_n)\pi_n - \mathcal{K}'(\pi_n v)\pi_n\|_\infty \\
&\quad + \|\mathcal{K}'(\pi_n v)\pi_n - \mathcal{K}'(v)\|_\infty \le c_6 \|\mathcal{K}'(\pi_n \tilde{v}_n) - \mathcal{K}'(\pi_n v)\|_\infty \\
&\quad + \|\mathcal{K}'(\pi_n v)\pi_n - \mathcal{K}'(v)\pi_n\|_\infty + \|\mathcal{K}'(v)\pi_n - \mathcal{K}'(v)\|_\infty \\
&\le c_6 c_2 c_\ell \|\pi_n \tilde{v}_n - \pi_n v\|_\infty + c_6 \|\mathcal{K}'(\pi_n v) - \mathcal{K}'(v)\|_\infty + \|\mathcal{K}'(v)(\pi_n - I)\|_\infty \\
&\le c_6^2 c_2 c_\ell \|\tilde{v}_n - v\|_\infty + c_6 c_2 c_\ell \|(\pi_n - I)v\|_\infty + \|\mathcal{K}'(v)(\pi_n - I)\|_\infty. \qquad (3.2)
\end{aligned}$$

Now, when $\mathcal{K}$ has smooth kernels, $\alpha = 0$, from (2.4) and orthogonality of π_n, we have

$$\begin{aligned}
&|\mathcal{K}'(v)(I - \pi_n)y(\xi)| \\
&= \Big| \int_0^1 \ell(\xi, s(\xi,\eta)) \frac{\partial k}{\partial v}(\xi, s(\xi,\eta), v(s(\xi,\eta)))(I - \pi_n)y(s(\xi,\eta))\, d\eta \Big| \\
&= \Big| \langle \ell(\xi, s(\xi,\cdot)) \frac{\partial k}{\partial v}(\xi, s(\xi,\cdot), v(s(\xi,\cdot))), (I - \pi_n)y(s(\xi,\cdot)) \rangle \Big| \\
&= \Big| \langle (I - \pi_n)\ell(\xi, s(\xi,\cdot)) \frac{\partial k}{\partial v}(\xi, s(\xi,\cdot), v(s(\xi,\cdot))), (I - \pi_n)y(s(\xi,\cdot)) \rangle \Big| \\
&\le \|(I - \pi_n)\ell(\xi, s(\xi,\cdot)) \frac{\partial k}{\partial v}(\xi, s(\xi,\cdot), v(s(\xi,\cdot)))\|_{L^2} \|(I - \pi_n)y\|_{L^2} \\
&\le \|(I - \pi_n)\ell(\xi, s(\xi,\cdot)) \frac{\partial k}{\partial v}(\xi, s(\xi,\cdot), v(s(\xi,\cdot)))\|_\infty (1 + c_6)\|y\|_\infty \\
&\le C h^r (1 + c_6)\|y\|_\infty,
\end{aligned}$$

which gives

$$\|\mathcal{K}'(v)(I - \pi_n)\|_\infty \le C h^r (1 + c_6) = \mathcal{O}(h^r). \qquad (3.3)$$

Hence for smooth kernels, $\alpha = 0$, using (2.4), (2.7) and (3.3), from (3.2), we have

$$\begin{aligned}
\|\mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n - \mathcal{K}'(v)\|_\infty &\le c_6^2 c_2 c_\ell \mathcal{O}(h^{2r}) + c_6 c_2 c_\ell c_7 \mathcal{O}(h^r) + \mathcal{O}(h^r) \\
&= \mathcal{O}(h^r) \to 0 \quad \text{as} \quad n \to \infty. \qquad (3.4)
\end{aligned}$$

Next, if $\mathcal{K}$ has weakly singular kernels, $0 < \alpha < 1$, $\ell(\xi, s(\xi,\eta)) = (1-\eta)^{-\alpha}\xi^{1-\alpha}$, considering $H(\xi, s(\xi,\eta)) = \ell(\xi, s(\xi,\eta)) \frac{\partial k}{\partial v}(\xi, s(\xi,\eta), v(s(\xi,\eta)))$, and similar as Theorem 3.2 of [19], we have that $\exists$ a polynomial $v_\xi \in \mathbb{P}_r$ such that

$$\|H(\xi, s(\xi,\cdot)) - v_\xi\|_{L^1} = \mathcal{O}(h^{1-\alpha}),$$

where $\mathbb{P}_r$ is the set of polynomials of degree $\le r - 1$. Then, using orthogonality of π_n, we get

$$\begin{aligned}
&|\mathcal{K}'(v)(I - \pi_n)y(\xi)| \\
&= \Big| \int_0^1 \ell(\xi, s(\xi,\eta)) \frac{\partial k}{\partial v}(\xi, s(\xi,\eta), v(s(\xi,\eta)))(I - \pi_n)y(s(\xi,\eta))\, d\eta \Big|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\langle H(\xi, s(\xi,\cdot)), (I-\pi_n)y(s(\xi,\eta))\rangle\right| \\
&= \left|\langle H(\xi, s(\xi,\cdot)) - v_\xi, (I-\pi_n)y(s(\xi,\eta))\rangle\right| \\
&\le \|H(\xi, s(\xi,\cdot)) - v_\xi\|_{L^1}\|(I-\pi_n)y\|_\infty \le (1+c_6)\|y\|_\infty \mathcal{O}(h^{1-\alpha}),
\end{aligned}
$$

which gives

$$
\|\mathcal{K}'(v)(I-\pi_n)\|_\infty = \mathcal{O}(h^{1-\alpha}). \tag{3.5}
$$

Thus, for $0<\alpha<1$, using (2.7) and (3.5), from (3.2), it follows that

$$
\begin{aligned}
\|\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n - \mathcal{K}'(v)\|_\infty &\le c_6^2 c_2 c_\ell \mathcal{O}(h^{r+1-\alpha}) + c_6 c_2 c_\ell \mathcal{O}(h^r) + \mathcal{O}(h^{1-\alpha}) \\
&= \mathcal{O}(h^{1-\alpha}) \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{3.6}
$$

Hence from (3.4) and (3.6), it implies $\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n$ is norm convergent to $\mathcal{K}'(v)$. This proves the lemma. □

Now, let 1 is not an eigenvalue of $\mathcal{K}'(v)$. Then, from [1] and Lemma 1, $\|(I-\mathcal{K}'(v))^{-1}\|_\infty$ is bounded gives $\|(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\|_\infty$ is uniformly bounded, i.e., there exists $\mathcal{L}>0$ such that $\|(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\|_\infty \le \mathcal{L} < \infty$ for enough large n.

Using this we can prove that $\mathcal{A}_n'(y)$ is Lipschitz continuous. For any $y_1, y_2 \in \mathbb{X}$, we obtain

$$
\begin{aligned}
\|\mathcal{A}_n'(y_1) - \mathcal{A}_n'(y_2)\|_\infty &= \|(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\{-I+\mathcal{K}'(y_1)+I-\mathcal{K}'(y_2)\}\|_\infty \\
&\le \|(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\|_\infty\|\mathcal{K}'(y_1)-\mathcal{K}'(y_2)\|_\infty \le \mathcal{L}c_2c_\ell\|y_1-y_2\|_\infty.
\end{aligned}
$$

Thus, $\mathcal{A}_n'(y)$ is $\mathcal{L}c_2c_\ell$-Lipschitz.

Next, we prove the following lemma that helps for proving our main theorem regarding convergence rates of Scheme (2.8).

Lemma 2. *Let v be the exact solution of (2.3). Then, under the scheme (3.1), for $(j+1)$-th Newton-iterated approximate solution $\tilde{v}_n^{(j+1)}$, there holds*

$$
\|v-\tilde{v}_n^{(j+1)}\|_\infty \le \left(C\|v-\tilde{v}_n^{(j)}\|_\infty + \mathcal{L}\|\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n - \mathcal{K}'(v)\|_\infty\right)\|v-\tilde{v}_n^{(j)}\|_\infty.
$$

Proof. Since v is the exact solution of (2.3), it follows that $\mathcal{A}_n(v)=v$. Then, from mean value theorem with $0<\theta<1$, we have

$$
\begin{aligned}
v-\tilde{v}_n^{(j+1)} &= \mathcal{A}_n(v)-\mathcal{A}_n(\tilde{v}_n^{(j)}) = \mathcal{A}_n'(v+\theta(v-\tilde{v}_n^{(j)}))(v-\tilde{v}_n^{(j)}) \\
&= \mathcal{A}_n'(v+\theta(v-\tilde{v}_n^{(j)}))(v-\tilde{v}_n^{(j)}) - \mathcal{A}_n'(v)(v-\tilde{v}_n^{(j)}) + \mathcal{A}_n'(v)(v-\tilde{v}_n^{(j)}).
\end{aligned}
$$

Now, from the Lipschitz continuity of $\mathcal{A}_n'$, it follows that

$$
\begin{aligned}
\|v-\tilde{v}_n^{(j+1)}\|_\infty \le& \|\mathcal{A}_n'(v+\theta(v-\tilde{v}_n^{(j)})) - \mathcal{A}_n'(v)\|_\infty\|v-\tilde{v}_n^{(j)}\|_\infty \\
&+ \|\mathcal{A}_n'(v)\|_\infty\|v-\tilde{v}_n^{(j)}\|_\infty
\end{aligned}
$$

$$\leq C\|v-\tilde{v}_n^{(j)}\|_\infty^2+\|\mathcal{A}_n'(v)\|_\infty\|v-\tilde{v}_n^{(j)}\|_\infty$$
$$\leq \left(C\|v-\tilde{v}_n^{(j)}\|_\infty+\|\mathcal{A}_n'(v)\|_\infty\right)\|v-\tilde{v}_n^{(j)}\|_\infty. \qquad (3.7)$$

Note that

$$\begin{aligned}\mathcal{A}_n'(v)&=I+(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}(-I+\mathcal{K}'(v))\\&=(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\left(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n-I+\mathcal{K}'(v)\right)\\&=-(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\left(\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n-\mathcal{K}'(v)\right).\end{aligned}$$

Then, from uniform boundedness of $(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}$, we get

$$\begin{aligned}\|\mathcal{A}_n'(v)\|_\infty&=\|(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\left(\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n-\mathcal{K}'(v)\right)\|_\infty\\&\leq\|(I-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n)^{-1}\|_\infty\|\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n-\mathcal{K}'(v)\|_\infty\\&\leq\mathcal{L}\|\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n-\mathcal{K}'(v)\|_\infty.\end{aligned}$$

Using this in (3.7), we obtain

$$\|v-\tilde{v}_n^{(j+1)}\|_\infty\leq\left(C\|v-\tilde{v}_n^{(j)}\|_\infty+\mathcal{L}\|\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n-\mathcal{K}'(v)\|_\infty\right)\|v-\tilde{v}_n^{(j)}\|_\infty.$$

Hence the proof. □

Now, we prove the following main theorem for convergence rates of the scheme defined by (2.8) based upon iterated-Galerkin operator.

Theorem 1. *Let v be the exact solution of (2.3) and $\tilde{v}_n^{(0)}$ be the iterated-Galerkin approximation $\tilde{v}_n$ defined in (2.6). Then, under the Newton-iteration scheme defined by (2.8) for the iterated-Galerkin operator, for $(j+1)$-th Newton iterated approximation $\tilde{v}_n^{(j+1)}$, there holds*

$$\|v-\tilde{v}_n^{(j+1)}\|_\infty=\begin{cases}\mathcal{O}(h^{(j+3)r}), & \text{for } \alpha=0,\\ \mathcal{O}(h^{r+(j+2)(1-\alpha)}), & \text{for } 0<\alpha<1.\end{cases}$$

Proof. From (3.1), for the Newton iteration scheme (2.8), we have

$$\begin{aligned}&\mathcal{K}'(v)-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n\\&=\mathcal{K}'(v)-\mathcal{K}'(v)\pi_n+\mathcal{K}'(v)\pi_n-\mathcal{K}'(\pi_nv)\pi_n+\mathcal{K}'(\pi_nv)\pi_n-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})\pi_n\\&=\mathcal{K}'(v)(I-\pi_n)+[\mathcal{K}'(v)-\mathcal{K}'(\pi_nv)]\pi_n+[\mathcal{K}'(\pi_nv)-\mathcal{K}'(\pi_n\tilde{v}_n^{(0)})]\pi_n.\end{aligned}$$

Then, using the estimates (2.4), (2.7), (3.3) and (3.5), and the Lipschitz continuity of $\mathcal{K}'(v)$, we obtain

$$\begin{aligned}
&\|\mathcal{K}'(v) - \mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n\|_\infty \leq \|\mathcal{K}'(v)(I - \pi_n)\|_\infty \\
&\quad + \|[\mathcal{K}'(v) - \mathcal{K}'(\pi_n v)]\pi_n\|_\infty \qquad + \|[\mathcal{K}'(\pi_n v) - \mathcal{K}'(\pi_n \tilde{v}_n^{(0)})]\pi_n\|_\infty \\
&\leq \|\mathcal{K}'(v)(I - \pi_n)\|_\infty + c_2 c_\ell \|v - \pi_n v\|_\infty c_6 + c_2 c_\ell \|\pi_n v - \pi_n \tilde{v}_n^{(0)}\|_\infty c_6 \\
&\leq \|\mathcal{K}'(v)(I - \pi_n)\|_\infty + c_6 c_2 c_\ell \|(I - \pi_n)v\|_\infty + c_6^2 c_2 c_\ell \|v - \tilde{v}_n^{(0)}\|_\infty \\
&= \begin{cases} \mathcal{O}(h^r), & \text{for } \alpha = 0, \\ \mathcal{O}(h^{(1-\alpha)}), & \text{for } 0 < \alpha < 1. \end{cases}
\end{aligned}$$

Using this, from Lemma 2, we get

$$\begin{aligned}
\|v - \tilde{v}_n^{(j+1)}\|_\infty &\leq \left(C\|v - \tilde{v}_n^{(j)}\|_\infty + \mathcal{L}\|\mathcal{K}'(\pi_n \tilde{v}_n^{(0)})\pi_n - \mathcal{K}'(v)\|_\infty \right) \|v - \tilde{v}_n^{(j)}\|_\infty \\
&\leq \left(C\|v - \tilde{v}_n^{(j)}\|_\infty + \mathcal{L} \begin{cases} \mathcal{O}(h^r) \\ \mathcal{O}(h^{(1-\alpha)}) \end{cases} \right) \|v - \tilde{v}_n^{(j)}\|_\infty.
\end{aligned}$$

Then, for $j = 0$, from (2.7), it follows that

$$\begin{aligned}
\|v - \tilde{v}_n^{(1)}\|_\infty &\leq \left(C\|v - \tilde{v}_n^{(0)}\|_\infty + \mathcal{L} \begin{cases} \mathcal{O}(h^r) \\ \mathcal{O}(h^{(1-\alpha)}) \end{cases} \right) \|v - \tilde{v}_n^{(0)}\|_\infty \\
&= \begin{cases} \mathcal{O}(h^{3r}), & \text{for } \alpha = 0, \\ \mathcal{O}(h^{r+2(1-\alpha)}), & \text{for } 0 < \alpha < 1. \end{cases}
\end{aligned}$$

Similarly for $j = 1$, we get

$$\begin{aligned}
\|v - \tilde{v}_n^{(2)}\|_\infty &\leq \left(C\|v - \tilde{v}_n^{(1)}\|_\infty + \mathcal{L} \begin{cases} \mathcal{O}(h^r) \\ \mathcal{O}(h^{(1-\alpha)}) \end{cases} \right) \|v - \tilde{v}_n^{(1)}\|_\infty \\
&= \begin{cases} \mathcal{O}(h^{4r}), & \text{for } \alpha = 0, \\ \mathcal{O}(h^{r+3(1-\alpha)}), & \text{for } 0 < \alpha < 1. \end{cases}
\end{aligned}$$

Similarly, proceeding further, it can be seen that under the scheme (2.8), in each step of reiteration, for smooth kernels, $\alpha = 0$, the order is increasing by h^r, whereas for weakly singular kernels, $0 < \alpha < 1$, the order increases by $h^{1-\alpha}$ for every step of reiteration. Thus in general we will get (it can be proved at ease using induction)

$$\|v - \tilde{v}_n^{(j+1)}\|_\infty \leq \begin{cases} \mathcal{O}(h^{(j+3)r}), & \text{for } \alpha = 0, \\ \mathcal{O}(h^{r+(j+2)(1-\alpha)}), & \text{for } 0 < \alpha < 1. \end{cases}$$

Hence the theorem. □

Remark 1. From Theorem 1, we note that the superconvergence rates are improving in each step of reiterations for the proposed scheme (2.8) for iterated-Galerkin operator.

4 Numerical results

Here numerical examples are given to justify the reliability and efficiency of our proposed technique. We provide the errors in finding iterated-Galerkin approximation $\tilde{v}_n^{(0)}$ and the corresponding first reiterated approximation for the iterated-Galerkin operator $\tilde{v}_n^{(1)}$ respectively, in infinity norm. Then we denote

$$\|v - \tilde{v}_n^{(0)}\|_\infty = \mathcal{O}(h^{\lambda_i}), \quad \text{and} \quad \|v - \tilde{v}_n^{(1)}\|_\infty = \mathcal{O}(h^{\mu_i}),$$

where $i = 1$ for smooth kernels, and $i = 2$ for weakly singular kernels, respectively, and v denotes the exact solution. We have performed all the numerical algorithms on Matlab (R2017a).

We choose the space of piecewise constant ($r = 1$) functions as approximating subspace. Then for $r = 1$, for smooth kernels, the expected orders of convergence will be $\lambda_1 = 2, \mu_1 = 3$ that we calculate in Table 1 of Example 1. Also for $\alpha = \frac{1}{2}, r = 1$, the expected rates for weakly singular kernels are $\lambda_2 = 1.5, \mu_2 = 2$ which we compute in the Table 2 of Example 2.

Example 1. Consider the following non-linear Volterra integral equation

$$v(\xi) = \int_0^\xi k(\xi, s, v(s))\, ds + f(\xi),$$

with smooth kernel $k(\xi, s, v(s)) = -s[v(s)]^3$, the function $f(\xi) = \xi + \frac{\xi^5}{5}$, and exact solution $v(\xi) = \xi$.

Table 1. Numerical results for Example 1.

n	$\|v - v_n^{(0)}\|_\infty$	λ_1	$\|v - v_n^{(1)}\|_\infty$	μ_1
2	$4.28181707 \times 10^{-2}$	–	$3.18558778 \times 10^{-3}$	–
4	$1.00182785 \times 10^{-2}$	2.10	$4.95673661 \times 10^{-4}$	2.68
8	$2.76579432 \times 10^{-3}$	1.86	$7.26741030 \times 10^{-5}$	2.77
16	$7.47153027 \times 10^{-4}$	1.89	$9.85115481 \times 10^{-6}$	2.88
32	$1.95350117 \times 10^{-4}$	1.94	$1.26030843 \times 10^{-6}$	2.96

From Table 1, we can see that the orders are matching well with the expected orders. Also, from Figure 1, we see that the errors in the reiterated approximation for the iterated-Galerkin operator $\tilde{v}_n^{(1)}$, when solving Example 1, are lower than those in iterated-Galerkin approximation $\tilde{v}_n^{(0)}$. Thus the reiterated approximation $\tilde{v}_n^{(1)}$ improves upon iterated-Galerkin approximation $\tilde{v}_n^{(0)}$. In [22], authors have solved Example 1 using the Legendre spectral Galerkin and Legendre spectral iterated Galerkin methods and obtained minimum errors of 6.33×10^{-3} and 5.02×10^{-5}, respectively, in the infinity norm (see, Table 3 of [22]), whereas from Table 1 of Example 1, in the reiterated approximation for the iterated-Galerkin operator $\tilde{v}_n^{(1)}$, we obtain minimum error of 1.26×10^{-6}.

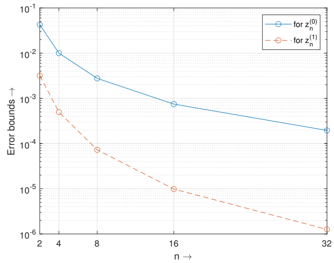

Figure 1. Error bounds for Example 1.

Example 2. Consider the following Volterra-Urysohn integral equation of second kind with weakly singular kernel

$$v(\xi) = \int_0^{\xi} (\xi - s)^{-\frac{1}{2}} k(\xi, s, v(s))\, ds + f(\xi), \quad \xi \in [0, 1],$$

where $k(\xi, s, v(s)) = v(s)^2$, $f(\xi) = \xi^4 - \frac{65536}{109395}\xi^{\frac{17}{2}}$ and the exact solution $v(\xi) = \xi^4$.

Table 2. Numerical results for Example 2.

n	$\|v - v_n^{(0)}\|_\infty$	λ_2	$\|v - v_n^{(1)}\|_\infty$	μ_2
2	$4.98669684 \times 10^{-1}$	–	$3.24034264 \times 10^{-1}$	–
4	$2.37780247 \times 10^{-1}$	1.07	$1.00771037 \times 10^{-1}$	1.69
8	$9.41916752 \times 10^{-2}$	1.34	$2.80842139 \times 10^{-2}$	1.84
16	$3.42077422 \times 10^{-2}$	1.46	$7.35471314 \times 10^{-3}$	1.93
32	$1.20050043 \times 10^{-2}$	1.51	$1.86604743 \times 10^{-3}$	1.98

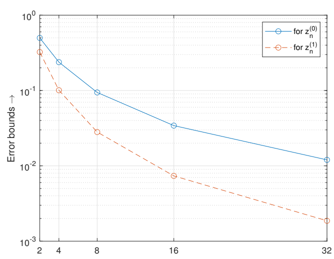

Figure 2. Error bounds for Example 2.

From Table 2, for weakly singular kernels, we see that the orders are matching well with the expected orders. Again, from Figure 2, we observe that the errors in the reiterated approximation $\tilde{v}_n^{(1)}$ are lower than those in the iterated-Galerkin approximation $\tilde{v}_n^{(0)}$. Thus the reiterated approximation $\tilde{v}_n^{(1)}$ improves upon iterated-Galerkin approximation $\tilde{v}_n^{(0)}$.

5 Conclusions

We developed Newton-iteration scheme for iterated-Galerkin operator for solving non-linear Volterra Urysohn integral equations for both smooth kernels and weakly singular kernels, and obtain superconvergence results. In each step of Newton iterations, we enhance the convergence results of the iterated Galerkin method for both smooth kernels and weakly singular kernels.

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