ISSN: 1392-6292/ eISSN: 1648-3510

VILNIUS TECH Vilnius Gediminas Technical University

MATHEMATICAL MODELLING and ANALYSIS

2025
Volume 30
Issue 2
Pages 362–385
https://doi.org/10.3846/mma.2025.22230

Approximation of a system of nonlinear Carrier wave equations by approximating the Carrier terms with their integral sums

Le Thi Phuong Ngoc[a], Nguyen Vu Dzung[b,c,d] and Nguyen Thanh Long[b,c]

[a] *University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam*
[b] *Faculty of Mathematics and Computer Science, University of Science, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam*
[c] *Vietnam National University, Ho Chi Minh City, Vietnam*
[d] *Cao Thang Technical College, 65 Huynh Thuc Khang Street, Ben Nghe Ward, District 1, Ho Chi Minh City, Vietnam*

Article History:

- received September 17, 2024
- revised February 15, 2025
- accepted February 25, 2025

Abstract. This paper is concerned with the approximation of a system of nonlinear Carrier wave equations (CEs) by approximating the Carrier terms with their integral sums. At first, under suitable conditions, the linear approximate method, the Galerkin method, and compactness arguments provide the unique existence of a weak solution (u^n, v^n) of the problem (P_n), for each $n \in \mathbb{N}$, for a system of nonlinear wave equations related to Maxwell fluid between two infinite coaxial circular cylinders. Next, we prove that $\{(u^n, v^n)\}_n$ converges to the weak solution $(u, v)$of the problem for a system of CEs in a suitable function space. This proof is done by using the compactness lemma of Aubin-Lions and the method of continuity with a priori estimates. We end the paper with a remark related to open problems.

Keywords: system of nonlinear wave equations; the helical flows of Maxwell fluid; Kirchhoff-Carrier equations, Faedo-Galerkin method; linearization method; approximating the Carrier terms.

AMS Subject Classification: 35L20; 35L70; 35Q72.

Corresponding author. E-mail: `longnt2@gmail.com`

1 Introduction

In this paper, we consider the following initial boundary value problem

$$
(P_n)\begin{cases}
u_{tt} - a_1(S_n[u](t))(u_{xx} + \frac{1}{x}u_x - \frac{1}{x^2}u) \\
\qquad = f(x,t,u,v,u_t,v_t,u_x,v_x),\ x \in \Omega = (1,R),\ 0 < t < T, \\
v_{tt} - a_2(S_n[v](t))(v_{xx} + \frac{1}{x}v_x) \\
\qquad = g(x,t,u,v,u_t,v_t,u_x,v_x),\ x \in \Omega,\ 0 < t < T, \\
u_x(1,t) - b_1 u(1,t) = v_x(1,t) = u(R,t) = v(R,t) = 0, \\
(u(x,0), v(x,0)) = (\tilde{u}_0(x), \tilde{v}_0(x)), \\
(u_t(x,0), v_t(x,0)) = (\tilde{u}_1(x), \tilde{v}_1(x)),
\end{cases}
$$

$b_1 > 0, R > 1$ are given constants, $a_1, a_2, \tilde{u}_0, \tilde{u}_1, \tilde{v}_0, \tilde{v}_1, f, g$ are given functions,

$$S_n[u](t)=((R-1)/n)\sum_{i=1}^{n} x_{n,i}u^2(x_{n,i},t),\, S_n[v](t)=((R-1)/n)\sum_{i=1}^{n} x_{n,i}v^2(x_{n,i},t),$$

in which $x_{n,i} = 1 + (R-1)(2i-1)/2n, i = \overline{1,n}, \forall n \in \mathbb{N}$. By the fact that if the functions $x \mapsto xu^2(x,t)$ and $x \mapsto xv^2(x,t)$ are continuous on $[1, R]$, then the integral sums $S_n[u](t)$ and $S_n[v](t)$ converge to $\|u(t)\|_0^2 = \int_1^R xu^2(x,t)dx$ and $\|v(t)\|_0^2 = \int_1^R xv^2(x,t)dx$, respectively, Problem (P_n) will be formally led to the following problem

$$\begin{cases} u_{tt} - a_1(\left\|u(t)\right\|_0^2)(u_{xx} + \dfrac{1}{x}u_x - \dfrac{1}{x^2}u) \\ \qquad = f(x,t,u,v,u_t,v_t,u_x,v_x),\ x \in \Omega = (1,R),\ 0 < t < T, \\ v_{tt} - a_2(\left\|v(t)\right\|_0^2)(v_{xx} + \dfrac{1}{x}v_x) \\ \qquad = g(x,t,u,v,u_t,v_t,u_x,v_x),\ x \in \Omega,\ 0 < t < T, \\ u_x(1,t) - b_1u(1,t) = v_x(1,t) = u(R,t) = v(R,t) = 0, \\ (u(x,0), v(x,0)) = (\tilde{u}_0(x), \tilde{v}_0(x)), \\ (u_t(x,0), v_t(x,0)) = (\tilde{u}_1(x), \tilde{v}_1(x)). \end{cases} \tag{1.2}$$

Problems (P_n) and (1.2) here will be investigated from mathematical point of view in the existing literature for Maxwell fluid between two infinite coaxial circular cylinders. It is well known that there is a great interest of theoretical and applied scientists relating to the fluid flows in the neighborhood of translating or oscillating bodies, in which, Maxwell fluid has received special attention; see for [6]– [10], [21] and the references therein. In [9], M. Jamil and C. Fetecau studied the following problem

$$\begin{cases} \lambda u_{tt} + u_t = \nu(u_{xx} + \dfrac{1}{x}u_x - \dfrac{1}{x^2}u),\ 1 < x < R,\ t > 0, \\ \lambda V_{tt} + V_t = \nu(V_{xx} + \frac{1}{x}V_x),\ 1 < x < R,\ t > 0, \\ u_x(1,t) - u(1,t) = \dfrac{F}{\mu}t,\ V_x(1,t) = \dfrac{G}{\mu}t,\ t > 0, \\ u(R,t) = V(R,t) = 0,\ t > 0, \\ u(x,0) = u_t(x,0) = V(x,0) = V_t(x,0) = 0,\ 1 < x < R, \end{cases} \tag{1.3}$$

where λ, μ, ν, F, G are the given constants, this is a mathematical model describing the helical flows of Maxwell fluid in the annular region between two infinite coaxial circular cylinders of radii 1 and $R > 1$. The authors have obtained an exact solution for Problem (1.3) by means of finite Hankel transforms and presented under series form in terms of Bessel functions $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$, $J_2(x)$ and $Y_2(x)$, satisfying all imposed initial and boundary conditions.

Afterward, in [10], M. Jamil et al. studied the problem in the form

$$\begin{cases} \lambda u_{tt} + u_t = \nu(u_{xx} + \frac{1}{x}u_x - \frac{1}{x^2}u),\ R_1 < x < R_2,\ t > 0, \\ \lambda V_{tt} + V_t = \nu(V_{xx} + \frac{1}{x}V_x),\ R_1 < x < R_2,\ t > 0, \\ u_x(R_1,t) - \frac{1}{R_1}u(R_1,t) = \frac{k_1}{\mu}t^2,\ \ V_x(R_1,t) = \frac{k_2}{\mu}t^2,\ t > 0, \\ u(R_1,t) = V(R_2,t) = 0,\ t > 0, \\ u(x,0) = u_t(x,0) = V(x,0) = V_t(x,0) = 0,\ R_1 < x < R_2, \end{cases}$$

where λ, μ, ν, k_1, k_2 are the given constants.

Problem (1.2) has its origin in the canonical model of Kirchhoff and Carrier which describes small vibrations of an elastic streched string. In [11], G.R. Kirchhoff first investigated the following nonlinear vibration of an elastic string

$$\rho h u_{tt} = (P_0 + \frac{Eh}{2L}\int_0^L \left\|\frac{\partial u}{\partial y}(y,t)\right\|^2 dy)u_{xx},$$

where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ is the mass density, h is the cross-section area, L is the length, E is the Young modulus, P_0 is the initial axial tension. And G.F. Carrier in [2] established a model of the type

$$u_{tt} - (P_0 + P_1 \textstyle\int_0^L u^2(y,t)dy)u_{xx} = 0,$$

where P_0, P_1 are given constants, which models vibrations of an elastic string when changes in tension are not small.

It is also well known that, for during last decades, initial-boundary value problems of the Kirchhoff-Carrier model have been studied extensively and obtained many importan results. By using different methods together with various techniques in functional analysis, several results concerning the existence and the properties of solutions such as blow-up, decay, stability have been established. Among the works of the Kirchhoff-Carrier type we can cite, for example, M.M. Cavalcanti et al. [4,5], N.A. Larkin [12], N.T. Long et al. [14], L.A. Medeiros [15], J.Y. Park and J.J. Bae [18], M.L. Santos [19] and the references given therein. A survey of the results about the mathematical aspects of Kirchhoff model can be found in L.A. Medeiros et al. [16,17].

It is important to have in mind that, an approximate solution of Problem (1.2) can be obtained via the solution of Problem (P_n), in other words, we can study Problems (P_n) and (1.2) in a new approach, which is approximation of a system of nonlinear Carrier wave equations Problem (1.2) by approximating the Carrier terms with their integral sums. We shall prove that for each $n \in \mathbb{N}$, Problem (P_n) has a unique weak solution (u^n, v^n) and then we continue to prove that $\{(u^n, v^n)\}_n$ converges to the weak solution (u, v) of Problem (1.2) in a suitable function space. With this approach, we aim to avoid the integral calculation for nonlocal terms in the integral form, such as the two Carrier terms $\left\|u(t)\right\|_0^2 = \int_1^R xu^2(x,t)dx$ and $\left\|v(t)\right\|_0^2 = \int_1^R xv^2(x,t)dx$, by replacing them with corresponding integral sums, namely $S_n[u](t)$ and $S_n[v](t)$. We note that if we do not replace the integral sums $S_n[u](t)$ and $S_n[v](t)$, but instead keep the Carrier terms $\left\|u(t)\right\|_0^2$ and $\left\|v(t)\right\|_0^2$, we would still proceed in the usual way by linearizing through an iterative sequence $\{(u_m, v_m)\}$, where the two Carrier terms would also be linearized by a previous iteration step into the terms $\left\|u_{m-1}(t)\right\|_0^2$ and $\left\|v_{m-1}(t)\right\|_0^2$. However, these two terms would still require integral calculations. The integral sums $S_n[u](t)$ and $S_n[v](t)$ are the rectangle rule approximations of the two Carrier terms, respectively. Obviously, we also can use other integral sums, such as the trapezoidal rule or Simpson's rule. The choice of integral sums that converge quickly to the function under the integral sign, and even allow for the evaluation of the convergence rate, requires the smoothness of the function under the integral sign. At the end of this paper (Remark 4), we will discuss how to choose the integral sums to approximate the solution (u, v) and how to estimate the error in approximating the Carrier terms and their integral sums. On the basis of the aforementioned works and the above ideas, in this paper, we study the existence with the relation of solutions of Problems (P_n) and (1.2). To the best of our knowledge, there are relatively few results related to approximation problems (P_n), with nonlinear expressions containing integral sums which approximates the Carrier terms, to have the approximation of the solutions of Problem (1.2).

This paper is structured as follows. Section 2 is devoted to preliminaries. In Section 3, we propose hypotheses in order to state and prove two theorems on the existence and uniqueness of a local weak solution of Problem (P_n), for each $n \in \mathbb{N}$. In Section 4, we prove that the solution of Problem (P_n) converges to the solution of Problem (1.2) in a sense as in Theorem 3 below. In the proofs of results obtained here, the main tools of functional analysis such as the linear approximate method, the Galerkin method, the arguments of continuity with priori estimates, compactness arguments including the compactness lemma of Aubin-Lions are employed. We end the paper with a remark related to open problems (Remark 5).

2 Preliminaries

In this paper, we put $\Omega = (1, R)$, $Q_T = \Omega \times (0, T)$, $T > 0$, and denote the norm in the space L^2 by $\|\cdot\|$. The notations of the function spaces used here, such as $L^2 \equiv L^2(\Omega)$, $H^1 \equiv H^1(\Omega)$, are standard and can be found in H. Brezis [1] or J.L. Lions's book [13]. On H^1, we shall use the following norm $\|v\|_{H^1} = (\|v\|^2 + \|v_x\|^2)^{1/2}$. Considering the set $V = \{v \in H^1 : v(R) = 0\}$, then V is a closed subspace of H^1 and on V two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms. We note that L^2, H^1 are also the Hilbert spaces with respect to the corresponding scalar products $\langle u, v\rangle = \int_1^R xu(x)v(x)dx$, $\langle u, v\rangle + \langle u_x, v_x\rangle$. These scalar products induce the corresponding norms in L^2 and H^1 which are denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. We note more that V is continuously and densely embedded in L^2. Identifying L^2 with $(L^2)'$ (the dual of L^2), we have $V \hookrightarrow L^2 \hookrightarrow V'$, therefore, the notation $\langle\cdot,\cdot\rangle$ is also used for the pairing between H^1 and $(H^1)'$. Corresponding to the above norms and spaces, we have the following lemmas, the proofs of which can be found in the paper [21].

Lemma 1. *The following inequalities are fulfilled*

$$(i)\ \|v\| \le \|v\|_0 \le \sqrt{R}\,\|v\|\,,\ \textit{for all } v \in L^2,$$
$$(ii)\ \|v\|_{H^1} \le \|v\|_1 \le \sqrt{R}\,\|v\|_{H^1}\,,\ \textit{for all } v \in H^1.$$

Lemma 2. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \alpha_0 \|v\|_{H^1}\ \textit{ for all } v \in H^1,$$

where $\alpha_0 = \frac{1}{\sqrt{2(R-1)}}\left(1+\sqrt{1+16(R-1)^2}\right)^{0.5}$.

Lemma 3. *The imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and for all $v \in V$,*

$$(i)\ \|v\|_{C^0(\overline{\Omega})} \le \sqrt{R-1}\,\|v_x\| \le \sqrt{R-1}\,\|v_x\|_0\,,$$
$$(ii)\ \|v\|_0 \le \sqrt{\frac{R+1}{2}}(R-1)\,\|v_x\|_0\,,$$
$$(iii)\ \int_1^R x\,\|v(x)\|^\gamma\,dx \le \frac{R^2-1}{2}(\sqrt{R-1})^\gamma\,\|v_x\|_0^\gamma\,,\quad \textit{for all } \gamma > 0.$$

We set $a(u,w) = \langle u_x, w_x\rangle + b_1 u(1)w(1) + \langle \frac{1}{x}u, \frac{1}{x}w\rangle$, and

$$\begin{aligned} &b(v,\phi) = \langle v_x, \phi_x\rangle, \text{for all} u,v,w,\phi \in V, \\ &\|v\|_a = \sqrt{a(v,v)} = \left[\|v_x\|_0^2 + b_1 v^2(1) + \left\|v/x\right\|_0^2\right]^{1/2}, \\ &\|v\|_b = \sqrt{b(v,v)} = \|v_x\|_0\,,\ v \in V, \end{aligned} \tag{2.4}$$

with $b_1 > 0$ is given constant. Then, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the symmetric bilinear forms on $\mathbb{V}$. Moreover, it is not difficult to prove the following lemma.

Lemma 4. *The following inequalities are fulfilled*

$$(i)\ \|v_x\|_0 \le \|v\|_a \le a_1^*\,\|v_x\|_0\,,\textit{for all } v \in V,$$
$$(ii)\ \|v_x\|_0 \le \|v\|_1 \le \bar{a}_1^*\,\|v_x\|_0\,,\textit{for all } v \in V.$$

where $a_1^*{=}[1+(b_1{+}0.5(R^2-1)(R-1)]^{1/2}$, $\bar{a}_1^*{=}[1{+}0.5(R+1)2(R-1)^2]^{1/2}$.

Remark 1. On L^2, two norms $v \longmapsto \|v\|$ and $v \longmapsto \|v\|_0$ are equivalent. It is similar to two norms $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v\|_1$ on H^1, and five norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_1$, $v \longmapsto \|v_x\|$, $v \longmapsto \|v_x\|_0$ and $v \longmapsto \|v\|_a$ on V.

Lemma 5. *There exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue $\bar{\lambda}_j$ such that $0 < \bar{\lambda}_1 \le \bar{\lambda}_2 \le \ldots \le \bar{\lambda}_j \le \bar{\lambda}_{j+1} \le \ldots$, $\lim_{j\to+\infty}\bar{\lambda}_j = +\infty$, $a(w_j,w) = \bar{\lambda}_j\langle w_j, w\rangle$ for all $w \in V$, $j = 1,2,\ldots$. Furthermore, the sequence $\{w_j/\sqrt{\bar{\lambda}_j}\}_j$ is the Hilbert orthonormal base of V with respect to $a(\cdot,\cdot)$. On the other hand, w_j, $j = 1,2,\ldots$, satisfy the problem $L_1 w_j \equiv -(w_{jxx}+\frac{1}{x}w_{jx})+\frac{1}{x^2}w_j = \bar{\lambda}_j w_j$, in $(1,R)$, $w_{jx}(1) - b_1 w_j(1) = w_j(R) = 0$, $w_j \in C^\infty([1,R])$.*

The proof of Lemma 5 can be found in [20], Theorem 7.7, with $H = L^2$, $V = \{v \in H^1 : v(R) = 0\}$ and $a(\cdot,\cdot)$ defined by (2.4). Similarly, we have

Lemma 6. *There exists the Hilbert orthonormal base* $\{\phi_j\}$ *of* L^2 *consisting of the eigenfunctions* ϕ_j *corresponding to the eigenvalue* $\bar{\mu}_j$ *such that* $0 < \bar{\mu}_1 \leq \bar{\mu}_2 \leq \ldots \leq \bar{\mu}_j \leq \bar{\mu}_{j+1} \leq \ldots,\ \lim_{j\to+\infty} \bar{\mu}_j = +\infty,\ b(\phi_j,\phi) = \bar{\mu}_j\langle\phi_j,\phi\rangle$ *for all* $\phi \in V$, $j = 1,2,\ldots$. *Furthermore, the sequence* $\{\phi_j/\sqrt{\bar{\mu}_j}\}$*is the Hilbert orthonormal base of* V *with respect to the scalar product* $b(\cdot,\cdot)$. *On the other hand,* ϕ_j, $j = 1,2,\ldots,$ *satisfy the problem* $L_2\phi_j \equiv -(\phi_{jxx} + \frac{1}{x}\phi_{jx}) = \bar{\mu}_j\phi_j$, *in* $(1,R)$, $\phi_{jx}(1) = \phi_j(R) = 0$, $\phi_j \in C^\infty([1,R])$.

Lemma 7. *Put* $\|v\|_{H^2\cap V} = \sqrt{\|v_x\|_0^2 + \|v_{xx}\|_0^2}$ *and*

$$L_1 v \equiv -(v_{xx} + v_x/x) + v/x^2,\ L_2 v \equiv -(v_{xx} + v_x/x),\ v \in H^2 \cap V.$$

Then, there exist constants $\gamma_1, \bar{\gamma}_1, \gamma_2, \tilde{\gamma}_1, \tilde{\gamma}_2 > 0$ *such that, for all* $v \in H^2 \cap V$,

$$\begin{gathered}(i)\ \|L_1 v\|_0 \leq \bar{\gamma}_1 \|v\|_{H^2\cap V},\ \|L_2 v\|_0 \leq \sqrt{2}\,\|v\|_{H^2\cap V},\\ (ii)\gamma_1 \|v\|_{H^2\cap V}^2 \leq \|L_1 v\|_0^2 + \|v\|_a^2 \leq \tilde{\gamma}_1 \|v\|_{H^2\cap V}^2,\\ (iii)\gamma_2 \|v\|_{H^2\cap V}^2 \leq \|L_2 v\|_0^2 + \|v_x\|_0^2 \leq \tilde{\gamma}_2 \|v\|_{H^2\cap V}^2.\end{gathered}$$

The proof of this lemma is not difficult, so we omit it. To make it more concise, we rewrite the function spaces as follows

$$\mathbb{L}^2 = L^2 \times L^2,\ \mathbb{V} = V \times V,\ \mathbb{H}^2 \cap \mathbb{V} = (H^2 \cap V) \times (H^2 \cap V).$$

Remark 2. The weak formulation of the initial-boundary value problem (1.2) can be given in the following manner.

Definition 1. The weak solution of Problem (1.2) is the couple of functions (u,v) such that $(u,v) \in \bar{W}_T$, where the set $\bar{W}_T = \{(u,v) \in L^\infty(0,T;\mathbb{H}^2 \cap \mathbb{V}) : (u',v') \in L^\infty(0,T;\mathbb{V}), (u'',v'') \in L^\infty(0,T;\mathbb{L}^2)\}$, furthermore (u,v) satisfies the following variational equation, for all $(w,\phi) \in \mathbb{V}$, a.e., $t \in (0,T)$,

$$\begin{cases} \langle u''(t), w\rangle + a_1(\|u(t)\|_0^2)a(u(t),w) = \langle f[u,v](t), w\rangle, \\ \langle v''(t), \phi\rangle + a_2(\|v(t)\|_0^2)\langle v_x(t), \phi_x\rangle = \langle g[u,v](t), \phi\rangle, \end{cases}$$

with the initial conditions $(u(0),u'(0)) = (\tilde{u}_0,\tilde{u}_1)$ and $(v(0),v'(0)) = (\tilde{v}_0,\tilde{v}_1)$, where $f[u,v](t), g[u,v](t)$ are defined in the same form as follows

$$f[u,v](x,t) = f(x,t,u(x,t),v(x,t),u'(x,t),v'(x,t),u_x(x,t),v_x(x,t)). \quad (2.10)$$

Remark 3. (see [13]) The set $\bar{W}_T$ contains all the elements (u,v) which belong to $L^\infty(0,T;\mathbb{H}^2 \cap \mathbb{V}) \cap C([0,T];\mathbb{V}) \cap C^1([0,T];\mathbb{L}^2)$ satisfying

$$(u',v') \in L^\infty(0,T;\mathbb{V}) \cap C([0,T];\mathbb{L}^2), (u'',v'') \in L^\infty(0,T;\mathbb{L}^2).$$

3 Existence and uniqueness of approximation problems

This section is devoted to the study of existence and uniqueness of a local weak solution for the approximation problems (P_n). For each $n \in \mathbb{N}$, Problem (P_n) is considered with given constants $b_1 > 0, R > 1$, and the following assumptions for given functions $a_1, a_2, \tilde{u}_0, \tilde{u}_1, \tilde{v}_0, \tilde{v}_1, f, g$.

(A_1) $(\tilde{u}_0, \tilde{u}_1), (\tilde{v}_0, \tilde{v}_1) \in (V \cap H^2) \times V, \tilde{u}_{0x}(1) - b_1\tilde{u}_0(1) = \tilde{v}_{0x}(1) = 0;$

(A_2) $a_1, a_2 \in C^1(\mathbb{R}_+), a_i(z) \geq a_{i*} > 0, \forall z \geq 0, i = 1, 2;$

(A_3) $f, g \in C^1([1, R] \times [0, T^*] \times \mathbb{R}^6)$, such that $\forall (t, y_5, y_6) \in [0, T^*] \times \mathbb{R}^2$, $f(R, t, 0, 0, 0, 0, y_5, y_6) = g(R, t, 0, 0, 0, 0, y_5, y_6) = 0,$

Definition 2. The weak solution of Problem (P_n) is the couple of functions $(u, v) \in \bar{W}_T$ satisfying the following variational equation

$$\begin{cases} \langle u''(t), w\rangle + a_1(S_n[u](t))a(u(t), w) = \langle f[u, v](t), w\rangle, \\ \langle v''(t), \phi\rangle + a_2(S_n[v](t))\langle v_x(t), \phi_x\rangle = \langle g[u, v](t), \phi\rangle, \end{cases} \tag{3.1}$$

for all $(w, \phi) \in \mathbb{V}$, a.e., $t \in (0, T)$, together with the initial conditions

$$(u(0), u'(0)) = (\tilde{u}_0, \tilde{u}_1),\ (v(0), v'(0)) = (\tilde{v}_0, \tilde{v}_1), \tag{3.2}$$

where $f[u, v]$, $g[u, v]$ as in (2.10). Fixed $T^* > 0$, let $T \in (0, T^*]$, we define

$$W_T = \{(u, v) \in L^\infty(0, T; \mathbb{H}^2 \cap \mathbb{V}) : (u', v') \in L^\infty(0, T; \mathbb{V}), (u'', v'') \in L^2(0, T; \mathbb{L}^2)\},$$

then W_T is the Banach space with norm

$$\|(u, v)\|_{W_T} = \max\{\|(u, v)\|_{L^\infty(0,T;\mathbb{H}^2\cap\mathbb{V})}, \|(u', v')\|_{L^\infty(0,T;\mathbb{V})}, \left\|(u'', v'')\right\|_{L^2(0,T;\mathbb{L}^2)}\}.$$

For $M > 0$, we put $W(M, T) = \left\{v \in W_T : \|v\|_{W_T} \leq M\right\}$, and

$$W_1(M, T) = \{(u, v) \in W(M, T) : (u'', v'') \in L^\infty(0, T; \mathbb{L}^2)\}.$$

We construct the recurrent sequence, with $(u_0, v_0) = (\tilde{u}_0, \tilde{v}_0)$, and suppose that

$$(u_{m-1}, v_{m-1}) \in W_1(M, T),$$

then Problem (3.1)–(3.2) is associated with the problem: Find $(u_m, v_m) \in W_1(M, T)$, $(m \geq 1)$, satisfying the following linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + a_{1m}(t)a(u_m(t), w) = \langle F_m(t), w\rangle, \\ \langle v_m''(t), \phi\rangle + a_{2m}(t)\langle v_{mx}(t), \phi_x\rangle = \langle G_m(t), \phi\rangle,\ \forall (w, \phi) \in \mathbb{V}, \\ (u_m(0), u_m'(0)) = (\tilde{u}_0, \tilde{u}_1), (v_m(0), v_m'(0)) = (\tilde{v}_0, \tilde{v}_1), \end{cases} \tag{3.7}$$

where $F_m(x, t) = f[u_{m-1}, v_{m-1}](x, t), G_m(x, t) = g[u_{m-1}, v_{m-1}](x, t), a_{1m}(t) = a_1(S_n[u_{m-1}](t)), a_{2m}(t) = a_2(S_n[v_{m-1}](t))$, and

$$\begin{aligned} S_n[u](t) &= ((R-1)/n)\sum_{i=1}^{n} x_{n,i}u^2(x_{n,i}, t), S_n[v](t) \\ &= ((R-1)/n)\sum_{i=1}^{n} x_{n,i}v^2(x_{n,i}, t), \\ x_{n,i} &= 1 + (R-1)(2i-1)/2n, i = \overline{0, n}, \forall n \in \mathbb{N}. \end{aligned} \tag{3.8}$$

Theorem 1. *Let $T^* > 0$ and (A_1)–(A_3) hold. Then, there exist positive constants M, $T > 0$ such that, for $(u_0, v_0) = (\tilde{u}_0, \tilde{v}_0)$, there exists a recurrent sequence $\{(u_m, v_m)\} \subset W_1(M, T)$ defined by (3.7) and (3.8).*

Proof. To prove this theorem, we use the Faedo-Galerkin method. Consider Hilbert orthonormal bases $\{w_j\}, \{\phi_j\}$ on L^2 mentioned in Lemmas 5 and 6. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \; v_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} d_{mj}^{(k)}(t) \phi_j,$$

where $c_{mj}^{(k)}(t)$, $d_{mj}^{(k)}(t)$ satisfy the system of nonlinear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j \rangle + a_{1m}(t) a(u_m^{(k)}(t), w_j) = \langle F_m(t), w_j \rangle, \\ \langle \ddot{v}_m^{(k)}(t), \phi_j \rangle + a_{2m}(t) \langle v_{mx}^{(k)}(t), \phi_{jx} \rangle = \langle G_m(t), \phi_j \rangle, 1 \le j \le k, \\ (u_m^{(k)}(0), \dot{u}_m^{(k)}(0)) = (\tilde{u}_{0k}, \tilde{u}_{1k}), (v_m^{(k)}(0), \dot{v}_m^{(k)}(0)) = (\tilde{v}_{0k}, \tilde{v}_{1k}), \end{cases} \tag{3.10}$$

with $(\tilde{u}_{0k}, \tilde{u}_{1k}) = \sum_{j=1}^{k} (\alpha_j^{(k)}, \beta_j^{(k)}) w_j, (\tilde{v}_{0k}, \tilde{v}_{1k}) = \sum_{j=1}^{k} (\tilde{\alpha}_j^{(k)}, \tilde{\beta}_j^{(k)}) \phi_j$, further, $(\tilde{u}_{0k}, \tilde{u}_{1k}) \to (\tilde{u}_0, \tilde{u}_1)$ strongly in $(H^2 \cap V) \times V$, $(\tilde{v}_{0k}, \tilde{v}_{1k}) \to (\tilde{v}_0, \tilde{v}_1)$ strongly in $(H^2 \cap V) \times V$. By using Banach's contraction principle, it is not difficult to prove that the above fixed point equation admits a unique solution $(u_m^{(k)}, v_m^{(k)})$ on $[0, T]$. Taking the constant $M_* = \max\{M^2, \frac{1}{2}(R^2 - 1)(R - 1)M^2\}$, and noting that $\sum_{i=1}^{n} x_{n,i} = n(1 + R)/2$ and $(u_{m-1}, v_{m-1}) \in W_1(M, T)$, it implies that

$$\begin{aligned} &\left\| S_n[u_{m-1}](t) \right\| \le ((R-1)/n) \sum\nolimits_{i=1}^{n} x_{n,i} (R-1) M^2 \le M_*, \\ &\left\| S_n[v_{m-1}](t) \right\| \le M_*. \end{aligned}$$

Put $K_M(f, g) = \max\{K_M(f), K_M(g)\}$, $\tilde{K}_M(a_1, a_2) = \max\{\tilde{K}_M(a_1), \tilde{K}_M(a_2)\}$, with $K_M(f) = \|f\|_{C^1(A_M)} = \|f\|_{C^0(A_M)} + \sum_{i=1}^{8} \|D_i f\|_{C^0(A_M)}$,

$$\begin{aligned} &\|f\|_{C^0(A_M)} = \sup_{(x,t,y_1,\ldots,y_6) \in A_M} \left\| f(x, t, y_1, \ldots, y_6) \right\|, \\ &\tilde{K}_M(a_i) = \|a_i\|_{C^1([0,M_*])} = \|a_i\|_{C^0([0,M_*])} + \left\| a_i' \right\|_{C^0([0,M_*])}, \; i = 1, 2, \\ &A_M = [1, R] \times [0, T^*] \times [-R_1 M, R_1 M]^4 \times [-\alpha_0 M, \alpha_0 M]^2, \\ &R_1 = \sqrt{R-1}, \; \alpha_0 = \frac{1}{\sqrt{2(R-1)}} \left(1 + \sqrt{1 + 16(R-1)^2}\right)^{\frac{1}{2}} = \frac{\left(1 + \sqrt{1 + 16R_1^4}\right)^{1/2}}{\sqrt{2} R_1}, \\ &f = f(x, t, y_1, \ldots, y_6), \; D_1 f = \frac{\partial f}{\partial x}, \; D_2 f = \frac{\partial f}{\partial t}, \; D_{i+2} f = \frac{\partial f}{\partial y_i}, \; i = 1, \ldots, 6, \\ &S_m^{(k)}(t) = \|\dot{u}_m^{(k)}(t)\|_0^2 + \|\dot{u}_m^{(k)}(t)\|_a^2 + \|\dot{v}_m^{(k)}(t)\|_0^2 + \|\dot{v}_{mx}^{(k)}(t)\|_0^2 + a_{1m}(t) \left(\|u_m^{(k)}(t)\|_a^2 \right. \\ &\left. + \|L_1 u_m^{(k)}(t)\|_0^2 \right) + a_{2m}(t) \left(\|v_{mx}^{(k)}(t)\|_0^2 + \|L_2 v_m^{(k)}(t)\|_0^2 \right) + \int_0^t \left(\|\ddot{u}_m^{(k)}(s)\|_0^2 + \|\ddot{v}_m^{(k)}(s)\|_0^2 ds. \right. \end{aligned}$$

Then, it follows that

$$\begin{aligned}
\gamma_* \bar{S}_m^{(k)}(t) \le S_m^{(k)}(t) = S_m^{(k)}(0) + 2\int_0^t a_{1m}'(s)(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|L_1 u_m^{(k)}(s)\right\|_0^2)ds \\
+ 2\int_0^t a_{2m}'(s)(\left\|v_{mx}^{(k)}(s)\right\|_0^2 + \left\|L_2 v_m^{(k)}(s)\right\|_0^2)ds \\
+ 2\int_0^t [\langle F_m(s), \dot{u}_m^{(k)}(s)\rangle + \langle G_m(s), \dot{v}_m^{(k)}(s)]ds + 2\int_0^t \big[a(F_m(s), \dot{u}_m^{(k)}(s)) \\
+ \langle G_{mx}(s), \dot{v}_{mx}^{(k)}(s)\rangle\big]ds + \int_0^t (\|\ddot{u}_m^{(k)}(s)\|_0^2 + \|\ddot{v}_m^{(k)}(s)\|_0^2)ds,
\end{aligned}$$

where $\gamma_* = \min\{1, a_{1*}\gamma_1, a_{2*}\gamma_2\}$ and

$$\begin{aligned}
\bar{S}_m^{(k)}(t) = \|\dot{u}_m^{(k)}(t)\|_0^2 + \|\dot{u}_m^{(k)}(t)\|_a^2 + \|\dot{v}_m^{(k)}(t)\|_0^2 + \|\dot{v}_{mx}^{(k)}(t)\|_0^2 \\
+ \|u_m^{(k)}(t)\|_{H^2\cap V}^2 + \|v_m^{(k)}(t)\|_{H^2\cap V}^2 + \int_0^t \big(\|\ddot{u}_m^{(k)}(s)\|_0^2 + \|\ddot{v}_m^{(k)}(s)\|_0^2\big)ds.
\end{aligned}$$

To continue the estimate, we note that the following inequalities are fulfilled

$$\begin{aligned}
\left\|F_{mx}(t)\right\| \le& (1+6M)K_M(f,g),\ \left\|G_{mx}(t)\right\| \le (1+6M)K_M(f,g), \\
\left\|a_{1m}'(t)\right\| \le& \bar{a}_*(M),\ \left\|a_{2m}'(t)\right\| \le \bar{a}_*(M),
\end{aligned}$$

where $\bar{a}_*(M) = (R^2-1)(R-1)M^2\tilde{K}_M(a_1,a_2)$. Therefore, we have

$$\begin{aligned}
&2\int_0^t a_{1m}'(s)\big(\|u_m^{(k)}(s)\|_a^2 + \|L_1 u_m^{(k)}(s)\|_0^2\big)ds \le 2\tilde{\gamma}_1\bar{a}_*(M)\int_0^t \bar{S}_m^{(k)}(s)ds; \\
&\int_0^t a_{2m}'(s)\big(\|v_{mx}^{(k)}(s)\|_0^2 + \|L_2 v_m^{(k)}(s)\|_0^2\big)ds \le 2\tilde{\gamma}_2\bar{a}_*(M)\int_0^t \bar{S}_m^{(k)}(s)ds; \\
&2\int_0^t [\langle F_m(s), \dot{u}_m^{(k)}(s)\rangle + \langle G_m(s), \dot{v}_m^{(k)}(s)\rangle]ds \le T(R^2-1)K_M^2(f,g) + \int_0^t \bar{S}_m^{(k)}(s)ds;
\end{aligned}$$

$$\begin{aligned}
2\int_0^t [a(F_m(s), \dot{u}_m^{(k)}(s)) + \langle G_{mx}(s), \dot{v}_{mx}^{(k)}(s)\rangle]ds \\
\le 2\int_0^t \big[a_1^*\|F_{mx}(s)\|_0\|\dot{u}_m^{(k)}(s)\|_a + \|G_{mx}(s)\|_0\|\dot{v}_{mx}^{(k)}(s)\|_0\big]ds \\
\le T(a_1^{*2}+1)(1+6M)^2K_M^2(f,g) + \int_0^t \bar{S}_m^{(k)}(s)ds.
\end{aligned}$$

We note that, Equation $(3.10)_1$ can be rewritten as follows

$$\langle \ddot{u}_m^{(k)}(t), w_j\rangle + a_{1m}(t)\langle L_1 u_m^{(k)}(t), w_j\rangle = \langle F_m(t), w_j\rangle,\ 1 \le j \le k.$$

Then, it follows that $\|\ddot{u}_m^{(k)}(t)\|_0^2 \le 2\tilde{K}_M^2(a_1,a_2)\bar{\gamma}_1^2\bar{S}_m^{(k)}(t) + (R^2-1)K_M^2(f,g)$. Similarly, we get $\|\ddot{v}_m^{(k)}(t)\|_0^2 \leq 4\tilde{K}_M^2(a_1,a_2)\bar{S}_m^{(k)}(t)+(R^2-1)K_M^2(f,g)$. Hence,

$$\int_0^t \Big(\|\ddot{u}_m^{(k)}(s)\|_0^2 + \|\ddot{v}_m^{(k)}(s)\|_0^2\Big)ds$$
$$\leq 2T(R^2-1)K_M^2(f,g) + 2(\bar{\gamma}_1^2+2)\tilde{K}_M^2(a_1,a_2)\int_0^t \bar{S}_m^{(k)}(s)ds.$$

It leads to $\bar{S}_m^{(k)}(t) \leq \frac{1}{\gamma_*}S_m^{(k)}(0) + TD_1(M) + D_2(M)\int_0^t \bar{S}_m^{(k)}(s)ds$, where

$$\begin{aligned}
&D_1(M) = (1/\gamma_*)[3(R^2-1) + (a_1^{*2}+1)(1+6M)^2]K_M^2(f,g),\\
&D_2(M) = (2\gamma_*)\big[1 + (\tilde{\gamma}_1+\tilde{\gamma}_2)\bar{a}_*(M) + (\bar{\gamma}_1^2+2)\tilde{K}_M^2(a_1,a_2)\big],\\
&S_m^{(k)}(0) = \|\tilde{u}_{1k}\|_0^2 + \|\tilde{u}_{1k}\|_a^2 + \|\tilde{v}_{1k}\|_0^2 + \|\tilde{v}_{1kx}\|_0^2\\
&\quad + a_{1m}(0)(\|\tilde{u}_{0k}\|_a^2 + \|L_1\tilde{u}_{0k}\|_0^2) + a_{2m}(0)(\|\tilde{v}_{0kx}\|_0^2 + \|L_2\tilde{v}_{0k}\|_0^2),
\end{aligned}$$

with $a_{1m}(0)=a_1(((R-1)/n)\sum_{i=1}^n x_{n,i}\tilde{u}_0^2(x_{n,i}))$ and $a_{2m}(0)=a_2(((R-1)/n)\times\sum_{i=1}^n x_{n,i}\tilde{v}_0(x_{n,i}))$. Remark that $0 \leq ((R-1)/n)\sum_{i=1}^n x_{n,i}\tilde{u}_0^2(x_{n,i})$ and $((R-1)/n)\sum_{i=1}^n x_{n,i}\tilde{u}_0^2(x_{n,i}) \leq \frac{1}{2}(R^2-1)(R-1)\|\tilde{u}_{0x}\|_0^2 \equiv \rho(\tilde{u}_0)$,
$a_{1m}(0) = a_1(((R-1)/n)\sum_{i=1}^n x_{n,i}\tilde{u}_0^2(x_{n,i})) \leq \sup_{0\leq z\leq\rho(\tilde{u}_0)} a_1(z)$.
Simlarly, $a_{2m}(0) = a_2(((R-1)/n)\sum_{i=1}^n x_{n,i}\tilde{v}_0(x_{n,i})) \leq \sup_{0\leq z\leq\rho(\tilde{v}_0)} a_2(z)$, with $\rho(\tilde{v}_0) = \frac{1}{2}(R^2-1)(R-1)\|\tilde{v}_{0x}\|_0^2$. The above convergences of $(\tilde{u}_{0k},\tilde{u}_{1k})$ and $(\tilde{v}_{0k},\tilde{v}_{1k})$ lead to there exists a constant $M>0$ independent of k and m such that $2S_m^{(k)}(0) \leq \gamma_* M^2$, for all k and $m\in\mathbb{N}$. We choose $T\in(0,T^*]$, such that

$$\big(M^2/2+TD_1(M)\big)\exp(TD_2(M))\leq M^2, k_T = 4\sqrt{TC_1(M)}\exp(TC_2(M)) < 1,$$

with $C_1(M) = \frac{\bar{\gamma}_1^2+2}{a_*}\bar{a}_*^2(M)$, and $C_2(M) = \frac{1}{a_*}[1+2\bar{a}_*(M)+4(\bar{a}_1^*+1)K_M(f,g)]$. Using Gronwall's Lemma, we get $\bar{S}_m^{(k)}(t) \leq M^2$, for all $t\in[0,T]$, for all m and $k\in\mathbb{N}$. Hence, $u_m^{(k)} \in W(M,T)$, for all m and $k\in\mathbb{N}$. Therefore, there exists a subsequence of the sequence of $\{u_m^{(k)}\}$, with the same notation, such that

$$\begin{cases}
(u_m^{(k)}, v_m^{(k)}) \to (u_m, v_m) & \text{in} \quad L^\infty(0,T;\mathbb{H}^2\cap\mathbb{V}) \text{ weak*},\\
(\dot{u}_m^{(k)}, \dot{v}_m^{(k)}) \to (u_m', v_m') & \text{in} \quad L^\infty(0,T;\mathbb{V}) \text{ weak*},\\
(\ddot{u}_m^{(k)}, \ddot{v}_m^{(k)}) \to (u_m'', v_m'') & \text{in} \quad L^2(0,T;\mathbb{L}^2) \text{ weak},
\end{cases}$$

and $(u_m,v_m)\in W(M,T)$. Passing to limit in (3.10), we have (u_m,v_m) satisfying (3.7) in $\mathbb{L}^2$ weak. Moreover, $u_m'' = -a_{1m}(t)L_1u_m + F_m \in L^\infty(0,T;L^2)$ and $v_m'' = -a_{2m}(t)L_2v_m + G_m \in L^\infty(0,T;L^2)$, hence $(u_m,v_m)\in W_1(M,T)$, so the proof of Theorem 1 is completed. □

Next, we state and prove the main result in this section, where we consider the Banach space $W_1(T) = C([0,T];\mathbb{V})\cap C^1([0,T];\mathbb{L}^2)$, with respect to the norm $\|(u,v)\|_{W_1(T)} = \|(u,v)\|_{C([0,T];\mathbb{V})} + \|(u',v')\|_{C([0,T];\mathbb{L}^2)}$.

Theorem 2. *Let $T^* > 0$ and (A_1)–(A_3) hold. Then, there exist positive constants M, $T > 0$ such that: Problem (P_n) has a unique weak solution $(\bar{u}, \bar{v}) \in W_1(M,T)$. The recurrent sequence $\{(u_m, v_m)\}$ defined by(3.7)–(3.8) converges to the weak solution $(\bar{u}, \bar{v})$ of Problem (P_n) strongly in the space $W_1(T)$. Furthermore, we have the estimate*

$$\left\|(u_m, v_m) - (\bar{u}, \bar{v})\right\|_{W_1(T)} \leq C_T k_T^m, \forall m \in \mathbb{N}, \tag{3.11}$$

where $k_T \in (0,1)$ and C_T are chosen such that k_T, C_T depend only on T, f, g, a_1, a_2, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$.

Proof. Let $\bar{u}_m = u_{m+1} - u_m$, $\bar{v}_m = v_{m+1} - v_m$. Then $(\bar{u}_m, \bar{v}_m)$ satisfies

$$\begin{cases} \langle \bar{u}_m''(t), w\rangle + a_{1,m+1}(t) a(\bar{u}_m(t), w) \\ = -[a_{1,m+1}(t) - a_{1m}(t)]\langle L_1 u_m(t), w\rangle + \langle F_{m+1}(t) - F_m(t), w\rangle, \\ \langle \bar{v}_m''(t), \phi\rangle + a_{2,m+1}(t)\langle \bar{v}_{mx}(t), \phi_x\rangle = -[a_{2,m+1}(t) - a_{2m}(t)]\langle L_2 v_m(t), \phi\rangle \\ +\langle G_{m+1}(t) - G_m(t), \phi\rangle, \quad \forall (w,\phi) \in \mathbb{V}, \\ (\bar{u}_m(0), \bar{v}_m(0)) = (\bar{u}_m'(0), \bar{v}_m'(0)) = (0,0). \end{cases}$$

By taking $(w, \phi) = (\bar{u}_m'(t), \bar{v}_m'(t))$, after integrating in t, we get the estimation

$$\begin{aligned} &a_* \bar{Z}_m(t) \leq 2\int_0^t [\langle F_{m+1}(s) - F_m(s), \bar{u}_m'(s)\rangle + \langle G_{m+1}(s) - G_m(s), \bar{v}_m'(s)\rangle] ds \\ &-2\int_0^t [a_{1,m+1}(s) - a_{1m}(s)]\langle L_1 u_m(s), \bar{u}_m'(s)\rangle ds \\ &-2\int_0^t [a_{2,m+1}(s) - a_{2m}(s)]\langle L_2 v_m(s), \bar{v}_m'(s)\rangle ds \\ &+2\int_0^t a_{1,m+1}'(s) \left\|\bar{u}_m(s)\right\|_a^2 ds + 2\int_0^t a_{2,m+1}'(s) \left\|\bar{v}_{mx}(s)\right\|_0^2 ds, \end{aligned}$$

with $a_* = \min\{1, a_{1*}, a_{2*}\}$ and $\bar{Z}_m(t) = \left\|\bar{u}_m'(t)\right\|_0^2 + \left\|\bar{v}_m'(t)\right\|_0^2 + \left\|\bar{u}_m(t)\right\|_a^2 + \left\|\bar{v}_{mx}(t)\right\|_0^2$. We continue to estimate terms in the above estimation, based on the following inequalities

$$\begin{aligned} &\left\|F_{m+1}(t) - F_m(t)\right\|_0 \leq 2(\bar{a}_1^* + 1) K_M(f,g)\sqrt{\bar{Z}_m(t)}, \forall m \in \mathbb{N}, \forall t \in [0,T], \\ &\left\|G_{m+1}(t) - G_m(t)\right\|_0 \leq 2(\bar{a}_1^* + 1) K_M(f,g)\sqrt{\bar{Z}_m(t)}, \forall m \in \mathbb{N}, \forall t \in [0,T], \\ &\left\|a_{1,m+1}(t) - a_{1m}(t)\right\| \leq \frac{\bar{a}_*(M)}{M}\left\|(\bar{u}_{m-1}, \bar{v}_{m-1})\right\|_{W_1(T)}, \\ &\left\|a_{2,m+1}(t) - a_{2m}(t)\right\| \leq \frac{\bar{a}_*(M)}{M}\left\|(\bar{u}_{m-1}, \bar{v}_{m-1})\right\|_{W_1(T)}. \end{aligned}$$

$$2\int_0^t [\langle F_{m+1}(s)-F_m(s),\bar{u}_m'(s)\rangle + \langle G_{m+1}(s)-G_m(s),\bar{v}_m'(s)\rangle]ds$$
$$\leq 8(\bar{a}_1^*+1)K_M(f,g)\int_0^t \bar{Z}_m(s)ds;$$
$$-2\int_0^t \left[a_{1,m+1}(s)-a_{1m}(s)\right]\langle L_1u_m(s),\bar{u}_m'(s)\rangle ds$$
$$\leq T\bar{\gamma}_1^2\bar{a}_*^2(M)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^2+\int_0^t \bar{Z}_m(s)ds;$$
$$-2\int_0^t \left[a_{2,m+1}(s)-a_{2m}(s)\right]\langle L_2v_m(s),\bar{v}_m'(s)\rangle ds$$
$$\leq 2T\bar{a}_*^2(M)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^2+\int_0^t \bar{Z}_m(s)ds;$$
$$2\int_0^t a_{1,m+1}'(s)\left\|\bar{u}_m(s)\right\|_a^2 ds \leq 2\bar{a}_*(M)\int_0^t \bar{Z}_m(s)ds;$$
$$2\int_0^t a_{2,m+1}'(s)\left\|\bar{v}_{mx}(s)\right\|_0^2 ds \leq 2\bar{a}_*(M)\int_0^t \bar{Z}_m(s)ds.$$

Consequently, we obtain that

$$\bar{Z}_m(t)\leq TC^{(1)}(M)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^2+2C^{(2)}(M)\int_0^t \bar{Z}_m(s)ds.$$

By Gronwall's lemma, we get $\left\|(\bar{u}_m,\bar{v}_m)\right\|_{W_1(T)} \leq k_T\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}$, which implies that, for all $m,p\in\mathbb{N}$,

$$\left\|(u_m,v_m)-(u_{m+p},v_{m+p})\right\|_{W_1(T)} \leq \frac{\left\|(u_1,v_1)-(\tilde{u}_0,\tilde{v}_0)\right\|_{W_1(T)}}{1-k_T}k_T^m. \tag{3.12}$$

Then, $\{(u_m,v_m)\}$ is a Cauchy sequence in $W_1(T)$, so there exists $(\bar{u},\bar{v})$ such that $(u_m,v_m)\to(\bar{u},\bar{v})$ strongly in $W_1(T)$. Note that $(u_m,v_m)\in W_1(M,T)$, then there exists a subsequence $\{(u_{m_j},v_{m_j})\}$ of $\{(u_m,v_m)\}$ such that

$$\begin{cases} (u_{m_j},v_{m_j})\to(\bar{u},\bar{v}) & \text{in} \quad L^\infty(0,T;(H^2\cap V)\times(H^2\cap V)) \text{ weak*},\\ (u_{m_j}',v_{m_j}')\to(\bar{u}',\bar{v}') & \text{in} \quad L^\infty(0,T;\mathbb{V}) \text{ weak*},\\ (u_{m_j}'',v_{m_j}'')\to(\bar{u}'',\bar{v}'') & \text{in} \quad L^2(0,T;\mathbb{L}^2) \text{ weak}, \quad (\bar{u},\bar{v})\in W(M,T). \end{cases}$$

Since $\|F_m-f[\bar{u},\bar{v}]\|_{C([0,T];L^2)}\leq K_M(f)(\bar{a}_1^*+1)\|(u_{m-1}-\bar{u},v_{m-1}-\bar{v})\|_{W_1(T)}$, $F_m\to f[\bar{u},\bar{v}]$ strongly in $C([0,T];L^2)$. Similarly, $G_m\to g[\bar{u},\bar{v}]$ strongly in $C([0,T];L^2)$. We have

$$\|a_{1m}(t)-a_1((S_n[\bar{u}])(t))\|\leq\left(\bar{a}_*(M)/M\right)\|(u_{m-1}-\bar{u},v_{m-1}-\bar{v})\|_{W_1(T)},$$
$$\sup_{0\leq t\leq T}\left\|a_{1m}(t)-a_1((S_n[\bar{u}])(t))\right\|\leq\left(\bar{a}_*(M)/M\right)\left\|(u_{m-1}-\bar{u},v_{m-1}-\bar{v})\right\|_{W_1(T)}.$$

Hence, $a_{1m}\to a_1(S_n[\bar{u}])$ strongly in $C([0,T])$. By

$$\sup_{0\leq t\leq T}\left\|a_{2m}(t)-a_2((S_n[\bar{v}])(t))\right\|\leq\frac{\bar{a}_*(M)}{M}\left\|(u_{m-1}-\bar{u},v_{m-1}-\bar{v})\right\|_{W_1(T)},$$

similarly, we also have $a_{2m} \to a_2(S_n[\bar{v}])$ strongly in $C([0,T])$. Finally, passing to limit in (3.7), (3.8) as $m = m_j \to \infty$, there exists $(\bar{u}, \bar{v}) \in W(M,T)$ satisfying

$$\begin{cases} \langle \bar{u}''(t), w\rangle + a_1(S_n[\bar{u}](t))a(\bar{u}(t), w) = \langle f[\bar{u},\bar{v}](t), w\rangle, \\ \quad \langle \bar{v}''(t), \phi\rangle + a_2(S_n[\bar{v}](t))b(\bar{v}(t), \phi) = \langle g[\bar{u},\bar{v}](t), \phi\rangle, \end{cases} \tag{3.13}$$

for all $(w,\phi) \in \mathbb{V}$, a.e., $t \in (0,T)$, and the initial conditions

$$(\bar{u}(0), \bar{u}'(0)) = (\tilde{u}_0, \tilde{u}_1), \ (\bar{v}(0), \bar{v}'(0)) = (\tilde{v}_0, \tilde{v}_1).$$

On the other hand, from the assumption (A_2), we obtain from $(\bar{u},\bar{v}) \in W(M,T)$ and (3.13) that $\bar{u}'' = -a_1(S_n[\bar{u}](t))L_1\bar{u} + f[\bar{u},\bar{v}] \in L^\infty(0,T;L^2)$ and $\bar{v}'' = -a_2(S_n[\bar{v}](t))L_2\bar{v} + g[\bar{u},\bar{v}] \in L^\infty(0,T;L^2)$. Thus, we have $(\bar{u},\bar{v}) \in W_1(M,T)$. Now, let (u_1,v_1), $(u_2,v_2) \in W_1(M,T)$ be two weak solutions of (P_n). Then $(\hat{u},\hat{v}) = (u_1,v_1) - (u_2,v_2) = (u_1-u_2, v_1-v_2)$ satisfies the system

$$\begin{cases} \langle \hat{u}''(t), w\rangle + a_{11}(t)a(\hat{u}(t), w) + [a_{11}(t) - a_{12}(t)]\langle L_1u_2(t), w\rangle = \langle F_1(t) - F_2(t), w\rangle, \\ \langle \hat{v}''(t), \phi\rangle + a_{21}(t)\langle \hat{v}_x(t), \phi_x\rangle + [a_{21}(t) - a_{22}(t)]\langle L_2v_2(t), \phi\rangle = \langle G_1(t) - G_2(t), \phi\rangle, \\ (\hat{u}(0), \hat{v}(0)) = (\hat{u}'(0), \hat{v}'(0)) = (0,0), \end{cases}$$

for all $(w,\phi) \in \mathbb{V}$, a.e., $t \in (0,T)$, in which $F_j(x,t) = f[u_j,v_j](x,t)$, $G_j(x,t) = g[u_j,v_j](x,t)$, and $a_{1j}(t) = a_1(S_n[u_j](t))$, $a_{2j}(t) = a_2(S_n[v_j](t))$, $j=1,2$. By taking $(w,\phi) = (\hat{u}'(t), \hat{v}'(t))$ in the above system and integrating in t, we get

$$\begin{aligned} \bar{\gamma}_* \bar{Z}(t) \le & \int_0^t [a_{11}'(s) \left\|\hat{u}(s)\right\|_a^2 + a_{21}'(s) \left\|\hat{v}_x(s)\right\|_0^2] ds - 2\int_0^t [a_{11}(s) - a_{12}(s)] \\ & \times \langle L_1u_2(s), u'(s)\rangle ds - 2\int_0^t [a_{21}(s) - a_{22}(s)]\langle L_2v_2(s), v'(s)\rangle ds \\ & + 2\int_0^t [\langle F_1(s) - F_2(s), u'(s)\rangle + \langle G_1(s) - G_2(s), v'(s)\rangle] ds, \end{aligned}$$

where

$$\bar{\gamma}_* = \min\{1, a_{1*}, a_{2*}\}, \ \ \bar{Z}(t) = \|\hat{u}'(t)\|_0^2 + \|\hat{v}'(t)\|_0^2 + \|\hat{u}(t)\|_a^2 + \|\hat{v}_x(t)\|_0^2.$$

Put $\tilde{q}_M = (4/\bar{\gamma}_*)[(1+\sqrt{3}+\bar{\gamma}_1)\bar{a}_*(M) + 4\sqrt{2}(1+\bar{a}_1^*)K_M(f,g)]$. Then, we obtain that $\bar{Z}(t) \le \tilde{q}_M \int_0^t \bar{Z}(s)ds$. Using Gronwall's lemma, we verify that $Z(t) \equiv 0$, it leads to $(u_1,v_1) = (u_2,v_2)$. Passing to the limit in (3.12) as $p \to +\infty$ for fixed m, we get (3.11). Theorem 2 is proved completely. $\square$

4 Convergence of solutions of approximation problems

For each fixed $n \in \mathbb{N}$, Theorem 2 shows that Problem (P_n) has a unique weak solution depending on n, which we denote as (u^n, v^n). In this section, we will prove that the sequence $\{(u^n, v^n)\}_n$ converges to the weak solution (u,v) of Problem (1.2) in a suitable function space. We note more that the positive constants M, T chosen as above will be independent of n, m and k. Therefore, we obtain the estimate for the sequence $\{(u_m^{(k)}, v_m^{(k)})\}$ satisfying

$$(u_m^{(k)}, v_m^{(k)}) \in W_1(M,T), \text{ for all } n, m, k \in \mathbb{N},$$

where M and T are positive constants independent of n, m and k. Hence, the limitation (u^n, v^n) of $\{(u_m^{(k)}, v_m^{(k)})\}$, as $k \to +\infty$ and $m \to +\infty$ later, is the unique weak solution of (P_n) and satisfying

$$(u^n, v^n) \in W_1(M,T), \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

By (4.1), there exists a subsequence of $\{(u^n, v^n)\}$, with the same symbol, such that

$$\begin{cases} (u^n, v^n) \to (u_\infty, v_\infty) & \text{in} \quad L^\infty(0,T; \mathbb{H}^2 \cap \mathbb{V}) \text{ weak*}, \\ (\dot{u}^n, \dot{v}^n) \to (u'_\infty, v'_\infty) & \text{in} \quad L^\infty(0,T; \mathbb{V}) \text{ weak*}, \\ (\ddot{u}^n, \ddot{v}^n) \to (u''_\infty, v''_\infty) & \text{in} \quad L^2(0,T; \mathbb{L}^2) \text{ weak}, \\ (u_\infty, v_\infty) \in W(M,T). \end{cases} \tag{4.2}$$

Applying the compactness lemma of Aubin-Lions, there exists a subsequence of $\{(u^n, v^n)\}$, also with the same symbol, such that

$$\begin{cases} (u^n, v^n) \to (u_\infty, v_\infty) & \text{in} \quad C([0,T]; \mathbb{V}) \text{ strongly}, \\ (\dot{u}^n, \dot{v}^n) \to (u'_\infty, v'_\infty) & \text{in} \quad C([0,T]; \mathbb{L}^2) \text{ strongly}. \end{cases} \tag{4.3}$$

Because (u^n, v^n) is the unique weak solution of (P_n), we get $\int_0^T \langle \ddot{u}^n(t), w \rangle \varphi(t) dt + \int_0^T a_1(S_n[u^n](t)) a(u^n(t), w) \varphi(t) dt = \int_0^T \langle f[u^n, v^n](t), w \rangle \varphi(t) dt$, and

$$\begin{aligned} &\int_0^T \langle \ddot{v}^n(t), \phi \rangle \varphi(t) dt + \int_0^T a_2(S_n[v^n](t)) b(v^n(t), \phi) \varphi(t) dt \\ &= \int_0^T \langle g[u^n, v^n](t), \phi \rangle \varphi(t) dt, \end{aligned} \tag{4.4}$$

for all $(w, \phi) \in \mathbb{V}$, $\forall \varphi \in C_c^\infty(0,T)$. By $(4.2)_3$, it leads to

$$\begin{aligned} &\int_0^T \langle \ddot{u}^n(t), w \rangle \varphi(t) dt \to \int_0^T \langle u''_\infty(t), w \rangle \varphi(t) dt, \\ &\int_0^T \langle \ddot{v}^n(t), \phi \rangle \varphi(t) dt \to \int_0^T \langle v''_\infty(t), \phi \rangle \varphi(t) dt. \end{aligned} \tag{4.5}$$

From (4.3), we deduce that

$$\begin{aligned} &\int_0^T \langle f[u^n, v^n](t), w \rangle \varphi(t) dt \to \int_0^T \langle f[u_\infty, v_\infty](t), w \rangle \varphi(t) dt, \\ &\int_0^T \langle g[u^n, v^n](t), \phi \rangle \varphi(t) dt \to \int_0^T \langle g[u_\infty, v_\infty](t), \phi \rangle \varphi(t) dt. \end{aligned} \tag{4.6}$$

Indeed, we prove (4.6) as follows. With $(4.6)_1$, from the following inequality

$$\begin{aligned} &\left\| f[u^n, v^n](t) - f[u_\infty, v_\infty](t) \right\|_0 \\ &\le K_M(f)[\|u^n(t) - u_\infty(t)\|_0 + \|v^n(t) - v_\infty(t)\|_0 + \|\dot{u}^n(t) - u'_\infty(t)\|_0 \\ &+ \|\dot{v}^n(t) - v'_\infty(t)\|_0 + \|u_x^n(t) - u_{\infty x}(t)\|_0 + \|v_x^n(t) - v_{\infty x}(t)\|_0 \end{aligned}$$

$$\begin{aligned}&\leq (1+\sqrt{R-1})K_M(f)\\ &\quad\times\left[\|(u^n,v^n)-(u_\infty,v_\infty)\|_{C([0,T];\mathbb{V})}+\|(\dot{u}^n,\dot{v}^n)-(u'_\infty,v'_\infty)\|_{C([0,T];\mathbb{L}^2)}\right],\\ &f[u^n,v^n]\to f[u_\infty,v_\infty] \text{ in } L^\infty(0,T;L^2) \text{ strongly.}\end{aligned}\tag{4.7}$$

From (4.7), we deduce that $(4.6)_1$ is true. Similarly, we also have

$$g[u^n,v^n]\to g[u_\infty,v_\infty] \text{ in } L^\infty(0,T;L^2) \text{ strongly,}$$

from which we deduce that $(4.6)_2$ is true. Now, we have to show that

$$\begin{aligned}&\int_0^T a_1(S_n[u^n](t))a(u^n(t),w)\varphi(t)dt\to\int_0^T a_1(\left\|u_\infty(t)\right\|_0^2)a(u_\infty(t),w)\varphi(t)dt,\\ &\int_0^T a_2(S_n[v^n](t))b(v^n(t),\phi)\varphi(t)dt\to\int_0^T a_2(\left\|v_\infty(t)\right\|_0^2)b(v_\infty(t),\phi)\varphi(t)dt.\end{aligned}\tag{4.8}$$

Lemma 8. *The following properties are fulfilled*

(i) $\|S_n[u^n]-S_n[u_\infty]\|_{C([0,T])}\to 0$,

(ii) $\|S_n[u_\infty]-\|u_\infty(\cdot)\|_0^2\ \|_{C([0,T])}\leq (1/4n)(R-1)^2(1+2R)\bar{C}_R^2M^2\to 0$,

(iii) $\|S_n[u^n]-\|u_\infty(\cdot)\|_0^2\ \|_{C([0,T])}\to 0$,

as $n\to\infty$, *where* $\bar{C}_R=\sup\limits_{0\neq w\in H^2\cap V}\frac{\|w\|_{C^1([1,R])}}{\|w\|_{H^2\cap V}}$. *The results (i)–(iii) still hold with* (u^n,u_∞) *replaced by* (v^n,v_∞).

Proof. By u_∞, $u^n\in L^\infty(0,T;H^2\cap V)\cap C([0,T];V)\cap C^1([0,T];L^2)$, we first note that u_∞, $u^n\in C([0,T];V)$. From here, we deduce that the functions $t\mapsto \left\|u_\infty(t)\right\|_0$, $t\mapsto S_n[u_\infty](t)$ and $t\mapsto S_n[u^n](t)$ are continuous on $[0,T]$. Therefore, all three functions in (i), (ii), and (iii) are also continuous on $[0,T]$. Next, we note that, we have the following estimation

$$\begin{aligned}\|S_n[u^n](t)-S_n[u_\infty](t)\|&\leq((R-1)/n)\sum\nolimits_{i=1}^n x_{n,i}\left||u^n(x_{n,i},t)|^2-u_\infty^2(x_{n,i},t)\right|\\ &\leq (2/n)(R-1)^2M\sum\nolimits_{i=1}^n x_{n,i}\left\|u^n-u_\infty\right\|_{C([0,T];V)}\\ &=(R-1)(R^2-1)M\left\|u^n-u_\infty\right\|_{C([0,T];V)}.\end{aligned}\tag{4.9}$$

Then, by $(4.3)_1$, we deduce from (4.9) that

$$\left\|S_n[u^n]-S_n[u_\infty]\right\|_{C([0,T])}\leq (R-1)(R^2-1)M\left\|u^n-u_\infty\right\|_{C([0,T];V)}\to 0,$$

as $n\to\infty$. Thus, (i) is valid. In order to prove (ii), by simple calculations with $F\in C^1([a,b])$, we first note that the following inequality is true

$$\Big|\int_a^b F(x)dx-\frac{b-a}{n}\sum_{i=1}^n F(a+\frac{i(b-a)}{2n})\Big|\leq\frac{(b-a)^2}{4n}\left\|F'\right\|_{C([a,b])}.$$

Consequently, with $F \in C^1([1,R]), a=1, b=R,\ x_i = 1+\frac{i(R-1)}{n},\ x_i - x_{i-1} = ((R-1)/n),\ \frac{x_{i-1}+x_i}{2} = 1+\frac{i(R-1)}{2n} = x_{n,i}$, we obtain

$$\Big|\int_1^R F(x)dx - ((R-1)/n)\sum_{i=1}^n F(x_{n,i})\Big| \le \frac{(R-1)^2}{4n}\left\|F'\right\|_{C([1,R])}. \qquad (4.10)$$

Since the embeddings $H^2\cap V \hookrightarrow C^1([1,R])\cap V \hookrightarrow C^1([1,R])$ are continuous, there exists a constant $\bar{C}_R > 0$ such that $\|w\|_{C^1([1,R])} \le \bar{C}_R \|w\|_{H^2\cap V}$, for all $w \in H^2\cap V$. We also deduce that the following embeddings are continuous

$$L^\infty(0,T;H^2\cap V) \hookrightarrow L^\infty(0,T;C^1([1,R])\cap V) \hookrightarrow L^\infty(0,T;C^1([1,R])).$$

From this, with the property $u_\infty \in L^\infty(0,T;H^2\cap V) \hookrightarrow L^\infty(0,T;C^1([1,R]))$, we have that $x \longmapsto F(x,t) = xu_\infty^2(x,t)$ belongs to $C^1([1,R])$ for almost every $t\in[0,T]$. Applying (4.10), with $F(x,t) = xu_\infty^2(x,t)$, we obtain

$$\big|\|u_\infty(t)\|_0^2 - S_n[u_\infty](t)\big| \le (1/n)(R-1)^2\|F_x(t)\|_{C([1,R])}. \qquad (4.11)$$

On the other hand, by the estimate

$$\begin{aligned}\|F_x(t)\|_{C([1,R])} &= \|u_\infty^2(t) + 2xu_\infty(t)u_{\infty x}(t)\|_{C([1,R])}\\ &\le \|u_\infty(t)\|^2_{C([1,R])} + 2R\|u_\infty(t)\|_{C([1,R])}\|u_{\infty x}(t)\|_{C([1,R])}\\ &\le (1+2R)\|u_\infty(t)\|^2_{C^1([1,R])} \le (1+2R)\bar{C}_R^2\|u_\infty(t)\|^2_{H^2\cap V}\\ &\le (1+2R)\bar{C}_R^2\|u_\infty\|^2_{L^\infty(0,T;H^2\cap V)} \le (1+2R)\bar{C}_R^2 M^2,\end{aligned}$$

it follows from (4.11) that

$$\big|\|u_\infty(t)\|_0^2 - S_n[u_\infty](t)\big| \le (1/4n)(R-1)^2(1+2R)\bar{C}_R^2M^2.$$

Thus, (ii) is true. It follows from (i), (ii) that

$$\begin{aligned}&\|S_n[u^n] - \|u_\infty(\cdot)\|_0^2\|_{C([0,T])}\\ &\quad\le \|S_n[u^n]-S_n[u_\infty]\|_{C([0,T])}+\|S_n[u_\infty]-\|u_\infty(\cdot)\|_0^2\|_{C([0,T])} \to 0,\end{aligned}$$

as $n\to\infty$. Hence, (iii) also holds. Therefore, Lemma 8 is proved. □

Lemma 9. *The following convergence is valid*

(i) $\|a_1(S_n[u^n]) - a_1(\|u_\infty(\cdot)\|_0^2)\|_{C([0,T])} \to 0,$ *as* $n\to\infty$,
(ii) $\|a_2(S_n[v^n]) - a_2(\|v_\infty(\cdot)\|_0^2)\|_{C([0,T])} \to 0,$ *as* $n\to\infty$.

Proof. Due to

$$\big|a_1(S_n[u^n](t)) - a_1(\|u_\infty(t)\|_0^2)\big| \le \tilde{K}_M(a_1)\big|S_n[u^n](t) - \|u_\infty(t)\|_0^2\big|, \qquad (4.12)$$

hence it follows from (4.12) and Lemma 8 (iii) that

$$\big\|a_1(S_n[u^n])-a_1(\|u_\infty(\cdot)\|_0^2)\big\|_{C([0,T])} \le \tilde{K}_M(a_1)\big\|S_n[u^n]-\|u_\infty(\cdot)\|_0^2\big\|_{C([0,T])} \to 0,$$

as $n\to\infty$. Moreover, (ii) is similar to (i). Thus, Lemma 9 is proved. □

Now, we continue the proof of (4.8). By the following inequality

$$
\begin{aligned}
&\Big|\int_0^T a_1(S_n[u^n](t))a(u^n(t),w)\varphi(t)dt - \int_0^T a_1(\|u_\infty(t)\|_0^2)a(u_\infty(t),w)\varphi(t)dt\Big|\\
&\quad\le \Big|\int_0^T [a_1(S_n[u^n](t)) - a_1(\|u_\infty(t)\|_0^2)]a(u^n(t),w)\varphi(t)dt\Big|\\
&\qquad + \Big|\int_0^T a_1(\|u_\infty(t)\|_0^2)a(u^n(t)-u_\infty(t),w)\varphi(t)dt\Big|\\
&\quad\le M\|w\|_a\|\varphi\|_{L^2(0,T)}\sqrt{T}\big\|a_1(S_n[u^n]) - a_1(\|u_\infty(\cdot)\|_0^2)\big\|_{C([0,T])}\\
&\qquad + \tilde{K}_M(a_1)\|w\|_a\|\varphi\|_{L^1(0,T)}\|u^n - u_\infty\|_{C([0,T];V)} \to 0, \ as\ n\to\infty.
\end{aligned}
$$

Combining $(4.3)_1$ and Lemma 9, we get $(4.8)_1$. Using a similar approach, we also have that $(4.8)_2$ is true. Finally, by (4.5), (4.6) and (4.8), giving $n\to\infty$ in (4.4), we obtain that $(u_\infty, v_\infty)\in W(M,T)$ satisfies the equation

$$
\begin{aligned}
&\int_0^T \langle u_\infty''(t),w\rangle\varphi(t)dt + \int_0^T a_1(\big\|u_\infty(t)\big\|_0^2)a(u_\infty(t),w)\varphi(t)dt\\
&= \int_0^T \langle f[u_\infty,v_\infty](t),w\rangle\varphi(t)dt,\\
&\int_0^T \langle v_\infty''(t),\phi\rangle\varphi(t)dt + \int_0^T a_2(\big\|v_\infty(t)\big\|_0^2)b(v_\infty(t),\phi)\varphi(t)dt\\
&= \int_0^T \langle g[u_\infty,v_\infty](t),\phi\rangle\varphi(t)dt,
\end{aligned}
\tag{4.13}
$$

for all $(w,\phi)\in\mathbb{V}$, $\forall\varphi\in C_c^\infty(0,T)$, together with the initial conditions

$$
(u_\infty(0),u_\infty'(0)) = (\tilde{u}_0,\tilde{u}_1),\ (v_\infty(0),v_\infty'(0)) = (\tilde{v}_0,\tilde{v}_1).
$$

Consequently,

$$
\begin{cases}
\langle u_\infty''(t),w\rangle + a_1(\big\|u_\infty(t)\big\|_0^2)a(u_\infty(t),w) = \langle f[u_\infty,v_\infty](t),w\rangle,\\
\langle v_\infty''(t),\phi\rangle + a_2(\big\|v_\infty(t)\big\|_0^2)b(v_\infty(t),\phi) = \langle g[u_\infty,v_\infty](t),\phi\rangle,\ \forall(w,\phi)\in\mathbb{V},\\
(u_\infty(0),u_\infty'(0)) = (\tilde{u}_0,\tilde{u}_1),\ (v_\infty(0),v_\infty'(0)) = (\tilde{v}_0,\tilde{v}_1),
\end{cases}
$$

and $(u_\infty,v_\infty)\in W(M,T)$. Furthermore, (4.13) implies that

$$
u_\infty'' = -a_1(\big\|u_\infty(t)\big\|_0^2)L_1u_\infty + f[u_\infty,v_\infty]\in L^\infty(0,T;L^2),
$$

$$
v_\infty'' = -a_2(\big\|v_\infty(t)\big\|_0^2)L_2v_\infty + g[u_\infty,v_\infty]\in L^\infty(0,T;L^2),
$$

so $(u_\infty,v_\infty)\in W_1(M,T)$, hence $(u_\infty,v_\infty)\in W_1(M,T)$ is a weak solution of (1.2). Moreover, we can verify that this weak solution (u_∞,v_∞) is unique.

By uniqueness of (1.2), we have $(u,v)=(u_\infty,v_\infty)$. We note more that, the sequence $\{(u^n,v^n)\}$ converges to (u,v) in the sense as in (4.2) and (4.3).

The above result leads to the following theorem.

Theorem 3. *Let (A_1)–(A_3) hold. Then there exist positive constants M, T such that*

(i) Problem (1.2) has a unique weak solution $(u,v) \in W_1(M,T)$.

(ii) The solution sequence $\{(u^n,v^n)\}_n$ of Problem (P_n) converges to the weak solution (u,v) of Problem(1.2) in the sense

$$\begin{cases} (u^n,v^n) \to (u,v) & in \quad L^\infty(0,T;\mathbb{H}^2\cap\mathbb{V}) \ weak^*, \\ (\dot{u}^n,\dot{v}^n) \to (u',v') & in \quad L^\infty(0,T;\mathbb{V}) \ weak^*, \\ (\ddot{u}^n,\ddot{v}^n) \to (u'',v'') & in \quad L^2(0,T;\mathbb{L}^2) \ weak, \\ (u^n,v^n) \to (u,v) & in \quad W_1(T) \ strongly. \end{cases}$$

(iii) Furthermore, we have the following estimates

$$\begin{aligned} &\|(u^n,v^n)-(u,v)\|_{W_1(T)} \le C_T\,\|E_n\|_{C([0,T])}\,,\ \forall n\in\mathbb{N}, \qquad (4.14)\\ &E_n(t)=\left|S_n[u^n](t)-\|u(t)\|_0^2\right|+\left|S_n[v^n](t)-\left\|v(t)\right\|_0^2\right|,\\ &\|E_n\|_{C([0,T])}\le\left\|S_n[u^n]-\|u(\cdot)\|_0^2\right\|_{C([0,T])}+\left\|S_n[v^n]-\|v(\cdot)\|_0^2\right\|_{C([0,T])}\to 0, \end{aligned}$$

as $n\to\infty$, and C_T is a constant depending only on $T, a_1, a_2, f, g, \tilde{u}_0, \tilde{u}_1, \tilde{v}_0, \tilde{v}_1$.

(iv) On the other hand, if $T>0$ is chosen small enough, we have the following estimate

$$\|(u^n,v^n)-(u,v)\|_{W_1(T)} \le C_T^*/n,\ \forall n\in\mathbb{N}, \qquad (4.15)$$

where, C_T^* is a constant depending only on T, a_1, a_2, f, g, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$.

Proof. It remains to prove (iii) and (iv).

We set $(\bar{u}_n,\bar{v}_n)=(u^n,v^n)-(u,v)=(u^n-u,\ v^n-v)$ and

$$\begin{aligned} &\bar{f}_n(t)=f[u^n,v^n](t)-f[u,v](t),\ \bar{g}_n(t)=g[u^n,v^n](t)-g[u,v](t),\\ &\bar{A}_1^n(t)=a_1(S_n[u^n](t))-a_1(\|u(t)\|_0^2),\ \bar{A}_2^n(t)=a_2(S_n[v^n](t))-a_2(\|v(t)\|_0^2),\\ &\tilde{a}_{1n}(t)=a_1(S_n[u^n](t)),\ \tilde{a}_{2n}(t)=a_2(S_n[v^n](t)), \qquad (4.16) \end{aligned}$$

then, $(\bar{u}_n,\bar{v}_n)\in\bar{W}_T$ satisfies the variational problem

$$\begin{cases} \langle\bar{u}_n''(t),w\rangle+\tilde{a}_{1n}(t)a(\bar{u}_n(t),w)+\bar{A}_1^n(t)a(u(t),w)=\langle\bar{f}_n(t),w\rangle,\\ \langle\bar{v}_n''(t),\phi\rangle+\tilde{a}_{2n}(t)b(\bar{v}_n(t),\phi)+\bar{A}_2^n(t)b(v(t),\phi)=\langle\bar{g}_n(t),\phi\rangle,\forall(w,\phi)\in\mathbb{V},\\ (\bar{u}_n(0),\bar{u}_n'(0))=(\bar{v}_n(0),\bar{v}_n'(0))=(0,0). \end{cases} \qquad (4.17)$$

Taking $(w,\phi)=(\bar{u}_n'(t),\bar{v}_n'(t))$ in $(4.17)_{1,2}$ and then integrating in t, we have

$$\begin{aligned} \hat{a}_*\bar{X}_n(t)\le&\int_0^t(\tilde{a}_{1n}'(s)\left\|\bar{u}_n(s)\right\|_a^2+\tilde{a}_{2n}'(s)\left\|\bar{v}_{nx}(s)\right\|_0^2)ds\\ &+2\int_0^t\bar{A}_1^n(s)\langle L_1u(s),\bar{u}_n'(s)\rangle+\bar{A}_2^n(s)\langle L_2v(s),\bar{v}_n'(s)\rangle ds \qquad (4.18)\\ &+2\int_0^t\left[\langle\bar{f}_n(s),\bar{u}_n'(s)\rangle+\langle\bar{g}_n(s),\bar{v}_n'(s)\rangle\right]ds, \end{aligned}$$

where $\hat{a}_* = \min\{1, a_{1*}, a_{2*}\}$ and

$$\bar{X}_n(t) = \left\|(\bar{u}'_n(t), \bar{v}'_n(t))\right\|^2_{\mathbb{L}^2} + \left\|(\bar{u}_n(t), \bar{v}_n(t))\right\|^2_{\mathbb{V}}.$$

First, we need to evaluate the terms $\tilde{a}'_{1n}(t)$, $\tilde{a}'_{2n}(t)$, $\bar{A}^n_1(t)$, $\bar{A}^n_2(t)$, $\bar{f}_n(t)$, $\bar{g}_n(t)$ as follows. Estimating $\tilde{a}'_{1n}(t)$, $\tilde{a}'_{2n}(t)$: Note that

$$\begin{aligned}\tilde{a}'_{1n}(t) &= a'_1(S_n[u^n](t))\frac{d}{dt}S_n[u^n](t)\\ &= 2a'_1(S_n[u^n](t))((R-1)/n)\sum_{i=1}^{n} x_{n,i}u^n(x_{n,i},t)\dot{u}^n(x_{n,i},t),\end{aligned}$$

hence

$$\begin{aligned}&\left\|\tilde{a}'_{1n}(t)\right\| \leq 2\tilde{K}_M(a_1)((R-1)/n)\sum\nolimits_{i=1}^{n} x_{n,i}(R-1)\left\|u^n_x(t)\right\|_0\left\|\dot{u}^n_x(t)\right\|_0\\ &\leq 2\tilde{K}_M(a_1)((R-1)/n)\frac{n(1+R)}{2}(R-1)M^2\\ &= \tilde{K}_M(a_1)(R-1)(R^2-1)M^2 \leq 2\tilde{K}_M(a_1)M_* \leq 2\tilde{K}_M(a_1,a_2)M_*. \qquad (4.19)\end{aligned}$$

Similarly,

$$\left\|\tilde{a}'_{2n}(t)\right\| \leq 2\tilde{K}_M(a_2)M_* \leq 2\tilde{K}_M(a_1,a_2)M_*.$$

Estimating $\bar{A}^n_1(t)$, $\bar{A}^n_2(t)$: by (4.16) we deduce that

$$\begin{aligned}&\|\bar{A}^n_1(t)\| \leq \tilde{K}_M(a_1)\left\|S_n[u^n](t) - \|u(t)\|^2_0\right\| \leq \tilde{K}_M(a_1,a_2)\left\|S_n[u^n] - \|u(\cdot)\|^2_0\right\|_{C([0,T])},\\ &\left\|\bar{A}^n_2(t)\right\| \leq \tilde{K}_M(a_1,a_2)\left\|S_n[v^n] - \|v(\cdot)\|^2_0\right\|_{C([0,T])}.\end{aligned}$$

Estimating $\bar{f}_n(t)$, $\bar{g}_n(t)$: we have

$$\begin{aligned}&\left\|\bar{f}_n(t)\right\|_0 \leq K_M(f)\left[\|\bar{u}_n(t)\|_0 + \|\bar{v}_n(t)\|_0\right.\\ &\qquad \left. + \left\|\bar{u}'_n(t)\right\|_0 + \left\|\bar{v}'_n(t)\right\|_0 + \left\|\bar{u}_{nx}(t)\right\|_0 + \left\|\bar{v}_{nx}(t)\right\|_0\right] \qquad (4.20)\\ &\quad \leq 2(1+\sqrt{R-1})K_M(f)\sqrt{\bar{X}_n(t)} \leq 2(1+\sqrt{R-1})K_M(f,g)\sqrt{\bar{X}_n(t)},\\ &\|\bar{g}_n(t)\|_0 \leq \|g[u^n,v^n](t) - g[u,v](t)\|_0 \leq 2(1+\sqrt{R-1})K_M(f,g)\sqrt{\bar{X}_n(t)}. \qquad (4.21)\end{aligned}$$

From the estimates (4.19)–(4.21), we evaluate the terms on the right-hand side of (4.18) as follows.

$$\begin{aligned}&\int_0^t (\tilde{a}'_{1n}(s)\left\|\bar{u}_n(s)\right\|^2_a + \tilde{a}'_{2n}(s)\left\|\bar{v}_{nx}(s)\right\|^2_0)ds\\ &\leq 2\tilde{K}_M(a_1,a_2)M_*\int_0^t \bar{X}_n(s)ds \equiv \eta^{(1)}_M\int_0^t \bar{X}_n(s)ds\\ &2\int_0^t \bar{A}^n_1(s)\langle L_1u(s), \bar{u}'_n(s)\rangle + \bar{A}^n_2(s)\langle L_2v(s), \bar{v}'_n(s)\rangle ds\\ &\leq \tilde{K}^2_M(a_1,a_2)(\bar{\gamma}_1+\sqrt{2})^2M^2\int_0^T E^2_n(s) + \int_0^t \bar{X}_n(s)ds\end{aligned}$$

$$
\begin{aligned}
&\leq \tilde{K}_M^2(a_1,a_2)(\bar{\gamma}_1+\sqrt{2})^2 M^2 T \left\| E_n \right\|_{C([0,T])}^2 + \int_0^t \bar{X}_n(s)ds \\
&\equiv T\eta_M^{(2)} \left\| E_n \right\|_{C([0,T])}^2 + \int_0^t \bar{X}_n(s)ds; \\
&2\int_0^t \left[\langle \bar{f}_n(s), \bar{u}_n'(s) \rangle + \langle \bar{g}_n(s), \bar{v}_n'(s) \rangle \right] ds \\
&\leq 2\int_0^t \left[\left\| \bar{f}_n(s) \right\|_0 \left\| \bar{u}_n'(s) \right\|_0 + \left\| \bar{g}_n(s) \right\|_0 \left\| \bar{v}_n'(s) \right\|_0 \right] ds \\
&\leq 8(1+\sqrt{R-1}) K_M(f,g) \int_0^t \bar{X}_n(s)ds \equiv \eta_M^{(3)} \int_0^t \bar{X}_n(s)ds.
\end{aligned}
$$

It follows from (4.16) and (4.20) that

$$
\bar{X}_n(t) \leq \frac{T\eta_M^{(2)}}{\hat{a}_*} \left\| E_n \right\|_{C([0,T])}^2 + \frac{1}{\hat{a}_*}(1+\eta_M^{(1)}+\eta_M^{(3)}) \int_0^t \bar{X}_n(s)ds. \tag{4.22}
$$

Using Gronwall's lemma, it follows from (4.22) that

$$
\bar{X}_n(t) \leq \frac{T\eta_M^{(2)}}{\hat{a}_*} \exp\left[\frac{T}{\hat{a}_*}(1+\eta_M^{(1)}+\eta_M^{(3)}) \right] \left\| E_n \right\|_{C([0,T])}^2 \equiv \eta_M^{(4)}(T) \left\| E_n \right\|_{C([0,T])}^2. \tag{4.23}
$$

We deduce from (4.23) that

$$
\left\| (u^n,v^n)-(u,v) \right\|_{W_1(T)} = \left\| (\bar{u}_n,\bar{v}_n) \right\|_{W_1(T)} \leq 4\sqrt{\eta_M^{(4)}(T)} \left\| E_n \right\|_{C([0,T])}.
$$

Hence, (4.14) holds. From Lemma 9, we have

$$
\begin{aligned}
&\left\| S_n[u^n] - \|u(\cdot)\|_0^2 \right\|_{C([0,T])} \leq \left\| S_n[u^n] - S_n[u] \right\|_{C([0,T])} + \left\| S_n[u] - \|u(\cdot)\|_0^2 \right\|_{C([0,T])} \\
&\leq (R-1)(R^2-1)M \left\| u^n - u \right\|_{C([0,T];V)} + \frac{(R-1)^2}{4n}(1+2R)\bar{C}_R^2 M^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left\| S_n[v^n] - \|v(\cdot)\|_0^2 \right\|_{C([0,T])} \\
&\quad \leq (R-1)(R^2-1)M \left\| v^n - v \right\|_{C([0,T];V)} + \frac{(R-1)^2}{4n}(1+2R)\bar{C}_R^2 M^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\| E_n \right\|_{C([0,T])} &\leq \left\| S_n[u^n] - \|u(\cdot)\|_0^2 \right\|_{C([0,T])} + \left\| S_n[v^n] - \|v(\cdot)\|_0^2 \right\|_{C([0,T])} \\
&\leq (R-1)(R^2-1)M \left\| (u^n,v^n)-(u,v) \right\|_{W_1(T)} + \frac{(R-1)^2}{2n}(1+2R)\bar{C}_R^2 M^2.
\end{aligned}
$$

From here, we deduce that

$$
\begin{aligned}
&\|(u^n,v^n)-(u,v)\|_{W_1(T)} \leq 4\big(\eta_M^{(4)}(T)\big)^{1/2} \|E_n\|_{C([0,T])} \\
&\quad \leq 4\big(\eta_M^{(4)}(T)\big)^{1/2}(R-1)(R^2-1)M\|(u^n,v^n)-(u,v)\|_{W_1(T)} \\
&\quad + 4\sqrt{\eta_M^{(4)}(T)}\frac{(R-1)^2}{2n}(1+2R)\bar{C}_R^2 M^2.
\end{aligned} \tag{4.24}
$$

By

$$\lim_{T\to 0_+}\sqrt{\eta_M^{(4)}(T)}=\lim_{T\to 0_+}\left(\frac{T\eta_M^{(2)}}{\hat{a}_*}\exp\left[\frac{T}{\hat{a}_*}(1+\eta_M^{(1)}+\eta_M^{(3)})\right]\right)^{1/2}=0,$$

we deduce that there exists a sufficiently small $T>0$ such that

$$4\sqrt{\eta_M^{(4)}(T)}(R-1)(R^2-1)M<1. \tag{4.25}$$

Combining (4.24) and (4.25), (4.15) holds. Theorem 3 is proved. □

Remark 4. We would like to discuss the possibility of replacing the integral sums $S_n[u](t)$ and $S_n[v](t)$ with trapezoidal rule or Simpson's rule. These rules provide better approximations of the two Carrier terms $\|u(t)\|_0^2=\int_1^R xu^2(x,t)dx$ and $\|v(t)\|_0^2=\int_1^R xv^2(x,t)dx$ if (u,v) are sufficiently smooth in terms of the variable x. To gain a clearer perspective, we return to the beginning with a function F defined on $[a,b]$ instead of $[1,R]$.

(i) For the trapezoidal rule, it allows us to approximate the integral

$$\int_a^b F(x)dx\approx\frac{b-a}{2n}\sum_{i=1}^{n}(F(x_{i-1})+F(x_i))$$
$$=\frac{b-a}{n}[\frac{F(x_0)+F(x_n)}{2}+\sum_{i=1}^{n-1}F(x_i)],\ x_i=a+\frac{i(b-a)}{n},\ i=\overline{0,n}.$$

If $F\in C^2([a,b])$, then the error of this approximation is estimated by

$$\left|\int_a^b F(x)dx-\frac{b-a}{2n}\sum_{i=1}^{n}(F(x_{i-1})+F(x_i))\right|\le\frac{(b-a)^3}{12n^2}\left\|F''\right\|_{C([a,b])}.$$

(ii) For the Simpson's rule, it allows us to approximate the integral

$$\int_a^b F(x)dx\approx\frac{b-a}{6n}\sum_{i=1}^{n}\left[f(x_{2i-2})+4f(x_{2i-1})+f(x_{2i})\right],$$
$$x_i=a+i(b-a)/(2n),\quad i=\overline{0,2n}.$$

If $F\in C^4([a,b])$, then the error of this approximation is estimated by

$$\left|\int_a^b f(x)dx-\frac{b-a}{6n}\sum_{i=1}^{n}\left[f(x_{2i-2})+4f(x_{2i-1})+f(x_{2i})\right]\right|$$
$$\le\frac{1}{90n^4}\left(\frac{b-a}{2}\right)^5\|f^{(4)}\|_{C([a,b])}.$$

Now, we try to replace the integral sums $S_n[u](t)$ and $S_n[v](t)$ with the trapezoidal formulas as follows

$$\tilde{S}_n[u](t)=\frac{R-1}{2n}\sum_{i=1}^{n}[x_{i-1}u^2(x_{i-1},t)+x_iu^2(x_i,t)],$$
$$\tilde{S}_n[v](t)=\frac{R-1}{2n}\sum_{i=1}^{n}\left[x_{i-1}v^2(x_{i-1},t)+x_iv^2(x_i,t)\right],\ x_i=1+\frac{i(R-1)}{n},\ i=\overline{0,n},$$

for all $n \in \mathbb{N}$. Therefore, in order to estimate the error of the trapezoidal formulas with the integrals $\int_1^R xu^2(x,t)dx$ and $\int_1^R xv^2(x,t)dx$, it requires that the functions $x \longmapsto xu^2(x,t)$ and $x \longmapsto xv^2(x,t)$ belong to $C^2([1,R])$ for almost every $t \in [0,T]$. Meanwhile, we only have $u,\ v \in L^\infty(0,T;H^2 \cap V) \hookrightarrow L^\infty(0,T;C^1([1,R]))$. Similarly, if we replace $S_n[u](t)$ and $S_n[v](t)$ with the Simpson's formulas, it would require that $x \longmapsto xu^2(x,t)$ and $x \longmapsto xv^2(x,t)$ belong to $C^4([1,R])$ for almost every $t \in [0,T]$. For this reason, in order to approximate $S_n[u](t)$ and $S_n[v](t)$, we use the rectangle rule to approximate $\int_1^R xu^2(x,t)dx$ and $\|v(t)\|_0^2 = \int_1^R xv^2(x,t)dx$, which is consistent with the smoothness of the solution (u,v).

Remark 5. We can consider Problem (P_n) with $S_n[u](t)$, $S_n[v](t)$ replaced by the following integral sums respectively

$$S_n[u_x](t) = ((R-1)/n)\sum_{i=1}^n x_{n,i}u_x^2(x_{n,i},t),$$
$$S_n[v_x](t) = ((R-1)/n)\sum_{i=1}^n x_{n,i}v_x^2(x_{n,i},t),$$

in which $x_{n,i} = 1+\frac{(R-1)(2i-1)}{2n}$, $i=0,\dots,n$, $\forall n \in \mathbb{N}$. This leads to the following open problem

$$(\bar{P}_n)\begin{cases} u_{tt} - a_1(S_n[u_x](t))\big(u_{xx} + \big(1/x\big)u_x - \big(1/x^2\big)u\big) \\ \qquad = f(x,t,u,v,u_t,v_t,u_x,v_x), 1<x<R, 0<t<T, \\ v_{tt} - a_2(S_n[v_x](t))(v_{xx} + \big(1/x\big)v_x) \\ \qquad = g(x,t,u,v,u_t,v_t,u_x,v_x), 1<x<R, 0<t<T, \\ u_x(1,t) - b_1u(1,t) = v_x(1,t) = u(R,t) = v(R,t) = 0, \\ (u(x,0),v(x,0)) = (\tilde{u}_0(x),\tilde{v}_0(x)),\ (u_t(x,0),v_t(x,0)) = (\tilde{u}_1(x),\tilde{v}_1(x)), \end{cases}$$

where $b_1 > 0$, $R > 1$ are given constants, and a_1, a_2, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$, f and g are given functions, with two questions:

(i) What is the sufficient condition for the unique existence of a weak solution (u^n, v^n) of Problem $(\bar{P}_n)$?

(ii) Does the sequence $\{(u_m, v_m)\}_n$ converge to a weak solution (u,v) of the correspondence problem $(\bar{P}_\infty)$ in a certain sense?

Acknowledgements

The authors wish to express their sincere thanks to the editor and the referees for the valuable comments and suggestions for the improvement of the paper.

References

[1] H. Brezis. *Functional Analysis, Sobolev spaces and partial differential equations.* Springer New York Dordrecht Heidelberg London, 2010. https://doi.org/10.1007/978-0-387-70914-7.

[2] G.F. Carrier. On the non-linear vibrations problem of elastic string. *Quarterly of Applied Mathematics*, **3**:157–165, 1945. https://doi.org/10.1090/qam/12351.

[3] M.M. Cavalcanti, V.N.D. Cavalcanti and J.A. Soriano. Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation. *Advances Differential Equations*, **6**(6):701–730, 2001. https://doi.org/10.57262/ade/1357140586.

[4] M.M. Cavalcanti, V.N.Domingos Cavalcanti, J.S. Prates Filho and J.A. Soriano. Existence and exponential decay for a Kirchhoff-Carrier model with viscosity. *Journal of Mathematical Analysis and Applications*, **226**(1):40–60, 1998. https://doi.org/10.1006/jmaa.1998.6057.

[5] M.M. Cavalcanti, V.N.Domingos Cavalcanti, J.A. Soriano and J.S. Prates Filho. Existence and asymptotic behaviour for a degenerate Kirchhoff-Carrier model with viscosity and nonlinear boundary conditions. *Revista Matemática Complutense*, **14**(1):177–203, 2001. https://doi.org/10.5209/rev_REMA.2001.v14.n1.17054.

[6] C. Fetecau, C. Fetecau, M. Jamil and A. Mahmood. Flow of fractional Maxwell fluid between coaxial cylinders. *Archive of Applied Mechanics*, **81**(8):1153–1163, 2011. https://doi.org/10.1007/s00419-011-0536-x.

[7] C. Fetecau, C. Fetecau, M. Khan and D. Vieru. Decay of a potential vortex in a generalized Oldroyd-B fluid. *Applied Mathematics and Computation*, **205**(1):497–506, 2008. https://doi.org/10.1016/j.amc.2008.08.017.

[8] T. Hayat, C. Fetecau and M. Sajid. On MHD transient flow of a Maxwell fluid in a porous medium and rotating frame. *Physics Letters A*, **372**(10):1639–1644, 2008. https://doi.org/10.1016/j.physleta.2007.10.036.

[9] M. Jamil and C. Fetecau. Helical flows of Maxwell fluid between coaxial cylinders with given shear stresses on the boundary. *Nonlinear Analysis: Real World Applications*, **11**(5):4302–4311, 2010. https://doi.org/10.1016/j.nonrwa.2010.05.016.

[10] M. Jamil, C. Fetecau, N.A. Khan and A. Mahmood. Some exact solutions for helical flows of Maxwell fluid in an annular pipe due to accelerated shear stresses. *International Journal of Chemical Reactor Engineering*, **9**(1), 2011. https://doi.org/10.1515/1542-6580.2486.

[11] G.R. Kirchhoff. *Vorlesungen über Mathematische Physik: Mechanik.* Teubner, Leipzig, 1986.

[12] N.A. Larkin. Global regular solutions for the nonhomogeneous Carrier equation. *Mathematical Problems in Engineering*, **8**(1):15–31, 2002. https://doi.org/10.1080/10241230211382.

[13] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites nonlinéaires.* Dunod, Gauthier - Villars, Paris, 1969.

[14] N.T. Long, A.P.N. Dinh and T.N. Diem. Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator. *Journal of Mathematical Analysis and Applications*, **267**(1):116–134, 2002. https://doi.org/10.1006/jmaa.2001.7755.

[15] L.A. Medeiros. On some nonlinear perturbation of Kirchhoff-Carrier operator. *Computational and Applied Mathematics*, **13**(3):225–233, 1994.

[16] L.A. Medeiros, J. Limaco and S.B. Menezes. Vibrations of elastic strings: Mathematical aspects, part one. *Journal of Computational Analysis and Applications*, **4**(2):91–127, 2002. https://doi.org/10.1023/A:1012934900316.

[17] L.A. Medeiros, J. Limaco and S.B. Menezes. Vibrations of elastic strings: Mathematical aspects, part two. *Journal of Computational Analysis and Applications*, **4**(3):211–263, 2002. https://doi.org/10.1023/A:1013151525487.

[18] J.Y. Park and J.Ja Bae. On coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term. *Applied Mathematics and Computation*, **129**(1):87–105, 2002. https://doi.org/10.1016/S0096-3003(01)00031-5.

[19] M.L. Santos, J. Ferreira, D.C. Pereira and C.A. Raposo. Global existence and stability for wave equation of Kirchhoff type with memory condition at the boundary. *Nonlinear Analysis: Theory, Methods & Applications*, **54**(5):959–976, 2003. https://doi.org/10.1016/S0362-546X(03)00121-4.

[20] R.E. Showalter. *Hilbert space methods for partial differential equations.* Electronic Journal of Differential Equations, 1994. https://doi.org/10.58997/ejde.mon.01.

[21] L.X. Truong, L.T.P. Ngoc, C.H. Hoa and N.T. Long. On a system of nonlinear wave equations associated with the helical flows of Maxwell fluid. *Nonlinear Analysis: Real World Applications*, **12**(6):3356–3372, 2011. https://doi.org/10.1016/j.nonrwa.2011.05.033.