ISSN: 1392-6292/ eISSN: 1648-3510

MATHEMATICAL
MODELLING and ANALYSIS

2025
Volume 30
Issue 2
Pages 186–202
https://doi.org/10.3846/mma.2025.22109

A joint discrete limit theorem for Epstein and Hurwitz zeta-functions

Hany Gerges[a], Antanas Laurinčikas[a] and Renata Macaitienė[b,c] ✉

[a] *Faculty of Mathematics and Informatics, Vilnius University, Lithuania*
[b] *Šiauliai Academy, Vilnius University, Lithuania*
[c] *Faculty of Business and Technologies, Šiauliai State Higher Education Institution, Lithuania*

Article History:
- received August 26, 2024
- revised October 7, 2024
- accepted November 5, 2024

Abstract. In the paper, we obtain a joint limit theorem on weak convergence for probability measure defined by discrete shifts of the Epstein and Hurwitz zeta-functions. The limit measure is explicitly given. For the proof, some linear independence restriction is required. The proved theorem extends and continues Bohr–Jessen's classical results on probabilistic characterization of value distribution for the Riemann zeta-function.

Keywords: Epstein zeta-function; Hurwitz zeta-function; limit theorem; Haar probability measure; weak convergence.

AMS Subject Classification: 11M06; 11M46.

✉ Corresponding author. E-mail: renata.macaitiene@sa.vu.lt

1 Introduction

In analytic number theory and mathematics in general, the important role belongs to zeta-functions. Zeta-functions are functions of a complex variable $s = \sigma + it$ defined in some half-plane by Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad \sigma > \sigma_0,$$

or their modifications. The name "zeta-functions" comes from the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

which was already studied with real s by L. Euler, though, a powerful potential of $\zeta(s)$ was opened by B. Riemann [35] in connection to distribution of prime numbers in the set of all natural numbers $\mathbb{N}$.

After the proof of the asymptotic distribution law of prime numbers p, see [11,18],

$$\sum_{p\le x} 1 \sim \int_2^x \frac{\mathrm{d}u}{\log u}, \quad x \to \infty,$$

it was observed that the function $\zeta(s)$ appears in solving other theoretical and practical problems, however, its value distribution is rather complicated. This suggested to H. Bohr to apply probabilistic methods in the theory of $\zeta(s)$ [5]. Bohr's ideas were realized in the joint works with B. Jessen [6,7]. Since this time, a probabilistic approach occupies a significant place in the theory of zeta-functions and their applications. The results obtained are stated as limit theorems on weakly convergent probability measures, see, for example, [1,23,24,25,36] and [2,27,31].

In [29], the first limit theorem for the Epstein zeta-function was proven. Let $\mathbb{Z}$ denote the set of all integer numbers, and Q be a positive definite $n \times n$ matrix. The Epstein zeta-function $\zeta(s;Q)$ is defined, for $\sigma > \frac{n}{2}$, by the Dirichlet type series

$$\zeta(s;Q) = \sum_{\underline{x}\in\mathbb{Z}^n\setminus\{\underline{0}\}} (\underline{x}^{\mathrm{T}} Q \underline{x})^{-s},$$

and has analytic continuation to the whole complex plane, except for a simple pole at the point $s = \frac{n}{2}$ with residue $\pi^{\frac{n}{2}} \left(\Gamma(\frac{n}{2})\sqrt{\det Q}\right)^{-1}$, where, as usual, $\Gamma(s)$ is the Euler gamma-function. The function $\zeta(s;Q)$ was introduced in [14] as an example of the most general zeta-function satisfying the functional equation proved by Riemann for $\zeta(s)$ in [35].

The function $\zeta(s;Q)$ is an attractive analytic object, and has been investigated by many mathematicians in [3,9,15,19,20,33]. The function is used for practical applications, see, for example, [12,13,17].

In [16], a probabilistic limit theorem was obtained for a pair $\underline{\zeta}(s,\alpha;Q) = \big(\zeta(s;Q), \zeta(s,\alpha)\big)$, where $\zeta(s,\alpha)$ is the classical Hurwitz zeta-function with parameter α, $0 < \alpha \le 1$. Recall that $\zeta(s,\alpha)$, for $\sigma > 1$, is defined by the series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s,\alpha)$ was introduced in [21], its theory, including limit theorems is also given in [28].

The statement of a result from [16] requires some notations and definitions. Let $\mathbb{P}$ be the set of all prime numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\Omega = \Omega_1 \times \Omega_2$, where

$$\Omega_1 = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\} \quad \text{and} \quad \Omega_2 = \prod_{m\in\mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}.$$

Then, the set Ω is a compact topological group, hence, on $(\Omega, \mathcal{B}(\Omega))$ (where $\mathcal{B}(\mathbb{X})$ denotes the Borel σ-field of the space $\mathbb{X}$), the probability Haar measure m_H can be defined, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Here,

m_H is the product of Haar measures m_{H1} and m_{H2}, where m_{H1} is the probability Haar measure on the coordinate space $(\Omega_1, \mathcal{B}(\Omega_1))$ and m_{H2} is the probability Haar measure on $(\Omega_2, \mathcal{B}(\Omega_2))$. We remind that the measure m_H is invariant with respect to shifts by points from Ω, i.e., $m_H(A) = m_H(\omega A) = m_H(A\omega)$, for every $A \in \mathcal{B}(\Omega)$ and all $\omega \in \Omega$. Let $\omega = (\omega_1, \omega_2)$ be elements of Ω, where $\omega_1 = (\omega_1(p) : p \in \mathbb{P}) \in \Omega_1$ and $\omega_2 = (\omega_2(m) : m \in \mathbb{N}_0) \in \Omega_2$.

In order to define a certain $\mathbb{C}^2$-valued random element on the space $(\Omega,\ \mathcal{B}(\Omega),\ m_H)$, suppose that $\underline{x}^{\mathrm{T}} Q \underline{x} \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^n \backslash \{0\}$. Under the restriction, it follows that $\zeta(s; Q) = \zeta(s; E_Q) + \zeta(s; F_Q)$, where

$$\zeta(s; E_Q) = \sum_{m=1}^{\infty} \frac{e_Q(m)}{m^s} \quad \text{and} \quad \zeta(s; F_Q) = \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}, \quad \sigma > \frac{n}{2},$$

are zeta-functions of certain Eisenstein series $E_Q(s) = \sum_{m=0}^{\infty} e_Q(m) \mathrm{e}^{2\pi i m s}$ and of a certain cusp form $F_Q(s) = \sum_{m=1}^{\infty} f_Q(m) \mathrm{e}^{2\pi i m s}$, respectively [15]. Moreover, for even $n \geq 4$, the Eisenstein series $E_Q(s)$ is a modular form of weight $\frac{n}{2}$ and level q, where $q \in \mathbb{N}$ is such that $q(2Q)^{-1}$ is an integral matrix [22]. This decomposition together with [19, 20] and [22] implies that, for $\sigma > \frac{n-1}{2}$,

$$\zeta(s; Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s, \chi_k) L\left(s - \frac{n}{2} + 1, \hat{\chi}_l\right) + \sum_{m=1}^{\infty} \frac{b_Q(m)}{m^s},$$

where $a_{kl} \in \mathbb{C}$, k and l are positive divisors of q, χ_k and $\hat{\chi}_l$ are Dirichlet characters modulo $\frac{q}{k}$ and $\frac{q}{l}$, respectively, and $L(s, \chi_k)$ and $L(s, \hat{\chi}_l)$ are Dirichlet L-functions. Besides, the series with coefficients $b_Q(m)$ is absolutely convergent for $\sigma > \frac{n-1}{2}$. We recall that the Dirichlet L-function $L(s, \chi)$ with character χ modulo q, for $\sigma > 1$, is given by

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ if χ is the principal character.

Let $\underline{\sigma} = (\sigma_1, \sigma_2)$. Now, for $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $\mathbb{C}^2$-valued random element

$$\underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) = \big(\zeta(\sigma_1, \omega_1; Q), \zeta(\sigma_2, \omega_2, \alpha)\big),$$

where

$$\begin{aligned} \zeta(\sigma_1, \omega_1; Q) &= \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl} \omega_1(k) \omega_1(l)}{k^{\sigma_1} l^{\sigma_1}} L(\sigma_1, \omega_1, \chi_k) L\left(\sigma_1 - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\right) \\ &+ \sum_{m=1}^{\infty} \frac{b_Q(m) \omega_1(m)}{m^{\sigma_1}} \end{aligned}$$

and

$$\zeta(\sigma_2, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m + \alpha)^{\sigma_2}}.$$

Here,

$$L(\sigma_1, \omega_1, \chi_k) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi_k(p)\omega_1(p)}{p^{\sigma_1}}\right)^{-1},$$
$$L\left(\sigma_1 - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\right) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\hat{\chi}_l(p)\omega_1(p)}{p^{\sigma_1 - \frac{n}{2} + 1}}\right)^{-1},$$
$$\omega_1(m) = \prod_{\substack{p^r \mid m \\ p^{r+1} \nmid m}} \omega_1^r(p), \ m \in \mathbb{N}.$$

We note that the second product is convergent for $\sigma_1 > \frac{n-1}{2}$ for almost all $\omega_1 \in \Omega_1$. Denote by $P_{\underline{\zeta},\underline{\sigma},Q,\alpha}$ the distribution of the random element $\underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q)$, i.e.,

$$P_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = m_H \left\{\omega \in \Omega : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}^2).$$

Set

$$\underline{\zeta}(\underline{\sigma} + it, \alpha; Q) = \left(\zeta(\sigma_1 + it; Q), \zeta(\sigma_2 + it, \alpha)\right),$$

denote by meas$A$ the Lebesgue measure on $\mathbb{R}$, and define

$$P_{T,\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = \frac{1}{T}\text{meas}\left\{t \in [0, T] : \underline{\zeta}(\underline{\sigma} + it, \alpha; Q) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}^2).$$

The main result of [16] is the following theorem.

Theorem 1. *Suppose that the set*

$$\left\{ (\log p : p \in \mathbb{P}), \left(\log(m + \alpha) : m \in \mathbb{N}_0\right) \right\}$$

is linearly independent over the field of rational numbers $\mathbb{Q}$, $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$ are fixed. Then $P_{T,\underline{\zeta},\underline{\sigma},Q,\alpha}$ converges weakly to the measure $P_{\underline{\zeta},\underline{\sigma},Q,\alpha}$ as $T \to \infty$.

Theorem 1 is of continuous type, since t in the definition of $P_{T,\underline{\zeta},\underline{\sigma},Q,\alpha}$ takes arbitrary values from $[0, T]$. The purpose of this paper is to prove a discrete version of Theorem 1 with the values t from a certain discrete set.

For positive h_1 and h_2, define the set

$$L(\alpha, h_1, h_2, \pi) = \left\{ (h_1 \log p : p \in \mathbb{P}), \left(h_2 \log(m + \alpha) : m \in \mathbb{N}_0\right), 2\pi\right\}.$$

We note that $L(\alpha, h_1, h_2, \pi)$ is a multiset, it can have identical elements. Denote by $\#A$ the cardinality of a set A, and let $\underline{h} = (h_1, h_2)$. For $N \in \mathbb{N}_0$ and $A \in \mathcal{B}(\mathbb{C}^2)$, set

$$P^{\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = \frac{1}{N+1}\#\left\{0 \le k \le N : \left(\zeta(\sigma_1 + ikh_1; Q), \zeta(\sigma_2 + ikh_2, \alpha)\right) \in A\right\}.$$

In this paper, we will prove the following statement.

Theorem 2. *Suppose that the set $L(\alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, and $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$ are fixed. Then $P^{\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$ converges weakly to the measure $P_{\underline{\zeta},\underline{\sigma},Q,\alpha}$ as $N \to \infty$.*

For the proof of Theorem 2, we will apply the Fourier transform method. The identification of the limit measure is based on the ergodic theory. By the Nesterenko theorem [34], the numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. This means that there are no polynomials $p(s_1, s_2) \not\equiv 0$ with rational coefficients such that $p(\pi, \mathrm{e}^{\pi}) = 0$. From this, it follows easily that the set $L(\frac{1}{\pi}, h_1, h_2, \pi)$ with rational h_1 and h_2 is linearly independent over $\mathbb{Q}$.

2 Case of Ω

In this section, we prove a limit lemma for the probability measure

$$P^{\underline{h}}_{N,\Omega,\alpha}(A) \stackrel{\text{def}}{=} \frac{1}{N+1}\#\Big\{0 \le k \le N : \big((p^{-ikh_1} : p \in \mathbb{P}), \big((m+\alpha)^{-ikh_2} : m \in \mathbb{N}_0)\big) \in A\Big\}, \quad A \in \mathcal{B}(\Omega).$$

Lemma 1. *Suppose that the set $L(\alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then $P^{\underline{h}}_{N,\Omega,\alpha}$ converges weakly to the Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ as $N \to \infty$.*

Proof. We use the Fourier transform approach. It is sufficient to show that the Fourier transform of the measure $P^{\underline{h}}_{N,\Omega,\alpha}$, as $N \to \infty$, converges to that of the measure m_H. The characters of the group Ω are of the form

$$\prod_{p\in\mathbb{P}}^{o} \omega_1^{k_{1p}}(p) \prod_{m\in\mathbb{N}_0}^{o} \omega_2^{k_{2m}}(m),$$

where the sign "o" indicates that only a finite number of $k_{1p} \in \mathbb{Z}$ and $k_{2m} \in \mathbb{Z}$ are not zeros. Hence, the Fourier transform $f^{\underline{h}}_{N,\Omega,\alpha}(\underline{k}_1, \underline{k}_2)$, where $\underline{k}_1 = \big(k_{1p} : k_{1p} \in \mathbb{Z}, p \in \mathbb{P}\big)$, $\underline{k}_2 = (k_{2m} : k_{2m} \in \mathbb{Z}, m \in \mathbb{N}_0)$, of $P^{\underline{h}}_{N,\Omega,\alpha}$ is given by

$$\begin{aligned} f^{\underline{h}}_{N,\Omega,\alpha}(\underline{k}_1, \underline{k}_2) &= \int_{\Omega} \bigg(\prod_{p\in\mathbb{P}}^{o} \omega_1^{k_{1p}}(p) \prod_{m\in\mathbb{N}_0}^{o} \omega_2^{k_{2m}}(m)\bigg) \mathrm{d}P^{\underline{h}}_{N,\Omega,\alpha} \\ &= \frac{1}{N+1}\sum_{k=0}^{N}\prod_{p\in\mathbb{P}}^{o} p^{-ikk_{1p}h_1} \prod_{m\in\mathbb{N}_0}^{o} (m+\alpha)^{-ikk_{2m}h_2} \\ &= \frac{1}{N+1}\sum_{k=0}^{N} \exp\bigg\{ -ik\Big(h_1 \sum_{p\in\mathbb{P}}^{o} k_{1p}\log(p) + h_2 \sum_{m\in\mathbb{N}_0}^{o} k_{2m}\log(m+\alpha)\Big)\bigg\}. \end{aligned} \tag{2.1}$$

Since the set $L(\alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, we have

$$H \stackrel{\text{def}}{=} h_1 \sum_{p\in\mathbb{P}}^{o} k_{1p}\log p + h_2 \sum_{m\in\mathbb{N}_0}^{o} k_{2m}\log(m+\alpha) \neq 2\pi r, \quad r \in \mathbb{Z},$$

for $(k_1, k_2) \neq (\underline{0}, \underline{0})$, with $\underline{0} = (0, \dots, 0, \dots)$. Therefore, in view of (2.1),

$$f^{\underline{h}}_{N,\Omega,\alpha}(\underline{k}_1, \underline{k}_2) = \frac{1 - \mathrm{e}^{-i(N+1)H}}{(N+1)(1 - \mathrm{e}^{-iH})}. \tag{2.2}$$

Obviously, $f^{\underline{h}}_{N,\Omega,\alpha}(\underline{0}, \underline{0}) = 1$. This and (2.2) yield

$$\lim_{N\to\infty} f^{\underline{h}}_{N,\Omega,\alpha}(\underline{k}_1, \underline{k}_2) = \begin{cases} 1, & \text{if } (\underline{k}_1, \underline{k}_2) = (\underline{0}, \underline{0}), \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

and the lemma is proved as the right-hand side of (2.3) is the Fourier transform of m_H. $\square$

3 Case of absolute convergence

In this section, we will apply Lemma 1 to obtain a limit lemma for absolutely convergent series connected to the functions $\zeta(s; Q)$ and $\zeta(s, \alpha)$.

Let $\theta > \frac{1}{2}$ be a fixed number, and $M \in \mathbb{N}$. Define the arithmetic functions

$$v_M(m) = \exp\left\{ -\left(\frac{m}{M}\right)^\theta \right\}, \quad m \in \mathbb{N},$$

$$v_M(m, \alpha) = \exp\left\{ -\left(\frac{m+\alpha}{M}\right)^\theta \right\}, \quad m \in \mathbb{N}_0.$$

Moreover, let

$$L_M\left(s - \frac{n}{2} + 1, \hat{\chi}_l\right) = \sum_{m=1}^{\infty} \frac{\hat{\chi}_l(m) v_M(m)}{m^{s-\frac{n}{2}+1}},$$

$$\zeta_M(s, \alpha) = \sum_{m=0}^{\infty} \frac{v_M(m, \alpha)}{(m+\alpha)^s},$$

$$L_M\left(s - \frac{n}{2} + 1, \hat{\chi}_l, \omega_1\right) = \sum_{m=1}^{\infty} \frac{\hat{\chi}_l(m)\omega_1(m) v_M(m)}{m^{s-\frac{n}{2}+1}},$$

$$\zeta_M(s, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m) v_M(m, \alpha)}{(m+\alpha)^s}.$$

All above series are absolutely convergent in every half-plane $\sigma > \sigma_0$ with finite σ_0, whereas $v_M(m)$ and $v_M(m, \alpha)$ decrease exponentially to zero with respect to m. For $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$, set

$$\underline{\zeta}_M(\underline{\sigma} + ik\underline{h}, \alpha; Q) = \left(\zeta_M(\sigma_1 + ikh_1; Q), \zeta_M(\sigma_2 + ikh_2, \alpha)\right),$$

where

$$\zeta_M(s; Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s, \chi_k) L_M\left(s - \frac{n}{2} + 1, \hat{\chi}_l\right) + \sum_{m=1}^{\infty} \frac{b_Q(m)}{m^s},$$

and

$$\underline{\zeta}_M(\underline{\sigma}+ik\underline{h},\alpha,\omega;Q)=\left(\zeta_M(\sigma_1+ikh_1,\omega_1;Q),\zeta_M(\sigma_2+ikh_2,\alpha,\omega_2)\right),$$

where

$$\zeta_M(s,\omega_1;Q)=\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{a_{kl}\omega_1(k)\omega_1(l)}{k^s l^s}L(s,\chi_k,\omega_1)L_M\left(s-\frac{n}{2}+1,\hat{\chi}_l,\omega_1\right)$$
$$+\sum_{m=1}^{\infty}\frac{b_Q(m)\omega_1(m)}{m^s}.$$

We notice that all functions that occur in $\underline{\zeta}_M(\underline{\sigma},\alpha;Q)$ are given by absolutely convergent series for $\sigma_1>\frac{n-1}{2}$ and $\sigma_2>\frac{1}{2}$.

In this section, we will obtain the weak convergence for

$$P^{\underline{h}}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}(A)\stackrel{\text{def}}{=}\frac{1}{N+1}\left\{0\le k\le N:\underline{\zeta}_M(\underline{\sigma}+ik\underline{h},\alpha;Q)\in A\right\},\quad A\in\mathcal{B}(\mathbb{C}^2),$$
$$P^{\underline{h},\Omega}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}(A)\stackrel{\text{def}}{=}\frac{1}{N+1}\left\{0\le k\le N:\underline{\zeta}_M(\underline{\sigma}+ik\underline{h},\alpha,\omega;Q)\in A\right\},\ A\in\mathcal{B}(\mathbb{C}^2),$$

as $N\to\infty$.

Before the statement of a limit lemma, we introduce the mapping $u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}:\Omega\to\mathbb{C}^2$ given, for $\sigma_1>\frac{n-1}{2}$ and $\sigma_2>\frac{1}{2}$, by the formula

$$u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}(\omega)=\underline{\zeta}_M(\underline{\sigma},\alpha,\omega;Q).$$

The absolute convergence of series defining $\underline{\zeta}_M(\underline{\sigma},\alpha,\omega;Q)$ ensures the continuity of $u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$. Hence, the mapping $u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$ is $(\mathcal{B}(\Omega),\mathcal{B}(\mathbb{C}^2))$-measurable. Therefore, the probability measure $U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}=m_H\left(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\right)^{-1}$ on $(\mathbb{C}^2,\mathcal{B}(\mathbb{C}^2))$, where, for $A\in\mathcal{B}(\mathbb{C}^2)$,

$$U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}(A)=m_H\left(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\right)^{-1}(A)=m_H\left(\left(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\right)^{-1}A\right),$$

can be defined.

Lemma 2. *Suppose that the hypotheses of Theorem 2 are fulfilled. Then,* $P^{\underline{h}}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}$ *and* $P^{\underline{h},\Omega}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}$ *both converge weakly to the same probability measure* $U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$ *as* $N\to\infty$.

Proof. The definitions of the measures $P^{\underline{h}}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}$ and $P^{\underline{h}}_{N,\Omega,\alpha}$, and mapping $u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$ imply that, for every $A\in\mathcal{B}(\mathbb{C}^2)$,

$$\begin{aligned}P^{\underline{h}}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}(A)=&\frac{1}{N+1}\Big\{0\le k\le N:\Big((p^{-ikh_1}:p\in\mathbb{P}),\\&\big((m+\alpha)^{-ikh_2}:m\in\mathbb{N}_0\big)\Big)\in(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}})^{-1}A\Big\}\\=&P^{\underline{h}}_{N,\Omega,\alpha}\left((u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}})^{-1}A\right).\end{aligned}$$

Thus,

$$P^{\underline{h}}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha} = P^{\underline{h}}_{N,\Omega,\alpha}\left(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\right)^{-1}.$$

This and continuity of $u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$ show that the principle of preservation of weak convergence under mappings is applicable, see [4], Theorem 5.1. Therefore, in view of Lemma 1, we obtain that $P^{\underline{h}}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}$ converges weakly to $m_H(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}})^{-1}$ as $N \to \infty$.

Now, consider the case of $P^{\underline{h},\Omega}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}$. For $\hat{\omega} \in \Omega$, let the mapping $\hat{u}^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}} : \Omega \to \mathbb{C}^2$ be defined by

$$\hat{u}^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}(\omega) = \underline{\zeta}_M(\underline{\sigma}, \alpha, \omega\hat{\omega}; Q).$$

Then, by the definition of $u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$,

$$\hat{u}^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}(\omega) = u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}(a(\omega)) \tag{3.1}$$

with $a : \Omega \to \Omega$ given by $a(\omega) = \omega\hat{\omega}$. Repeating the above arguments shows that $P^{\underline{h},\Omega}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}$ converges weakly to the measure $m_H(\hat{u}^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}})^{-1}$ as $N \to \infty$. It is well known that the Haar measure is invariant with respect to shifts by elements from Ω, i.e.,

$$m_H(A) = m_H(\omega A) = m_H(A\omega)$$

for all $A \in \mathcal{B}(\Omega)$ and $\omega \in \Omega$. Therefore, in view of (3.1), we have

$$m_H\left(\hat{u}^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\right)^{-1} = \left(m_H a^{-1}\right)\left(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\right)^{-1} = m_H\left(u^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\right)^{-1}.$$

Thus, the measure $P^{\underline{h},\Omega}_{N,M,\underline{\zeta},\underline{\sigma},Q,\alpha}$, as $N \to \infty$, converges weakly to $U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$ as well. The lemma is proved. □

4 Approximation result

In this section, we will show that $\underline{\zeta}_M(\underline{\sigma}, \alpha; Q)$ and $\underline{\zeta}_M(\underline{\sigma}, \alpha, \omega; Q)$ are close to $\underline{\zeta}(\underline{\sigma}, \alpha; Q)$ and $\underline{\zeta}(\underline{\sigma}, \alpha, \omega; Q)$, respectively, in the mean. We note that $\underline{\zeta}(\underline{\sigma} + ik\underline{h}, \alpha; Q)$ and $\underline{\zeta}(\underline{\sigma}+ik\underline{h}, \alpha, \omega; Q)$ are defined similarly to $\underline{\zeta}_M(\underline{\sigma}+ik\underline{h}, \alpha; Q)$ and $\underline{\zeta}_M(\underline{\sigma} + ik\underline{h}, \alpha, \omega; Q)$, respectively. Denote by ρ the usual metric in $\mathbb{C}^2$. Then the following statement is valid.

Lemma 3. *Suppose that $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$ are fixed. Then,*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=1}^{N} \rho\left(\underline{\zeta}(\underline{\sigma} + ik\underline{h}, \alpha; Q), \underline{\zeta}_M(\underline{\sigma} + ik\underline{h}, \alpha; Q)\right) = 0, \tag{4.1}$$

and, if the set $\left\{(h_2 \log(m + \alpha) : m \in \mathbb{N}_0), 2\pi\right\}$ is linearly independent over $\mathbb{Q}$, for almost all $\omega \in \Omega$,

$$\lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=1}^{N} \rho\left(\underline{\zeta}(\underline{\sigma}+ik\underline{h}, \alpha, \omega; Q), \underline{\zeta}_M(\underline{\sigma}+ik\underline{h}, \alpha, \omega; Q)\right) = 0. \tag{4.2}$$

Proof. Equality (4.1) follows from the equalities

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=1}^{N}\left|\zeta(\sigma_1+ikh_1;Q)-\zeta_M(\sigma_1+ikh_1;Q)\right|=0$$

and

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=1}^{N}\left|\zeta(\sigma_2+ikh_2,\alpha)-\zeta_M(\sigma_2+ikh_2,\alpha)\right|=0.$$

The first of them has been proven in [30]. For its proof, the integral representation for the function

$$L_M\left(s,\chi\right)=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}L\left(s+z,\chi\right)l_M(z)\mathrm{d}z$$

with

$$l_M(s)=\frac{1}{\theta}\Gamma\left(\frac{s}{\theta}\right)M^s$$

has been used. Moreover, the discrete mean square estimate for the Dirichlet L-function $L(s,\chi)$, with $\sigma>\frac{1}{2}$, $\tau\in\mathbb{R}$ and $h>0$,

$$\sum_{k=0}^{N}\left|L(\sigma+ikh+i\tau,\chi)\right|^2\ll_h N(1+|\tau|),$$

which follows from the continuous mean square estimate with application of the Gallagher lemma [32] connecting continuous and discrete mean square of some functions, has been essentially applied.

The second equality has been obtained in [8], Lemma 5.

Similarly, Equality (4.2) is a consequence of the equalities

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=1}^{N}\left|\zeta(\sigma_1+ikh_1,\omega_1;Q)-\zeta_M(\sigma_1+ikh_1,\omega_1;Q)\right|=0$$

and

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=1}^{N}\left|\zeta(\sigma_2+ikh_2,\alpha,\omega_2)-\zeta_M(\sigma_2+ikh_2,\alpha,\omega_2)\right|=0,$$

that are valid for almost all ω_1 and ω_2, respectively. The first of them is given in [30], the second follows from [26], Lemma 2.6. □

5 Relative compactness

Let $\{P\}$ be a family of probability measures in the space $(\mathbb{X},\mathcal{B}(\mathbb{X}))$. In the theory of the weak convergence of probability measures, the notion of relative compactness of families of probability measures often is useful. Recall that the family $\{P\}$ is relatively compact if every sequence $\{P_n\} \subset \{P\}$ possesses a subsequence $\{P_{n_r}\}$ weakly convergent to a certain probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ as $r \to \infty$. However, it is not easy to prove relative compactness. Therefore, it is a simple notion, tightness of $\{P\}$, which implies its relative compactness. Thus, $\{P\}$ is tight if, for every $\epsilon > 0$, there is a compact set $K \subset \mathbb{X}$ such that

$$P(K) > 1 - \epsilon$$

for all $P \in \{P\}$.

Lemma 4. *Under the hypotheses of Theorem 2, the sequence* $\{U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}} : M \in \mathbb{N}\}$ *is tight.*

Proof. Denote the marginal measures of $U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$ by

$$U^{Q}_{M,\sigma_1}(A) = U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}(A \times \mathbb{C}), \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$U^{\alpha}_{M,\sigma_2}(A) = U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}(\mathbb{C} \times A), \quad A \in \mathcal{B}(\mathbb{C}).$$

Then, $U^{Q}_{M,\sigma_1} = m_{1H}(u^{Q}_{M,\sigma_1})^{-1}$ and $U^{\alpha}_{M,\sigma_2} = m_{2H}(u^{\alpha}_{M,\sigma_2})^{-1}$, where m_{jH} is the probability Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, 2$. Also, $u^{Q}_{M,\sigma_1} : \Omega_1 \to \mathbb{C}$ is given by

$$u^{Q}_{M,\sigma_1}(\omega_1) = \zeta_M(\sigma_1, \omega_1; Q)$$

and $u^{\alpha}_{M,\sigma_2} : \Omega_2 \to \mathbb{C}$ by

$$u^{Q}_{M,\sigma_2}(\omega_2) = \zeta_M(\sigma_2, \omega_2; \alpha).$$

In the proof of Lemma 8 from [30], it was obtained that the measure U^{Q}_{M,σ_1} is tight. Therefore, for every $\epsilon > 0$, there is a compact set $K_1 \subset \mathbb{C}$ such that

$$U^{Q}_{M,\sigma_1}(K_1) > 1 - \epsilon/2 \tag{5.1}$$

for all $M \in \mathbb{N}$. Similarly, in the proof of Lemma 2.7 from [26], it was proved that there exists a compact set $K_2 \subset \mathbb{C}$ such that

$$U^{\alpha}_{M,\sigma_2}(K_2) > 1 - \epsilon/2 \tag{5.2}$$

for all $M \in \mathbb{N}$. Note that in [26] the linear independence over $\mathbb{Q}$ for the set $\{\log(m + \alpha) : m \in \mathbb{N}_0, \frac{2\pi}{h_2}\}$ is required, which is ensured by the linear independence of the set $L(\alpha, h_1, h_2, \pi)$. Let $K = K_1 \times K_2$. Then K is a compact set in $\mathbb{C}^2$. Moreover,

$$\begin{aligned} U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\left(\mathbb{C}^2 \setminus K\right) &\leq U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\left(\mathbb{C} \setminus K_1, \mathbb{C}\right) + U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\left(\mathbb{C}, \mathbb{C} \setminus K_2\right) \\ &= U^{Q}_{M,\sigma_1}\left(\mathbb{C} \setminus K_1\right) + U^{\alpha}_{M,\sigma_2}\left(\mathbb{C} \setminus K_2\right) \end{aligned}$$

for all $M \in \mathbb{N}$. Therefore, in view of (5.1) and (5.2),

$$U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\left(\mathbb{C}^2 \setminus K\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $M \in \mathbb{N}$, and this proves the lemma. □

6 Proof of Theorem 2: the first step

For $\omega \in \Omega$ and $A \in \mathcal{B}(\mathbb{C}^2)$, define

$$P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : \big(\zeta(\sigma_1 + ikh_1, \omega_1; Q), \\ \zeta(\sigma_2 + ikh_2, \omega_2, \alpha)\big) \in A\Big\}.$$

In this section, we will prove that the measures $P^{\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$ and $P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$ have the same limit measure in the sense of the weak convergence. More precisely, the following lemma is true.

Lemma 5. *Under the hypotheses of Theorem 2, on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ there exists a probability measure $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}$ such that both measures $P^{\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$ and $P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$ converge weakly to $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}$ as $N \to \infty$.*

Proof. Let the random variable θ_N be defined on a certain probability space $(\Pi, \mathcal{A}, \nu)$, and let

$$\mu\{\theta_N = k\} = 1/(N+1), \quad k = 0, 1, \ldots, N.$$

Consider the $\mathbb{C}^2$-valued random elements

$$X^{\underline{h}}_{N,M,\underline{\zeta}} = X^{\underline{h}}_{N,M,\underline{\zeta}}(\underline{\sigma}, \alpha; Q) = \underline{\zeta}_M(\underline{\sigma} + i\underline{h}\theta_N, \alpha; Q)$$
$$X^{\underline{h}}_{N,\underline{\zeta}} = X^{\underline{h}}_{N,\underline{\zeta}}(\underline{\sigma}, \alpha; Q) = \underline{\zeta}(\underline{\sigma} + i\underline{h}\theta_N, \alpha; Q),$$

and the $\mathbb{C}^2$-valued random element $Y_{M,\underline{\zeta}} = Y_{M,\underline{\zeta}}(\underline{\sigma}, \alpha; Q)$ with the distribution $U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}$. Then, denoting by $\xrightarrow{D}$ the convergence in distribution, by Lemma 2, we have

$$X^{\underline{h}}_{N,M,\underline{\zeta}} \xrightarrow[N\to\infty]{D} Y_{M,\underline{\zeta}}. \tag{6.1}$$

Now, we will apply Lemma 4. Since the sequence $\{U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}} : M \in \mathbb{N}\}$ is tight, by the Prokhorov theorem, see [4], Theorem 6.1, this sequence is relatively compact. Hence, there exists a subsequence $\{U^{Q,\alpha}_{M_r,\underline{\zeta},\underline{\sigma}}\} \subset \{U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\}$ and a probability measure $U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$ on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ satisfying the relation

$$Y_{M_r,\underline{\zeta}} \xrightarrow[r\to\infty]{D} U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}. \tag{6.2}$$

The latter relation shows that $Y_{M_r,\underline{\zeta}}$ converges in distribution to a random element having the distribution $U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$. Having in mind the above definitions, we obtain, for every $\epsilon > 0$,

$$\begin{aligned} &\nu\Big\{\rho\left(X^{\underline{h}}_{N,\underline{\zeta}}, X^{\underline{h}}_{N,M_r,\underline{\zeta}}\right) \ge \epsilon\Big\} \\ = &\frac{1}{N+1}\#\Big\{0 \le k \le N : \rho\left(\underline{\zeta}(\underline{\sigma} + ik\underline{h}, \alpha; Q), \underline{\zeta}_{M_r}(\underline{\sigma} + ik\underline{h}, \alpha; Q)\right) \ge \epsilon\Big\} \end{aligned}$$

$$\leq \quad \frac{1}{\epsilon(N+1)} \sum_{k=0}^{N} \rho\left(\underline{\zeta}(\underline{\sigma}+ik\underline{h},\alpha;Q), \underline{\zeta}_{M_r}(\underline{\sigma}+ik\underline{h},\alpha;Q)\right).$$

Thus, in view of Lemma 3,

$$\lim_{r\to\infty} \limsup_{N\to\infty} \nu\Big\{\rho\left(X^{\underline{h}}_{N,\underline{\zeta}}, X^{\underline{h}}_{N,M_r,\underline{\zeta}}\right) \geq \epsilon\Big\} = 0.$$

The latter equality, relations (6.1) and (6.2), show that, for the above random elements, Theorem 4.2 from [4] is applicable. This procedure leads to the relation

$$X^{\underline{h}}_{N,\underline{\zeta}} \xrightarrow[N\to\infty]{D} U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}, \tag{6.3}$$

and we have the weak convergence for $P^{\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$ to $U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$ as $N \to \infty$.

Now, we deal with the measure $P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$. First, we observe that relation (6.3) implies that the measure $U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$ is independent of the sequence $\{M_r\}$. From this and relative compactness of $\{U^{Q,\alpha}_{M,\underline{\zeta},\underline{\sigma}}\}$, it follows that

$$Y_{M,\underline{\zeta}} \xrightarrow[M\to\infty]{D} U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}. \tag{6.4}$$

Similarly to the random elements $X^{\underline{h}}_{N,M,\underline{\zeta}}$ and $X^{\underline{h}}_{N,\underline{\zeta}}$, introduce

$$X^{\Omega,\underline{h}}_{N,M,\underline{\zeta}} = X^{\Omega,\underline{h}}_{N,M,\underline{\zeta}}(\underline{\sigma},\alpha,\omega;Q) = \underline{\zeta}_M(\underline{\sigma}+i\underline{h}\theta_N,\alpha,\omega;Q)$$
$$X^{\Omega,\underline{h}}_{N,\underline{\zeta}} = X^{\Omega,\underline{h}}_{N,\underline{\zeta}}(\underline{\sigma},\alpha,\omega;Q) = \underline{\zeta}(\underline{\sigma}+i\underline{h}\theta_N,\alpha,\omega;Q).$$

Then, by Lemma 2,

$$X^{\Omega,\underline{h}}_{N,M,\underline{\zeta}} \xrightarrow[N\to\infty]{D} Y_{M,\underline{\zeta}}, \tag{6.5}$$

and, in virtue of Lemma 3, for $\epsilon > 0$,

$$\lim_{M\to\infty} \limsup_{N\to\infty} \nu\Big\{\rho\left(X^{\Omega,\underline{h}}_{N,\underline{\zeta}}, X^{\Omega,\underline{h}}_{N,M,\underline{\zeta}}\right) \geq \epsilon\Big\}$$
$$\leq \lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{\epsilon(N+1)} \sum_{k=0}^{N} \rho\left(\underline{\zeta}(\underline{\sigma}+ik\underline{h},\alpha,\omega;Q), \underline{\zeta}_M(\underline{\sigma}+ik\underline{h},\alpha,\omega;Q)\right) = 0.$$

Thus, this equality, (6.4), (6.5) and Theorem 4.2 of [4] give the weak convergence for $P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}$ to $U^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$ as $N \to \infty$. The lemma is proved. □

7 Proof of Theorem 2: the second step

In this section, we identify the measure $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}$ in Lemma 5. For this, some elements of ergodic theory will be applied. Let

$$e_{\underline{h},\alpha} = \Big(\big(p^{-ih_1} : p \in \mathbb{P}\big), \big((m+\alpha)^{-ih_2} : m \in \mathbb{N}_0\big)\Big).$$

Clearly, we have $e_{\underline{h},\alpha} \in \Omega$. Define the transformation $T_{\underline{h},\alpha} : \Omega \to \Omega$ by

$$T_{\underline{h},\alpha}(\omega) = e_{\underline{h},\alpha}\omega, \quad \omega \in \Omega.$$

Since the Haar measure m_H is invariant with respect to shifts by elements of Ω, the transformation $T_{\underline{h},\alpha}$ is measurable measure preserving.

Recall that a set $A \in \mathcal{B}(\Omega)$ is invariant with respect to $T_{\underline{h},\alpha}$ if the sets $A_{\underline{h},\alpha} = T_{\underline{h},\alpha}(A)$ and A can differ one from another at most by a set of m_H-measure zero. All invariant sets constitute the σ-subfield of $\mathcal{B}(\Omega)$. The transformation $T_{\underline{h},\alpha}$ is called ergodic if the σ-field of invariant sets consists only from sets having m_H-measure zero or one.

In what follows, the ergodicity $T_{\underline{h},\alpha}$ plays an important role.

Lemma 6. *Suppose that the set $L(\alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then, $T_{\underline{h},\alpha}$ is ergodic.*

Proof. Let A be an invariant set with respect to $T_{\underline{h},\alpha}$, and I_A its indicator function. In the proof of Lemma 1, it was noted that the characters $\chi(\omega)$ of Ω are given by

$$\chi(\omega) = \prod_{p\in\mathbb{P}}^{o} \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}^{o} \omega_2^{l_m}(m),$$

where only a finite number of integers k_p and l_m are non-zeros. First, suppose that $\chi(\omega) \not\equiv 1$ for all $\omega \in \Omega$. Then,

$$\begin{aligned}\chi(e_{\underline{h},\alpha}) &= \prod_{p\in\mathbb{P}}^{o} p^{-ikh_1} \prod_{m\in\mathbb{N}_0}^{o} (m+\alpha)^{-ilh_2} \\ &= \exp\Big\{ - ih_1 \sum_{p\in\mathbb{P}}^{o} k_p \log(p) - ih_2 \sum_{m\in\mathbb{N}_0}^{o} l_m \log(m+\alpha)\Big\}.\end{aligned}$$

Since the set $L(\alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$,

$$-ih_1 \sum_{p\in\mathbb{P}}^{o} k_p \log(p) - ih_2 \sum_{m\in\mathbb{N}_0}^{o} l_m \log(m+\alpha) \neq 2\pi r, \quad r \in \mathbb{Z},$$

for $k_p \not\equiv 0$ and $l_m \not\equiv 0$. Therefore,

$$\chi(e_{\underline{h},\alpha}) \neq 1. \tag{7.1}$$

By the invariance of A, for almost all $\omega \in \Omega$,

$$I_A(T_{\underline{h},\alpha}) = I_A(\omega). \tag{7.2}$$

Denoting by $\hat{f}$ the Fourier transform of f, using the invariance of the measure m_H and multiplicativity of characters, we obtain by (7.2) that

$$\hat{I}_A(\chi) = \int_\Omega I_A(\omega)\chi(\omega)\mathrm{d}m_H = \chi(e_{\underline{h},\alpha}) \int_\Omega I_A(\omega)\chi(\omega)\mathrm{d}m_H = \chi(e_{\underline{h},\alpha})\hat{I}_A(\chi).$$

Hence, in view of (7.1),

$$\hat{I}_A(\chi) = 0. \tag{7.3}$$

Now, let $\chi(\omega) \equiv 1$, and $\hat{I}_A(\chi) = a$ with arbitrary χ. Then, by orthogonality of characters,

$$\hat{a}(\chi) = \int_\Omega a(\omega)\chi(\omega)\mathrm{d}m_H = a\int_\Omega \chi(\omega)\mathrm{d}m_H = \begin{cases} a, & \text{if } \chi(\omega) \equiv 1, \\ 0, & \text{otherwise.} \end{cases}$$

This and (7.3) show that $\hat{I}_A(\chi) = \hat{a}$. Hence, $I_A(\omega) = a$ for almost all $\omega \in \Omega$. Thus, $I_A(\omega) = 1$ or $I_A(\omega) = 0$ for almost all $\omega \in \Omega$. From this, we have $m_H(A) = 1$ or $m_H(A) = 0$. The lemma is proved. □

For convenience of application, we recall the classical Birkhoff-Khinchine ergodic theorem, see, for example, [10]. Denote by $\mathbb{E}X$ the expectation of the random element X.

Lemma 7. *Let g be a measurable measure preserving ergodic transformation on the space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), m)$. Then, for every function $g \in L^1(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), m)$,*

$$\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{k=0}^{n} f\left(g^k(\hat{\omega})\right) = \mathbb{E}f$$

for almost all $\hat{\omega} \in \hat{\Omega}$.

Proof of Theorem 2. We have to show that $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}$ in Lemma 5 coincides with $P_{\underline{\zeta},\underline{\sigma},Q,\alpha}$.

Let A be a continuity set of the measure $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}$. Then, in view of Lemma 5 and the equivalent of the weak convergence in terms of continuity sets, see [4], Theorem 2.1,

$$\lim_{N\to\infty} P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A). \tag{7.4}$$

On $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable

$$\xi(\omega) = \begin{cases} 1, & \text{if } \ \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\mathbb{E}\xi = \int_\Omega \xi \mathrm{d}m_H = m_H\left\{\omega \in \Omega : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\right\} = P_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A). \tag{7.5}$$

Moreover, Lemmas 6 and 7 imply

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{n} \xi\left(T_{\underline{h},\alpha}(\omega)\right) = \mathbb{E}\xi$$

for almost all $\omega \in \Omega$. The definitions of ξ and $T_{\underline{h},\alpha}$ show that

$$\frac{1}{N+1}\sum_{k=0}^{n}\xi\left(T_{\underline{h},\alpha}(\omega)\right) = \frac{1}{N+1}\#\left\{\omega\in\Omega : \underline{\zeta}(\underline{\sigma}+ik\underline{h}, \alpha, \omega; Q)\in A\right\} = P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}(A).$$

Therefore, Equality (7.5) yields

$$\lim_{N\to\infty} P^{\Omega,\underline{h}}_{N,\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = P_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A)$$

for almost all $\omega \in \Omega$. This and (7.4) show that $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = P_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A)$ for all continuity sets of the measure $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}$. Since all continuity sets form the determining class, we obtain that $P^{\underline{h}}_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A) = P_{\underline{\zeta},\underline{\sigma},Q,\alpha}(A)$ for all $A \in \mathcal{B}(\mathbb{C}^2)$. The theorem is proved. □

Acknowledgements

The research of R. Macaitienė was funded by the Research Council of Lithuania (LMTLT), agreement No. S-MIP-22-81.

References

[1] B. Bagchi. *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series.* PhD Thesis, Indian Statistical Institute, Calcutta, 1981.

[2] A. Balčiūnas, V. Garbaliauskienė, V. Lukšienė, R. Macaitienė and A. Rimkevičienė. Joint discrete approximation of analytic functions by Hurwitz zeta-functions. *Math. Model. Anal.*, **27**(1):88–100, 2022. https://doi.org/10.3846/mma.2022.15068.

[3] P. Bateman and E. Grosswald. On Epstein's zeta function. *Acta Arith.*, **9**:365–373, 1964. https://doi.org/0.4064/aa-9-4-365-373.

[4] P. Billingsley. *Convergence of Probability Measures.* Willey, New York, 1968. https://doi.org/10.1002/9780470316962.

[5] H. Bohr. Zur Theorie der Riemann'schen Zetafunktion im kritischen Streifen. *Acta Math.*, **40**:67–100, 1916. https://doi.org/10.1007/bf02418541.

[6] H. Bohr and B. Jessen. Über die Wertverteilung der Riemanschen Zetafunktion, Erste mitteilung. *Acta Math.*, **54**:1–35, 1930. https://doi.org/10.1007/BF02547516.

[7] H. Bohr and B. Jessen. Über die Wertverteilung der Riemanschen Zetafunktion, Zweite Mitteilung. *Acta Math.*, **58**:1–55, 1932. https://doi.org/10.1007/BF02547773.

[8] E. Buivydas, A. Laurinčikas, J. Rašytė and R. Macaitienė. Discrete universality theorems for the Hurwitz zeta-function. *J. Approx. Th.*, **183**:1–13, 2014. https://doi.org/10.1016/j.jat.2014.03.006.

[9] S. Chowla and A. Selberg. On Epstein's zeta-function. *Journal für die Reine und Angewandte Mathematik*, **227**:86–110, 1967. https://doi.org/10.1515/crll.1967.227.86.

[10] H. Cramér and M. R. Leadbetter. *Stationary and Related Stochastic Processes.* Wiley, New York, 1967. https://doi.org/10.2307/2344096.

[11] C.-J. de la Vallée Poussin. Recherches analytiques la thorie des nombres premiers. *Ann. Soc. Scient. Bruxelles*, **20**:183–256, 1896.

[12] E. Elizalde. Multidimensional extension of the generalized Chowla–Selberg formula. *Comm. Math. Phys.*, **198**:83–95, 1998. https://doi.org/10.1007/s002200050472.

[13] E. Elizalde. *Ten physical applications of spectral zeta functions.* Lecture Notes Physics, vol. 855, Springer, Berlin, Germany, 2012. https://doi.org/10.1007/978-3-642-29405-1.

[14] P. Epstein. Zur Theorie allgemeiner Zetafunktionen. *Math. Ann.*, **56**:615–644, 1903. https://doi.org/10.1007/BF01444309.

[15] O. M. Fomenko. Order of the Epstein zeta-function in the critical strip. *J. Math. Sci.*, **110**:3150–3163, 2002. https://doi.org/10.1023/A:1015432614102.

[16] H. Gerges, A. Laurinčikas and R. Macaitienė. A joint limit theorem for Epstein and Hurwitz zeta-functions. *Mathematics*, **12**(13):article no. 1922, 2024. https://doi.org/10.3390/math12131922.

[17] M. L. Glasser and I. J. Zucker. Lattice sums in theoretical chemistry. *Theoretical Chemistry: Advances and Perspectives (Eds. D. Henderson / H. Eyring)*, **5**:67–139, 1980. https://doi.org/10.1016/B978-0-12-681905-2.50008-6.

[18] J. Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bull. Soc. Math. France*, **24**:199–220, 1896. https://doi.org/10.24033/bsmf.545.

[19] E. Hecke. Über Modulfunktionen und die Dirichletchen Reihen mit Eulerscher Produktentwicklung. I. *Math. Ann.*, **114**:1–28, 1937. https://doi.org/10.1007/BF01594160.

[20] E. Hecke. Über Modulfunktionen und die Dirichletchen Reihen mit Eulerscher Produktentwicklung. II. *Math. Ann.*, **114**:316–351, 1937. https://doi.org/10.1007/BF01594180.

[21] A. Hurwitz. Einige Eigenschaften der Dirichlet'schen Funktionen $f(s) = \sum\left(\frac{D}{n}\frac{1}{n^s}\right)$ die bei der Bestimmung der Klassenzahlen binärer quadratischer Formen auftreten. *Zeitschr. für Math. und Phys.*, **27**:86–101, 1882. https://doi.org/10.1007/978-3-0348-4161-0_3.

[22] H. Iwaniec. *Topics in Classical Automorphic Forms, Graduate Studies in Mathematics.* American Mathematical Society: Providence, RI, Volume 17, 1997. https://doi.org/10.1090/gsm/017.

[23] D. Joyner. *Distribution theorems of L-functions. Ditman Research Notes in Math.* Longman Scientific, Harlow, 1986.

[24] E. Kowalski. *An Introduction to Probabilistic Number Theory.* Cambridge Univ. Press, 2021. https://doi.org/10.1017/9781108888226.

[25] A. Laurinčikas. *Limit Theorems for the Riemann Zeta-Function.* Kluwer, Dordrecht, 1996. https://doi.org/10.1007/978-94-017-2091-5.

[26] A. Laurinčikas. A discrete universality theorem for the Hurwitz zeta-function. *J. Number Th.*, **143**:232–247, 2014. https://doi.org/10.1016/j.jnt.2014.04.013.

[27] A. Laurinčikas. On value distribution of certain Beurling zeta-functions. *Mathematics*, **12**(459):1–15, 2024. https://doi.org/10.3390/axioms13040251.

[28] A. Laurinčikas and R. Garunkštis. *The Lerch Zeta-Function.* Kluwer, Dordrecht, 2002. https://doi.org/10.1007/978-94-017-6401-8.

[29] A. Laurinčikas and R. Macaitienė. A Bohr-Jessen type theorem for the Epstein zeta-function. *Results in Math.*, **73**(4):147–163, 2018. https://doi.org/10.1007/s00025-018-0909-3.

[30] A. Laurinčikas and R. Macaitienė. A Bohr-Jessen type theorem for the Epstein zeta-function. II. *Results in Math.*, **75**(25):1–16, 2020. https://doi.org/10.1007/s00025-019-1151-3.

[31] A. Laurinčikas and D. Šiaučiūnas. Generalized limit theorem for Mellin transform of the Riemann zeta-function. *Axioms*, **13**(251):1–17, 2024. https://doi.org/10.3390/axioms13040251.

[32] H. L. Montgomery. *Topics in Multiplicative Number Theory*. Springer-Verlag, Berlin, Heidelberg, New York, 1971. https://doi.org/10.1016/j.jnt.2021.03.021.

[33] T. Nakamura and Ł. Pańkowski. On zeros and c-values of Epstein zeta-functions. *Šiauliai Math. Semin.*, **8**:181–195, 2013.

[34] Yu. V. Nesterenko. Modular functions and transcendence questions. *Sb. Math.*, **187**:1319–1348, 1996. https://doi.org/10.1070/SM1996v187n09ABEH000158.

[35] B. Riemann. Über die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsberichte der Berliner Akademie*, pp. 671–680, 1859.

[36] J. Steuding. *Value-Distribution of L-Functions*. Lecture Notes Math., vol. 1877, Springer, Berlin, Heidelberg, 2007. https://doi.org/10.1007/978-3-540-44822-8.