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Analysis study of hybrid Caputo-Atangana-Baleanu fractional pantograph system under integral boundary conditions

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Abstract. This manuscript investigates the qualitative analysis of a new hybrid fractional pantograph system involving Atangana-Baleanu-Caputo derivatives, complemented by hybrid integral boundary conditions. Dhage's fixed point theorem is employed to investigate the existence theorem of the solutions, while uniqueness is proven by using Perov's approach and Lipschitz's matrix. The Hyers-Ulam ($\mathbb{HU}$) stability is also demonstrated using the Lipschitz's matrix and techniques from nonlinear analysis. Finally, illustrative example is enhanced to examine the effectiveness of the obtained results

Keywords: Caputo-Atangana-Baleanu operator; coupled hybrid fractional differential system; pantograph problem; Dhage and Perov techniques; Lipschitzian's matrix.

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1 Introduction

Recently, the fractional differential equations ($\mathbb{FDE}$)s have developed as an exciting and widely appreciated research subject. Various research papers are referring to $\mathbb{FDE}$s and inclusions with vary boundary conditions published in the literature [9, 10, 11], for instance, studied qualitative results of fractional pantograph problem involving the Caputo-Hadamard derivatives with Dirichlet boundary conditions [21]. Moreover, authors discussed extensively several non-local ($\mathbb{F}\mathcal{O}$)s (fractional operators) with applications on $\mathbb{FDE}$s [19, 30]. Additionally, the fixed point theorems have played key roles in the study $\mathbb{FDE}$s [2, 15, 16, 17].

In 2016. Atangana and Baleanu [7] defined a novel $\mathbb{F}\mathcal{O}$ of non-singular kernel includes Mittag-Leffler function in the Caputo sense, which so-called $\mathcal{ABC}$-$\mathbb{F}\mathcal{O}$. Recently, these operators attracted the interest of several researchers to consider many problems under these operators. For example, the works [29], established qualitative properties of $\mathcal{ABC}$-fractional hybrid system and thermostat dynamics model by utilizing $\mathbb{FPT}$s. The fractional calculus offers a powerful tool for modeling complex biological systems. Kucche and Sutar [14], calculated approximations for the $\mathcal{ABC}$-fractional derivative at the extreme positions. Also, the same authors studied hybrid fractional differential equations via the $\mathcal{ABC}$-fractional derivative [25].

The pantograph is a tool utilized to collect electricity from overhead wires on electric trains. The pantograph problem is a differential problem endowed by delay. The pantograph equation (P$\mathbb{E}$) has several applications in various areas, such as applied and pure mathematics, physics, probability, electrodynamics, quantum mechanics, control systems, and number theory. Recently, there has been increasing interest in the study of various P$\mathbb{E}$s, such as the implicit pantograph system [3, 26], sequential P$\mathbb{E}$ [13], integro-differential pantograph problem of variable fractional order [24], and the pantograph boundary value problems considered in these papers [6] and references cited therein. Furthermore, the $\mathbb{HU}$-stability is a method for analyzing the behavior of solutions to $\mathbb{FDE}$s.

Recently, Aljoudi [5] investigated the qualitative theories for a second-order $\mathcal{CF}-$ fractional order of coupled differential problems with four points boundary conditions by the Banach and Krasnoselskii theorems:

$$\begin{cases} {}^{\mathcal{CF}}\mathfrak{D}_0^{\sigma_1}\mathtt{x}(u) = \mathtt{Q}_1(u, \mathtt{x}(u), \mathtt{y}(u)), & u \in \mathcal{J} := [0, \mathfrak{z}], \\ {}^{\mathcal{CF}}\mathfrak{D}_0^{\sigma_2}\mathtt{y}(u) = \mathtt{Q}_2(u, \mathtt{x}(u), \mathtt{y}(u)), & 1 < \sigma_1, \sigma_2 \le 2, \\ \mathtt{x}(0) = \mathtt{y}(0) = 0, \\ \mathtt{x}(\mathfrak{z}) = \lambda_1 \mathtt{y}(t_1), \quad \mathtt{y}(\mathfrak{z}) = \lambda_2 \mathtt{x}(t_2), & t_i \in (0, \mathfrak{z}), \lambda_i > 0, i = 1, 2. \end{cases} \tag{1.1}$$

Furthermore, Boutiara *et al.* combined Lipschitz's matrix with contraction techniques in generalized metric space to investigate sufficient conditions of solutions for a coupled (p, q)-fractional differential system [8]. Moreover, Zhao and Jiang applied Dhage's fixed point principle to study the existence result of mild solutions for a coupled hybrid fractional system via $\mathcal{ABC}$-$\mathbb{F}\mathcal{O}$ [29]. Also in 2022, the authors used the Leray-Schauder and Banach theorems to establish the qualitative results of the following couple of second-order $\mathcal{ABC}$-fractional pantograph system ($\mathbb{FPS}$):

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_a^{\sigma_1}\mathtt{x}(u) = \mathtt{Q}_1\big(u, \mathtt{x}(\mu u), \mathtt{y}(u)\big), \\ {}^{\mathcal{ABC}}\mathfrak{D}_a^{\sigma_2}\mathtt{y}(u) = \mathtt{Q}_2\big(u, \mathtt{x}(u), \mathtt{y}(\mu u)\big), \\ \mathtt{x}(a) = \mathtt{y}(a) = 0, \quad \mathtt{x}(\mathfrak{z}) = \lambda_1 \mathtt{y}(t_1), \quad \mathtt{y}(\mathfrak{z}) = \lambda_2 \mathtt{x}(t_2), \end{cases} \tag{1.2}$$

for $u \in [a, \mathfrak{z}]$, where $\mu \in (0,1)$, $1 < \sigma_i \leq 2$, $t_i \in (0, \mathfrak{z})$, $\lambda_i > 0$, $i = 1, 2$ [4]. Very recently in 2023, Shah *et al.* studied the qualitative theorems for a coupled of $\mathcal{CF}$-fractional system under integral boundary conditions by the Perov and Schauder fixed point theorems ($\mathbb{FPT}$s) [23]. Moreover, the extremal solutions established via upper and lower solution techniques together with a monotone iterative approach for the following coupled system:

$$\begin{cases} {}^{\mathcal{CF}}\mathfrak{D}_0^{\sigma_1}\mathtt{x}(u) = -\mathtt{Q}_1\big(u, \mathtt{x}(u), \mathtt{y}(u)\big), \\ {}^{\mathcal{CF}}\mathfrak{D}_0^{\sigma_2}\mathtt{y}(u) = -\mathtt{Q}_2\big(u, \mathtt{x}(u), \mathtt{y}(u)\big), \\ \mathtt{x}(1) = \displaystyle\int_0^1 \omega_1(v)\mathtt{y}(v)\,\mathrm{d}v, \quad \mathtt{y}(1) = \int_0^1 \omega_2(v)\mathtt{x}(v)\,\mathrm{d}v. \end{cases} \tag{1.3}$$

and initial condition $\mathtt{x}(0) = \mathtt{y}(0) = 0$, for $u \in [0, 1]$, and $1 < \sigma_1, \sigma_2 \leq 2$ [23].

Motivated by the above research articles, this manuscript investigates the existence and uniqueness theorems along with the $\mathbb{HU}$ stability by utilizing Dhage and Perov $\mathbb{FPT}$s as well as Lipschitz's matrix for a hybrid $\mathcal{ABC}$-$\mathbb{FPS}$ subjected to hybrid integral boundary conditions which is given by:

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_1}\left[\frac{\mathtt{x}(u)}{\chi_1(u,\mathtt{x}(u),\mathtt{y}(u))}\right] + \mathtt{Q}_1\left(u, \mathtt{x}(u), \mathtt{x}(\mu u), \mathtt{y}(u)\right) = 0, \\ {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_2}\left[\frac{\mathtt{y}(u)}{\chi_2(u,\mathtt{x}(u),\mathtt{y}(u))}\right] + \mathtt{Q}_2\left(u, \mathtt{x}(u), \mathtt{y}(u), \mathtt{y}(\mu u)\right) = 0, \\ \frac{\mathtt{x}(\mathfrak{z})}{\chi_1(\mathfrak{z},\mathtt{x}(\mathfrak{z}),\mathtt{y}(\mathfrak{z}))} = \displaystyle\int_0^{\mathfrak{z}} \omega_1\left(v, \mathtt{y}(v)\right)\,\mathrm{d}v, \frac{\mathtt{y}(\mathfrak{z})}{\chi_2(\mathfrak{z},\mathtt{x}(\mathfrak{z}),\mathtt{y}(\mathfrak{z}))} = \int_0^{\mathfrak{z}} \omega_2\left(v, \mathtt{x}(v)\right)\,\mathrm{d}v, \end{cases} \tag{1.4}$$

and $\mathtt{x}(0) = \mathtt{y}(0) = 0$, for $u \in \mathcal{J} =: [0, \mathfrak{z}]$, $\mu \in (0,1)$, ${}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma}$ is $\mathcal{ABC}$-fractional derivative of order $\sigma = \{\sigma_1, \sigma_2\} \in (1, 2]$, and functions $\mathtt{Q}_1, \mathtt{Q}_2 : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$, $\chi_i : \mathcal{J} \times \mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$, $\omega_i : \mathcal{J} \times \mathbb{R} \to \mathbb{R}, (i = 1, 2)$, are continuous on $\mathcal{J}$. Here, we declare that the coupled hybrid system considered in this paper is new in the form of $\mathcal{ABC}$-$\mathbb{FPS}$ supported by hybrid integral boundary conditions. Moreover, our strategy is adopting Dhage's technique to study the existence of solutions in the space of Banach algebra. Furthermore, we combined Perov's approach in metric space with Lipschitz's matrix to establish the uniqueness and $\mathbb{HU}$–stability. Additionally, the hybrid system (1.4) covers some problems which don't consider yet and includes several existing studies in the literature as follows:

i) The hybrid system (1.4) becomes as problem (1.1), if we replace ${}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_i}$ by

${}^{\mathcal{CF}}\mathfrak{D}_0^{\sigma_i}$, and by putting $\chi_i = 1, \mu = 1$, and

$$\int_0^{\mathfrak{z}} \omega_1(v, \mathrm{y}(v))\,\mathrm{d}v = \lambda_1 \mathrm{y}(t_1), \qquad \int_0^{\mathfrak{z}} \omega_2(v, \mathrm{x}(v))\,\mathrm{d}v = \lambda_2 \mathrm{x}(t_2); \tag{1.5}$$

ii) The hybrid system (1.4) coincides problem (1.2), if we take $\chi_i = 1$, and consider Equation (1.5);

iii) The hybrid system (1.4) returns to problem (1.3), if we replace ${}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_i}$ by ${}^{\mathcal{CF}}\mathfrak{D}_0^{\sigma_i}$, and by putting $\chi_i = \mu = 1$, and $\omega_1(v, \mathrm{y}(v)) = \omega_1(v)\mathrm{y}(v)$, $\omega_2(v, \mathrm{x}(v)) = \omega_2(v)\mathrm{x}(v)$.

The rest of this work is organized as: Various essential preliminaries are supplied in Section 2. Qualitative properties of solution for system (1.4) are discussed by using Dhage and Perov techniques along with Lipschitzian's matrix in Section 3. Furthermore, the $\mathbb{HU}$ stability result is established in Section 4. Finally, one concrete example is examined to check the validity of major theorems in Section 5.

2 Preliminaries

Herein, we will present various essential elementary definitions and theorems linked to non-linear analysis and $\mathcal{ABC}$-fractional calculus. Let the Banach space $\mathfrak{C}(\mathcal{J})$ equipped with the norm $\|\mathrm{x}\| = \sup_{u\in\mathcal{J}} |\mathrm{x}(u)|$. Moreover, we define a Banach algebra subject to the multiplication by $(\mathrm{x}\cdot\mathrm{y})(u) = \mathrm{x}(u)\cdot\mathrm{y}(u)$, for $\mathrm{x}, \mathrm{y} \in \mathfrak{C}(\mathcal{J})$, $u \in \mathcal{J}$. Let $\varSigma = \mathfrak{C}(\mathcal{J}) \times \mathfrak{C}(\mathcal{J})$ be a product Banach space with the norm $\|(\mathrm{x}, \mathrm{y})\| = \|\mathrm{x}\| + \|\mathrm{y}\|$, and can be represented as Banach algebra too. The multiplication of two vectors of $\varSigma$ is given by:

$$((\mathrm{x}, \mathrm{y}) \cdot (\tilde{\mathrm{x}}, \tilde{\mathrm{y}}))(u) = (\mathrm{x}, \mathrm{y})(u) \cdot (\tilde{\mathrm{x}}, \tilde{\mathrm{y}})(u) = \big(\mathrm{x}(u)\cdot\tilde{\mathrm{x}}(u), \mathrm{y}(u)\cdot\tilde{\mathrm{y}}(u)\big),$$

for each $(\mathrm{x}, \mathrm{y}), (\tilde{\mathrm{x}}, \tilde{\mathrm{y}}) \in \varSigma$, and $u \in \mathcal{J}$. Also, consider a metric $\mathfrak{d}$ on the space $\varSigma$ given by $\mathfrak{d}\left((\mathrm{x}, \mathrm{y}), (\tilde{\mathrm{x}}, \tilde{\mathrm{y}})\right) = \big(\|\mathrm{x} - \tilde{\mathrm{x}}\| \; \|\mathrm{y} - \tilde{\mathrm{y}}\|\big)$. Obviously, $\mathfrak{d}$ is a vector-valued metric on $\varSigma$.

The $\mathcal{ABC}-$derivative of fractional order $\sigma \in (0, 1]$, for a function $\mathcal{Z}(u) \in \mathfrak{L}^1(\mathcal{J}, \mathbb{R})$ is given by,

$$\left({}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma}\mathcal{Z}\right)(u) = \tfrac{\mho(\sigma)}{1-\sigma} \int_0^u \mathbb{E}_\sigma\left(-\sigma \tfrac{(u-v)^\sigma}{1-\sigma}\right) \mathcal{Z}'(v)\,\mathrm{d}v, \qquad u > 0,$$

where $\mathbb{E}_\sigma$ is the Mittag-Leffler function defined as

$$\mathbb{E}_\sigma(z) = \sum_{k=0}^{\infty} \tfrac{z^k}{\Gamma(\sigma k+1)},$$

such that $Re(\sigma) > 0, z \in \mathbb{C}$, and $\Gamma(\cdot)$ is the Gamma function and $\mho(\sigma)$ is known by the normalization function admits $\mho(0) = \mho(1) = 1$ [7]. Note that the $\mathcal{ABC}-$derivative convert to $\mathcal{FCF}-$derivative if $\sigma = 1$ in the kernel $\mathbb{E}_\sigma$. Furthermore, the σ^{th} $\mathcal{AB}-$integral is expressed by:

$$\left({}^{\mathcal{AB}}\mathfrak{I}_0^\sigma \mathcal{Z}\right)(u) = \frac{1-\sigma}{\mho(\sigma)}\mathcal{Z}(u) + \frac{\sigma}{\mho(\sigma)}\left({}^{R}\mathfrak{I}_0^\sigma \mathcal{Z}\right)(u), \qquad u > 0,$$

such that ${}^{R}\mathfrak{I}_0^\sigma$ is $\mathcal{RL}-$fractional integral of order $\sigma \in (0,1]$, defined as,

$$\left({}^{R}\mathfrak{I}_0^\sigma \mathcal{Z}\right)(u) = \int_0^u \frac{(u-v)^{\sigma-1}}{\Gamma(\sigma)}\mathcal{Z}(v)\,\mathrm{d}v.$$

Definition 1. ([1])Let $\mathcal{Z}^n \in \mathfrak{L}^1(\mathcal{J},\mathbb{R})$, $n < \vartheta \leq n+1$, $n \in \mathbb{N}$ and $\theta = \vartheta - n$ Then, $\mathcal{ABC}$-fractional derivative verifying,

$$\left({}^{\mathcal{ABC}}\mathfrak{D}_0^\vartheta \mathcal{Z}\right)(u) = \left({}^{\mathcal{ABC}}\mathfrak{D}_0^\theta \mathcal{Z}^{(n)}\right)(u),$$

and the $\mathcal{AB}-$fractional integral verifying $\left({}^{\mathcal{AB}}\mathfrak{I}_0^\vartheta \mathcal{Z}\right)(u) = \left(\mathfrak{I}_0^n\ {}^{\mathcal{AB}}\mathfrak{I}_0^\theta \mathcal{Z}\right)(u)$, such that $\mathfrak{I}_0^n$ is n^{th} integral.

Lemma 1. ([7]) *For $n < \vartheta \leq n+1$, $n \in \mathbb{N}$, the following identity holds:*

$${}^{\mathcal{AB}}\mathfrak{I}_0^\vartheta \left({}^{\mathcal{ABC}}\mathfrak{D}_0^\vartheta \mathcal{Z}\right)(u) = \mathcal{Z}(u) + \sum_{i=0}^{n} b_i u^i, \qquad b_i \in \mathbb{R}.$$

Definition 2. ([28]) Let $\varrho(\mathbb{D})$ be a spectral radius of the square matrix $\mathbb{D}$, then $\mathbb{D}$ tends to zero if and only if $\varrho(\mathbb{D}) < 1$, that is for $|\Lambda| < 1$ and $\det(\mathbb{D} - \Lambda\mathbb{I}) = 0$ for all $\Lambda \in \mathbb{C}$ and unit matrix $\mathbb{I} \in \mathbb{D}_{n\times n}(\mathbb{R})$.

Theorem 1. ([28]) *Let $\mathbb{D}$ be square matrix of non-negative components. Then, the characteristics to be mentioned below are equivalent:* i) $\mathbb{D}^n \to 0$ *as* $n \to \infty$; ii) $\varrho(\mathbb{D}) < 1$; iii) *the matrix* $(\mathbb{I}-\mathbb{D})$ *is non-singular and* $(\mathbb{I}-\mathbb{D})^{-1} = \mathbb{I}+\mathbb{D}+\ldots+\mathbb{D}^n+\ldots$; (iv) *the matrix* $\mathbb{I}-\mathbb{D}$ *is non-singular and* $(\mathbb{I}-\mathbb{D})^{-1}$ *is a non-negative.*

Definition 3. ([20, 22]) Let $(\check{\Pi}, \mathfrak{d})$ be a generalized metric space, then the operator $\Psi\colon \check{\Pi} \to \check{\Pi}$ is contractive, means that for all $s,t \in \check{\Pi}$, $\mathfrak{d}\left(\Psi(s),\Psi(t)\right) \leq \mathbb{D}\ \mathfrak{d}(s,t)$, if there is a matrix $\mathbb{D}$ convergence to zero.

Theorem 2. (Perov's $\mathbb{FPT}$ [18, 20]) *Consider $(\check{\Pi}, \mathfrak{d})$ is a generalized complete metric space. If $\Psi\colon \check{\Pi} \to \check{\Pi}$ be a contractive mapping with Lipschitz's matrix $\mathbb{D}$, then Ψ has exactly one fixed point u_0, for all $\forall u \in \check{\Pi}$, and*

$$\forall k \in \mathbb{N}, \quad \mathfrak{d}\left(\Psi^k(u), u_0\right) \leq \mathbb{D}^k(\mathbb{I}-\mathbb{D})^{-1}\ \mathfrak{d}\left(u,\Psi(u)\right).$$

Theorem 3. (Dhage's $\mathbb{FPT}$ [12]) *Assume that Σ be a Banach algebra and D be a bounded closed convex nonempty subset of Σ. Let two operators $E_1 : \Sigma \to \Sigma$ and $E_2 : D \to \Sigma$ satisfy the following:* (i) *E_1 is Lipschitz with constant X,* (ii) *E_2 is continuous and compact,* (iii) $u = E_1uE_2v \implies u \in D, \forall v \in D$; (iv) *$XZ < 1$, with $Z = \|E_2(D)\| = \sup\{\|E_2(u)\| : u \in D\}$. Then, the mapping equation $u = E_1u\,E_2u$ possesses a solution.*

3 Existence and uniqueness results

We present an equivalent equation to the hybrid $\mathcal{ABC}$-$\mathbb{F}$PS (1.4).

Lemma 2. *For $\vartheta : \mathcal{J} \to \mathbb{R}$, the following hybrid $\mathcal{ABC}$-fractional equation:*

$$ {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_1}\left[\frac{\mathtt{x}(u)}{\chi_1(u,\mathtt{x}(u),\mathtt{y}(u))}\right]+\vartheta(u)=0, \qquad \sigma_1\in(1,2], \tag{3.1}$$

$$ \mathtt{x}(0)=0, \quad \frac{\mathtt{x}(\mathfrak{z})}{\chi_1(\mathfrak{z},\mathtt{x}(\mathfrak{z}),\mathtt{y}(\mathfrak{z}))}=\int_0^{\mathfrak{z}}\omega_1\left(v,\mathtt{y}(v)\right)\,\mathrm{d}v,$$

possesses a solution formed by

$$ \mathtt{x}(u)=\chi_1\left(u,\mathtt{x}(u),\mathtt{y}(u)\right)\left[\int_0^{\mathfrak{z}}\frac{u\omega_1}{\mathfrak{z}}\left(v,\mathtt{y}(v)\right)\,\mathrm{d}v-\frac{2-\sigma_1}{\mho(\sigma_1-1)}\left[\int_0^u\vartheta(v)\,\mathrm{d}v\right.\right.$$
$$ \left.\left.-\int_0^{\mathfrak{z}}\frac{u\vartheta(v)}{\mathfrak{z}}\,\mathrm{d}v\right]-\frac{\sigma_1-1}{\mho(\sigma_1-1)}\left[({}^R\mathfrak{I}_0^{\sigma_1}\vartheta)(u)-\frac{u}{\mathfrak{z}}\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(\mathfrak{z})\right]\right]. \tag{3.2}$$

Proof. Using ${}^{\mathcal{AB}}\mathfrak{I}_0^{\sigma_1}$ on both sides of (3.1) and using Lemma 1, one gets

$$ \frac{\mathtt{x}(u)}{\chi_1\left(u,\mathtt{x}(u),\mathtt{y}(u)\right)}=a_1+a_2u-\int_0^u\frac{(2-\sigma_1)\vartheta(v)}{\mho(\sigma_1-1)}\,\mathrm{d}v-\frac{\sigma_1-1}{\mho(\sigma_1-1)}\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(u), \tag{3.3}$$

where $a_1,a_2\in\mathbb{R}$. Then, due to the boundary condition $\mathtt{x}(0)=0$, we have $a_1=0$. So, we have

$$ \frac{\mathtt{x}(u)}{\chi_1\left(u,\mathtt{x}(u),\mathtt{y}(u)\right)}=a_2u-\int_0^u\frac{(2-\sigma_1)\vartheta(v)}{\mho(\sigma_1-1)}\,\mathrm{d}v-\frac{\sigma_1-1}{\mho(\sigma_1-1)}\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(u).$$

Next, applying the second boundary condition, $\frac{\mathtt{x}(\mathfrak{z})}{\chi_1(\mathfrak{z},\mathtt{x}(\mathfrak{z}),\mathtt{y}(\mathfrak{z}))}=\int_0^{\mathfrak{z}}\omega_1(v,\mathtt{y}(v))\mathrm{d}v$, we find

$$ \int_0^{\mathfrak{z}}\omega_1\left(v,\mathtt{y}(v)\right)\,\mathrm{d}v=a_2\mathfrak{z}-\int_0^{\mathfrak{z}}\frac{(2-\sigma_1)\vartheta(v)}{\mho(\sigma_1-1)}\,\mathrm{d}v-\frac{\sigma_1-1}{\mho(\sigma_1-1)}\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(u),$$

it follows

$$ a_2=\int_0^{\mathfrak{z}}\frac{\omega_1\left(v,\mathtt{y}(v)\right)}{\mathfrak{z}}\,\mathrm{d}v+\int_0^{\mathfrak{z}}\frac{(2-\sigma_1)\vartheta(v)}{\mho(\sigma_1-1)\mathfrak{z}}\,\mathrm{d}v+\frac{\sigma_1-1}{\mho(\sigma_1-1)\mathfrak{z}}\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(\mathfrak{z}).$$

Finally, by substituting the values of a_1 and a_2 in (3.3), we obtain,

$$ \frac{\mathtt{x}(u)}{\chi_1\left(u,\mathtt{x}(u),\mathtt{y}(u)\right)}=\int_0^{\mathfrak{z}}\frac{u\omega_1\left(v,\mathtt{y}(v)\right)}{\mathfrak{z}}\,\mathrm{d}v+\int_0^{\mathfrak{z}}\frac{u(2-\sigma_1)\vartheta(v)}{\mho(\sigma_1-1)\mathfrak{z}}\,\mathrm{d}v+\frac{u(\sigma_1-1)}{\mho(\sigma_1-1)\mathfrak{z}}\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(\mathfrak{z})$$
$$ -\int_0^u\frac{(2-\sigma_1)\vartheta(v)}{\mho(\sigma_1-1)}\,\mathrm{d}v-\frac{\sigma_1-1}{\mho(\sigma_1-1)}({}^R\mathfrak{I}_0^{\sigma_1}\vartheta)(u)$$
$$ =\frac{u}{\mathfrak{z}}\int_0^{\mathfrak{z}}\omega_1\left(v,\mathtt{y}(v)\right)\,\mathrm{d}v-\frac{2-\sigma_1}{\mho(\sigma_1-1)}\left[\int_0^u\vartheta(v)\,\mathrm{d}v-\int_0^{\mathfrak{z}}\frac{u\vartheta(v)}{\mathfrak{z}}\,\mathrm{d}v\right]$$
$$ -\frac{\sigma_1-1}{\mho(\sigma_1-1)}\left[\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(u)-\frac{u}{\mathfrak{z}}\left({}^R\mathfrak{I}_0^{\sigma_1}\vartheta\right)(\mathfrak{z})\right].$$

Hence, it follows the required Equation (3.2). □

Based on Lemma 2, we conclude the next important lemma.

Lemma 3. *The solution of hybrid* $\mathcal{ABC}-\mathbb{FPS}$ (1.4) *is formed by,*

$$\begin{cases} \mathtt{x}(u) = \chi_1(u, \mathtt{x}(u), \mathtt{y}(u))\Big(\int_0^{\mathfrak{z}} \frac{u\omega_1(v,\mathtt{y}(v))}{\mathfrak{z}}\,\mathrm{d}v \\ \quad -\frac{2-\sigma_1}{\mho(\sigma_1-1)}\Big[\int_0^u \widehat{\mathtt{Q}}_{1;\mu}(v,\mathtt{x})\,\mathrm{d}v - \int_0^{\mathfrak{z}} \frac{u\widehat{\mathtt{Q}}_{1;\mu}(v,\mathtt{x})}{\mathfrak{z}}\,\mathrm{d}v\Big] \\ \quad -\frac{\sigma_1-1}{\mho(\sigma_1-1)}\Big[{}^{R}\mathfrak{I}_0^{\sigma_1}\widehat{\mathtt{Q}}_{1;\mu}(u,\mathtt{x}) - \frac{u}{\mathfrak{z}}\,{}^{R}\mathfrak{I}_0^{\sigma_1}\widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z},\mathtt{x})\Big]\Big), \\ \mathtt{y}(u) = \chi_2(u, \mathtt{x}(u), \mathtt{y}(u))\Big(\int_0^{\mathfrak{z}} \frac{u\omega_2(v,\mathtt{x}(v))}{\mathfrak{z}}\,\mathrm{d}v \\ \quad -\frac{2-\sigma_2}{\mho(\sigma_2-1)}\Big[\int_0^u \widehat{\mathtt{Q}}_{2;\mu}(v,\mathtt{y})\,\mathrm{d}v - \int_0^{\mathfrak{z}} \frac{u\widehat{\mathtt{Q}}_{2;\mu}(v,\mathtt{y})}{\mathfrak{z}}\,\mathrm{d}v\Big] \\ \quad -\frac{\sigma_2-1}{\mho(\sigma_2-1)}\Big[{}^{R}\mathfrak{I}_0^{\sigma_2}\widehat{\mathtt{Q}}_{2;\mu}(u,\mathtt{y}) - \frac{u}{\mathfrak{z}}\,{}^{R}\mathfrak{I}_0^{\sigma_2}\widehat{\mathtt{Q}}_{2;\mu}(\mathfrak{z},\mathtt{y})\Big]\Big), \end{cases} \tag{3.4}$$

with $\widehat{\mathtt{Q}}_{1;\mu}(u,\mathtt{x}) := \mathtt{Q}_1(u,\mathtt{x}(u),\mathtt{x}(\mu u),\mathtt{y}(u))$, $\widehat{\mathtt{Q}}_{2;\mu}(u,\mathtt{y}) := \mathtt{Q}_1(u,\mathtt{x}(u),\mathtt{x}(u),\mathtt{y}(\mu u))$.

Before establishing the qualitative theorems, we need to present the following assumptions:

$\mathbb{H}_1$) There exist functions $L_{\mathtt{Q}_1}, L_{\mathtt{Q}_2}, L_{\omega_i} \in \mathfrak{C}(\mathcal{J},\mathbb{R}^+)$ such that $|\omega_i(u,\mathtt{x})| \le L_{\omega_i}(u)$, $|\mathtt{Q}_1(u,\mathtt{x},\mathtt{y},\mathtt{z})| \le L_{\mathtt{Q}_1}(u)$, $|\mathtt{Q}_2(u,\mathtt{x},\mathtt{y},\mathtt{z})| \le L_{\mathtt{Q}_2}(u)$, for each $\mathtt{x},\mathtt{y},\mathtt{z} \in \mathbb{R}$, and $u \in \mathcal{J}$;

$\mathbb{H}_2$) There are constants $A_{\chi_i} > 0, (i = 1, 2)$, where

$$|\chi_i(u,\mathtt{x}_1,\mathtt{y}_1) - \chi_i(u,\mathtt{x}_2,\mathtt{y}_2)| \le A_{\chi_i}\left(|\mathtt{x}_1-\mathtt{x}_2| + |\mathtt{y}_1-\mathtt{y}_2|\right),$$

for each $\mathtt{x}_i,\mathtt{y}_i \in \mathbb{R}$, and $u \in \mathcal{J}, (i = 1, 2)$;

$\mathbb{H}_3$) There exist functions $L_{\chi_i} \in \mathfrak{C}(\mathcal{J},\mathbb{R}^+)$ such that $|\chi_i(u,\mathtt{x},\mathtt{y})| \le L_{\chi_i}(u)$, for each $\mathtt{x},\mathtt{y} \in \mathbb{R}$, and $u \in \mathcal{J}$;

$\mathbb{H}_4$) There exist $A_{\mathtt{Q}_1}, A_{\mathtt{Q}_2} > 0$, such that for each $\mathtt{x}_i, \grave{\mathtt{x}}_i \in \mathbb{R}$, $i = 1, 2, 3$, and $u \in \mathcal{J}$, $k = 1, 2$,

$$\left|\mathtt{Q}_k\left(u,\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3\right) - \mathtt{Q}_k\left(u,\grave{\mathtt{x}}_1,\grave{\mathtt{x}}_2,\grave{\mathtt{x}}_3\right)\right| \le A_{\mathtt{Q}_k}\Big(\sum_{i=1}^{3}\left|\mathtt{x}_i-\grave{\mathtt{x}}_i\right|\Big);$$

$\mathbb{H}_5$) There exist $A_{\omega_i} > 0, (i = 1, 2)$, where $|\omega_i(u,\mathtt{x}_1)-\omega_i(u,\mathtt{x}_2)| \le A_{\omega_i}|\mathtt{x}_1-\mathtt{x}_2|$, for each $\mathtt{x}_i,\mathtt{y}_i \in \mathbb{R}$, and $u \in \mathcal{J}$, $i = 1, 2$.

For simplicity of analysis, we set

$$\begin{aligned} \Lambda_1 &= \mathfrak{z}L^*_{\omega_1} + \frac{2\mathfrak{z}L^*_{\mathtt{Q}_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|} + \frac{2\mathfrak{z}^{\sigma_1}L^*_{\mathtt{Q}_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|}, \\ \Lambda_2 &= \mathfrak{z}L^*_{\omega_2} + \frac{2\mathfrak{z}L^*_{\mathtt{Q}_2}(2-\sigma_2)}{|\mho(\sigma_2-1)|} + \frac{2\mathfrak{z}^{\sigma_2}L^*_{\mathtt{Q}_2}(\sigma_2-1)}{\Gamma(1+\sigma_2)|\mho(\sigma_2-1)|}, \end{aligned} \tag{3.5}$$

and for $i = 1, 2$, $L^*_{\mathtt{Q}_i} = \sup_{u\in\mathcal{J}}\{L_{\mathtt{Q}_i}(u)\}$, $L^*_{\omega_i} = \sup_{u\in\mathcal{J}}\{L_{\omega_i}(u)\}$, $L^*_{\chi_i} = \sup_{u\in\mathcal{J}}\{L_{\chi_i}(u)\}$. Next, we will prove the existence theorem by employing Dhage's $\mathbb{FPT}$.

Theorem 4. *Let the conditions* $(\mathbb{H}_1)$ *and* $(\mathbb{H}_2)$ *are fulfilled, and if*

$$(A_{\chi_1} + A_{\chi_2}) < 1, \quad (A_{\chi_1} + A_{\chi_2})(\Lambda_1 + \Lambda_2) < 1, \tag{3.6}$$

then, the hybrid $\mathcal{ABC}$–$\mathbb{F}$PS (1.4) *admits at least one solution in* Σ.

Proof. Let the ball $\mathfrak{B}_\rho = \{(\mathtt{x},\mathtt{y}) \in \Sigma : \|(\mathtt{x},\mathtt{y})\| \le \rho\}$, with

$$\|\chi_1^0\|\Lambda_1 + \|\chi_2^0\|\Lambda_2 \le \rho[1-(A_{\chi_1}+A_{\chi_2})](\Lambda_1+\Lambda_2),$$

where $\|\chi_i^0\| = \sup_{u\in\mathcal{J}} |\chi_i(u,0,0)|$ $(i=1,2)$. Obviously, $\mathfrak{B}_\rho$ is convex, bounded, and closed subset of Σ. Now, we define the operators $\mathbb{E} = (\mathbb{E}_1,\mathbb{E}_2) : \Sigma \to \Sigma$, and $\mathbb{F} = (\mathbb{F}_1,\mathbb{F}_2) : \mathfrak{B}_\rho \to \Sigma$, as follows,

$$\begin{aligned}
&\big(\mathbb{E}_1(\mathtt{x},\mathtt{y})\big)(u) = \chi_1\big(u,\mathtt{x}(u),\mathtt{y}(u)\big),\ \big(\mathbb{E}_2(\mathtt{x},\mathtt{y})\big)(u) = \chi_2\big(u,\mathtt{x}(u),\mathtt{y}(u)\big),\\
&\big(\mathbb{F}_1\big(\mathtt{x},\mathtt{y}\big)\big)(u) = \int_0^{\mathfrak{z}} \tfrac{u\omega_1(v,\mathtt{y}(v))}{\mathfrak{z}}\,\mathrm{d}v - \tfrac{2-\sigma_1}{\mho(\sigma_1-1)}\Big[\int_0^u \widehat{\mathtt{Q}}_{1;\mu}(v,\mathtt{x})\,\mathrm{d}v\\
&\quad - \tfrac{u}{\mathfrak{z}}\int_0^{\mathfrak{z}} \widehat{\mathtt{Q}}_{1;\mu}(v,\mathtt{x})\,\mathrm{d}v\Big] - \tfrac{\sigma_1-1}{\mho(\sigma_1-1)}\Big[{}^R\mathfrak{I}_0^{\sigma_1}\widehat{\mathtt{Q}}_{1;\mu}(u,\mathtt{x}) - \tfrac{u}{\mathfrak{z}}{}^R\mathfrak{I}_0^{\sigma_1}\widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z},\mathtt{x})\Big],\\
&(\mathbb{F}_2(\mathtt{x},\mathtt{y}))(u) = \int_0^{\mathfrak{z}} \tfrac{u\omega_2(v,\mathtt{x}(v))}{\mathfrak{z}}\,\mathrm{d}v - \tfrac{2-\sigma_2}{|\mho(\sigma_2-1)|}\Big[\int_0^u \widehat{\mathtt{Q}}_{2;\mu}(v,\mathtt{y})\,\mathrm{d}v\\
&\quad - \tfrac{u}{\mathfrak{z}}\int_0^{\mathfrak{z}} \widehat{\mathtt{Q}}_{2;\mu}(v,\mathtt{y})\,\mathrm{d}v\Big] - \tfrac{\sigma_2-1}{|\mho(\sigma_2-1)|}\Big[{}^R\mathfrak{I}_0^{\sigma_2}\widehat{\mathtt{Q}}_{2;\mu}(u,\mathtt{y}) - \tfrac{u}{\mathfrak{z}}{}^R\mathfrak{I}_0^{\sigma_2}\widehat{\mathtt{Q}}_{2;\mu}(\mathfrak{z},\mathtt{y})\Big].
\end{aligned}$$

Hence, the pantograph integral system (3.4) deform into the following coupled of mapping equations

$$\mathbb{E}(\mathtt{x},\mathtt{y})(u)\cdot\mathbb{F}(\mathtt{x},\mathtt{y})(u) = (\mathtt{x},\mathtt{y})(u). \tag{3.7}$$

Thus for $u \in \mathcal{J}$, one has

$$\mathbb{E}_1(\mathtt{x},\mathtt{y})(u)\cdot\mathbb{F}_1(\mathtt{x},\mathtt{y})(u) = \mathtt{x}(u), \quad \mathbb{E}_2(\mathtt{x},\mathtt{y})(u)\cdot\mathbb{F}_2(\mathtt{x},\mathtt{y})(u) = \mathtt{y}(u).$$

Now, we split our proof into a sequence of procedures. Firstly, we prove that $E = (\mathbb{E}_1,\mathbb{E}_2)$ is Lipschitz mapping. Let $(\mathtt{x}_i,\mathtt{y}_i) \in \Sigma$, $i=1,2$, and by using $(\mathbb{H}_2)$, we find,

$$\begin{aligned}
&|\mathbb{E}_1(\mathtt{x}_1,\mathtt{y}_1)(u) - \mathbb{E}_1(\mathtt{x}_2,\mathtt{y}_2)(u)| = |\chi_1(u,\mathtt{x}_1(u),\mathtt{y}_1(u)) - \chi_1(u,\mathtt{x}_2(u),\mathtt{y}_2(u))|\\
&\quad \le A_{\chi_1}\left(|\mathtt{x}_1(u)-\mathtt{x}_2(u)| + |\mathtt{y}_1(u)-\mathtt{y}_2(u)|\right) \le A_{\chi_1}\left(\|\mathtt{x}_1-\mathtt{x}_2\| + \|\mathtt{y}_1-\mathtt{y}_2\|\right).
\end{aligned}$$

Thus, for $k=1,2$,

$$\big\|\mathbb{E}_k(\mathtt{x}_1,\mathtt{x}_1) - \mathbb{E}_k(\grave{\mathtt{x}}_1,\grave{\mathtt{x}}_2)\big\| \le A_{\chi_k}\sum_{i=1}^2 \big\|\mathtt{x}_i - \grave{\mathtt{x}}_i\big\|.$$

Now, by definition of the operator $\mathbb{E}$, we have

$$\begin{aligned}
\big\|\mathbb{E}(\mathtt{x}_1,\mathtt{y}_1) - \mathbb{E}(\mathtt{x}_2,\mathtt{y}_2)\big\| &= \big\|\big(\mathbb{E}_1(\mathtt{x}_1,\mathtt{y}_1),\mathbb{E}_2(\mathtt{x}_1,\mathtt{y}_1)\big) - \big(\mathbb{E}_1(\mathtt{x}_2,\mathtt{y}_2),\mathbb{E}_2(\mathtt{x}_2,\mathtt{y}_2)\big)\big\|\\
&= \big\|\big(\mathbb{E}_1(\mathtt{x}_1,\mathtt{y}_1) - \mathbb{E}_1(\mathtt{x}_2,\mathtt{y}_2),\mathbb{E}_2(\mathtt{x}_1,\mathtt{y}_1) - \mathbb{E}_2(\mathtt{x}_2,\mathtt{y}_2)\big)\big\|\\
&= \big\|\mathbb{E}_1(\mathtt{x}_1,\mathtt{y}_1) - \mathbb{E}_1(\mathtt{x}_2,\mathtt{y}_2)\big\| + \big\|\mathbb{E}_2(\mathtt{x}_1,\mathtt{y}_1) - \mathbb{E}_2(\mathtt{x}_2,\mathtt{y}_2)\big\|\\
&\le (A_{\chi_1}+A_{\chi_2})\Big(\big\|\mathtt{x}_1-\mathtt{x}_2\big\| + \big\|\mathtt{y}_1-\mathtt{y}_2\big\|\Big).
\end{aligned}$$

In view of the condition (3.6), we have $X = (A_{\chi_1} + A_{\chi_2}) < 1$. Hence, the mapping $\mathbb{E}$ is an Lipschitzian on Σ with Lipschitz's constant $(A_{\chi_1} + A_{\chi_2})$. Secondly, we prove that $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2) : \mathfrak{B}_\rho \to \Sigma$, is a continuous and compact mapping. In order to show the continuity of $\mathbb{F}$, let the sequence $\{(\mathtt{x}_n, \mathtt{y}_n)\}_{n\in\mathbb{N}}$ convergence to $(\mathtt{x}, \mathtt{y})$ in $\mathfrak{B}_\rho$ as $n \to \infty$. Then, based on continuity of the functions $\mathtt{Q}_1, \mathtt{Q}_2, \omega_1, \omega_2$, and by utilizing the Lebesgue-dominated convergence theorem, we get,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{F}_1(\mathtt{x}_n, \mathtt{y}_n)(u) = & \int_0^{\mathfrak{z}} \lim_{n\to\infty} \tfrac{u\omega_1(v,\mathtt{y}_n(v))}{\mathfrak{z}}\, dv - \tfrac{2-\sigma_1}{\mho(\sigma_1-1)} \Big[\int_0^u \lim_{n\to\infty} \widehat{\mathtt{Q}}_{1;\mu}(v, \mathtt{x}_n)\, dv \\ & - \int_0^{\mathfrak{z}} \lim_{n\to\infty} \tfrac{u\widehat{\mathtt{Q}}_{1;\mu}(v,\mathtt{x}_n)}{\mathfrak{z}}\, dv \Big] - \tfrac{\sigma_1-1}{\mho(\sigma_1-1)} \Big[{}^R\mathfrak{I}_0^{\sigma_1} \lim_{n\to\infty} \widehat{\mathtt{Q}}_{1;\mu}(u, \mathtt{x}_n) \\ & - \tfrac{u}{\mathfrak{z}}\, {}^R\mathfrak{I}_0^{\sigma_1} \lim_{n\to\infty} \widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z}, \mathtt{x}_n) \Big] = \mathbb{F}_1(\mathtt{x}, \mathtt{y})(u).\end{aligned}$$

Likewise, one has $\lim_{n\to\infty} \mathbb{F}_2(\mathtt{x}_n, \mathtt{y}_n)(u) = \mathbb{F}_2(\mathtt{x}, \mathtt{y})(u)$. Hence,

$$\lim_{n\to\infty} \mathbb{F}(\mathtt{x}_n, \mathtt{y}_n) = \lim_{n\to\infty} (\mathbb{F}_1(\mathtt{x}_n, \mathtt{y}_n), \mathbb{F}_2(\mathtt{x}_n, \mathtt{y}_n)) = (\mathbb{F}_1(\mathtt{x}, \mathtt{y}), \mathbb{F}_2(\mathtt{x}, \mathtt{y})) = \mathbb{F}(\mathtt{x}, \mathtt{y}).$$

Next, we prove uniformly bounded and equicontinuous of a set $\mathbb{F}(\mathfrak{B}_\rho)$ in Σ. Regarding uniformly bounded, let $(\mathtt{x}, \mathtt{y}) \in \mathfrak{B}_\rho$ and by using $(\mathbb{H}_1)$, we obtain,

$$\begin{aligned}|(\mathbb{F}_1(\mathtt{x}, \mathtt{y}))(u)| \le & \int_0^{\mathfrak{z}} \tfrac{uL_{\omega_1}(v)}{\mathfrak{z}}\, dv + \tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|} \Big[\int_0^u L_{\mathtt{Q}_1}(v)\, dv + \int_0^{\mathfrak{z}} \tfrac{uL_{\mathtt{Q}_1}(v)}{\mathfrak{z}}\, dv \Big] \\ & + \tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|} \Big[{}^R\mathfrak{I}_0^{\sigma_1} L_{\mathtt{Q}_1}(u) + \tfrac{u}{\mathfrak{z}}\, {}^R\mathfrak{I}_0^{\sigma_1} L_{\mathtt{Q}_1}(\mathfrak{z}) \Big] \le \mathfrak{z}L^*_{\omega_1} + \tfrac{2\mathfrak{z}L^*_{\mathtt{Q}_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|} + \tfrac{2\mathfrak{z}^{\sigma_1}L^*_{\mathtt{Q}_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|}.\end{aligned}$$

Hence, $\|\mathbb{F}_1(\mathtt{x}, \mathtt{y})\| \le \mathfrak{z}L^*_{\omega_1} + \frac{2\mathfrak{z}L^*_{\mathtt{Q}_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|} + \frac{2\mathfrak{z}^{\sigma_1}L^*_{\mathtt{Q}_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|} = \Lambda_1$. Similarly,

$$\|\mathbb{F}_2(\mathtt{x}, \mathtt{y})\| \le \mathfrak{z}L^*_{\omega_2} + \frac{2\mathfrak{z}L^*_{\mathtt{Q}_2}(2-\sigma_2)}{|\mho(\sigma_2-1)|} + \frac{2\mathfrak{z}^{\sigma_2}L^*_{\mathtt{Q}_2}(\sigma_2-1)}{\Gamma(1+\sigma_2)|\mho(\sigma_2-1)|} = \Lambda_2.$$

Therefore, $\|\mathbb{F}(\mathtt{x}, \mathtt{y})\| = \|\mathbb{F}_1(\mathtt{x}, \mathtt{y})\| + \|\mathbb{F}_2(\mathtt{x}, \mathtt{y})\| \le \Lambda_1 + \Lambda_2 < \infty$. Thus, it follows that $\mathbb{F}$ is uniformly bounded mapping on $\mathfrak{B}_\rho$. Now, we are ready to show that $\mathbb{F}$ is equicontinious. For any $u_1, u_2 \in \mathcal{J}, u_1 < u_2$, $(\mathtt{x}, \mathtt{y}) \in \mathfrak{B}_\rho$ and by using $(\mathbb{H}_1)$, we find,

$$\begin{aligned}\big|(\mathbb{F}_1(\mathtt{x}, \mathtt{y}))(u_2) - (\mathbb{F}_1(\mathtt{x}, \mathtt{y}))(u_1)\big| \le & \int_0^{\mathfrak{z}} \tfrac{(u_2-u_1)|\omega_1(v,\mathtt{y}(v))|}{\mathfrak{z}}\, dv \\ & + \tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|} \Big[\int_{u_1}^{u_2} \big|\widehat{\mathtt{Q}}_{1;\mu}(v, \mathtt{x})\big|\, dv + \int_0^{\mathfrak{z}} \tfrac{(u_2-u_1)|\widehat{\mathtt{Q}}_{1;\mu}(v,\mathtt{x})|}{\mathfrak{z}}\, dv \Big] \\ & + \tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|} \Big[\int_0^{u_1} \tfrac{(u_2-v)^{\sigma_1-1} - (u_1-v)^{\sigma_1-1}}{\Gamma(\sigma_1)} \big|\widehat{\mathtt{Q}}_{1;\mu}(v, \mathtt{x})\big|\, dv \\ & + \int_{u_1}^{u_2} \tfrac{(u_2-v)^{\sigma_1-1}}{\Gamma(\sigma_1)} \big|\widehat{\mathtt{Q}}_{1;\mu}(v, \mathtt{x})\big|\, dv + \tfrac{u_2-u_1}{\mathfrak{z}}\, {}^R\mathfrak{I}_0^{\sigma_1} \big|\widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z}, \mathtt{x})\big| \Big] \\ \le & (u_2 - u_1)L^*_{\omega_1} + \tfrac{2(2-\sigma_1)}{|\mho(\sigma_1-1)|}(u_1 - u_2)L^*_{\mathtt{Q}_1} \\ & + \tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|} \Big[\tfrac{L^*_{\mathtt{Q}_1}}{\Gamma(\sigma_1+1)} \big[2(u_2 - u_1)^{\sigma_1} + u_2^{\sigma_1} - u_1^{\sigma_1}\big] + \tfrac{\mathfrak{z}^{\sigma_1}L^*_{\mathtt{Q}_1}(u_2-u_1)}{\mathfrak{z}\Gamma(1+\sigma_1)} \Big].\end{aligned}$$

Then, as $u_2 \to u_1$, we have $|(\mathbb{F}_1(\mathtt{x},\mathtt{y}))(u_2) - (\mathbb{F}_1(\mathtt{x},\mathtt{y}))(u_1)| \to 0$. In the same manner, we obtain $|(\mathbb{F}_2(\mathtt{x},\mathtt{y}))(u_2) - (\mathbb{F}_2(\mathtt{x},\mathtt{y}))(u_1)| \to 0$, as $u_2 \to u_1$. Consequently, one finds $|(\mathbb{F}(\mathtt{x},\mathtt{y}))(u_2) - (\mathbb{F}(\mathtt{x},\mathtt{y}))(u_1)| \to 0$, as $u_2 \to u_1$. Hence, $\mathbb{F}$ is an equicontinuous mapping on $\mathfrak{B}_\rho$, implies that $\mathbb{F}$ is relatively compact. Based on Arzelà-Ascoli's theorem,we deduce that $\mathbb{F}$ is completely continuous. Thirdly, we show that the property (iii) of Theorem 3 is verified. Let $(\mathtt{x},\mathtt{y}) \in \mathfrak{B}_\rho$ satisfy (3.7), that means $\big(\mathbb{E}_1(\mathtt{x},\mathtt{y}) \cdot \mathbb{F}_1(\mathtt{x},\mathtt{y}), \mathbb{E}_2(\mathtt{x},\mathtt{y}) \cdot \mathbb{F}_2(\mathtt{x},\mathtt{y})\big) = (\mathtt{x},\mathtt{y})$. Then, by applying $(\mathbb{H}_1)$ and $(\mathbb{H}_2)$, we have,

$$\begin{aligned}
|\mathtt{x}(u)| &= |\mathbb{E}_1(\mathtt{x},\mathtt{y})| \cdot |\mathbb{F}_1(\mathtt{x},\mathtt{y})| \le \big|\chi_1(u,\mathtt{x}(u),\mathtt{y}(u))\big| \Big(\int_0^{\mathfrak{z}} \tfrac{u|\omega_1(v,\mathtt{y}(v))|}{\mathfrak{z}}\, dv \\
&\quad + \tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|} \Big[\int_0^u |\mathtt{Q}_1(v,\mathtt{x}(v),\mathtt{x}(\mu v),\mathtt{y}(v))|\, dv + \int_0^{\mathfrak{z}} \tfrac{u|\widehat{\mathtt{Q}}_{1;\mu}(v,\mathtt{x})|}{\mathfrak{z}}\, dv \Big] \\
&\quad + \tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|} \Big[{}^R\mathfrak{I}_0^{\sigma_1} |\widehat{\mathtt{Q}}_{1;\mu}(u,\mathtt{x})| + \tfrac{u}{\mathfrak{z}}\, {}^R\mathfrak{I}_0^{\sigma_1} |\widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z},\mathtt{x})| \Big] \Big) \\
&\le |\chi_1(u,\mathtt{x}(u),\mathtt{y}(u)) - \chi_1(u,0,0)| + |\chi_1(u,0,0)| \\
&\quad \times \left(\mathfrak{z} L^*_{\omega_1} + \frac{2\mathfrak{z} L^*_{\mathtt{Q}_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|} + \frac{2\mathfrak{z}^{\sigma_1} L^*_{\mathtt{Q}_1}(\sigma_1-1)}{\varGamma(1+\sigma_1)|\mho(\sigma_1-1)|} \right) \\
&\le \Big[A_{\chi_1}(\|\mathtt{x}\| + \|\mathtt{y}\|) + \|\chi_1^0\| \Big] \varLambda_1 \Rightarrow \|\mathtt{x}\| \le \Big[A_{\chi_1}(\|\mathtt{x}\| + \|\mathtt{y}\|) + \|\chi_1^0\| \Big] \varLambda_1.
\end{aligned}$$

By the same above procedures, we get,

$$\|\mathtt{y}\| \le \Big[A_{\chi_2}(\|\mathtt{x}\| + \|\mathtt{y}\|) + \|\chi_2^0\| \Big] \varLambda_2.$$

Therefore, we obtain,

$$\begin{aligned}
\|(\mathtt{x},\mathtt{y})\| &= \Big\| \big(\mathbb{E}_1(\mathtt{x},\mathtt{y}) \cdot \mathbb{F}_1(\mathtt{x},\mathtt{y}), \mathbb{E}_2(\mathtt{x},\mathtt{y}) \cdot \mathbb{F}_2(\mathtt{x},\mathtt{y}) \big) \Big\| \\
&= \big\| (\mathbb{E}_1(\mathtt{x},\mathtt{y}) \cdot \mathbb{F}_1(\mathtt{x},\mathtt{y}) \big\| + \big\| \mathbb{E}_2(\mathtt{x},\mathtt{y}) \cdot \mathbb{F}_2(\mathtt{x},\mathtt{y}) \big\| \\
&\le \Big(A_{\chi_1}(\|\mathtt{x}\| + \|\mathtt{y}\|) + \|\chi_1^0\| \Big) \varLambda_1 + \Big(A_{\chi_2}(\|\mathtt{x}\| + \|\mathtt{y}\|) + \|\chi_2^0\| \Big) \varLambda_2 \\
&\le (\varLambda_1 A_{\chi_1} + \varLambda_2 A_{\chi_2}) \|(\mathtt{x},\mathtt{y})\| + \varLambda_1 \|\chi_1^0\| + \varLambda_2 \|\chi_2^0\| \\
&\le (A_{\chi_1} + A_{\chi_2})(\varLambda_1 + \varLambda_2) \|(\mathtt{x},\mathtt{y})\| + \varLambda_1 \|\chi_1^0\| + \varLambda_2 \|\chi_2^0\|.
\end{aligned}$$

Therefore,

$$\|(\mathtt{x},\mathtt{y})\| \le \tfrac{\|\chi_1^0\|\varLambda_1 + \|\chi_2^0\|\varLambda_2}{(1-(A_{\chi_1}+A_{\chi_2})(\varLambda_1+\varLambda_2)} \le \rho.$$

That means the property (iii) of Theorem 3 verified. Fourthly, we establish that the property (iv) of Theorem 3 is hold. In fact, by uniform bounded of $\mathbb{F}$, we have,

$$\begin{aligned}
Z &= \sup \Big\{ \|\mathbb{F}(\mathtt{x},\mathtt{y})\| \,:\, (\mathtt{x},\mathtt{y}) \in \mathfrak{B}_\rho \Big\} \\
&\le \sup \Big\{ \|\mathbb{F}_1(\mathtt{x},\mathtt{y})\| + \|\mathbb{F}_2(\mathtt{x},\mathtt{y})\| \,:\, (\mathtt{x},\mathtt{y}) \in \mathfrak{B}_\rho \Big\} \le (\varLambda_1 + \varLambda_2).
\end{aligned}$$

Hence, we get $XZ = (A_{\chi_1} + A_{\chi_2})(\Lambda_1 + \Lambda_2) < 1$, so the property (iv) of Theorem 3 verified. According to all above conclusions, the Dhage's Theorem 3, implies that the hybrid $\mathcal{ABC}$-$\mathbb{FPS}$ (1.4) possesses a solution in Σ. □

In what follows, the uniqueness of solution for a coupled hybrid system (1.4) will establish by utilizing Perov's $\mathbb{FPT}$.

Theorem 5. *Let the conditions* $(\mathbb{H}_1)$–$(\mathbb{H}_5)$ *are fulfilled. If* $\mathbb{M}$ *is positive square matrix of spectral radius* $\varrho(\mathbb{M}) < 1$, *such that*

$$\mathbb{M} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}, \tag{3.8}$$

where

$$\begin{aligned}
e_{11} &= \mathfrak{z} A_{\chi_1} L^*_{\omega_1} + \frac{2\mathfrak{z}(2A_{\mathtt{Q}_1} L^*_{\chi_1} + A_{\chi_1} L^*_{\mathtt{Q}_1})(2-\sigma_1)}{|\mho(\sigma_1 - 1)|} + \frac{2\mathfrak{z}^{\sigma_1}(2A_{\mathtt{Q}_1} L^*_{\chi_1} + A_{\chi_1} L^*_{\mathtt{Q}_1})(\sigma_1 - 1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1 - 1)|}, \\
e_{12} &= \mathfrak{z} A_{\chi_1} L^*_{\omega_1} + A_{\omega_1} L^*_{\chi_1} + \frac{2\mathfrak{z}(2A_{\mathtt{Q}_1} L^*_{\chi_1} + A_{\chi_1} L^*_{\mathtt{Q}_1})(2-\sigma_1)}{|\mho(\sigma_1 - 1)|} + \frac{2\mathfrak{z}^{\sigma_1}(2A_{\mathtt{Q}_1} L^*_{\chi_1} + A_{\chi_1} L^*_{\mathtt{Q}_1})(\sigma_1 - 1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1 - 1)|}; \\
e_{21} &= \mathfrak{z} A_{\chi_2} L^*_{\omega_2} + A_{\omega_2} L^*_{\chi_2} + \frac{2\mathfrak{z}(2A_{\mathtt{Q}_2} L^*_{\chi_2} + A_{\chi_2} L^*_{\mathtt{Q}_2})(2-\sigma_2)}{|\mho(\sigma_2 - 1)|} \\
&\quad + \frac{2\mathfrak{z}^{\sigma_2}(2A_{\mathtt{Q}_2} L^*_{\chi_2} + A_{\chi_2} L^*_{\mathtt{Q}_2})(\sigma_2 - 1)}{\Gamma(1+\sigma_2)|\mho(\sigma_2 - 1)|}, \\
e_{22} &= \mathfrak{z} A_{\chi_2} L^*_{\omega_2} + \frac{2\mathfrak{z}(2A_{\mathtt{Q}_2} L^*_{\chi_2} + A_{\chi_2} L^*_{\mathtt{Q}_2})(2-\sigma_2)}{|\mho(\sigma_2 - 1)|} + \frac{2\mathfrak{z}^{\sigma_2}(2A_{\mathtt{Q}_2} L^*_{\chi_2} + A_{\chi_2} L^*_{\mathtt{Q}_2})(\sigma_2 - 1)}{\Gamma(1+\sigma_2)|\mho(\sigma_2 - 1)|}.
\end{aligned}$$

Then, the hybrid $\mathcal{ABC}$–$\mathbb{FPS}$ (1.4) *admits one solution in* Σ.

Proof. We define the operator $\Pi : \Sigma \to \Sigma$ by $\Pi(\mathrm{x}, \mathrm{y}) = (\Pi_1, \Pi_2)(\mathrm{x}, \mathrm{y})$, where,

$$\begin{aligned}
\big(\Pi_1(\mathrm{x}, \mathrm{y})\big)(u) &= \chi_1\big(u, \mathrm{x}(u), \mathrm{y}(u)\big)\Big(\int_0^{\mathfrak{z}} \frac{u\omega_1(v, \mathrm{y}(v))}{\mathfrak{z}} \, \mathrm{d}v \\
&\quad - \frac{2-\sigma_1}{\mho(\sigma_1 - 1)} \Big[\int_0^u \widehat{\mathtt{Q}}_{1;\mu}(v, \mathrm{x}) \, \mathrm{d}v - \int_0^{\mathfrak{z}} \frac{u\widehat{\mathtt{Q}}_{1;\mu}(v, \mathrm{x})}{\mathfrak{z}} \, \mathrm{d}v \Big] \\
&\quad - \frac{\sigma_1 - 1}{\mho(\sigma_1 - 1)} \Big[{}^{R}\mathfrak{I}_0^{\sigma_1} \widehat{\mathtt{Q}}_{1;\mu}(u, \mathrm{x}) - \tfrac{u}{\mathfrak{z}} {}^{R}\mathfrak{I}_0^{\sigma_1} \widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z}, \mathrm{x}) \Big] \Big),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\big(\Pi_2(\mathrm{x}, \mathrm{y})\big)(u) &= \chi_2\big(u, \mathrm{x}(u), \mathrm{y}(u)\big)\Big(\int_0^{\mathfrak{z}} \frac{u\omega_2(v, \mathrm{x}(v))}{\mathfrak{z}} \, \mathrm{d}v \\
&\quad - \frac{2-\sigma_2}{\mho(\sigma_2 - 1)} \Big[\int_0^u \widehat{\mathtt{Q}}_{2;\mu}(v, \mathrm{y}) \, \mathrm{d}v - \int_0^{\mathfrak{z}} \frac{u\widehat{\mathtt{Q}}_{2;\mu}(v, \mathrm{y})}{\mathfrak{z}} \, \mathrm{d}v \Big] \\
&\quad - \frac{\sigma_2 - 1}{\mho(\sigma_2 - 1)} \Big[{}^{R}\mathfrak{I}_0^{\sigma_2} \widehat{\mathtt{Q}}_{2;\mu}(u, \mathrm{y}) - \tfrac{u}{\mathfrak{z}} {}^{R}\mathfrak{I}_0^{\sigma_2} \widehat{\mathtt{Q}}_{2;\mu}(\mathfrak{z}, \mathrm{y}) \Big] \Big).
\end{aligned} \tag{3.10}$$

Our proof based on the Perov's $\mathbb{FPT}$, so we prove that Π has exactly one fixed point, which represents a solution of the system (1.4). Now, for any

$(\mathrm{x}_1,\mathrm{y}_1),(\mathrm{x}_2,\mathrm{y}_2)\in\Sigma$, $u\in\mathcal{J}$, and using $(\mathbb{H}_1)$–$(\mathbb{H}_5)$, we have,

$$
\begin{aligned}
&\left\|\big(\Pi_1(\mathrm{x}_1,\mathrm{y}_1)\big)(u)-\big(\Pi_1(\mathrm{x}_2,\mathrm{y}_2)\big)(u)\right\| \le \Big|\chi_1(u,\mathrm{x}_1(u),\mathrm{y}_1(u))\\
&\quad-\chi_1(u,\mathrm{x}_2(u),\mathrm{y}_2(u))\Big|\Big(\int_0^{\mathfrak{z}}\tfrac{u|\omega_1(v,\mathrm{y}_2(v))|}{\mathfrak{z}}\,\mathrm{d}v+\tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|}\Big[\int_0^{u}|\widehat{\mathtt{Q}}_{1;\mu}(v,\mathrm{x}_1)|\,\mathrm{d}v\\
&\qquad+\int_0^{\mathfrak{z}}\tfrac{u\widehat{\mathtt{Q}}_{1;\mu}(v,\mathrm{x}_2)}{\mathfrak{z}}\,\mathrm{d}v\Big]+\tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|}\Big[{}^{R}\mathfrak{I}_0^{\sigma_1}|\widehat{\mathtt{Q}}_{1;\mu}(u,\mathrm{x}_2)|+\tfrac{u}{\mathfrak{z}}{}^{R}\mathfrak{I}_0^{\sigma_1}|\widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z},\mathrm{x}_2)|\Big]\Big)\\
&\qquad+|\chi_1(u,\mathrm{x}_1(u),\mathrm{y}_1(u))|\Big\{\int_0^{\mathfrak{z}}\tfrac{u|\omega_1(v,\mathrm{y}_1(v))-\omega_1(v,\mathrm{y}_2(v))|}{\mathfrak{z}}\,\mathrm{d}v\\
&\qquad+\tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|}\Big[\int_0^{u}|\widehat{\mathtt{Q}}_{1;\mu}(v,\mathrm{x}_1)-\widehat{\mathtt{Q}}_{1;\mu}(v,\mathrm{x}_2)|\,\mathrm{d}v+\int_0^{\mathfrak{z}}\tfrac{u|\widehat{\mathtt{Q}}_{1;\mu}(v,\mathrm{x}_1)-\widehat{\mathtt{Q}}_{1;\mu}(v,\mathrm{x}_2)|}{\mathfrak{z}}\,\mathrm{d}v\Big]\\
&\quad+\tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|}\Big[{}^{R}\mathfrak{I}_0^{\sigma_1}|\widehat{\mathtt{Q}}_{1;\mu}(u,\mathrm{x}_1)-\widehat{\mathtt{Q}}_{1;\mu}(u,\mathrm{x}_2)|+\tfrac{u}{\mathfrak{z}}{}^{R}\mathfrak{I}_0^{\sigma_1}|\widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z},\mathrm{x}_1)-\widehat{\mathtt{Q}}_{1;\mu}(\mathfrak{z},\mathrm{x}_2)|\Big]\Big\}\\
&\quad\le A_{\chi_1}\left(|\mathrm{x}_1(u)-\mathrm{x}_2(u)|+|\mathrm{y}_1(u)-\mathrm{y}_2(u)|\right)\Big\{\int_0^{\mathfrak{z}}\tfrac{uL_{\omega_1}(v)}{\mathfrak{z}}\,\mathrm{d}v+\tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|}\\
&\quad\times\Big[\int_0^{u}L_{\mathtt{Q}_1}(v)\,\mathrm{d}v+\int_0^{\mathfrak{z}}\tfrac{uL_{\mathtt{Q}_1}(v)}{\mathfrak{z}}\,\mathrm{d}v\Big]+\tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|}\Big[{}^{R}\mathfrak{I}_0^{\sigma_1}L_{\mathtt{Q}_1}(u)+\tfrac{u}{\mathfrak{z}}{}^{R}\mathfrak{I}_0^{\sigma_1}L_{\mathtt{Q}_1}(\mathfrak{z})\Big]\Big\}\\
&\quad+L_{\chi_1}(u)\Big\{\int_0^{\mathfrak{z}}\tfrac{uA_{\omega_1}|\mathrm{y}_1(v)-\mathrm{y}_2(v)|}{\mathfrak{z}}\,\mathrm{d}v+\tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|}\Big[\int_0^{u}A_{\mathtt{Q}_1}\Big(|\mathrm{x}_1(v)-\mathrm{x}_2(v)|\\
&\quad+|\mathrm{x}_1(\mu v)-\mathrm{x}_2(\mu v)|+|\mathrm{y}_1(v)-\mathrm{y}_2(v)|\Big)\,\mathrm{d}v\\
&\qquad+\int_0^{\mathfrak{z}}\tfrac{u}{\mathfrak{z}}A_{\mathtt{Q}_1}\Big(|\mathrm{x}_1(v)-\mathrm{x}_2(v)|+|\mathrm{x}_1(\mu v)-\mathrm{x}_2(\mu v)|+|\mathrm{y}_1(v)-\mathrm{y}_2(v)|\Big)\,\mathrm{d}v\Big]\\
&\qquad+\tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|}\Big[{}^{R}\mathfrak{I}_0^{\sigma_1}\Big[A_{\mathtt{Q}_1}\Big(|\mathrm{x}_1(u)-\mathrm{x}_2(u)|+|\mathrm{x}_1(\mu u)-\mathrm{x}_2(\mu u)|\\
&\qquad+|\mathrm{y}_1(u)-\mathrm{y}_2(u)|\Big)\Big]\\
&\quad+\tfrac{u}{\mathfrak{z}}{}^{R}\mathfrak{I}_0^{\sigma_1}\Big[A_{\mathtt{Q}_1}\Big(|\mathrm{x}_1(\mathfrak{z})-\mathrm{x}_2(\mathfrak{z})|+|\mathrm{x}_1(\mu\mathfrak{z})-\mathrm{x}_2(\mu\mathfrak{z})|+|\mathrm{y}_1(\mathfrak{z})-\mathrm{y}_2(\mathfrak{z})|\Big)\Big]\Big]\Big\}\\
&\quad\le A_{\chi_1}\left(\|\mathrm{x}_1-\mathrm{x}_2\|+\|\mathrm{y}_1-\mathrm{y}_2\|\right)\Big[\mathfrak{z}L^*_{\omega_1}+\tfrac{2\mathfrak{z}L^*_{\mathtt{Q}_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|}+\tfrac{2\mathfrak{z}^{\sigma_1}L^*_{\mathtt{Q}_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|}\Big]\\
&\qquad+L^*_{\chi_1}\Big(A_{\omega_1}\|\mathrm{y}_1-\mathrm{y}_2\|+\tfrac{2\mathfrak{z}(2-\sigma_1)}{|\mho(\sigma_1-1)|}\Big[A_{\mathtt{Q}_1}\left(2\|\mathrm{x}_1-\mathrm{x}_2\|+\|\mathrm{y}_1-\mathrm{y}_2\|\right)\Big]\\
&\qquad+\tfrac{2\mathfrak{z}^{\sigma_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|}\Big[A_{\mathtt{Q}_1}\left(2\|\mathrm{x}_1-\mathrm{x}_2\|+\|\mathrm{y}_1-\mathrm{y}_2\|\right)\Big]\Big)\\
&\quad\le\left(\|\mathrm{x}_1-\mathrm{x}_2\|+\|\mathrm{y}_1-\mathrm{y}_2\|\right)\Big[\mathfrak{z}L^*_{\omega_1}A_{\chi_1}+\tfrac{2\mathfrak{z}L^*_{\mathtt{Q}_1}A_{\chi_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|}+\tfrac{2\mathfrak{z}^{\sigma_1}L^*_{\mathtt{Q}_1}A_{\chi_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|}\Big]\\
&\qquad+L^*_{\chi_1}A_{\omega_1}\|\mathrm{y}_1-\mathrm{y}_2\|+\tfrac{4\mathfrak{z}L^*_{\chi_1}A_{\mathtt{Q}_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|}\left(\|\mathrm{x}_1-\mathrm{x}_2\|+\|\mathrm{y}_1-\mathrm{y}_2\|\right)\\
&\qquad+\tfrac{4\mathfrak{z}^{\sigma_1}L^*_{\chi_1}A_{\mathtt{Q}_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|}\left(\|\mathrm{x}_1-\mathrm{x}_2\|+\|\mathrm{y}_1-\mathrm{y}_2\|\right)\le e_{11}\|\mathrm{x}_1-\mathrm{x}_2\|+e_{12}\|\mathrm{y}_1-\mathrm{y}_2\|.
\end{aligned}
$$

Therefore,

$$\left\|\Pi_1(\mathrm{x}_1,\mathrm{y}_1)-\Pi_1(\mathrm{x}_2,\mathrm{y}_2)\right\|\le e_{11}\|\mathrm{x}_1-\mathrm{y}_2\|+e_{12}\|\mathrm{x}_1-\mathrm{y}_2\|. \tag{3.11}$$

Similarly, we find

$$\left\|\Pi_2(\mathrm{x}_1,\mathrm{y}_1) - \Pi_2(\mathrm{x}_2,\mathrm{y}_2)\right\| \leq e_{21}\|\mathrm{x}_1 - \mathrm{y}_2\| + e_{22}\|\mathrm{x}_1 - \mathrm{y}_2\|. \tag{3.12}$$

Hence, by (3.11) and (3.12), yield that

$$\begin{pmatrix} \left\|\Pi_1(\mathrm{x}_1,\mathrm{y}_1) - \Pi_1(\mathrm{x}_2,\mathrm{y}_2)\right\| \\ \left\|\Pi_2(\mathrm{x}_1,\mathrm{y}_1) - \Pi_2(\mathrm{x}_2,\mathrm{y}_2)\right\| \end{pmatrix} \leq \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} \left\|\mathrm{x}_1 - \mathrm{x}_2\right\| \\ \left\|\mathrm{y}_1 - \mathrm{y}_2\right\| \end{pmatrix},$$

which implies that,

$$\mathfrak{d}\left(\Pi(\mathrm{x}_1,\mathrm{y}_1), \Pi(\mathrm{x}_2,\mathrm{y}_2)\right) \leq \mathbb{M}\,\mathfrak{d}\left((\mathrm{x}_1,\mathrm{y}_1),(\mathrm{x}_2,\mathrm{y}_2)\right).$$

Since $\varrho(\mathbb{M}) < 1$, then each conditions of the Perov's theorem are satisfied. Thus, the hybrid $\mathcal{ABC}$-$\mathbb{F}$PS (1.4) possesses a unique solution in Σ. □

4 $\mathbb{HU}$ stability

Throughout this section, we will establish the $\mathbb{HU}$ stability of hybrid $\mathcal{ABC}$–$\mathbb{F}$PS (1.4), by using its integral solution which given by:

$$\mathrm{x}(u) = \Pi_1(\mathrm{x},\mathrm{y})(u), \qquad \mathrm{y}(u) = \Pi_2(\mathrm{x},\mathrm{y})(u),$$

such that Π_1 and Π_2 are defined in (3.9) and (3.10), respectively. Additionally, we suppose that operators $\mathbb{K}_1, \mathbb{K}_2 : \Sigma \to \mathfrak{C}(\mathcal{J},\mathbb{R})$ satisfy the following identities:

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_1}\left[\frac{\tilde{\mathrm{x}}(u)}{\chi_1(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u))}\right] + \mathsf{Q}_1(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{x}}(\mu u),\tilde{\mathrm{y}}(u)) = \mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u), \\ {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_2}\left[\frac{\tilde{\mathrm{y}}(u)}{\chi_2(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u))}\right] + \mathsf{Q}_2(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u),\tilde{\mathrm{y}}(\mu u)) = \mathbb{K}_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u), \end{cases} \tag{4.1}$$

for $u \in \mathcal{J}$. Additionally, we suppose that, for $u \in \mathcal{J}$, and some $\epsilon_1, \epsilon_2 > 0$,

$$\left\|\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)\right\| \leq \epsilon_1, \qquad \left\|\mathbb{K}_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)\right\| \leq \epsilon_2.$$

Definition 4 [[27]]. The hybrid $\mathcal{ABC}$–$\mathbb{F}$PS (1.4) is $\mathbb{HU}$ stable if there are constants $\varsigma_i > 0, (i = 1, 2, 34)$ where $\forall \epsilon_1, \epsilon_2 > 0$ and for all solution $(\tilde{\mathrm{x}},\tilde{\mathrm{y}}) \in \Sigma$ of inequality (4.1), there is a solution $(\mathrm{x},\mathrm{y}) \in \Sigma$ of (1.4), where for $u \in \mathcal{J}$,

$$\begin{cases} \left\|\tilde{\mathrm{x}}(u) - \mathrm{x}(u)\right\| \leq \varsigma_1\epsilon_1 + \varsigma_2\epsilon_2, \\ \left\|\tilde{\mathrm{y}}(u) - \mathrm{y}(u)\right\| \leq \varsigma_3\epsilon_1 + \varsigma_4\epsilon_2. \end{cases}$$

Theorem 6. *Consider the assumptions of Theorem 5 hold. Then, the hybrid* $\mathcal{ABC}$–$\mathbb{F}$PS (1.4) *is* $\mathbb{HU}$ *stable.*

Proof. Let $(\mathrm{x},\mathrm{y}) \in \Sigma$ is the solution of hybrid $\mathcal{ABC}$-$\mathbb{F}$PS (1.4) verifying (3.9) and (3.10). Also, assume that $(\tilde{\mathrm{x}},\tilde{\mathrm{y}})$ is any solution verifying (4.1) and

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_1}\left[\frac{\tilde{\mathrm{x}}(u)}{\chi_1(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u))}\right] + \mathsf{Q}_1(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{x}}(\mu u),\tilde{\mathrm{y}}(u)) - \mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u) = 0, \\ {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_2}\left[\frac{\tilde{\mathrm{y}}(u)}{\chi_2(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u))}\right] + \mathsf{Q}_2(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u),\tilde{\mathrm{y}}(\mu u)) - \mathbb{K}_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u) = 0, \end{cases}$$

for $u \in \mathcal{J}$. So,

$$\begin{aligned}\tilde{\mathrm{x}}(u) &= \Pi_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u) + \chi_1(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u))\Big(\tfrac{2-\sigma_1}{\mho(\sigma_1-1)}\Big[\int_0^u \mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(v)\,\mathrm{d}v \\ &\quad + \int_0^{\mathfrak{z}} \tfrac{u\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(v)}{\mathfrak{z}}\,\mathrm{d}v\Big] + \tfrac{\sigma_1-1}{\mho(\sigma_1-1)}\Big[{}^R\mathfrak{I}_0^{\sigma_1}\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)+\tfrac{u}{\mathfrak{z}}{}^R\mathfrak{I}_0^{\sigma_1}\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)\Big]\Big),\\ \tilde{\mathrm{y}}(u) &= \Pi_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)+\chi_2(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u))\Big(\tfrac{2-\sigma_2}{\mho(\sigma_2-1)}\Big[\int_0^u \mathbb{K}_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(v)\,\mathrm{d}v \\ &\quad + \int_0^{\mathfrak{z}} \tfrac{u\mathbb{K}_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(v)}{\mathfrak{z}}\,\mathrm{d}v\Big] + \tfrac{\sigma_2-1}{\mho(\sigma_2-1)}\Big[{}^R\mathfrak{I}_0^{\sigma_2}\mathbb{K}_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)+\tfrac{u}{\mathfrak{z}}{}^R\mathfrak{I}_0^{\sigma_2}\mathbb{K}_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)\Big]\Big).\end{aligned}$$

It implies that

$$\begin{aligned}&\big\|\tilde{\mathrm{x}}(u) - \Pi_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)\big\| \le |\chi_1(u,\tilde{\mathrm{x}}(u),\tilde{\mathrm{y}}(u))|\Big\{\tfrac{2-\sigma_1}{|\mho(\sigma_1-1)|}\Big[\int_0^u |\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(v)|\,\mathrm{d}v \\ &\quad + \int_0^{\mathfrak{z}} \tfrac{u|\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(v)|}{\mathfrak{z}}\,\mathrm{d}v\Big] + \tfrac{\sigma_1-1}{|\mho(\sigma_1-1)|}\Big[{}^R\mathfrak{I}_0^{\sigma_1}|\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)| + \tfrac{u}{\mathfrak{z}}{}^R\mathfrak{I}_0^{\sigma_1}|\mathbb{K}_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(\mathfrak{z})|\Big]\Big\} \\ &\le L^*_{\chi_1}\Big[\tfrac{2\mathfrak{z}\epsilon_1(2-\sigma_1)}{|\mho(\sigma_1-1)|} + \tfrac{2\mathfrak{z}^{\sigma_1}\epsilon_1(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|}\Big] \le L^*_{\chi_1}\mathbb{V}_1\ \epsilon_1, \end{aligned} \tag{4.2}$$

and

$$\big\|\tilde{\mathrm{y}}(u) - \Pi_2(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)\big\| \le L^*_{\chi_2}\Big[\tfrac{2\mathfrak{z}\epsilon_2(2-\sigma_2)}{|\mho(\sigma_2-1)|} + \tfrac{2\mathfrak{z}^{\sigma_2}\epsilon_1(\sigma_2-1)}{\Gamma(1+\sigma_2)|\mho(\sigma_2-1)|}\Big] \le L^*_{\chi_2}\mathbb{V}_2\ \epsilon_2, \tag{4.3}$$

where

$$\mathbb{V}_i := \frac{2\mathfrak{z}(2-\sigma_i)}{|\mho(\sigma_i-1)|} + \frac{2\mathfrak{z}^{\sigma_i}(\sigma_i-1)}{\Gamma(1+\sigma_i)|\mho(\sigma_i-1)|}, \quad i=1,2.$$

Then, according to inequalities (4.2) and (4.3), we have

$$\begin{aligned}\big\|\tilde{\mathrm{x}}(u) - \mathrm{x}(u)\big\| &= \big\|\tilde{\mathrm{x}}(u) - \Pi_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u) + \Pi_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u) - \mathrm{x}(u)\big\| \\ &\le \big\|\tilde{\mathrm{x}}(u) - \Pi_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u)\big\| + \big\|\Pi_1(\tilde{\mathrm{x}},\tilde{\mathrm{y}})(u) - \Pi_1(\mathrm{x},\mathrm{y})(u)\big\| \\ &\le L^*_{\chi_1}\mathbb{V}_1\ \epsilon_1 + \big(e_{11}\|\tilde{\mathrm{x}}-\mathrm{x}\| + e_{12}\|\tilde{\mathrm{y}}-\mathrm{y}\|\big).\end{aligned}$$

Hence, we find

$$\|\tilde{\mathrm{x}}-\mathrm{x}\| \le L^*_{\chi_1}\mathbb{V}_1\ \epsilon_1 + \big(e_{11}\|\tilde{\mathrm{x}}-\mathrm{x}\| + e_{12}\|\tilde{\mathrm{y}}-\mathrm{y}\|\big).$$

Similarly, we get $\|\tilde{\mathrm{y}}-\mathrm{y}\| \le L^*_{\chi_2}\mathbb{V}_2\ \epsilon_2 + \big(e_{21}\|\tilde{\mathrm{x}}-\mathrm{x}\| + e_{22}\|\tilde{\mathrm{y}}-\mathrm{y}\|\big)$. These inequalities, can be reformulate as

$$(\mathbb{I}-\mathbb{M})\begin{pmatrix}\|\tilde{\mathrm{x}}-\mathrm{x}\|\\ \|\tilde{\mathrm{y}}-\mathrm{y}\|\end{pmatrix} \le \begin{pmatrix}L^*_{\chi_1}\mathbb{V}_1\epsilon_1\\ L^*_{\chi_2}\mathbb{V}_2\epsilon_2\end{pmatrix}, \tag{4.4}$$

where the matrix $\mathbb{M}$ is given by (3.8) with spectral radius $\varrho(\mathbb{M}) < 1$. Thus, by Theorem 1, we conclude that $(\mathbb{I}-\mathbb{M})$ is nonsingular, and $(\mathbb{I}-\mathbb{M})^{-1}$ has non-negative components. Hence, (4.4) is equivalent to the following format:

$$\begin{pmatrix}\|\tilde{\mathrm{x}}-\mathrm{x}\|\\ \|\tilde{\mathrm{y}}-\mathrm{y}\|\end{pmatrix} \le (\mathbb{I}-\mathbb{M})^{-1}\begin{pmatrix}L^*_{\chi_1}\mathbb{V}_1\epsilon_1\\ L^*_{\chi_2}\mathbb{V}_2\epsilon_2\end{pmatrix},$$

which follows that,

$$\|\tilde{\mathrm{x}} - \mathrm{x}\| \leq q_1 L^*_{\chi_1} \mathbb{V}_1 \epsilon_1 + q_2 L^*_{\chi_2} \mathbb{V}_2 \epsilon_2,$$
$$\|\tilde{\mathrm{y}} - \mathrm{y}\| \leq q_3 L^*_{\chi_1} \mathbb{V}_1 \epsilon_1 + q_4 L^*_{\chi_2} \mathbb{V}_2 \epsilon_2,$$

such that $q_j, j = 1, 2, 3, 4$ are the components of $(\mathbb{I} - \mathbb{M})^{-1}$. Hence, the hybrid $\mathcal{ABC}$–$\mathbb{F}$PS (1.4) is $\mathbb{HU}$ stable. □

5 Illustrative application

We present in this section one concrete example to illustrate the validity of the main results. In this regard, first in Example 1, we check the accuracy of our outcomes for different values of σ_1.

Example 1. Let the hybrid $\mathcal{ABC}$–$\mathbb{F}$PS is given as follows:

$$\begin{cases} {}^{\mathcal{ABC}}\mathfrak{D}_0^{\sigma_1}\left[\dfrac{\mathrm{x}(u)}{\frac{1}{4} + \frac{\sqrt{u}}{16\left(1+|\mathrm{x}(u)|\right)} + \frac{e^u}{16}\left|\cos^{-1}\left(\mathrm{y}(u)\right)\right|}\right] \\ \quad = -\dfrac{1}{25}\tan^{-1}\left(\mathrm{x}(u)\right) - \dfrac{\left|\mathrm{x}\left(u/3\right)\right|}{49\left(1+\left|\mathrm{x}\left(u/3\right)\right|\right)} - \dfrac{u}{25}\sin^{-1}\left(\mathrm{y}(u)\right), \\ {}^{\mathcal{ABC}}\mathfrak{D}_0^{5/3}\left[\dfrac{\mathrm{y}(u)}{\frac{1}{7}u + \frac{\sqrt{u}}{81}\left|\tan^{-1}\left(\mathrm{x}(u)\right)\right| + \frac{\sqrt{u}|\mathrm{y}(u)|}{81\left(3+|\mathrm{y}(u)|\right)}}\right] \\ \quad = -\dfrac{1}{16+\sqrt{u}}\sin^{-1}\left(\mathrm{x}(u)\right) - \dfrac{|\mathrm{y}(u)|}{9\left(1+|\mathrm{y}(u)|\right)} - \dfrac{1}{16}\tan^{-1}\left(\mathrm{y}\left(u/3\right)\right), \end{cases} \tag{5.1}$$

for $u \in \mathcal{J}$, $\mathfrak{z} = 1$, via conditions $\mathrm{x}(0) = \mathrm{y}(0) = 0$,

$$\begin{cases} \dfrac{\mathrm{x}(1)}{\frac{1}{4}+\frac{1}{16}\frac{1}{(1+|\mathrm{x}(1)|)}+\frac{e^1}{16}|\cos^{-1}(\mathrm{y}(1))|} = \displaystyle\int_0^1 \left(\frac{e^v}{12} + \frac{1}{144}\frac{\mathrm{y}(v)}{(1+\mathrm{y}(v))}\right) dv, \\ \dfrac{\mathrm{y}(1)}{\frac{1}{7}+\frac{1}{81}|\tan^{-1}(\mathrm{x}(1))|+\frac{1}{81}\frac{|\mathrm{y}(1)|}{(3+|\mathrm{y}(1)|)}} = \displaystyle\int_0^1 \left(\frac{\sqrt{v}}{5} + \frac{1}{14}\frac{\mathrm{x}(v)}{(3+\mathrm{x}(v))}\right) dv, \end{cases}$$

where $\sigma_1 = \left\{\frac{20}{17}, \frac{10}{9}, \frac{20}{19}, 2\right\} \subset (1, 2]$, $\sigma_2 = \frac{5}{3} \in (1, 2]$, $\mu = \frac{1}{3} \in (0, 1)$,

$$\chi_1\left(u, \mathrm{x}(u), \mathrm{y}(u)\right) = \tfrac{1}{4} + \tfrac{\sqrt{u}}{16}\tfrac{1}{(1+|\mathrm{x}(u)|)} + \tfrac{e^u}{16}|\cos^{-1}(\mathrm{y}(u))|,$$
$$\chi_2\left(u, \mathrm{x}(u), \mathrm{y}(u)\right) = \frac{u}{7} + \tfrac{\sqrt{u}}{81}|\tan^{-1}\left(\mathrm{x}(u)\right)| + \frac{\sqrt{u}}{81}\frac{|\mathrm{y}(u)|}{(3+|\mathrm{y}(u)|)},$$
$$\omega_1\left(u, \mathrm{y}(u)\right) = \frac{e^u}{12} + \frac{\mathrm{y}(u)}{144(1+\mathrm{y}(u))}, \quad \omega_2\left(u, \mathrm{x}(u)\right) = \frac{\sqrt{u}}{5} + \frac{\mathrm{x}(u)}{14(3+\mathrm{x}(u))},$$

and

$$\widehat{\mathtt{Q}}_{1;\mu}(u,\mathrm{x}) = \tfrac{1}{25}\tan^{-1}\left(\mathrm{x}(u)\right) + \frac{\left|\mathrm{x}\left(u/3\right)\right|}{49\left(1+\left|\mathrm{x}\left(u/3\right)\right|\right)} + \tfrac{u}{25}\sin^{-1}\left(\mathrm{y}(u)\right),$$

$$\widehat{\mathtt{Q}}_{2;\mu}(u,\mathrm{y}) = \tfrac{1}{16+\sqrt{u}}\sin^{-1}\left(\mathrm{x}(u)\right) + \tfrac{1}{9}\tfrac{|\mathrm{y}(u)|}{(1+|\mathrm{y}(u)|)} + \tfrac{1}{16}\tan^{-1}\left(\mathrm{y}\left(\tfrac{u}{3}\right)\right).$$

Thus, we get

$$|\mathtt{Q}_1(u,\mathrm{x},\tilde{\mathrm{y}},\mathrm{y})| \le \frac{1}{25}+\frac{1}{49}+\frac{u}{25},\quad \mathtt{Q}_2(u,\mathrm{x},\tilde{\mathrm{y}},\mathrm{y})| \le \frac{1}{16+\sqrt{u}}+\frac{25}{144},$$

here $L_{\mathtt{Q}_1}(u) = \frac{1}{25}+\frac{1}{49}+\frac{u}{25}$, $L^*_{\mathtt{Q}_1} = \frac{123}{1225}$, $L_{\mathtt{Q}_2}(u) = \frac{1}{16+\sqrt{u}}+\frac{25}{144}$, $L^*_{\mathtt{Q}_2} = \frac{17}{72}$. Also,

$$|\omega_1(u,\mathrm{x})| = \left|\frac{e^u}{12}+\frac{1}{144}\frac{\mathrm{y}(u)}{(1+\mathrm{y}(u))}\right| \le L_{\omega_1}(u),$$

$$|\omega_2(u,\mathrm{x})| = \left|\frac{\sqrt{u}}{5}+\frac{1}{14}\frac{\mathrm{x}(u)}{(3+\mathrm{x}(u))}\right| \le L_{\omega_2}(u),$$

then $L_{\omega_1}(u) = \frac{e^u}{12}+\frac{1}{144}$, and $L_{\omega_2}(u) = \frac{\sqrt{u}}{5}+\frac{1}{14}$, which these follow that $L^*_{\omega_1} \simeq 0.233468$, $L^*_{\omega_2} = \frac{19}{70}$, and

$$|\chi_1(u,\mathrm{x}_1,\mathrm{y}_1)-\chi_1(u,\mathrm{x}_2,\mathrm{y}_2)| \le \frac{\sqrt{u}}{16}\left|\frac{1}{1+|\mathrm{x}_1(u)|}-\frac{1}{1+|\mathrm{x}_2(u)|}\right| + \frac{e^u}{16}\Big|\left|\cos^{-1}\left(\mathrm{y}_1(u)\right)\right| - \left|\cos^{-1}\left(\mathrm{y}_2(u)\right)\right|\Big| \le A_{\chi_1}\left(|\mathrm{x}_1-\mathrm{x}_2|+|\mathrm{y}_1-\mathrm{y}_2|\right),$$

$$|\chi_2(u,\mathrm{x}_1,\mathrm{y}_1)-\chi_2(u,\mathrm{x}_2,\mathrm{y}_2)| \le \frac{\sqrt{u}}{81}\left|\frac{\sqrt{u}}{81}\right|\left|\tan^{-1}\left(\mathrm{x}_1(u)\right)\right| - \left|\tan^{-1}\left(\mathrm{x}_2(u)\right)\right|\Big| + \frac{\sqrt{u}}{81}\left|\frac{|\mathrm{y}_1(u)|}{(3+|\mathrm{y}_1(u)|)}-\frac{|\mathrm{y}_2(u)|}{(3+|\mathrm{y}_2(u)|)}\right| \le A_{\chi_2}\left(|\mathrm{x}_1-\mathrm{x}_2|+|\mathrm{y}_1-\mathrm{y}_2|\right).$$

Thus, $A_{\chi_1} = \frac{e}{16}$, $A_{\chi_2} = \frac{1}{81}$. Next, by Mathematica software, employing (3.5), geting data and taking nonmalized function $\mho(\sigma) = 1.25-(\sigma-0.5)^2$, we have

$$\Lambda_1 = \mathfrak{z}L^*_{\omega_1} + \frac{2\mathfrak{z}L^*_{\mathtt{Q}_1}(2-\sigma_1)}{|\mho(\sigma_1-1)|} + \frac{2\mathfrak{z}^{\sigma_1}L^*_{\mathtt{Q}_1}(\sigma_1-1)}{\Gamma(1+\sigma_1)|\mho(\sigma_1-1)|} \simeq \begin{cases} 0.483284, & \sigma_1 = 20/17,\\ 0.461306, & \sigma_1 = 10/9,\\ 0.445807, & \sigma_1 = 20/19,\\ 0.333876, & \sigma_1 = 2,\end{cases}$$

and

$$\Lambda_2 = \mathfrak{z}L^*_{\omega_2} + \frac{2\mathfrak{z}L^*_{\mathtt{Q}_2}(2-\sigma_2)}{|\mho(\sigma_2-1)|} + \frac{2\mathfrak{z}^{\sigma_2}L^*_{\mathtt{Q}_2}(\sigma_2-1)}{\Gamma(1+\sigma_2)|\mho(\sigma_2-1)|} \simeq 3.57124.$$

Then,

$$(A_{\chi_1}+A_{\chi_2})(\Lambda_1+\Lambda_2) \simeq \left\{\begin{array}{ll} 0.73889, & \sigma_1 = 20/17,\\ 0.73488, & \sigma_1 = 10/9,\\ 0.73206, & \sigma_1 = 20/19,\\ 0.71166, & \sigma_1 = 2,\end{array}\right\} < 1.$$

Thus, each conditions of Theorem 4 hold. Hence, the hybrid $\mathcal{ABC}$–$\mathbb{F}$PS (5.1) possesses at least one solution in Σ.

6 Conclusions

The $\mathbb{PE}$ has several applications in various fields, such as physics, applied and pure mathematics, probability, electrodynamics, quantum mechanics, control systems, and number theory. Throughout this work, we inspected a hybrid $\mathcal{ABC}$-$\mathbb{F}$PS under hybrid integral boundary conditions (1.4). Dhage's $\mathbb{FPT}$ was utilized to discuss the existence theorem of the hybrid $\mathcal{ABC}$–$\mathbb{F}$PS (1.4). Furthermore, the uniqueness theorem and $\mathbb{HU}$-stability of solutions for the proposed system were established by Lipschitz's matrix and Perov's $\mathbb{FPT}$. At the end, one example was provided to interpret the effectiveness of essential findings. The hybrid system (1.4) covers various problems that haven't been studied yet and includes several research studies existing in the literature as problems (1.1), (1.2), and (1.3).

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