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Analyzing Helmholtz phenomena for mixed boundary values via graphically controlled contractions

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Abstract. Helmholtz's problem helps us to completely understand how sound behaves in a cylinder that is closed from one of its ends and opened at another. This paper aims to employ some novel convergence results to the Helmholtz problem with mixed boundary conditions and demonstrate the existence and uniqueness of the solution by applying graph-controlled contractions. For this purpose, we enunciate graphically Reich type and graphically Ćirić type contractions in the realm of graphical-controlled metric type spaces. In our study, we showcase the existence and uniqueness of fixed point results by employing these graphical contractions. This is demonstrated through extensive examples that a graphical-controlled metric-type space is distinct from traditional controlled metric-type spaces. We also exhibit an example of a graphically Reich contraction that is not a classical Reich contraction. Similarly, a decent example of graphical Ćirić contraction is presented, which is distinct from the classical Ćirić contraction. Concrete illustrative examples solidify our theoretical framework.

Keywords: directed graph, Helmholtz phenomena, fixed-point, mixed boundary conditions graphic-contractions.

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1 Introduction

Stefan Banach's noteworthy work [10] established the groundwork for the field of fixed point theory. Ensuing scientists developed these underlying ideas, extending comprehension and expanding the hypothesis' applications. Stanislaw Saks' contribution [28] was compelling, as he presented novel viewpoints on multivalued mappings and the topological qualities of fixed point sets. These revolutionary works, combined with Banach's foundational research, established a solid framework that has encouraged significant investigation and various applications in the subject of fixed point theory; for instance, see [4, 26, 27]. In 1989, Bakhtin [9] introduced the notion of a b-metric space, subsequently elaborated upon by Czerwik in 1993 [13]. These spaces generalize metric spaces by modifying the triangle inequality condition with a multiplicative constant. In 2017, Kamran [19] advanced this concept by presenting extended b-metric spaces, hence expanding their applicability. In 2018, Mlaiki [21] introduced the concept of controlled metric type spaces, expanding upon these foundations. This framework included a control mechanism to manage the distance between points, resulting in novel fixed-point outcomes. Subsequently, to enhance this structure, Abdeljawad [1] proposed the notion of double controlled metric type spaces by incorporating an additional function into the triangular property. This improvement established a more robust framework for demonstrating the existence and uniqueness of fixed points. In 2021, Ahmad et al. [3] furthered this field by presenting double controlled partial metric type spaces. Their research examined the ramifications of this novel framework and developed notable fixed-point outcomes. These subsequent developments demonstrate the gradual evolution of metric-type spaces, increasing their relevance in fixed-point theory and associated mathematical fields (also see [6, 25]).

In the realm of fixed point theory and its applications, the literature presents a rich tapestry of research that explores various aspects of contractions in metric spaces with graphs, ordered metric spaces, and modular function spaces. In 2007, Jachymski [18] examined the contraction principle for mappings on a metric space endowed with a graph, providing foundational insights into the interplay between graph theory and metric space contractions. Consequently, Nieto et al. [22] in 2007 delved into fixed point theorems within ordered abstract spaces, offering pivotal results that bridge order theory and fixed point principles. Building on these themes, O'Regan et al. [23] in 2008 extended fixed point theorems to generalized contractions in ordered metric spaces, further enriching the theoretical framework.

In 2010, Harjani and Sadarangani [17] investigated generalized contractions in partially ordered metric spaces, with applications to ordinary differential equations, highlighting practical implications in differential equations. In 2012, Aleomraninejad et al. [7] contributed by exploring fixed point results in metric spaces with a graph, emphasizing the significance of graphical structures in fixed point theory. This focus on graphical aspects continued with Dinevari and Frigon [15] in 2013, examined multivalued contractions on metric spaces with a graph, providing new insights into multivalued mappings and their fixed points.

Furthering this line of inquiry, Beg and Butt [11] in 2013 investigated set-valued graph contractive mappings, expanding the scope of fixed point results to set-valued mappings. In 2015, Alfuraidan [8] explored fixed points of multivalued mappings in modular function spaces with a graph, integrating modular function spaces into the fixed point discourse. In 2017, Mirmostafaee and Alireza [20] examined coupled fixed points for mappings on a b-metric

space with a graph, introducing coupled fixed point results in the context of b-metric spaces. In the same year, Shukla [31] presented a generalized setting in fixed point theory through graphical metric spaces, offering a comprehensive framework that unifies various fixed point results under a graphical metric space setting. Lastly, Chuensupantharat [12] in 2019 investigated graphic contraction mappings via graphical b-metric spaces, providing applications that underscore the practical utility of these theoretical advancements. In their 2024 paper, Dubey et al. [16] presented the notion of graphical symmetric spaces, which enhanced conventional metric spaces by integrating a graph framework. They provided fixed-point theorems for particular contractive mappings in these spaces and utilized their results to prove the existence of positive solutions for fractional periodic boundary value problems. In their 2023 study, Shukla et al. [29] investigated fixed-point theorems within graphical cone metric spaces, integrating graph theory with cone metric fields, establishing fixed-points for specific contractive mappings. Further, researchers utilized graph-based fixed-point theorems to address systems of initial value problems, emphasizing the practical implications of their theoretical discoveries.
Motivated by the growing literature on graphical structures and their applications, we dig into fixed point theorems inside the framework of graphical-controlled metric type spaces, utilizing graphically Reich type and graphically Ćirić type contractions. This exploration is driven by the acknowledgment that conventional fixed-point theory, while strong, often fails to address the complexities in graph-structured spaces.
Our method leads to useful conclusions: the existence and uniqueness of fixed points of graphical-controlled metric type spaces that are not necessarily graphical-controlled metric type spaces. Graphically controlled metric type spaces may not satisfy conditions of controlled metric type spaces. We give explicit examples to show that graphically Reich and graphically Ćirić contractions have distinctive properties different from that of the traditional Reich contractions and Ćirić contractions, respectively. It is not only that we develop new theories, but also results become more reliable and strong.
In addition, we illustrate our theoretical results for the Helmholtz phenomenon under mixed boundary conditions. This application highlights the significance of our work in different physical contexts – from acoustics to electromagnetics and mechanical vibrations – and demonstrates how the Helmholtz equation describes the behaviour of sound waves in cylinders, the distribution of electric or magnetic fields in waveguides or resonant cavities, and the dynamics of vibrating structures.

2 Preliminaries

In this section, we outline some fundamental concepts and definitions that are critical for the subsequent analysis of this article.

In 2008, Jachymski [18] described a scenario, where $\mathcal{W}$ is a nonempty set and Δ is the diagonal of $\mathcal{W} \times \mathcal{W}$. He defined a directed graph $\Phi = (\mathbb{V}(\Phi), \mathbb{E}(\Phi))$ free from parallel edges, where $\mathbb{V}(\Phi)$ is the vertex set of Φ, aligning with the set $\mathcal{W}$, and $\mathbb{E}(\varPhi)$ is the edge set of $\varPhi$, encompassing all the loops of $\varPhi$, such that $\varDelta \subseteq \mathbb{E}(\varPhi)$. The graph $\varPhi^{-1}$ is gained by reversing the trajectory of $\mathbb{E}(\varPhi)$. If $\varPhi$ owns symmetric edges is portrayed as $\breve{G}$, in such a way that

$$\mathbb{E}(\breve{G}) = \mathbb{E}(\varPhi^{-1}) \cup \mathbb{E}(\varPhi).$$

Assume that, $\gimel$ and $\hbar$ are vertices in the directed graph $\varPhi$. A path in $\varPhi$ is a sequence $\{\gimel_j\}_{j=0}^{m}$ of $(m+1)$ vertices, whereby $\gimel_0 = \gimel, \quad \gimel_m = \hbar$ accompanied by $(\gimel_{j-1}, \gimel_j) \in \mathbb{E}(\varPhi)$, where $j = 1, 2, \ldots, m$. A graph $\varPhi$, is referred as connected when any two vertices can be connected by a group of edges. An undirected graph $\varPhi$, is weakly connected if any two vertices within it can be joined by a sequence of edges. A graph $\varphi = \big(\mathbb{V}(\varphi), \mathbb{E}(\varphi)\big)$ is a subgraph of $\varPhi = (\mathbb{V}(\varPhi), \mathbb{E}(\varPhi))$ if all the vertices $\mathbb{V}(\varphi) \subseteq \mathbb{V}(\varPhi)$ and all the edges $\mathbb{E}(\varphi) \subseteq \mathbb{E}(\varPhi)$ meet these requirements in the same order. For further explanation, we recommend [2,34]. In 2007, Shukla [31] presented the notion that $\gimel_{\varPhi}^{l} = \{\hbar \in \mathcal{W} : \text{there is a directed path from } \gimel \text{ to } \hbar \in \varPhi \text{ with length } l\ \}$. A relation $\mathbb{P}$ on $\mathcal{W}$ such that $(\gimel\mathbb{P}\hbar)_{\varPhi}$ signifies a path from $\gimel$ to $\hbar$ in $\varPhi$ suggests that if $\wp \in (\gimel\mathbb{P}\hbar)$, then $\wp$ lies on the path $(\gimel\mathbb{P}\hbar)_{\varPhi}$. Moreover, a sequence $\{\gimel_n\} \subset \mathcal{W}$ is supposed to be $\varPhi$-termwise connected ($\varPhi$-$\mathbb{J}\mathcal{WC}$) if $(\gimel_n\mathbb{P}\gimel_{n+1})_{\varPhi}$ for all $n \in \mathbb{N}$. From this point forward, all graphs are considered directed unless otherwise stated.

Definition 1. [33] Let $\eth : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\mathcal{L}_g : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ be functions on a nonempty set $\mathcal{W}$ linked with graph $\varPhi$. Suppose, for all $\gimel, \hbar, \wp \in \mathcal{W}$ with $(\gimel, \hbar) \in \mathcal{W} \times \mathcal{W}$ the underneath axioms hold:

1. If $\mathcal{L}_g(\gimel, \hbar) = 0$, then $\gimel = \hbar$.
2. Iff $\gimel = \hbar$, then $\mathcal{L}_g(\gimel, \hbar) = 0$.
3. $\mathcal{L}_g(\gimel, \hbar) = \mathcal{L}_g(\hbar, \gimel)$, for all $\gimel \in \mathcal{W}$.
4. $(\gimel\mathbb{P}\hbar)_{\varPhi}$, $\wp \in (\gimel\mathbb{P}\hbar)_{\varPhi} \Rightarrow \mathcal{L}_g(\gimel, \hbar) \leq \eth(\gimel, \wp)\mathcal{L}_g(\gimel, \wp) + \eth(\wp, \hbar)\mathcal{L}_g(\wp, \hbar)$.
   Then, $(\mathcal{W}, \mathcal{L}_g)$ is referred to as a graphical-controlled metric type space.

Remark 1. It is crucial to keep in mind that not all graphical-controlled metric-type spaces are controlled metric-type spaces [21]. The accompanying example confirms our remark.

Example 1. Let $\mathcal{W} = \{0, 1, 4, 6, 8, 10, 12, 14\}$ and $\eth : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ be a mapping such that

$$\eth(\gimel, \hbar) = \frac{1}{\gimel\hbar + 2} + \frac{1}{\gimel\hbar + 3} + 1.9803.$$

Assume that $\mathcal{L}_g : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ is defined as:

$$\mathcal{L}_g(\gimel, \hbar) = \begin{cases} |\gimel - \hbar|^2, & \text{if } \gimel \neq \hbar, \\ 0, & \text{if } \gimel = \hbar. \end{cases}$$

The vertex set $\mathcal{W} = \mathbb{V}(\varPhi)$ and the edge set is designed in the following way

$$\mathbb{E}(\varPhi)=\varDelta\cup\left\{\begin{array}{l}(0,1),(0,4),(0,6),(0,8),(0,10),(0,12),(0,14),(1,4),(1,6),(1,8),\\(1,10),(1,12),(1,14),(4,6),(4,8),(4,10),(4,12),(4,14),(6,8),\\(6,10),(6,12),(6,14),(8,10),(8,12),(8,14),(10,12),(12,14)\end{array}\right\},$$

as shown in Figure 1. Clearly, $(\mathcal{W}, \mathcal{L}_g)$ is graphical-controlled metric type space

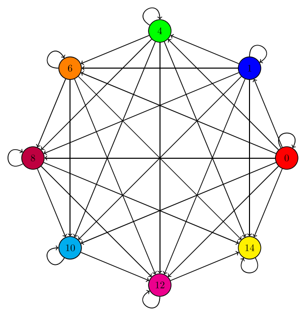

Figure 1. Graph of Example 1.

but not controlled metric type space for

$$\mathcal{L}_g(10,14) \leq \eth(10,12)\mathcal{L}_g(10,12) + \eth(12,14)\mathcal{L}_g(12,14), \quad 16 \nleq 15.955.$$

Therefore, a graphical-controlled metric space is not always the same as a standard controlled metric type space.

Definition 2. [33] Let $(\mathcal{W}, \mathcal{L}_g)$ is graphical-controlled metric type space then,

(i) a sequence $\{\daleth_n\}$ converges to some $\daleth$ in $\mathcal{W}$, if for each positive ϵ, there is some positive N_ϵ such that $\mathcal{L}_g(\daleth_n, \daleth) < \epsilon$ for each $n \geq N_\epsilon$. It can be written as

$$\lim_{n\to\infty} \daleth_n = \daleth.$$

(ii) a sequence $\{\daleth_n\}$ is referred to as Cauchy sequence, if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\mathcal{L}_g(\daleth_n, \daleth_m) < \epsilon$ for all $m, n \geq N_\epsilon$.

(iii) $(\mathcal{W}, \mathcal{L}_g)$ is said to be $\varPhi$-complete if every Cauchy sequence is convergent in $\mathcal{W}$.

3 Graphical convergence results

In this section, we will consider the graph $\varPhi$ to be a weighted graph and explore some fixed points in relation to graphical-controlled metric spaces $(\mathcal{W}, \mathcal{L}_g)$. Let $\daleth_0 \in \mathcal{W}$ be the starting value of the sequence $\{\daleth_n\}$, then define $\{\daleth_n\}$ as a $\mathbb{J}-$Picard sequence $(\mathbb{J} - \mathcal{PS})$ if $\daleth_n = \mathbb{J}\daleth_{n-1}$ for all $n \in \mathbb{N}$.

DEFINITION 3. Let $\mathbb{P}$ be a relation on $\mathcal{W}$ defined as follows:
The notation $(\gimel\mathbb{P}\hbar)_{\Phi}$ indicates that there exists a directed path from $\gimel$ to $\hbar$ in the graph Φ.
If $\wp \in (\gimel\mathbb{P}\hbar)_{\Phi}$, it is understood that $\wp$ lies along the path $(\gimel\mathbb{P}\hbar)_{\Phi}$.
A sequence $\{\gimel_n\} \subseteq \mathcal{W}$ is called Φ-termwise connected (denoted as $\Phi - \mathbb{J}\mathcal{WC}$) if $(\gimel_n\mathbb{P}\gimel_{n+1})_{\Phi}$ holds for all $n \in \mathbb{N}$.
A graph $\Phi = (\mathbb{V}(\Phi), \mathbb{E}(\Phi))$ is said to satisfy property $(\mathbb{P})$ if every Φ-termwise connected $\mathbb{J}$-Picard sequence $\{\gimel_n\}$ is convergent in $\mathcal{W}$. That is, there exists a limit $\gimel' \in \mathcal{W}$ such that either

$$(\gimel_n, \gimel') \in \mathbb{E}(\Phi) \quad \text{or} \quad (\gimel', \gimel_n) \in \mathbb{E}(\Phi) \quad \text{for all } n > n_0.$$

DEFINITION 4. We say $\mathbb{J} : \mathcal{W} \to \mathcal{W}$ is a graphical Reich contraction on $(\mathcal{W}, \mathcal{L}_g)$ linked with a graph Φ encompassing all loops satisfying:

1. For all $\gimel, \hbar \in \mathcal{W}$

$$\text{if } (\gimel, \hbar) \in \mathbb{E}(\Phi) \Longrightarrow (\mathbb{J}\gimel, \mathbb{J}\hbar) \in \mathbb{E}(\Phi), \tag{3.1}$$

($\mathbb{J}$ preserves edges of the graph Φ).

2. There exist non-negative constants such that $\pi_1 + \pi_2 + \pi_3 < 1$, for all $\gimel, \hbar \in \mathcal{W}$ with $(\gimel, \hbar) \in \mathbb{E}(\Phi)$, we achieve

$$\mathcal{L}_g(\mathbb{J}\gimel, \mathbb{J}\hbar) \leq \pi_1\mathcal{L}_g(\gimel, \hbar) + \pi_2\mathcal{L}_g(\gimel, \mathbb{J}\gimel) + \pi_3\mathcal{L}_g(\hbar, \mathbb{J}\hbar). \tag{3.2}$$

Theorem 1. *Let $\mathbb{J} : \mathcal{W} \to \mathcal{W}$ be a graphical Reich contraction on a $\Phi-$complete graphical-controlled metric type space $(\mathcal{W}, \mathcal{L}_g)$. Assume that the graph Φ demonstrates the property $(\mathbb{P})$, $\gimel_0 \in \mathcal{W}$ with $\mathbb{J}\gimel_0 \in [\gimel_0]^l_{\Phi}$ for certain $l \in \mathbb{N}$. Then there exists $\gimel' \in \mathcal{W}$ to such an extent that the $\mathbb{J} - \mathcal{PS}\{\gimel_n\}$ is $\Phi - \mathbb{J}\mathcal{WC}$ and converges to $\gimel'$.*

Proof. Let $\gimel_0 \in \mathcal{W}$, with the result that $\mathbb{J}\gimel_0 \in [\gimel_0]^l_{\Phi}$ for certain $l \in \mathbb{N}$. Adopting $\gimel_0$ be the starting point derived from $\mathbb{J}$ Picards sequence $\{\gimel_n\}$, $\exists$ a path $\{\hbar_j\}^l_{j=0}$, to such an extent that $\gimel_0 = \hbar_0$, $\mathbb{J}\gimel_0 = \hbar_1$ and $(\hbar_{j-1}, \hbar_j) \in \mathbb{E}(\Phi)$ for $j = 1, 2, \ldots, l$. Employing Equation (3.1) of Definition 3.2, we find that $(\mathbb{J}\hbar_{j-1}, \mathbb{J}\hbar_j) \in \mathbb{E}(\Phi)$ for $j = 1, 2, \ldots, l$. This indicates that $\{\mathbb{J}\hbar_j\}^l_{j=0}$ is a path from $\mathbb{J}\hbar_0 = \mathbb{J}\gimel_0 = \gimel_1$ to $\mathbb{J}\hbar_1 = \mathbb{J}^2\gimel_0 = \gimel_2$ characterized by a length l with the result that $\gimel_2 \in [\gimel_1]^l_{\Phi}$. Proceeding in this manner, we deduce that $\{\mathbb{J}^n\hbar_j\}^l_{j=0}$ is a path from $\mathbb{J}^n\gimel_0 = \mathbb{J}^n\hbar_0 = \gimel_n$ to $\mathbb{J}^n\gimel_1 = \mathbb{J}^n\mathbb{J}\hbar_0 = \gimel_{n+1}$ of length l and hence $\gimel_{n+1} \in [\gimel_n]^l_{\Phi}$ for all $m \in \mathbb{N}$. This confirms that $\{\gimel_n\}$ is a $\Phi - \mathbb{J}\mathcal{WC}$ sequence, which shows that

$$(\mathbb{J}^n_{j-1}\gimel, \mathbb{J}^n_j\gimel) \in \mathbb{E}(\Phi) \text{ for } j = 1, 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Then, with the help of inequality(3.2), we derive

$$\begin{aligned}
\mathcal{L}_g(\mathbb{J}^n_{j-1}\gimel, \mathbb{J}^n\gimel_j) &= \mathcal{L}_g(\mathbb{J}(\mathbb{J}^{n-1}\gimel_{j-1}), \mathbb{J}(\mathbb{J}^{n-1}\gimel_j)) \\
&\le \pi_1\mathcal{L}_g(\mathbb{J}^{n-1}\gimel_{j-1}, \mathbb{J}^{n-1}\gimel_j) + \pi_2\mathcal{L}_g(\mathbb{J}^{n-1}\gimel_{j-1}, \mathbb{J}(\mathbb{J}^{n-1}\gimel_{j-1})) \\
&\quad + \pi_3\mathcal{L}_g(\mathbb{J}^{n-1}\gimel_j, \mathbb{J}(\mathbb{J}^{n-1}\gimel_j)) = \pi_1\mathcal{L}_g(\mathbb{J}^{n-1}\gimel_{j-1}, \mathbb{J}^{n-1}\gimel_j) \\
&\quad + \pi_2\mathcal{L}_g(\mathbb{J}^{n-1}\gimel_{j-1}, \mathbb{J}^{n-1}\gimel_j) + \pi_3\mathcal{L}_g(\mathbb{J}^n\gimel_{j-1}, \mathbb{J}^n\gimel_j), \\
\mathcal{L}_g(\mathbb{J}^n\gimel_{j-1}, \mathbb{J}^n\gimel_j) &\le \left(\frac{\pi_1+\pi_2}{1-\pi_3}\right)\mathcal{L}_g(\mathbb{J}^{n-1}\gimel_{j-1}, \mathbb{J}^{n-1}\gimel_j).
\end{aligned}$$

Given that $\pi_1+\pi_2+\pi_3 < 1$, set $\frac{\pi_1+\pi_2}{1-\pi_3} = \eta$, where $\eta \in [0,1)$ the above inequality reduces to

$$\mathcal{L}_g(\mathbb{J}^n\gimel_{j-1}, \mathbb{J}^n\gimel_j) \le \eta d(\mathbb{J}^{n-1}\gimel_{j-1}, \mathbb{J}^{n-1}\gimel_j).$$

Repeating the same procedure until we arrive at

$$\mathcal{L}_g(\mathbb{J}^n\gimel_{j-1}, \mathbb{J}^n\gimel_j) \le \eta^n\mathcal{L}_g(\gimel_{j-1}, \gimel_j).$$

The next step is to prove that $\{\gimel_n\}$ is a Cauchy sequence, for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
\mathcal{L}_g(\gimel_n, \gimel_m) &\le \eth(\gimel_n, \gimel_{n+1})\mathcal{L}_g(\gimel_n, \gimel_{n+1}) + \eth(\gimel_{n+1}, \gimel_m)\mathcal{L}_g(\gimel_{n+1}, \gimel_m) \\
&\le \eth(\gimel_n, \gimel_{n+1})\mathcal{L}_g(\gimel_n, \gimel_{n+1}) + \eth(\gimel_{n+1}, \gimel_m)\eth(\gimel_{n+1}, \gimel_{n+2})\mathcal{L}_g(\gimel_{n+1}, \gimel_{n+2}) \\
&\quad + \eth(\gimel_{n+1}, \gimel_m)\eth(\gimel_{n+2}, \gimel_m)\mathcal{L}_g(\gimel_{n+2}, \gimel_m) \\
&\le \eth(\gimel_n, \gimel_{n+1})\mathcal{L}_g(\gimel_n, \gimel_{n+1}) + \eth(\gimel_{n+1}, \gimel_m)\eth(\gimel_{n+1}, \gimel_{n+2})\mathcal{L}_g(\gimel_{n+1}, \gimel_{n+2}) \\
&\quad + \eth(\gimel_{n+1}, \gimel_m)\eth(\gimel_{n+2}, \gimel_m)\eth(\gimel_{n+2}, \gimel_{n+3})\mathcal{L}_g(\gimel_{n+2}, \gimel_{n+3}) + \eth(\gimel_{n+1}, \gimel_m) \\
&\quad \times \eth(\gimel_{n+2}, \gimel_m)\eth(\gimel_{n+3}, \gimel_m)\mathcal{L}_g(\gimel_{n+3}, \gimel_m) \\
&\le \eth(\gimel_n, \gimel_{n+1})\mathcal{L}_g(\gimel_n, \gimel_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\eth(\gimel_j, \gimel_m)\right)\eth(\gimel_i, \gimel_{i+1})\mathcal{L}_g(\gimel_i, \gimel_{i+1}) \\
&\quad + \prod_{k=n+1}^{m-1}\eth(\gimel_k, \gimel_m)\mathcal{L}_g(\gimel_{m-1}, \gimel_m) \le \eth(\gimel_n, \gimel_{n+1})\eta^n\mathcal{L}_g(\gimel_0, \gimel_1) + \sum_{i=n+1}^{m-2} \\
&\quad \times \left(\prod_{j=n+1}^{i}\eth(\gimel_j, \gimel_m)\right)\eth(\gimel_i, \gimel_{i+1})\eta^i\mathcal{L}_g(\gimel_0, \gimel_1) + \prod_{k=n+1}^{m-1}\eth(\gimel_k, \gimel_m)\eta^{m-1}\mathcal{L}_g(\gimel_0, \gimel_1) \\
&\le \eth(\gimel_n, \gimel_{n+1})\eta^n\mathcal{L}_g(\gimel_0, \gimel_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\eth(\gimel_j, \gimel_m)\right)\eth(\gimel_i, \gimel_{i+1})\eta^i\mathcal{L}_g(\gimel_0, \gimel_1) \\
&\quad + \prod_{k=n+1}^{m-1}\eth(\gimel_k, \gimel_m)\eth(\gimel_{m-1}, \gimel_m)\eta^{m-1}\mathcal{L}_g(\gimel_0, \gimel_1) \\
&= \eth(\gimel_n, \gimel_{n+1})\eta^n\mathcal{L}_g(\gimel_0, \gimel_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\eth(\gimel_j, \gimel_m)\right)\eth(\gimel_i, \gimel_{i+1})\eta^i\mathcal{L}_g(\gimel_0, \gimel_1) \\
&\le \eth(\gimel_n, \gimel_{n+1})\eta^n\mathcal{L}_g(\gimel_0, \gimel_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\eth(\gimel_j, \gimel_m)\right)\eth(\gimel_i, \gimel_{i+1})\eta^i\mathcal{L}_g(\gimel_0, \gimel_1).
\end{aligned}$$

Assume that

$$S_l = \sum_{i=0}^{l} \left(\prod_{j=0}^{i} \eth(\gimel_j, \gimel_m) \right) \eth(\gimel_i, \gimel_{i+1}) \eta^i .$$

Then, we obtain

$$\mathcal{L}_g(\gimel_n, \gimel_m) \leq \mathcal{L}_g(\gimel_0, \gimel_1)[\eta^n \eth(\gimel_n, \gimel_{n+1}) + (S_{m-1} - S_n)]. \tag{3.3}$$

Using ratio test, we have

$$\pi_i = \prod_{j=0}^{i} \eth(\gimel_j, \gimel_m) \eth(\gimel_i, \gimel_{i+1}) \eta^i, \text{ where } \frac{\pi_{i+1}}{\pi_i} < \frac{1}{k},$$

By taking limit as $n, m \to \infty$, inequality (3.3) becomes

$$\lim_{n \to \infty} \mathcal{L}_g(\gimel_n, \gimel_m) = 0,$$

this indicates that $\{\gimel_n\}$ represents a Cauchy sequence in a $\Phi-$complete graphical-controlled metric type $(\mathcal{W}, \mathcal{L}_g)$. As a result, $\{\gimel_n\} \to \gimel^{'} \in \mathcal{W}$. Therefore, $(\gimel_n, \gimel^{'}) \in \mathbb{E}(\Phi)$ or $(\gimel^{'}, \gimel_n) \in \mathbb{E}(\Phi)$ for all $n > n_0$,

$$\lim_{n \to \infty} \mathcal{L}_g(\gimel_n, \gimel^{'}) = 0,$$

which demonstrates that $\gimel_n$ converges to $\gimel^{'}$. □

Definition 5. Let $\mathbb{J} : \mathcal{W} \to \mathcal{W}$ be a selfmap on a graphical-controlled metric type space $(\mathcal{W}, \mathcal{L}_g)$. We say that a triplet $(\mathcal{W}, \mathcal{L}_g, \mathbb{J})$ is congruent with the property $(\mathbb{P})$, if associated with two limiting value $\gimel' \in \mathcal{W}$ and $\hbar' \in \mathbb{J}(\mathcal{W})$, where $\{\gimel_n\}$ term wise connected Picards sequence as a result, we have $\gimel' = \hbar'$.

Theorem 2. *Assuming the conditions of Theorem (1) are met and furthermore the triplet* $(\mathcal{W}, \mathcal{L}_g, \mathbb{J})$ *manifests the property* $(\mathbb{P})$, *then* $\mathbb{J}$ *possess a fixed point.*

Proof. According to Theorem 1, which attests to the fact that the $\mathbb{J}-$Picards sequence with initial point $\gimel_0$ converges to both $\gimel'$ and Tv'. Since Φ is connected, thus $(\gimel^{'} \mathbb{P} \mathbb{J} \gimel^{'})_\Phi \in \mathbb{E}(\Phi)$ or $(\mathbb{J} \gimel^{'} P x^{'})_\Phi \in \mathbb{E}(\Phi)$, then we gain

$$\begin{aligned}
\mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'}) &\leq \eth(\gimel^{'}, \gimel_n) \mathcal{L}_g(\gimel^{'}, \gimel_n) + \eth(\gimel_n, \mathbb{J}\gimel^{'}) \mathcal{L}_g(\gimel_n, \mathbb{J}\gimel^{'}) \\
&= \eth(\gimel^{'}, \gimel_n) \mathcal{L}_g(\gimel^{'}, \gimel_n) + \eth(\gimel_n, \mathbb{J}\gimel^{'}) \mathcal{L}_g(\mathbb{J}\gimel_{n-1}, \mathbb{J}\gimel^{'}),
\end{aligned}$$

using inequality (3.2), we have

$$\begin{aligned}
\mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'}) &\leq \eth(\gimel^{'}, \gimel_n) \mathcal{L}_g(\gimel^{'}, \gimel_n) + \eth(\gimel_n, \mathbb{J}\gimel^{'})[\pi_1 \mathcal{L}_g(\gimel_{n-1}, \gimel^{'}) \\
&\quad + \pi_2 \mathcal{L}_g(\gimel_{n-1}, \mathbb{J}\gimel_{n-1}) + \pi_3 \mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'})] \\
&= \eth(\gimel^{'}, \gimel_n) \mathcal{L}_g(\gimel^{'}, \gimel_n) + \pi_1 \eth(\gimel_n, \mathbb{J}\gimel^{'}) \mathcal{L}_g(\gimel_{n-1}, \gimel^{'}) \\
&\quad + \pi_2 \eth(\gimel_n, \mathbb{J}\gimel^{'}) \mathcal{L}_g(\gimel_{n-1}, \gimel_n) + \pi_3 \eth(\gimel_n, \mathbb{J}\gimel^{'}) \mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'}).
\end{aligned}$$

After simple calculation, we derive

$$\mathcal{L}_g(\beth', \mathbb{J}\beth') \leq \left(\frac{\eth(\beth', \beth_n)\mathcal{L}_g(\beth', \beth_n) + \eth(\beth_n, \mathbb{J}\beth')[\pi_1 \mathcal{L}_g(\beth_{n-1}, \beth') + \pi_2 \mathcal{L}_g(\beth_{n-1}, \beth_n)]}{1 - \pi_3 \eth(\beth_n, \mathbb{J}\beth')} \right).$$

Taking $\lim\limits_{n\to\infty}$, we have

$$\mathcal{L}_g(\beth', \mathbb{J}\beth') = 0.$$

Hence, $\beth' = \mathbb{J}\beth'$ which demonstrate that $\beth'$ is fixed point of $\mathbb{J}$. □

Example 2. Let $\mathcal{W} = \{0, 4, 8, 12, 16, 20\}$ and $\eth : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ be a mapping such that

$$\eth(\beth, \hbar) = \frac{1}{\beth\hbar + 2} + \frac{1}{\beth\hbar + 3} + 1.9803.$$

Assume that $\mathcal{L}_g : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ is defined as:

$$\mathcal{L}_g(\beth, \hbar) = \begin{cases} |\beth - \hbar|^2, & \text{if } \beth \neq \hbar, \\ 0, & \text{if } \beth = \hbar. \end{cases}$$

The vertex set $\mathcal{W} = \mathbb{V}(\Phi)$ and edge set is designed as

$$\mathbb{E}(\Phi) = \Delta \cup \left\{ \begin{array}{c} (0,4), (0,8), (0,12), (0,16), (0,20), (4,8), (4,12), \\ (4,16), (4,20), (8,12), (8,20), (12,16), (16,20) \end{array} \right\},$$

as shown in Figure 2.

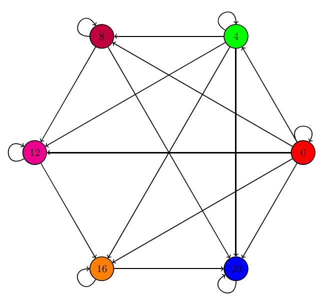

Figure 2. Related graph with vertex set $\mathbb{V}(\Phi) = \mathcal{W}$.

Clearly, $(\mathcal{W}, \mathcal{L}_g)$ is graphical-controlled metric type space but not controlled metric type space, since

$$\begin{aligned} \mathcal{L}_g(12, 20) &\leq \eth(12, 16)\mathcal{L}_g(12, 16) + \eth(16, 20)\mathcal{L}_g(16, 20), \\ 64 &\nleq 63.63, \\ \mathcal{L}_g(8, 16) &\leq \eth(8, 12)\mathcal{L}_g(8, 12) + \eth(12, 16)\mathcal{L}_g(12, 16), \\ 64 &\nleq 63.859. \end{aligned}$$

For contraction, define a mapping $\mathbb{J} : \mathcal{W} \to \mathcal{W}$ as

$$\mathbb{J}\daleth = \begin{cases} 0, & \text{for } \daleth \in \{0, 4, 8, 12\}, \\ 4, & \text{for } \daleth \in \{16, 20\}. \end{cases}$$

Case (i): For $\daleth = 0$ and $\hbar = 16$, we have

$$\begin{aligned} \mathcal{L}_g(\mathbb{J}0, \mathbb{J}16) &\leq \pi_1 \mathcal{L}_g(0, 16) + \pi_2 \mathcal{L}_g(0, \mathbb{J}0) + \pi_3 \mathcal{L}_g(16, \mathbb{J}16), \\ 16 &\leq \pi_1(256) + \pi_2(0) + \pi_3(144). \end{aligned}$$

Case (ii): When $\daleth = 0$ and $\hbar = 20$, we get

$$\begin{aligned} \mathcal{L}_g(\mathbb{J}0, \mathbb{J}20) &\leq \pi_1 \mathcal{L}_g(0, 20) + \pi_2 \mathcal{L}_g(0, \mathbb{J}0) + \pi_3 \mathcal{L}_g(20, \mathbb{J}20), \\ 16 &\leq \pi_1(400) + \pi_2(0) + \pi_3(256). \end{aligned}$$

Case (iii): If $\daleth = 4$ and $\hbar = 16$, then

$$\begin{aligned} \mathcal{L}_g(\mathbb{J}4, \mathbb{J}16) &\leq \pi_1 \mathcal{L}_g(4, 16) + \pi_2 \mathcal{L}_g(4, \mathbb{J}4) + \pi_3 \mathcal{L}_g(16, \mathbb{J}16), \\ 16 &\leq \pi_1(144) + \pi_2(16) + \pi_3(144). \end{aligned}$$

Case (iv): For $\daleth = 4$ and $\hbar = 20$, then

$$\begin{aligned} \mathcal{L}_g(\mathbb{J}4, \mathbb{J}20) &\leq \pi_1 \mathcal{L}_g(4, 20) + \pi_2 \mathcal{L}_g(4, \mathbb{J}4) + \pi_3 \mathcal{L}_g(20, \mathbb{J}20), \\ 16 &\leq \pi_1(256) + \pi_2(16) + \pi_3(256). \end{aligned}$$

Case (v): If $\daleth = 8$ and $\hbar = 20$, then

$$\begin{aligned} \mathcal{L}_g(\mathbb{J}8, \mathbb{J}20) &\leq \pi_1 \mathcal{L}_g(8, 20) + \pi_2 \mathcal{L}_g(8, \mathbb{J}8) + \pi_3 \mathcal{L}_g(20, \mathbb{J}20), \\ 16 &\leq \pi_1(144) + \pi_2(64) + \pi_3(256). \end{aligned}$$

Case (vi): For $\daleth = 12$ and $\hbar = 16$, we have

$$\begin{aligned} \mathcal{L}_g(\mathbb{J}12, \mathbb{J}16) &\leq \pi_1 \mathcal{L}_g(12, 16) + \pi_2 \mathcal{L}_g(12, \mathbb{J}12) + \pi_3 \mathcal{L}_g(16, \mathbb{J}16), \\ 1 &\leq \pi_1(16) + \pi_2(144) + \pi_3(144). \end{aligned}$$

Thus all the cases are satisfied for $\pi_1 = \frac{1}{15}$, $\pi_2 = \frac{1}{18}$ and $\pi_3 = \frac{1}{19}$, where $\pi_1 + \pi_2 + \pi_3 < 1$. Consequently, for $\daleth = 8$ and $\hbar = 16$, we obtain

$$\begin{aligned} \mathcal{L}_g(\mathbb{J}8, \mathbb{J}16) &\leq \pi_1 \mathcal{L}_g(8, 16) + \pi_2 \mathcal{L}_g(8, \mathbb{J}8) + \pi_3 \mathcal{L}_g(16, \mathbb{J}16), \\ 16 &\leq \pi_1(64) + \pi_2(64) + \pi_3(144), \\ 16 &\nleq 15.4011, \end{aligned}$$

which shows that above contraction is "*graphically Reich contraction but not a Reich contraction*". Therefore, all the terms and conditions of Theorem 1 are satisfied and 0 is the unique fixed point of the mapping $\mathbb{J}$.

DEFINITION 6. We say $\mathbb{J} : \mathcal{W} \to \mathcal{W}$ is a graphical-Ćirić contraction on $(\mathcal{W}, \mathcal{L}_g)$ linked with a graph Φ encompassing all loops if the following properties are satisfied:

1. For all $\daleth, \hbar \in \mathcal{W}$

$$\text{if } (\daleth, \hbar) \in \mathbb{E}(\Phi) \Longrightarrow (\mathbb{J}\daleth, \mathbb{J}\hbar) \in \mathbb{E}(\Phi), \tag{3.4}$$

   i.e., $\mathbb{J}$ also preserves edges of Φ.

2. There exists a non-negative constant $\eta < 1$ such that

$$\mathcal{L}_g(\mathbb{J}\daleth, \mathbb{J}\hbar) \leq \eta \max\{\mathcal{L}_g(\daleth, \hbar), \mathcal{L}_g(\daleth, \mathbb{J}\daleth), \mathcal{L}_g(\hbar, \mathbb{J}\hbar), \mathcal{L}_g(\daleth, \mathbb{J}\hbar), \mathcal{L}_g(\hbar, \mathbb{J}\daleth)\}, \tag{3.5}$$

   for all $\daleth, \hbar \in \mathcal{W}$ with $(\daleth, \hbar) \in \mathbb{E}(\Phi)$.

Theorem 3. *Let $\mathbb{J} : \mathcal{W} \to \mathcal{W}$ be a graphical Ćirić contraction on a $\Phi-$complete graphical controlled metric type space $(\mathcal{W}, \mathcal{L}_g)$. Assume that, the graph Φ demonstrates the property $(\mathbb{P})$, $\daleth_0 \in \mathcal{W}$ with $\mathbb{J}\daleth_0 \in [\daleth_0]^l_\Phi$ for certain $l \in \mathbb{N}$, then there exists $\daleth' \in \mathcal{W}$ to such an extent that the $\mathbb{J} - \mathcal{PS}\{\daleth_n\}$ is $\Phi - \mathbb{J}\mathcal{WC}$ and converges to $\daleth'$.*

Proof. Let $\daleth_0 \in \mathcal{W}$, with the result that $\mathbb{J}\daleth_0 \in [\daleth_0]^l_\Phi$ for certain $l \in \mathbb{N}$. Adopting $\daleth_0$ be the starting point derived from $\mathbb{J}$ Picards sequence $\{\daleth_n\}$, $\exists$ a path $\{\hbar_j\}^l_{j=0}$, to such an extent that $\daleth_0 = \hbar_0$, $\mathbb{J}\daleth_0 = \hbar_1$ and $(\hbar_{j-1}, \hbar_j) \in \mathbb{E}(\Phi)$ for $j = 1, 2, \ldots, l$. Employing reference (3.4), we find that $(\mathbb{J}\hbar_{j-1}, \mathbb{J}\hbar_j) \in \mathbb{E}(\Phi)$ for $j = 1, 2, \ldots, l$. This indicates that $\{\mathbb{J}\hbar_j\}^l_{j=0}$ is a path from $\mathbb{J}\hbar_0 = \mathbb{J}\daleth_0 = \daleth_1$ to $\mathbb{J}\hbar_1 = \mathbb{J}^2\daleth_0 = \daleth_2$ characterized by a length l with the result that $\daleth_2 \in [\daleth_1]^l_\Phi$. Proceeding in this manner, we deduce that $\{\mathbb{J}^n\hbar_j\}^l_{j=0}$ is a path from $\mathbb{J}^n\daleth_0 = \mathbb{J}^n\hbar_0 = \daleth_n$ to $\mathbb{J}^n\daleth_1 = \mathbb{J}^n\mathbb{J}\hbar_0 = \daleth_{n+1}$ of length l and hence $\daleth_{n+1} \in [\daleth_n]^l_\Phi$ for all $m \in \mathbb{N}$. This confirms that $\{\daleth_n\}$ is a $\Phi - \mathbb{J}\mathcal{WC}$ sequence, which shows that

$$(\mathbb{J}^n_{j-1}\daleth, \mathbb{J}^n_j\daleth) \in \mathbb{E}(\Phi) \text{ for } j = 1, 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Then by using (3.5), we obtain

$$\begin{aligned}
\mathcal{L}_g(\mathbb{J}^n_{j-1}\daleth, \mathbb{J}^n\daleth_j) &= \mathcal{L}_g(\mathbb{J}(\mathbb{J}^{n-1}\daleth_{j-1}), \mathbb{J}(\mathbb{J}^{n-1}\daleth_j)) \\
&\leq \eta \max\{\mathcal{L}_g(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}^{n-1}\daleth_j), \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}(\mathbb{J}^{n-1}\daleth_{j-1})), \\
&\quad \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_j, \mathbb{J}(\mathbb{J}^{n-1}\daleth_j)), \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}(\mathbb{J}^{n-1}\daleth_j)), \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_j, \mathbb{J}(\mathbb{J}^{n-1}\daleth_{j-1}))\} \\
&= \eta \max\{\mathcal{L}_g(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}^{n-1}\daleth_j), \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}^{n-1}\daleth_j), \\
&\quad \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_j, \mathbb{J}^{n-1}\daleth_{j+1}), \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}^{n-1}\daleth_{j+1}), \mathcal{L}_g(\mathbb{J}^{n-1}\daleth_j, \mathbb{J}^{n-1}\daleth_j)\}.
\end{aligned}$$

If we choose maximum other than $\mathcal{L}_g(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}^{n-1}\daleth_j)$, it will be a contradiction. So, we achieve

$$\mathcal{L}_g(\mathbb{J}^n_{j-1}\daleth, \mathbb{J}^n\daleth_j) \leq \eta d(\mathbb{J}^{n-1}\daleth_{j-1}, \mathbb{J}^{n-1}\daleth_j).$$

Continuing in the similar fashion, we infer

$$\mathcal{L}_g(\mathbb{J}^n\daleth_{j-1}, \mathbb{J}^n\daleth_j) \leq \eta^n \mathcal{L}_g(\daleth_{j-1}, \daleth_j).$$

Our next step is to prove that $\{\daleth_n\}$ is a Cauchy sequence, for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
&\mathcal{L}_g(\daleth_n, \daleth_m) \leq \eth(\daleth_n, \daleth_{n+1})\mathcal{L}_g(\daleth_n, \daleth_{n+1}) + \eth(\daleth_{n+1}, \daleth_m)\mathcal{L}_g(\daleth_{n+1}, \daleth_m) \\
&\leq \eth(\daleth_n, \daleth_{n+1})\mathcal{L}_g(\daleth_n, \daleth_{n+1}) + \eth(\daleth_{n+1}, \daleth_m)\eth(\daleth_{n+1}, \daleth_{n+2})\mathcal{L}_g(\daleth_{n+1}, \daleth_{n+2}) \\
&\quad + \eth(\daleth_{n+1}, \daleth_m)\eth(\daleth_{n+2}, \daleth_m)\mathcal{L}_g(\daleth_{n+2}, \daleth_m) \\
&\leq \eth(\daleth_n, \daleth_{n+1})\mathcal{L}_g(\daleth_n, \daleth_{n+1}) + \eth(\daleth_{n+1}, \daleth_m)\eth(\daleth_{n+1}, \daleth_{n+2})\mathcal{L}_g(\daleth_{n+1}, \daleth_{n+2}) \\
&\quad + \eth(\daleth_{n+1}, \daleth_m)\eth(\daleth_{n+2}, \daleth_m)\eth(\daleth_{n+2}, \daleth_{n+3})\mathcal{L}_g(\daleth_{n+2}, \daleth_{n+3}) + \eth(\daleth_{n+1}, \daleth_m) \\
&\qquad \times \eth(\daleth_{n+2}, \daleth_m)\eth(\daleth_{n+3}, \daleth_m)\mathcal{L}_g(\daleth_{n+3}, \daleth_m) \\
&\leq \eth(\daleth_n, \daleth_{n+1})\mathcal{L}_g(\daleth_n, \daleth_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\eth(\daleth_j, \daleth_m)\right)\eth(\daleth_i, \daleth_{i+1})\mathcal{L}_g(\daleth_i, \daleth_{i+1}) \\
&+ \prod_{k=n+1}^{m-1}\eth(\daleth_k, \daleth_m)\mathcal{L}_g(\daleth_{m-1}, \daleth_m) \leq \eth(\daleth_n, \daleth_{n+1})\eta^n\mathcal{L}_g(\daleth_0, \daleth_1) + \sum_{i=n+1}^{m-2} \\
&\times \left(\prod_{j=n+1}^{i}\eth(\daleth_j, \daleth_m)\right)\eth(\daleth_i, \daleth_{i+1})\eta^i\mathcal{L}_g(\daleth_0, \daleth_1) + \prod_{k=n+1}^{m-1}\eth(\daleth_k, \daleth_m)\eta^{m-1}\mathcal{L}_g(\daleth_0, \daleth_1) \\
&\leq \eth(\daleth_n, \daleth_{n+1})\eta^n\mathcal{L}_g(\daleth_0, \daleth_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\eth(\daleth_j, \daleth_m)\right)\eth(\daleth_i, \daleth_{i+1})\eta^i\mathcal{L}_g(\daleth_0, \daleth_1) \\
&\quad + \prod_{k=n+1}^{m-1}\eth(\daleth_k, \daleth_m)\eth(\daleth_{m-1}, \daleth_m)\eta^{m-1}\mathcal{L}_g(\daleth_0, \daleth_1) \\
&= \eth(\daleth_n, \daleth_{n+1})\eta^n\mathcal{L}_g(\daleth_0, \daleth_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\eth(\daleth_j, \daleth_m)\right)\eth(\daleth_i, \daleth_{i+1})\eta^i\mathcal{L}_g(\daleth_0, \daleth_1) \\
&\leq \eth(\daleth_n, \daleth_{n+1})\eta^n\mathcal{L}_g(\daleth_0, \daleth_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\eth(\daleth_j, \daleth_m)\right)\eth(\daleth_i, \daleth_{i+1})\eta^i\mathcal{L}_g(\daleth_0, \daleth_1).
\end{aligned}$$

Assume that

$$S_l = \sum_{i=0}^{l}\left(\prod_{j=0}^{i}\eth(\daleth_j, \daleth_m)\right)\eth(\daleth_i, \daleth_{i+1})\eta^i.$$

Then, we obtain

$$\mathcal{L}_g(\daleth_n, \daleth_m) \leq \mathcal{L}_g(\daleth_0, \daleth_1)[\eta^n\eth(\daleth_n, \daleth_{n+1}) + (S_{m-1} - S_n)]. \tag{3.6}$$

Using ratio test, we have

$$\pi_i = \prod_{j=0}^{i}\eth(\daleth_j, \daleth_m)\eth(\daleth_i, \daleth_{i+1})\eta^i, \text{ where } \frac{\pi_{i+1}}{\pi_i} < \frac{1}{k},$$

By taking limit as $n, m \to \infty$, inequality (3.6) becomes

$$\lim_{n\to\infty}\mathcal{L}_g(\daleth_n, \daleth_m) = 0,$$

this indicates that $\{\gimel_n\}$ represents a Cauchy sequence in a $\Phi-$complete graphical-controlled metric type $(\mathcal{W}, \mathcal{L}_g)$. As a result, $\{\gimel_n\} \to \gimel^{'} \in \mathcal{W}$. Therefore, $(\gimel_n, \gimel^{'}) \in \mathbb{E}(\Phi)$ or $(\gimel^{'}, \gimel_n) \in \mathbb{E}(\Phi)$ for all $n > n_0$,

$$\lim_{n\to\infty} \mathcal{L}_g(\gimel_n, \gimel^{'}) = 0,$$

which demonstrates that $\gimel_n$ converges to $\gimel^{'}$. □

Theorem 4. *If the assumptions of Theorem 3 are valid and the triplet* $(\mathcal{W}, \mathcal{L}_g, \mathbb{J})$ *has the property* $(\mathbb{P})$*, then* $\mathbb{J}$ *has a fixed point.*

Proof. Theorem 3 validates that the $\mathbb{J}-$Picards sequence with initial point $\gimel_0$ converges to both $\gimel'$ and Tv'. Since Φ is connected so $(\gimel^{'}\mathbb{P}\mathbb{J}\gimel^{'})_{\Phi} \in \mathbb{E}(\Phi)$ or $(\mathbb{J}\gimel^{'}Px^{'})_{\Phi} \in \mathbb{E}(\Phi)$, then we have

$$\begin{aligned} \mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'}) &\leq \eth(\gimel^{'}, \gimel_n)\mathcal{L}_g(\gimel^{'}, \gimel_n) + \eth(\gimel_n, \mathbb{J}\gimel^{'})\mathcal{L}_g(\gimel_n, \mathbb{J}\gimel^{'}) \\ &= \eth(\gimel^{'}, \gimel_n)\mathcal{L}_g(\gimel^{'}, \gimel_n) + \eth(\gimel_n, \mathbb{J}\gimel^{'})\mathcal{L}_g(\mathbb{J}\gimel_{n-1}, \mathbb{J}\gimel^{'}), \end{aligned}$$

using inequality (3.5), we achieve

$$\begin{aligned} &\mathcal{L}_g(\mathbb{J}\gimel_{n-1}, \mathbb{J}\gimel^{'}) \leq \eta \max\{\mathcal{L}_g(\gimel_{n-1}, \gimel^{'}), \mathcal{L}_g(\gimel_{n-1}, \mathbb{J}\gimel_{n-1}), \mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'}), \mathcal{L}_g(\gimel_{n-1}, \mathbb{J}\gimel^{'}), \\ &\quad \mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel_{n-1})\} \leq \eta d(\gimel_{n-1}, \gimel^{'}). \end{aligned}$$

After substituting, we conclude

$$\mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'}) \leq \eth(\gimel^{'}, \gimel_n)\mathcal{L}_g(\gimel^{'}, \gimel_n) + \eta\varpi(\gimel_n, \mathbb{J}\gimel^{'})\mathcal{L}_g(\gimel_{n-1}, \gimel^{'}).$$

Taking $\lim\limits_{n\to\infty}$, we have

$$\mathcal{L}_g(\gimel^{'}, \mathbb{J}\gimel^{'}) = 0.$$

Hence, $\gimel^{'} = \mathbb{J}\gimel^{'}$ which demonstrate that $\gimel^{'}$ is fixed point of $\mathbb{J}$. □

Example 3. Let $\mathcal{W} = \{0, 1, 2, 4, 6, 8, 10, 12\}$ and $\eth : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ be a mapping such that

$$\eth(\gimel, \hbar) = \frac{1}{\gimel\hbar + 2} + \frac{1}{\gimel\hbar + 3} + 1.95.$$

Assume that $\mathcal{L}_g : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ is defined as:

$$\mathcal{L}_g(\gimel, \hbar) = \begin{cases} |\gimel - \hbar|^2, & \text{if } \gimel \neq \hbar, \\ 0, & \text{if } \gimel = \hbar. \end{cases}$$

The vertex set $\mathcal{W} = \mathbb{V}(\Phi)$ and the edge set is designed in the following way

$$\mathbb{E}(\Phi) = \Delta \cup \left\{ \begin{array}{c} (0,1), (0,2), (0,4), (0,6), (0,8), (0,10), (0,12), (1,2), \\ (1,4), (1,6), (1,8), (1,10), (1,12), (2,4), (2,6), (2,8), (2,10), \\ (2,12), (4,6), (4,8), (4,10), (6,8), (6,12), (8,10), (10,12), \end{array} \right\}$$

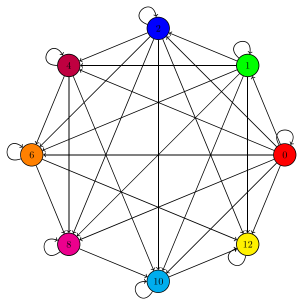

Figure 3. Graph with vertex set $\mathbb{V}(\Phi) = \{0, 1, 2, 4, 6, 8, 10, 12\}$.

as shown in Figure 3. *Clearly, $(\mathcal{W}, \mathcal{L}_g)$ is graphically-controlled metric-type space but not a controlled metric-type space.* For the following pair of vertices, we have

$$\begin{aligned}
\mathcal{L}_g(4,12) &\leq \eth(4,8)\mathcal{L}_g(4,8) + \eth(8,12)\mathcal{L}_g(8,12), \quad 64 \nleq 63.653,\\
\mathcal{L}_g(6,10) &\leq \eth(6,8)\mathcal{L}_g(6,8) + \eth(8,10)\mathcal{L}_g(8,10), \quad 16 \nleq 15.855,\\
\mathcal{L}_g(8,12) &\leq \eth(8,10)\mathcal{L}_g(8,10) + \eth(10,12)\mathcal{L}_g(10,12), \quad 16 \nleq 15.762.
\end{aligned}$$

For contractive conditions, define a mapping $\mathbb{J} : \mathcal{W} \to \mathcal{W}$ as

$$\mathbb{J}\gimel = \begin{cases} 0, & \text{for } \gimel \in \{4, 8\},\\ 1, & \text{for } \gimel = 1,\\ 10 & \text{for } \gimel = 12. \end{cases}$$

Case (i): For $\gimel = 1$ and $\hbar = 4$, we have

$$\begin{aligned}
\mathcal{L}_g(\mathbb{J}1, \mathbb{J}4) &\leq \eta \max\{\mathcal{L}_g(1,4), \mathcal{L}_g(1,\mathbb{J}1), \mathcal{L}_g(4,\mathbb{J}4), \mathcal{L}_g(1,\mathbb{J}4), \mathcal{L}_g(4,\mathbb{J}1)\},\\
1 &\leq \eta \max\{9, 0, 16, 1, 9\}, \quad 1 \leq \eta(16).
\end{aligned}$$

Case (ii): If $\gimel = 1$ and $\hbar = 8$, we gain

$$\begin{aligned}
\mathcal{L}_g(\mathbb{J}1, \mathbb{J}8) &\leq \eta \max\{\mathcal{L}_g(1,8), \mathcal{L}_g(1,\mathbb{J}1), \mathcal{L}_g(8,\mathbb{J}8), \mathcal{L}_g(1,\mathbb{J}8), \mathcal{L}_g(8,\mathbb{J}1)\},\\
1 &\leq \eta \max\{49, 0, 64, 1, 49\}, \quad 1 \leq \eta(64).
\end{aligned}$$

Case (iii): When $\gimel = 1$ and $\hbar = 12$, we achieve

$$\begin{aligned}
\mathcal{L}_g(\mathbb{J}1, \mathbb{J}12) &\leq \eta \max\{\mathcal{L}_g(1,12), \mathcal{L}_g(1,\mathbb{J}1), \mathcal{L}_g(12,\mathbb{J}12), \mathcal{L}_g(1,\mathbb{J}12), \mathcal{L}_g(12,\mathbb{J}1)\},\\
81 &\leq \eta \max\{121, 0, 4, 81, 121\}, \quad 81 \leq \eta(121).
\end{aligned}$$

all the cases are satisfied for $\eta = 0.69 \in (0, 1)$. Consequently, for $\gimel = 4$ and $\hbar = 12$,

$$\begin{aligned}
\mathcal{L}_g(\mathbb{J}4, \mathbb{J}12) &\leq \eta \max\{\mathcal{L}_g(4,12), \mathcal{L}_g(4,\mathbb{J}4), \mathcal{L}_g(12,\mathbb{J}12), \mathcal{L}_g(4,\mathbb{J}12), \mathcal{L}_g(12,\mathbb{J}4)\},\\
100 &\leq \eta \max\{64, 16, 4, 36, 144\}, \quad 100 \nleq \eta(144).
\end{aligned}$$

Similarly, if $\gimel = 8$ and $\hbar = 12$,

$$\mathcal{L}_g(\mathbb{J}8, \mathbb{J}12) \leq \eta \max\{\mathcal{L}_g(8,12), \mathcal{L}_g(8,\mathbb{J}8), \mathcal{L}_g(12,\mathbb{J}12), \mathcal{L}_g(8,\mathbb{J}12), \mathcal{L}_g(12,\mathbb{J}8)\},$$
$$100 \leq \eta \max\{16, 64, 4, 4, 144\}, \quad 100 \nleq \eta(144),$$

which shows that above contraction is a graphically Ciric contraction for $\eta = 0.69$ but not a Ciric contraction. Therefore, all the terms and conditions of Theorem 3 are satisfied and 1 is the unique fixed point of the mapping $\mathbb{J}$.

4 Application to Helmholtz phenomena with mixed BVP

In fixed-point analysis, different applications have been investigated across multiple abstract spaces. However, there is relatively limited work on the application of fixed-point theory within graphical systems. In this section, we focus on applying graphical fixed-point theory to study its applicability to Helmholtz phenomena with mixed boundary value problems (BVPs). For deeper insights into graphical structures and their larger applications, interested scholars may refer to [5, 14, 32].
The Helmholtz equation characterizes steady-state wave events in acoustics, electromagnetics, and quantum mechanics. We encounter a mixed boundary value problem (BVP) when we apply different boundary conditions (Dirichlet, Neumann, or Robin) to different parts of the domain boundary. These problems show up in waveguides, scattering phenomena, and Helmholtz resonators, where boundary conditions have a big effect on how waves behave. The method of separation of variables can yield analytical solutions in simple geometries, while Green's functions offer integral representations. Numerical techniques such as the Finite Element Method (FEM) and the Boundary Element Method (BEM) address intricate scenarios. The interaction of waves with boundaries results in resonance, diffraction, and mode-shaping phenomena. Helmholtz effects are very important in mixed boundary value problems for making wave propagation better and controlling resonances. Solutions are frequently articulated using eigenfunctions, such as Bessel functions or Fourier series. Comprehending these effects aids in the design of efficient acoustic, optical, and electromagnetic systems.

Helmholtz's problem provides us with a more rich understanding of the functioning of electricity and magnetism in wires and boxes. It also aids to understand the behaviour of these structures when they are vibrating. By comparing our results with those real-life examples, we can demonstrate that these are significant and useful for researchers in the field of fixed point theory. The Helmholtz condition with mixed boundary conditions,

$$\begin{aligned} &u''(\gimel) + \lambda u(\gimel) = f(\gimel), \\ &u(0) = 0, \quad u'(1) = 0, \end{aligned} \tag{4.1}$$

addresses a strong structure for analyzing standing wave peculiarities across different actual settings. The presence of the term $\frac{\pi^2}{4}$ as the eigenvalue λ means that we are managing a basic frequency or resonant mode, portrayed by a wavenumber $k = \frac{\pi}{2}$. This implies that the frequency λ is two times the length of the interval, fitting precisely into the domain $[0, 1]$ with a half-wavelength.

In acoustics, such an arrangement is frequently used to portray the way of behaving of sound waves in a cylinder with one end shut and the opposite end open. At $t = 0$, the Dirichlet boundary condition $u(0) = 0$, suggests a pressure node, where the displacement of air particles is zero. At $t = 1$, the Neumann boundary condition $u\prime(1) = 0$, shows a pressure antinode, where the gradient (and thus the speed of air particles) is zero. This setup is crucial for understanding how sound waves resound inside instruments like organ pipes or in building acoustics, where the plan of spaces can improve or dampen specific frequencies.

In the domain of electromagnetics, the condition models the circulation of electric or magnetic fields in waveguides or resonant cavities. The Boundary condition $u(0) = 0$ could address an impeccably conducting surface where the electric field should be zero, while $u\prime(1) = 0$ could mean where the magnetic field is tangentially zero, aligned with ideal magnetic conductors. This is basic in planning microwave cavities or optical resonators, where explicit methods of the electromagnetic field are supported to expand effectiveness and reduce losses.

Mechanical vibrations additionally utilize this condition to depict the elements of vibrating structures, like pillars or strings. Here, $u(0) = 0$ signifies a fixed end with no displacement, typical of clamped or pinned conditions, while $u\prime(1) = 0$, denotes a free end where the bending moment is absent, similar to a cantilever beam. Understanding these vibrational modes is fundamental in primary designing and material science to anticipate and control reverberation, which can result in either optimal performance in applications such as musical instruments and precision tools or undesirable oscillations that may lead to structural failure.

The fundamental or resonant mode depicted by this Helmholtz condition is significant in designing. It guarantees that energy is stored and moved proficiently, which is key for planning gadgets and designs that oversee wave development. This condition assists us with understanding how waves act with specific boundary conditions, which is significant for working on sound quality, upgrading electromagnetic fields, and keeping mechanical designs stable.
Let f be a continuous function from $[0, 1] \times \mathbb{R}^2 \to \mathbb{R}$, as investigated in 4.1. The problem under consideration has a solution $\daleth$ in the space $\mathcal{W} = \mathcal{C}[[0, 1], \mathbb{R}]$, represented by the integral equation

$$\daleth(\mathsf{t}) = \int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta)\mathsf{f}(\zeta, t, \daleth(t))dt, \quad \mathsf{t} \in [0, 1] \tag{4.2}$$

and $\dot{G}(\mathsf{t}, \zeta)$ is Green's function for any $\lambda, \zeta > 0$ expressed as

$$\dot{\mathsf{G}}(\mathsf{t}, \zeta) = \begin{cases} \frac{\sin\left(\sqrt{\lambda}t\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)} \sin\left(\sqrt{\lambda}(1-\zeta)\right), & 0 \le t \le \zeta, \\ \frac{\sin\left(\sqrt{\lambda}\zeta\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)} \sin\left(\sqrt{\lambda}(1-t)\right), & \zeta \le t \le 1. \end{cases} \tag{4.3}$$

Define a graphical-controlled metric type space $\mathcal{L}_g : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ as:

$$\mathcal{L}_g(\beth(\mathsf{t}), \hbar(\mathsf{t})) = \sup_{\mathsf{t}\in[0,1]} \left|\beth(\mathsf{t}) - \hbar(\mathsf{t})\right|^2.$$

Let $\eth : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ defined as $\eth(\beth(\mathsf{t}), \hbar(\mathsf{t})) = \exp\{\beth(\mathsf{t}) + \hbar(\mathsf{t}) + 1\}$ for all $\beth, \hbar \in \mathcal{W}$ and $\mathsf{t} \in [\pi, b]$. Clearly, $(\mathcal{W}, \mathcal{L}_g)$ is complete graphical-controlled metric type space. Before going to prove, we have to choose particular λ and ζ, for this we integrating the (4.3) over the domain of $[0, 1]$ in such a way

$$\int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta) = \frac{1}{\lambda \sin\left(\sqrt{\lambda}\right)} \left[\begin{array}{c} \sin\left(\sqrt{\lambda}(1-\zeta)\right)\left(1 - \cos\left(\sqrt{\lambda}\zeta\right)\right) \\ + \sin\left(\sqrt{\lambda}\zeta\right)\left(1 - \cos\left(\sqrt{\lambda}(1-\zeta)\right)\right). \end{array} \right]$$

For choosing specific $\lambda = \frac{\pi^2}{4}$ and $\zeta = \frac{1}{2}$. Furthermore, we also discuss their physical significance in the above paragraphs, after simple calculation, we achive

$$\int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta)dt = 4(\sqrt{2} - 1)/\pi^2.$$

A forthcoming theorem offers sufficient conditions for the existence and uniqueness of a solution to the problem 4.1.

Theorem 5. *Assume that the following conditions hold:*

1. *For each* $\mathsf{t} \in [0, 1]$ *and* $\beth, \hbar \in \mathcal{W}$*, we have*

$$\left|\mathsf{f}(\zeta, t, \beth(t)) - \mathsf{f}(\zeta, t, \hbar(t))\right|^2 \leq \left|\beth(t) - \hbar(t)\right|^2.$$

Consequently, the existence of a solution to the integral equation in 4.2 yields a solution to the Helmholtz problem described in 4.1.

Proof. Define an operator $\mathbb{J} : \mathcal{W} \to \mathcal{W}$, given by$\mathbb{J}\beth(\mathsf{t}) = \int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta)\mathsf{f}(\zeta, t, \beth(t))dt$, obviously $\mathbb{J}$ is well defined. Now, $(\beth, \hbar) \in \mathbb{E}(\Phi)$ for all $\beth, \hbar \in \mathcal{W}$, we have

$$\begin{aligned} \left|(\mathbb{J}\beth(\mathsf{t}) - \mathbb{J}\hbar(\mathsf{t})\right|^2 &= \left|\int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta)\mathsf{f}(\zeta, t, \beth(t))dt - \int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta)\mathsf{f}(\zeta, t, \hbar(t))dt\right|^2 \\ &= \left|\left(\int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta)dt\right)\left|\mathsf{f}(\zeta, t, \beth(t)) - \mathsf{f}(\zeta, t, \hbar(t))\right|\right|^2 \\ &\leq \left(\int_0^1 \dot{\mathsf{G}}(\mathsf{t}, \zeta)dt\right)^2 \left|\mathsf{f}(\zeta, t, \beth(t)) - \mathsf{f}(\zeta, t, \hbar(t))\right|^2 \\ &\leq \left|\mathsf{f}(\zeta, t, \beth(t)) - \mathsf{f}(\zeta, t, \hbar(t))\right|^2 \left(\frac{4(\sqrt{2}-1)}{\pi^2}\right)^2 \\ &\leq \left|\beth(t) - \hbar(t)\right|^2 \left(\frac{16(\sqrt{2}-1)^2}{\pi^4}\right) \end{aligned}$$

If $\frac{16(\sqrt{2}-1)^2}{\pi^4} = \eta \in (0, 1)$, above inequality can be expressed as

$$\left|(\mathbb{J}\beth(\mathsf{t}) - \mathbb{J}\hbar(\mathsf{t})\right|^2 \leq \eta \left|\beth(t) - \hbar(t)\right|^2.$$

From given hypothesis, we conclude that

$$\sup_{t\in[0,1]} \left|\mathbb{J}\beth(t) - \mathbb{J}\hbar(t)\right|^2 \leq \eta \sup_{t\in[0,1]} \left|\beth(t) - \hbar(t)\right|^2.$$

It can easily be observed that$\mathcal{L}_g(\mathbb{J}\beth, \mathbb{J}\hbar) \leq \eta d(\beth, \hbar)$, The aforementioned inequality can be presented in the following way

$$\mathcal{L}_g(\mathbb{J}\beth, \mathbb{J}\hbar) \leq \eta \max\{\mathcal{L}_g(\beth, \hbar), \mathcal{L}_g(\beth, \mathbb{J}\beth), \mathcal{L}_g(\hbar, \mathbb{J}\hbar), \mathcal{L}_g(\beth, \mathbb{J}\hbar), \mathcal{L}_g(\hbar, \mathbb{J}\beth)\}.$$

Consequently, all hypotheses of Theorem 3 are fulfilled. Therefore, the mixed boundary value problem 4.1 admits a unique solution in $\mathcal{W}$. □

5 Conclusions

In conclusion, we used Reich and Ćirić contractions in graphical control metric spaces to obtain significant fixed point findings. Our approach proved the existence and uniqueness of fixed points using graph-based contractions, demonstrating its potential for broader use. We showed the distinction between graphical-controlled and controlled metric-type spaces using specific examples. Additionally, we claimed that both graphically Reich and Ćirić contractions need not be their traditional counterparts. Our discoveries are now more extensive and significantly more substantial. This research also explored mixed boundary problems for the Helmholtz equation using graphic contractions of the Ćirić type, thereby mingling theoretical and applied mathematics. In acoustics, our findings demonstrate sound wave behavior in a cylinder with one end closed and the other open.

For further research, utilizing the concept propounded in this manuscript, one can research about the fixed-point analysis of complex-valued fractional order neural networks [24].

It would be intriguing to investigate whether the findings presented in this study can be generalized to multivalued mappings as done in [30].

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