ISSN: 1392-6292/ eISSN: 1648-3510

MATHEMATICAL MODELLING and ANALYSIS

2025
Volume 30
Issue 4
Pages 645–663
https://doi.org/10.3846/mma.2025.21979

Convergence analysis of a class of iterative methods: a unified approach

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Article History:

- received August 26, 2024
- revised January 14, 2025
- accepted January 24, 2025

Abstract. In this paper, we study the convergence of a class of iterative methods for solving the system of nonlinear Banach space valued equations. We provide a unified local and semi-local convergence analysis for these methods. The convergence order of these methods are obtained using the conditions on the derivatives of the involved operator up to order 2 only. Further, we provide the number of iterations required to obtain the given accuracy of the solution. Various numerical examples including integral equations and Caputo fractional differential equations are considered to show the performance of our methods.

Keywords: iterative methods; nonlinear equations; Caputo fractional operator; basin of attraction.

AMS Subject Classification: 65J15; 65J05; 65H10; 34A08.

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1 Introduction

Typically, most problems in Science and Engineering will be nonlinear and hard to solve analytically [1, 4]. So, one needs a numerical solution. Starting with Newton, many authors have given multiple higher order methods to solve such equations under various assumptions (See [2, 6, 10]). Here, we follow the following definition of order of convergence.

Definition 1. A sequence $\{x_n\}$ is said to converge to ξ^* in the sense of R-order at least $p > 0$ if

(i) $\lim_{n\to\infty} \|x_n - \xi^*\| = 0$,

(ii) there exists $\mathcal{E} \geq 0$ and $n_0 \in \mathbb{N}$ with $\|x_{n+1} - \xi^*\| \leq \mathcal{E}\|x_n - \xi^*\|^p, \forall n \geq n_0$.

In general, there are three types of convergence analysis for an iterative method, namely (i) local (ii) semi-local and (iii) global convergence [2]. Our interest in

Table 1. Special cases of (1.2) and (1.3).

Label	Parameter choice	Method
(a)	$a_0 = b_1 = b_2 = 1$ in (1.2)	Third-order method in [8]
(b)	$a_0 = b_1 = b_2 = c_2 = 1$ and $c_1 = c_3 = 0$ in (1.3)	Fifth-order method in [13]
(c)	$a_0 = b_1 = b_2 = c_3 = 1$ and $c_1 = c_2 = 0$ in (1.3)	Sixth-order method in [13]
(d)	$a_0 = \frac{1}{2}$, $b_1 = 0$, $b_2 = 1$ in (1.2)	Third-order method in [9,12]
(e)	$c_2 = 0$ and $c_3 \neq 0$ in (1.3)	Sixth-order method (See Remark 2)

this paper is to provide the unified local and semi-local convergence analysis for a class of iterative methods to find a solution ξ^* of the nonlinear equation

$$T(x) \quad = \quad 0, \tag{1.1}$$

where $T : \mathcal{G} \subset X \to Y$ is a nonlinear operator, X and Y are Banach spaces, and $\mathcal{G}$ is a nonempty open convex set. To be Precise, we consider the iterative methods of order three and five, defined for $x_0 \in \mathcal{G}$ and $n = 0, 1, \ldots$ as

$$\begin{aligned} y_n &= x_n - a_0 T'(x_n)^{-1} T(x_n), \\ x_{n+1} &= x_n - T'\left((b_1 x_n + b_2 y_n)/(b_1 + b_2)\right)^{-1} T(x_n), \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} y_n &= x_n - a_0 T'(x_n)^{-1} T(x_n), \\ z_n &= x_n - T'\left((b_1 x_n + b_2 y_n)/(b_1 + b_2)\right)^{-1} T(x_n), \\ x_{n+1} &= z_n - T'\left(\frac{c_1 x_n + c_2 y_n + c_3 z_n}{c_1 + c_2 + c_3}\right)^{-1} T(z_n), \end{aligned} \tag{1.3}$$

where a_0, b_1, b_2, c_1, c_2 and c_3 are constants such that $a_0 \neq 0$, $b_1 + b_2 \neq 0$, $c_1 + c_2 + c_3 \neq 0$, $b_1 + b_2 = 2a_0 b_2$ and $c_1 + c_2 = a_0 c_2$. Note that (1.2) and (1.3) cover some of the existing methods of order three, five and six, shown in Table 1. Third order convergence for the methods, labelled as (a) and (d) in Table 1 are obtained through Taylor series expansion [8, 9, 12]. Therefore, one needs the derivative of T up to order four. i.e, for the l-th order method, one needs the condition that $T^{(l+1)}$ exists to show the convergence order using Taylor series expansion. So, our interest is to obtain the corresponding order of convergence for the methods (1.2) and (1.3) without using Taylor series expansion.

The benefits of our work are listed below.

1. We study a class of methods that includes the existing methods with suitable choice of the parameters as in Table 1. Thus, our analysis gives the unified analysis for the methods in Table 1.

2. For local convergence, we use the assumptions on the derivative of involved operator up to order two only which is a better approach than the existing theory [8, 9, 12].

3. A prior number of iterations required to obtain a given error tolerence can be obtained using our analysis.

This paper is assembled as follows. Sections 2 and 3 contain the local convergence analysis of the methods (1.2) and (1.3), respectively. The semi-local convergence analysis of the methods (1.2) and (1.3) is given in Sections 4 and 5, respectively. Examples are discussed in Sections 6 and 7. Eventually, the conclusion is given in Section 8.

2 Local convergence analysis of (1.2)

Denote $\mathcal{B}(Y, X)$ by the set, which contains all bounded linear operators from Y to X. let $U(\xi^*, R)$ be the ball with radius R centred at ξ^*, where ξ^* be a simple solution of (1.1).
The following four assumptions are considered to prove the local convergence.

(i) $T'(\xi^*)^{-1} \in \mathcal{B}(Y, X)$.

(ii) $\left\|T'(\xi^*)^{-1}\left(T'(x) - T'(\xi^*)\right)\right\| \leq L\|x-\xi^*\|$ for some $L > 0$ and $\forall x \in \mathcal{G}_0$, where $\mathcal{G}_0 := \mathcal{G} \cap U\left(\xi^*, \frac{-3+\sqrt{13}}{L}\right)$.

(iii) $\left\|T'(\xi^*)^{-1}\left(T''(x) - T''(\xi^*)\right)\right\| \leq K\|x-\xi^*\|$ for $K > 0$ and $\forall x \in \mathcal{G}_0$.

(iv) $\left\|T'(\xi^*)^{-1} T''(x)\right\| \leq M$ for some $M > 0$ and $\forall x \in \mathcal{G}_0$.

By Banach lemma [1] and the assumption (ii), we have

$$\left\|T'(x)^{-1}T'(\xi^*)\right\| \leq \frac{1}{1 - L\|x-\xi^*\|}, \quad \forall x \in \mathcal{G}_0. \tag{2.1}$$

Mean value theorem [1] gives

$$T(w_2) - T(w_1) = \int_0^1 T'\left(w_1 + s(w_2 - w_1)\right) ds\,(w_2 - w_1), \ \forall w_1, w_2 \in \mathcal{G}. \tag{2.2}$$

Using (2.2), (2.1) and the assumption (ii), we get, for all $x \in \mathcal{G}_0$,

$$\begin{aligned}
&\|x - \xi^* - T'(x)^{-1}T(x)\| \\
&\quad = \left\|x - \xi^* - T'(x)^{-1}\int_0^1 T'\left(\xi^* + t(x-\xi^*)\right) dt\,(x-\xi^*)\right\| \\
&\quad \leq \|T'(x)^{-1}T'(\xi^*)\|\left\|\int_0^1 T'(\xi^*)^{-1}\left(T'(x) - T'(\xi^*) + T'(\xi^*)\right.\right. \\
&\quad \left.\left. - T'\left(\xi^* + t(x-\xi^*)\right) dt\right\|\|x-\xi^*\| \leq \frac{3L}{2\left(1 - L\|x-\xi^*\|\right)}\|x-\xi^*\|^2,
\end{aligned} \tag{2.3}$$

Next, consider the functions $\psi, h_1 : \left[0, \frac{-3+\sqrt{13}}{L}\right) \longrightarrow \mathbb{R}$ defined as

$$\psi(s) \quad = \big[(16+9Ls)K + 72ML\big]/\big[24\big(4-6Ls-L^2s^2\big)\big]$$

and $h_1(s) = \psi(s)s^2 - 1$. Note that $\frac{1}{24(4-6Ls-L^2s^2)}$ and $\big(16+9Ls\big)K+72ML$ are the non-decreasing continuous functions on $\left[0, \frac{-3+\sqrt{13}}{L}\right)$, and hence, we have $\psi(s)$ is non-decreasing continuous function on $\left[0, \frac{-3+\sqrt{13}}{L}\right)$. Also, observe that continuity of ψ implies h_1 is continuous, and $h_1'(s) = 2\psi(s)s + s^2\psi'(s) \geq 0$ since ψ is non-decreasing function (i.e, $\psi'(s) \geq 0$). Thus, the function h_1 is also continuous and non-decreasing on $\left[0, \frac{-3+\sqrt{13}}{L}\right)$. Moreover, we have $h_1(0) = -1$ and $\lim\limits_{s\to k_1{}^-} h_1(s) = +\infty$, where $k_1 = \frac{-3+\sqrt{13}}{L}$. By Intermediate Value Theorem (IVT), there is a $R_1^* \in (0, k_1)$ such that $h_1(R_1^*) = 0$. Then, we have $0 \leq \psi(s)s^2 < 1,\ \forall s \in (0, R_1)$, where $R_1 = \min\{R_1^*, \frac{2}{5L}\}$.

Theorem 1. *Suppose the assumptions (i)–(iv) hold. Then, the iterates* (x_n) *in (1.2) with* $x_0 \in U(\xi^*, R_1) \setminus \{\xi^*\}$ *is well defined and*

$$\|x_{n+1} - \xi^*\| \leq \psi(R_1)\,\|x_n - \xi^*\|^3, \quad n = 0, 1, \ldots. \tag{2.4}$$

Moreover, $\{x_n : n = 0, 1, \ldots\} \subset U(\xi^*, R_1)$ *and* $(x_n) \longrightarrow \xi^*$ *with third order convergence.*

Proof. We shall prove by induction. First, we show that $T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}$ is invertible on $U(\xi^*, R_1) \setminus \{\xi^*\}$ as follows. Using the assumption (ii), we get

$$\left\|T'(\xi^*)^{-1}\left[T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right) - T'(\xi^*)\right]\right\| \leq L\left\|\frac{b_1(x_0-\xi^*)+b_2(y_0-\xi^*)}{b_1+b_2}\right\|. \tag{2.5}$$

Observe that

$$\begin{aligned} b_2(y_0 - \xi^*) =& b_2(x_0 - \xi^* - a_0T'(x_0)^{-1}T(x_0)) \\ =& b_2(1-a_0)(x_0 - \xi^*) + b_2a_0[x_0 - \xi^* - T'(x_0)^{-1}T(x_0)]. \end{aligned} \tag{2.6}$$

Using (2.6), (2.3) and the relation $b_1 + b_2 = 2a_0b_2$ in (2.5), we get

$$\begin{aligned} \left\|T'(\xi^*)^{-1}\left[T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right) - T'(\xi^*)\right]\right\| &\leq \frac{L}{2}\Big[\|x_0 - \xi^*\| \\ + \|x_0 - \xi^* - T'(x_0)^{-1}T(x_0)\|\Big] &\leq \frac{L(2+L\|x_0-\xi^*\|)}{4(1-L\|x_0-\xi^*\|)}\|x_0 - \xi^*\|. \end{aligned}$$

Since $x_0 \in U(\xi^*, R_1) \setminus \{\xi^*\}$, we have

$$\left\|T'(\xi^*)^{-1}\left[T\prime\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right) - T\prime(\xi^*)\right]\right\| < 1.$$

Then, by Banach lemma [1], $T'(\xi^*)^{-1}T\prime\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)$ is invertible and

$$\left\|T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)\right\| \leq \frac{4(1-L\|x_0-\xi^*\|)}{4-6L\|x_0-\xi^*\|-L^2\|x_0-\xi^*\|^2}. \tag{2.7}$$

By using (2.2) in the second step of (1.2), we obtain

$$\begin{aligned}
&x_1-\xi^* = x_0-\xi^*-T'\left(\frac{bx_0+cy_0}{b_1+b_2}\right)^{-1}[T(x_0)-T(\xi^*)]\\
=&T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}\left[T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)-\int_0^1 T'\left(\xi^*+t\left(x_0-\xi^*\right)\right)dt\right](x_0-\xi^*)\\
=&T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}\int_0^1\left[T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)-T'\left(\xi^*+t\left(x_0-\xi^*\right)\right)\right]dt\,(x_0-\xi^*)\\
=&T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}\int_0^1\int_0^1 T''\Big(\xi^*+t\left(x_0-\xi^*\right)+\zeta\Big(\frac{b_1x_0+b_2y_0}{b_1+b_2}\\
&-\xi^*-t\left(x_0-\xi^*\right)\Big)\Big)\left[\frac{b_1x_0+b_2y_0}{b_1+b_2}-\xi^*-t\big(x_0-\xi^*\big)\right]dtd\zeta\,(x_0-\xi^*).
\end{aligned} \tag{2.8}$$

By simplifying and using the fact that $T'(\xi^*)T'(\xi^*)^{-1}$ is identity, we get

$$\begin{aligned}
x_1-\xi^* =& T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)T'(\xi^*)^{-1}\int_0^1\int_0^1 T''\Big(\xi^*+t\left(x_0-\xi^*\right)\\
&+\zeta\Big[\frac{b_1(x_0-\xi^*)+b_2(y_0-\xi^*)-t(b_1+b_2)(x_0-\xi^*)}{b_1+b_2}\Big]\Big)\\
&\times\left[\frac{b_1(x_0-\xi^*)+b_2(y_0-\xi^*)-t(b_1+b_2)(x_0-\xi^*)}{b_1+b_2}\right]dtd\zeta\,(x_0-\xi^*).
\end{aligned} \tag{2.9}$$

By denoting

$$T_{t,\zeta}=\xi^*+t\left(x_0-\xi^*\right)+\zeta\left(\frac{b_1(x_0-\xi^*)+b_2(y_0-\xi^*)-t(b_1+b_2)(x_0-\xi^*)}{b_1+b_2}\right) \tag{2.10}$$

and using (2.6) in (2.9), we get

$$\begin{aligned}
x_1-\xi^* =& T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)T'(\xi^*)^{-1}\int_0^1\int_0^1 T\prime\prime\left(T_{t,\zeta}\right)\\
&\times\left[\frac{b_1+b_2(1-a_0)-t(b_1+b_2)}{b_1+b_2}\right]dtd\zeta(x_0-\xi^*)^2\\
&+T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)\int_0^1\int_0^1 T'(\xi^*)^{-1}T\prime\prime\left(T_{t,\zeta}\right)dtd\zeta\\
&\times\frac{b_2a_0}{b_1+b_2}\left[x_0-\xi^*-T'(x_0)^{-1}T(x_0)\right](x_0-\xi^*).
\end{aligned} \tag{2.11}$$

Addition and subtraction of the term $T''(\xi^*)$ in the first term of (2.11) gives

$$x_1-\xi^* = T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)\int_0^1\int_0^1 T'(\xi^*)^{-1}[T''\left(T_{t,\zeta}\right)-T''(\xi^*)]$$

$$
\begin{aligned}
&\times\left[\frac{b_1+b_2(1-a_0)-t(b_1+b_2)}{b_1+b_2}\right]dtd\zeta(x_0-\xi^*)^2+T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}\\
&\times T'(\xi^*)T'(\xi^*)^{-1}T''(\xi^*)\left[\frac{2b_1+2b_2(1-a_0)-(b_1+b_2)}{2(b_1+b_2)}\right](x_0-\xi^*)^2\\
&+T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)\int_0^1\int_0^1 T'(\xi^*)^{-1}T''\left(T_{t,\zeta}\right)dtd\zeta\\
&\times\frac{b_2a_0}{b_1+b_2}\left[x_0-\xi^*-T'(x_0)^{-1}T(x_0)\right](x_0-\xi^*). \qquad (2.12)
\end{aligned}
$$

Since $b_1+b_2 = 2a_0b_2$, we get $\left[\frac{2b_1+2b_2(1-a_0)-(b_1+b_2)}{2(b_1+b_2)}\right] = 0$. So, the Equation (2.12) becomes

$$
\begin{aligned}
x_1-\xi^* &= T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)\Bigg[\int_0^1\int_0^1 T'(\xi^*)^{-1}[T''\left(T_{t,\zeta}\right)-T''(\xi^*)]\\
&\quad\times\left[\frac{1-2t}{2}\right]dtd\zeta(x_0-\xi^*)^2+\frac{1}{2}\int_0^1\int_0^1 T'(\xi^*)^{-1}T''\left(T_{t,\zeta}\right)dtd\zeta\\
&\quad\times\left[x_0-\xi^*-T'(x_0)^{-1}T(x_0)\right](x_0-\xi^*)\Bigg]. \qquad (2.13)
\end{aligned}
$$

From (2.10), (2.6) and $b_1+b_2=2a_0b_2$, it is observed that

$$
\begin{aligned}
&\|T_{t,\zeta}-\xi^*\|=\left\|t\left(x_0-\xi^*\right)+\zeta\Big(\frac{b_1(x_0-\xi^*)+b_2(y_0-\xi^*)-t(b_1+b_2)(x_0-\xi^*)}{b_1+b_2}\Big)\right\|\\
&\le t\|x_0-\xi^*\|+\frac{\zeta}{2}\Big[|1-2t|\,\|x_0-\xi^*\|+\|x_0-\xi^*-T'(x_0)^{-1}T(x_0)\|\Big]\\
&\le\left[t+\frac{\zeta}{2}|1-2t|+\frac{3L\zeta\|x_0-\xi^*\|}{4(1-L\|x_0-\xi^*\|)}\right]\|x_0-\xi^*\|. \qquad (2.14)
\end{aligned}
$$

Next, we estimate $\|x_1-\xi^*\|$ using (2.13) as

$$
\begin{aligned}
&\|x_1-\xi^*\|=\Bigg\|T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)\Bigg[\int_0^1\int_0^1 T'(\xi^*)^{-1}[T''\left(T_{t,\zeta}\right)-T''(\xi^*)]\\
&\times\left[\frac{1-2t}{2}\right]dtd\zeta(x_0-\xi^*)^2+\frac{1}{2}\int_0^1\int_0^1 T'(\xi^*)^{-1}T''\left(T_{t,\zeta}\right)dtd\zeta\\
&\times\left[x_0-\xi^*-T'(x_0)^{-1}T(x_0)\right](x_0-\xi^*)\Bigg]\Bigg\|\\
&\le\left\|T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(\xi^*)\right\|\Bigg[\int_0^1\int_0^1\|T'(\xi^*)^{-1}[T''\left(T_{t,\zeta}\right)-T''(\xi^*)]\|\\
&\times\frac{|1-2t|}{2}dtd\zeta\|x_0-\xi^*\|^2+\frac{1}{2}\int_0^1\int_0^1\|T'(\xi^*)^{-1}T''\left(T_{t,\zeta}\right)\|dtd\zeta\\
&\times\|x_0-\xi^*-T'(x_0)^{-1}T(x_0)\|\|x_0-\xi^*\|\Bigg]. \qquad (2.15)
\end{aligned}
$$

Using (2.7), (2.14), and the assumptions (iii) and (iv), we obtain

$$\begin{aligned}\|x_1-\xi^*\| &\le \frac{4(1-L\|x_0-\xi^*\|)}{4-6L\|x_0-\xi^*\|-L^2\|x_0-\xi^*\|^2}\Bigg[\int_0^1\int_0^1 K\bigg(t+\frac{\zeta}{2}|1-2t| \\ &+\frac{3L\zeta\|x_0-\xi^*\|}{4(1-L\|x_0-\xi^*\|)}\bigg)\frac{|1-2t|}{2}dtd\zeta\|x_0-\xi^*\|^3+\frac{3ML\,\|x_0-\xi^*\|^3}{4(1-L\|x_0-\xi^*\|)}\Bigg] \\ &\le \frac{\big(16+9L\|x_0-\xi^*\|\big)K+72ML}{24\Big[4-6L\|x_0-\xi^*\|-L^2\|x_0-\xi^*\|^2\Big]}\times\|x_0-\xi^*\|^3 \\ &\le \psi(\|x_0-\xi^*\|)\|x_0-\xi^*\|^3.\end{aligned}$$

Since $0\le\psi(s)s^2<1$ for all $s\in(0,R_1)$, we get $x_1\in U(\xi^*,R_1)$. Next, the replacements x_0 by x_n, y_0 by y_n and x_1 by x_{n+1} in this proof give $x_n\in U(\xi^*,R_1)$ for all $n=0,1,\ldots$. Thus, the inequality (2.4) holds and $(x_n)\longrightarrow\xi^*$ with convergence order three. □

Remark 1. The result regarding uniqueness is given in [14, Proposition 2.3].

3 Local convergence analysis of (1.3)

Consider the non-decreasing continuous functions $\psi_1,q_1:[0,d_1)\longrightarrow\mathbb{R}$ defined as

$$\psi_1(s)=\frac{M\,\psi(s)}{|a_0c_2+c_3|-L\left[\frac{3L|a_0c_2|}{2(1-Ls)}+\psi(s)s\,|c_3|\right]s^2}\left[\frac{3L\,|a_0c_2|}{2(1-Ls)}+\frac{|c_3-a_0c_2|}{2}\psi(s)s\right]$$

and $q_1(s)=\psi_1(s)s^4-1$, where d_1 be the smallest positive real solution of $|a_0c_2+c_3|-L\left(\frac{3L|a_0c_2|}{2(1-Ls)}+|c_3|\psi(s)s\right)s^2=0$. Then, we have $q_1(0)=-1$ and $\lim\limits_{s\to d_1^-}q_1(s)=+\infty$. By IVT, there exists smallest $r_1\in(0,d_1)$ such that $q_1(r_1)=0$. Let $R_2=\min\{R_1,r_1\}$. Then, we have $0\le\psi_1(s)s^4<1,\ \forall\ s\in[0,R_2)$.

Theorem 2. *Suppose the assumptions (i)–(iv) hold. Then, the iterates (x_n) in (1.3) with $x_0\in U(\xi^*,R_2)\setminus\{\xi^*\}$ is well defined and*

$$\|x_{n+1}-\xi^*\|\le\psi_1(R_2)\,\|x_n-\xi^*\|^5,\quad n=0,1,\ldots. \tag{3.1}$$

Moreover, $\{x_n:n=0,1,\ldots\}\subset U(\xi^*,R_2)$ and $(x_n)\longrightarrow\xi^*$ with convergence order five.

Proof. By considering $R_1=R_2$ and $z_n=x_{n+1}$ in Theorem 1, we get

$$\|z_n-\xi^*\|\quad\le\quad\psi(\|x_n-\xi^*\|)\|x_n-\xi^*\|^3,\quad n=0,1,\ldots. \tag{3.2}$$

Firstly we shall prove that $T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)^{-1}$ exists on $U(\xi^*,R_2)\setminus\{\xi^*\}$. Set $\mathcal{C}_n=\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}$. Then, using the inequality (2.3), $c_1+c_2=a_0c_2$,

$$c_2(y_n-\xi^*)=c_2(1-a_0)(x_n-\xi^*)+c_2a_0[x_n-\xi^*-T'(x_n)^{-1}T(x_n)] \tag{3.3}$$

and the assumption (ii), we get

$$\begin{aligned}\left\|T'(\xi^*)^{-1}T\prime(\mathcal{C}_n)-I\right\| &= \left\|T'(\xi^*)^{-1}\left[T\prime\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)-T\prime(\xi^*)\right]\right\|\\ &\leq L\frac{\left\|a_0c_2(x_n-\xi^*-T'(x_n)^{-1}T(x_n))+(z_n-\xi^*)c_3\right\|}{|a_0c_2+c_3|}\\ &\leq \frac{L}{|a_0c_2+c_3|}\left[\frac{3L|a_0c_2|}{2(1-L\|x_n-\xi^*\|)}+\psi(\|x_n-\xi^*\|)\right.\\ &\quad\left.\times\|x_n-\xi^*\|\,|c_3|\right]\|x_n-\xi^*\|^2.\end{aligned}$$

Since $R_2 = \min\{R_1, r_1\}$, we have $\left\|T'(\xi^*)^{-1}T\prime\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)-I\right\| < 1$. Then, by Banach lemma [1], $T'(\xi^*)^{-1}T\prime\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)$ is invertible and

$$\left\|T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)^{-1}T'(\xi^*)\right\| \leq \frac{1}{1-L\Upsilon_{x_n}}, \tag{3.4}$$

where $\Upsilon_{x_n} = \frac{\|x_n-\xi^*\|^2}{|a_0c_2+c_3|}\left[\frac{3L|a_0c_2|}{2(1-L\|x_n-\xi^*\|)}+\psi(\|x_n-\xi^*\|)\|x_n-\xi^*\|\,|c_3|\right]$. Note that

$$\begin{aligned}x_{n+1}-\xi^* &= z_n-\xi^*-T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)^{-1}[T(z_n)-T(\xi^*)]\\ &= T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)^{-1}\int_0^1\left[T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)\right.\\ &\quad\left.-T'(\xi^*+t(z_n-\xi^*))\right]dt\,(z_n-\xi^*) = T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)^{-1}\\ &\quad\times T'(\xi^*)T'(\xi^*)^{-1}\int_0^1\int_0^1 T''\left(W_{t,\zeta}\right)\frac{1}{c_1+c_2+c_3}\left[c_1(x_n-\xi^*)+c_2(y_n-\xi^*)\right.\\ &\quad\left.+(c_3-t(c_1+c_2+c_3))(z_n-\xi^*)\right]dtd\zeta(z_n-\xi^*),\end{aligned} \tag{3.5}$$

where

$$W_{t,\zeta}=\xi^*+t(z_n-\xi^*)+\zeta\frac{[c_1(x_n-\xi^*)+c_2(y_n-\xi^*)+(c_3-t(c_1+c_2+c_3))(z_n-\xi^*)]}{c_1+c_2+c_3}.$$

Using (3.3) and the relation $c_1+c_2=a_0c_2$ in (3.5), we get

$$\begin{aligned}x_{n+1}-\xi^* &= T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)^{-1}T'(\xi^*)\frac{1}{a_0c_2+c_3}\\ &\quad\times\left[\int_0^1\int_0^1 T'(\xi^*)^{-1}T''(W_{t,\zeta})c_2a_0[x_n-\xi^*-T'(x_n)^{-1}T(x_n)]dtd\zeta\right.\\ &\quad\left.+\int_0^1\int_0^1 T'(\xi^*)^{-1}T''(W_{t,\zeta})(c_3-t(a_0c_2+c_3))(z_n-\xi^*)dtd\zeta\right](z_n-\xi^*).\end{aligned} \tag{3.6}$$

Next, the estimation for $\|x_{n+1}-\xi^*\|$ can be obtained from the Equation (3.6) with the help of (3.4), (2.3), (3.2) and the assumption (iv) as

$$\begin{aligned}
&\|x_{n+1}-\xi^*\| \leq \left\| T'\left(\frac{c_1x_n+c_2y_n+c_3z_n}{c_1+c_2+c_3}\right)^{-1} T'(\xi^*)\right\| \frac{1}{|a_0c_2+c_3|}\\
&\times\Bigg[|a_0c_2|\int_0^1\int_0^1 \|T'(\xi^*)^{-1}T''(W_{t,\zeta})\|dtd\zeta\\
&\times\|x_n-\xi^*-T'(x_n)^{-1}T(x_n)\|+\frac{|c_3-a_0c_2|}{2}\\
&\times\int_0^1\int_0^1 \|T'(\xi^*)^{-1}T''(W_{t,\zeta})\|dtd\zeta\|z_n-\xi^*\|\Bigg]\|z_n-\xi^*\|\\
\leq&\frac{M}{1-L\Upsilon_{x_n}}\left[\frac{3L\,|a_0c_2|}{2(1-L\|x_n-\xi^*\|)}+\frac{|c_3-a_0c_2|}{2}\psi(\|x_n-\xi^*\|)\|x_n-\xi^*\|\right]\\
&\times\frac{1}{|a_0c_2+c_3|}\psi(\|x_n-\xi^*\|)\|x_n-\xi^*\|^5\leq\psi_1(\|x_n-\xi^*\|)\|x_n-\xi^*\|^5. \qquad (3.7)
\end{aligned}$$

Since $0\leq\psi_1(s)s^4<1\ \forall s\in(0,R_2)$, we get $x_n\in U(\xi^*,R_2)$ for all $n=0,1,\ldots$. Thus, the inequality (3.1) holds and $(x_n)\longrightarrow\xi^*$ with convergence order five. □

Remark 2. If $c_2=0$, the convergence order of the method (1.3) is six, which can be noticed from (3.7).

4 Semi-local convergence analysis of (1.2)

Firstly, we define the scalar sequences (majorizing sequences) $\{\alpha_n\}$ and $\{\beta_n\}$ using the idea given in [2,15] as $\alpha_0=0,\beta_0\geq 0$,

$$\begin{aligned}
&q_n=Q_1\left((|b_1|\alpha_n+|b_2|\beta_n)/|b_1+b_2|\right),\\
&\alpha_{n+1}=\beta_n+\frac{1}{|a_0|}\left[\frac{q_n+Q_1(\alpha_n)}{(1-q_n)}+|a_0-1|\right](\beta_n-\alpha_n),\\
&\tau_{n+1}=\left[\int_0^1 Q_2(\delta(\alpha_{n+1}-\alpha_n))d\delta+\left|1-\frac{1}{a_0}\right|(1+Q_1(\alpha_n))\right](\alpha_{n+1}-\alpha_n)\\
&\quad+1/|a_0|(1+Q_1(\alpha_n))(\alpha_{n+1}-\beta_n),\\
&\beta_{n+1}=\alpha_{n+1}+\frac{\tau_{n+1}}{|a_0|(1-Q_1(\alpha_{n+1}))}, \qquad (4.1)
\end{aligned}$$

where the functions $Q_1,Q_2:[0,\infty)\longrightarrow\mathbb{R}$ are non-decreasing and continuous such that $Q_1(0)=0$ and $Q_2(0)=0$.

Lemma 1. *Assume that there is $\lambda>0$ such that $Q_1(\alpha_n)<1$, $q_n<1$ and $\alpha_n\leq\lambda$ for all whole numbers n. Then, the scalar sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined in (4.1) are converge to some $R_3\in[\beta_0,\lambda]$ and $0\leq\alpha_n\leq\beta_n\leq\alpha_{n+1}\leq R_3$.*

Proof. Analogous to the proof of Lemma 4.1 in [15]. □

We shall show that the above scalar sequences are majorizing for (1.2) with the following assumptions.

(I) There exists $x_0 \in \mathcal{G}$, $\vartheta \geq 0$ such that $T'(x_0)^{-1} \in L(Y,X)$ and $\|T'(x_0)^{-1}T(x_0)\| \leq \vartheta$. Set $\mathcal{D}_0 = \mathcal{G} \cap U(x_0, R_3)$.

(II) $\left\|T'\left(x_0\right)^{-1}\left(T'(x) - T'\left(x_0\right)\right)\right\| \leq Q_1(\|x - x_0\|), \quad \forall x \in \mathcal{D}_0.$

(III) $\left\|T'\left(x_0\right)^{-1}\left(T'(x) - T'\left(y\right)\right)\right\| \leq Q_2(\|x - y\|), \quad \forall x, y \in \mathcal{D}_0.$

(IV) The conditions of Lemma 1 hold for $\lambda = R_3$.

(V) $\overline{U(x_0, R_3)} \subset \mathcal{G}$.

By Banach lemma [1] on the assumption (II), we get

$$\left\|T'(x)^{-1}T'\left(x_0\right)\right\| \leq \frac{1}{1 - Q_1(\|x - x_0\|)}, \quad \forall x \in \mathcal{D}_0.$$

The assumption (II) also gives

$$\|T'(x_0)^{-1}T'(x)\| = \|T'(x_0)^{-1}\left[T'(x) - T'(x_0)\right] + I\| \leq 1 + Q_1(\|x - x_0\|). \tag{4.2}$$

Theorem 3. *Suppose the conditions (I)–(V) hold. Then, the sequence $\{x_n\}$ in (1.2) with $x_0 \in U(x_0, R_3)$ satisfies*

$$\|y_n - x_n\| \leq \beta_n - \alpha_n, \text{ and} \tag{4.3}$$

$$\|x_{n+1} - y_n\| \leq \alpha_{n+1} - \beta_n. \tag{4.4}$$

Further, the sequence $\{x_n\} \longrightarrow \xi^* \in \overline{U(x_0, R_3)}$.

Proof. By induction, we shall prove (4.3) and (4.4). Set $\beta_0 = |a_0|\vartheta$. Observe that y_0 is well defined by (1.2). Also,

$$\|y_0 - x_0\| = |a_0| \left\|T'(x_0)^{-1}T\left(x_0\right)\right\| \leq |a_0|\,\vartheta = \beta_0 - \alpha_0 \leq R_3.$$

Thus, $y_0 \in \overline{U(x_0, R_3)}$ and (4.3) is true for $n = 0$. Set $B_n = T'\left(\frac{b_1x_n + b_2y_n}{b_1 + b_2}\right)$. Now, we shall show that B_0^{-1} exists on $U\left(x_0, R_3\right) \setminus \{x_0\}$ as follows.

Using the assumption (II), we get

$$\left\|T'(x_0)^{-1}\left[T'\left(\frac{b_1x_0 + b_2y_0}{b_1 + b_2}\right) - T'(x_0)\right]\right\| \leq Q_1\left(\frac{|b_2|\,\|y_0 - x_0\|}{|b_1 + b_2|}\right) \leq q_0 < 1. \tag{4.5}$$

By Banach lemma [1], $T'(x_0)^{-1}T'\left(\frac{b_1x_0 + b_2y_0}{b_1 + b_2}\right)$ is invertible and

$$\left\|T'\left(\frac{b_1x_0 + b_2y_0}{b_1 + b_2}\right)^{-1} T'(x_0)\right\| \leq \frac{1}{1 - q_0}. \tag{4.6}$$

Thus, we have $T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)$ is invertible on $U(x_0,R_3)\setminus\{x_0\}$. Next, from the second step of (1.2), we get

$$\begin{aligned}
x_1-y_0 &= -T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T(x_0)+a_0T'(x_0)^{-1}T(x_0)\\
&= -\left(T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}-T'(x_0)^{-1}\right)T(x_0)+(a_0-1)T'(x_0)^{-1}T(x_0)\\
&= \left[T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}\left(T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)-T'(x_0)\right)+(a_0-1)\right]\\
&\times T'(x_0)^{-1}T(x_0)\\
&= -\frac{1}{a_0}\left[T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}\left(T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)-T'(x_0)\right)\right.\\
&\quad\left.+(a_0-1)\right](y_0-x_0).
\end{aligned}\tag{4.7}$$

By (4.6) and (4.5), we get

$$\begin{aligned}
\|x_1-y_0\| &\le \frac{1}{|a_0|}\left[\left\|T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)^{-1}T'(x_0)\right\|\right.\\
&\quad\times\left.\left\|T'(x_0)^{-1}\left(T'\left(\frac{b_1x_0+b_2y_0}{b_1+b_2}\right)-T'(x_0)\right)\right\|+|a_0-1|\right]\|y_0-x_0\|\\
&\le \frac{1}{|a_0|}\left[\frac{q_0}{1-q_0}+|a_0-1|\right](\beta_0-\alpha_0)=\alpha_1-\beta_0.
\end{aligned}$$

Note that $\|x_1-x_0\|\le\|x_1-y_0\|+\|y_0-x_0\|\le\alpha_1-\beta_0+\beta_0-\alpha_0\le R_3$. Therefore, $x_1\in\overline{U(x_0,R_3)}$ and (4.4) is true for $n=0$.

As an induction hypothesis, assume that (4.3) and (4.4) are true, and $T'\left(\frac{b_1x_n+b_2y_n}{b_1+b_2}\right)^{-1}$ exists for all $n=0,1,2,\dots,k-1$. This gives $\|x_n-x_{n-1}\|\le\alpha_n-\alpha_{n-1}$ and $\|x_n-x_0\|\le\alpha_n-\alpha_0$ for all $n=1,2,\dots,k-1$.

Now, using (1.2) and (2.2), we have

$$\begin{aligned}
T(x_k) =& T(x_k)-T(x_{k-1})+T(x_{k-1})+\frac{1}{a_0}T'(x_{k-1})(x_k-x_{k-1})\\
&-\frac{1}{a_0}T'(x_{k-1})(x_k-x_{k-1})\\
=&\int_0^1 T'\Big(x_{k-1}+\delta(x_k-x_{k-1})\Big)d\delta\,(x_k-x_{k-1})-\frac{1}{a_0}T'(x_{k-1})(x_k-x_{k-1})\\
&+\frac{1}{a_0}T'(x_{k-1})(x_k-y_{k-1})\\
=&\left[\int_0^1\left[T'\Big(x_{k-1}+\delta(x_k-x_{k-1})\Big)-T'(x_{k-1})\right]d\delta+\left(1-\frac{1}{a_0}\right)T'(x_{k-1})\right]\\
&\times(x_k-x_{k-1})+\frac{1}{a_0}T'(x_{k-1})(x_k-y_{k-1}).
\end{aligned}\tag{4.8}$$

From the above Equation (4.8), we get the estimate for $\|T'(x_0)^{-1}T(x_k)\|$ using (4.2) and the assumptions (II)–(III) as

$$\begin{aligned}
\|T'(x_0)^{-1}T(x_k)\| \leq & \Bigg[\int_0^1 \left\| T'(x_0)^{-1} \Big[T'\Big(x_{k-1} + \delta(x_k - x_{k-1}) \Big) - T'(x_{k-1}) \Big] \right\| d\delta \\
& + \left| 1 - \frac{1}{a_0} \right| \|T'(x_0)^{-1}T'(x_{k-1})\| \Bigg] \|x_k - x_{k-1}\| \\
& + \frac{1}{|a_0|} \|T'(x_0)^{-1}T'(x_{k-1})\| \, \|x_k - y_{k-1}\| \\
\leq & \left[\int_0^1 Q_2(\delta \|x_k - x_{k-1}\|) \, d\delta + \left| 1 - \frac{1}{a_0} \right| \big(1 + Q_1(\|x - x_0\|) \big) \right] \\
& \times \|x_k - x_{k-1}\| + \frac{1}{|a_0|} \big(1 + Q_1(\|x_{k-1} - x_0\|) \big) \|x_k - y_{k-1}\| \leq \tau_k .
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
& \|y_k - x_k\| \leq \frac{1}{|a_0|} \|T'(x_k)^{-1}T(x_k)\| \leq \frac{1}{|a_0|} \|T'(x_k)^{-1}T'(x_0)\| \, \|T'(x_0)^{-1}T(x_k)\| \\
& \leq \frac{\tau_k}{|a_0|(1 - Q_1(\|x_k - x_0\|))} \leq \frac{\tau_k}{|a_0|(1 - Q_1(\alpha_k))} = \beta_k - \alpha_k \\
& \|y_k - x_0\| \leq \|y_k - x_k + x_k - x_0\| \leq \beta_k < R_3 .
\end{aligned}$$

So, $y_k \in U(x_0, R_3)$, and the inequality (4.3) is true for all k.

Now, we shall show that $T' \left(\frac{b_1 x_k + b_2 y_k}{b_1 + b_2} \right)^{-1}$ exists on $U(x_0, R_3) \setminus \{x_0\}$ as follows. Using the assumption (II), we get

$$\left\| T'(x_0)^{-1} \left[T' \left(\frac{b_1 x_k + b_2 y_k}{b_1 + b_2} \right) - T'(x_0) \right] \right\| \leq Q_1 \left(\frac{|b_1| \, \|x_k - x_0\| + |b_2| \, \|y_k - x_0\|}{|b_1 + b_2|} \right) \leq q_k < 1 .$$

By Banach lemma [1], $T'(x_0)^{-1} T' \left(\frac{b_1 x_k + b_2 y_k}{b_1 + b_2} \right)$ is invertible and

$$\left\| T' \left(\frac{b_1 x_k + b_2 y_k}{b_1 + b_2} \right)^{-1} T'(x_0) \right\| \leq \frac{1}{1 - q_k} .$$

Thus, we have $T' \left(\frac{b_1 x_k + b_2 y_k}{b_1 + b_2} \right)$ is invertible on $U(x_0, R_3) \setminus \{x_0\}$.

Next, from the second step of (1.2), we get

$$\begin{aligned}
x_{k+1}-y_k &= -T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)^{-1}T(x_k)+a_0T'(x_k)^{-1}T(x_k)\\
&= -\left(T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)^{-1}-T'(x_k)^{-1}\right)T(x_k)+(a_0-1)T'(x_k)^{-1}T(x_k)\\
&= \left[T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)^{-1}\left(T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)-T'(x_k)\right)+(a_0-1)\right]\\
&\quad\times T'(x_k)^{-1}T(x_k)=\frac{-1}{a_0}\left[T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)^{-1}\right.\\
&\quad\times\left.\left(T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)-T'(x_k)\right)+(a_0-1)\right](y_k-x_k). \qquad (4.9)
\end{aligned}$$

By applying (4.5) and (4.6) in (4.9), we get

$$\begin{aligned}
&\|x_{k+1}-y_k\|\leq\frac{1}{|a_0|}\left[\left\|T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)^{-1}T'(x_0)\right\|\right.\\
&\times\left.\left\|T'(x_0)^{-1}\left(T'\left(\frac{b_1x_k+b_2y_k}{b_1+b_2}\right)-T'(x_0)+T'(x_0)-T'(x_k)\right)\right\|+|a_0-1|\right]\\
&\times\|y_k-x_k\|\leq\frac{1}{|a_0|}\left[\frac{q_k+Q_1(\alpha_k)}{(1-q_k)}+|a_0-1|\right](\beta_k-\alpha_k)=\alpha_{k+1}-\beta_k. \qquad (4.10)
\end{aligned}$$

Notice that $\|x_{k+1}-x_0\|\leq\|x_{k+1}-y_k+y_k-x_0\|\leq\alpha_{k+1}<R_3$. Thus, $x_{k+1}\in U(x_0,R_3)$ and (4.4) is true for all $k=\mathbb{N}\cup\{0\}$. From Lemma 1, the scalar sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are Cauchy and so $\{x_n\}$ and $\{y_n\}$ are Cauchy by (4.3) and (4.4). Thus, $(x_n)\longrightarrow\xi^*\in U(x_0,R_3)$. □

5 Semi-local convergence analysis of (1.3)

Here, we define the majorizing sequence for (1.3) as $\alpha_0=0,\beta_0\geq 0$,

$$\begin{aligned}
\gamma_n &= \beta_n+\frac{1}{|a_0|}\left[\frac{q_n+Q_1(\alpha_n)}{(1-q_n)}+|a_0-1|\right](\beta_n-\alpha_n),\\
p_n &= \left(1+\int_0^1 Q_1\Big(\beta_n+\delta(\gamma_n-\beta_n)\Big)d\delta\right)(\gamma_n-\beta_n)\\
&\quad+\left(\int_0^1 Q_2(\delta(\gamma_n-\beta_n))d\delta+\left|1-\frac{1}{a_0}\right|(1+Q_1(\alpha_n))\right)(\beta_n-\alpha_n),\\
\alpha_{n+1} &= \gamma_n+\frac{p_n}{1-Q_1\left(\frac{|c_1|\,\alpha_n+|c_2|\,\beta_n+|c_3|\,\gamma_n}{|c_1+c_2+c_3|}\right)},\\
\beta_{n+1} &= \alpha_{n+1}+\frac{\tau_{n+1}}{|a_0|(1-Q_1(\alpha_{n+1}))}.
\end{aligned}$$

One can notice that the results of Lemma 1 is true with $Q_1(\alpha_n) < 1$, $q_n < 1$, $\alpha_n \leq \lambda$ and $Q_1\left(\frac{|c_1|\,\alpha_n+|c_2|\,\beta_n+|c_3|\,\gamma_n}{|c_1+c_2+c_3|}\right) < 1$.

Theorem 4. *Suppose the conditions (I)–(V) hold. Then, the sequence $\{x_n\}$ in (1.3) with $x_0 \in U(x_0, R_3)$ satisfies*

$$\|y_n - x_n\| \leq \beta_n - \alpha_n, \quad \|z_n - y_n\| \leq \gamma_n - \beta_n, \quad \|x_{n+1} - z_n\| \leq \alpha_{n+1} - \gamma_n.$$

Further, the sequence $(x_n) \longrightarrow \xi^* \in \overline{U(x_0, R_3)}$.

Proof. Note that the replacement of x_{k+1} by z_k in (4.10) gives

$$\|z_k - y_k\| \quad \leq \quad \frac{1}{|a_0|}\left[\frac{q_k}{(1-q_k)} + |a_0 - 1|\right](\beta_k - \alpha_k) = \gamma_k - \beta_k.$$

Now, we shall show that $T'\left(\frac{c_1x_k+c_2y_k+c_3z_k}{c_1+c_2+c_3}\right)^{-1}$ exists on $U(x_0, R_3)\setminus\{x_0\}$.

As before, set $\mathcal{C}_k = \frac{c_1x_k+c_2y_k+c_3z_k}{c_1+c_2+c_3}$. Then, we have

$$\begin{aligned}
&\left\|T'(x_0)^{-1}T\prime(\mathcal{C}_k) - I\right\| = \left\|T'(x_0)^{-1}\left[T\prime\left(\frac{c_1x_k + c_2y_k + c_3z_k}{c_1+c_2+c_3}\right) - T\prime(x_0)\right]\right\| \\
&\leq Q_1\left(\frac{|c_1|\,\alpha_k + |c_2|\,\beta_k + |c_3|\,\gamma_k}{|c_1+c_2+c_3|}\right) < 1.
\end{aligned}$$

Then, by Banach lemma [1], $T'(x_0)^{-1}T\prime\left(\frac{c_1x_k+c_2y_k+c_3z_k}{c_1+c_2+c_3}\right)$ is invertible and

$$\left\|T'\left(\frac{c_1x_k+c_2y_k+c_3z_k}{c_1+c_2+c_3}\right)^{-1}T'(x_0)\right\| \leq \frac{1}{1 - Q_1\left(\frac{|c_1|\,\alpha_k+|c_2|\,\beta_k+|c_3|\,\gamma_k}{|c_1+c_2+c_3|}\right)}. \quad (5.1)$$

Now, using (1.3) and (2.2), we get

$$\begin{aligned}
&T(z_k) = T(z_k) - T(y_k) + T(y_k) - T(x_k) + T(x_k) \\
&= \int_0^1 T'\Big(y_k + \delta(z_k - y_k)\Big)d\delta(z_k - y_k) + \int_0^1 T'\Big(x_k + \delta(y_k - x_k)\Big)d\delta \\
&\quad \times (y_k - x_k) - \frac{1}{a_0}T'(x_k)(y_k - x_k) \\
&= \int_0^1 \left[T'\Big(y_k + \delta(z_k - y_k)\Big) - T'(x_0) + T'(x_0)\right]d\delta(z_k - y_k) \\
&\quad + \left[\int_0^1 \left[T'\Big(x_k + \delta(y_k - x_k)\Big) - T'(x_k)\right]d\delta + \left(1 - \frac{1}{a_0}\right)T'(x_k)\right](y_k - x_k),
\end{aligned}$$

which yields (by (II), (III) and (4.2))

$$\begin{aligned}
&\|T'(x_0)^{-1}T(z_k)\| \leq \left(1 + \int_0^1 Q_1\Big(\|y_k - x_0\| + \delta\|z_k - y_k\|\Big)d\delta\right)\|z_k - y_k\| \\
&\quad + \left(\int_0^1 Q_2(\delta\|y_k - x_k\|)d\delta + \left|1 - \frac{1}{a_0}\right|\,\big(1 + Q_1(\|x_k - x_0\|)\big)\right)\|y_k - x_k\| \leq p_k. \quad (5.2)
\end{aligned}$$

Next, the inequalities (5.1) and (5.2) give

$$\begin{aligned}\|x_{k+1}-z_k\| &\le \left\|T'\left(\frac{c_1x_k+c_2y_k+c_3z_k}{c_1+c_2+c_3}\right)^{-1}T'(x_0)\right\| \|T'(x_0)^{-1}T(z_k)\| \\ &\le \frac{p_k}{1-Q_1\left(\frac{|c_1|\,\alpha_k+|c_2|\,\beta_k+|c_3|\,\gamma_k}{|c_1+c_2+c_3|}\right)}=\alpha_{k+1}-\gamma_k.\end{aligned}$$

Hence, we have $\|z_k-x_0\|\le\|z_k-y_k+y_k-x_0\|\le\gamma_k<R_3$ and $\|x_{k+1}-x_0\|\le\|x_{k+1}-z_k+z_k-x_0\|\le\alpha_{k+1}<R_3$. The rest follows as done in Theorem 3. $\square$

The result regarding uniqueness is given in [3, Proposition 2].

6 Examples

We considered the examples from integral equations, system of equations and fractional differential equations in this section.

Example 1. Consider $T: U(0,1)\subset C[0,1]\longrightarrow C[0,1]$ as

$$T(z)(s)=z(s)-\frac{1}{8}\int_0^1 e^{-(t+s)}\left[z(t)\right]^4 dt. \tag{6.1}$$

Note that $z^*(s)=0$ solves the Equation (6.1). The constants values are $M=\frac{3}{2}$, $L=\frac{1}{2}$ and $K=\frac{3}{2}$. Further, the convergence balls' radius considered in Sections 2 and 3 are shown in Table 2.

Table 2. Comparison table of convergence balls' radius for Example 1.

Parameter $(a_0,b_1,b_2,c_1,c_2,c_3)$	R_1	R_2
$(1,1,1,0,1,0)$	0.7338532901	0.6587983484
$(1,1,1,0,0,1)$	0.7338532901	0.7338532901
$(0.5,0,1,0,0,1)$	0.7338532901	0.7338532901
$(2,9,3,4,4,0)$	0.9517204676	0.8685170918
$(2,-9,-3,-4,-4,0)$	0.9517204676	0.8685170918

The Equation (6.1) is discretized with the help of Gauss-Legendre quadrature rule given in [11] and then the discretized system is solved using (1.2) and (1.3) by considering the initial point $x_0=-2\cdot(1,1,1,1,1,1,1,1)$. The obtained results are shown in Tables 3 and 4.

Table 3. Solution of (6.1) at each iteration using (1.2) when $a_0 = \frac{1}{2}$, $b_1 = 0$ and $b_2 = 1$.

Grids	$z_1(s)$	$z_2(s)$	$z_3(s)$	$z_4(s)$
s_1	-0.451919660302	0.000536980682946	1.10×10^{-15}	0
s_2	-0.416419319350	0.000494798412503	1.01×10^{-15}	0
s_3	-0.363625938602	0.000432068179368	8.82×10^{-16}	0
s_4	-0.306456804616	0.000364138581902	7.43×10^{-16}	0
s_5	-0.255096568085	0.000303111241622	6.18×10^{-16}	0
s_6	-0.214990381116	0.000255456205648	5.21×10^{-16}	0
s_7	-0.187734035120	0.000223069627739	4.55×10^{-16}	0
s_8	-0.172986674383	0.000205546495783	4.19×10^{-16}	0

Table 4. Solution of (6.1) using (1.3).

Grids	$z_1(s)$	$z_2(s)$
s_1	-0.00326613099205	0
s_2	-0.00300956157496	0
s_3	-0.00262801124162	0
s_4	-0.00221483629770	0
s_5	-0.00184364363885	0
s_6	-0.00155378667590	0
s_7	-0.00135679857336	0
s_8	-0.00125021588579	0

From Tables 3 and 4, it is noticed that only 4 and 2 iterations are required, respectively for the methods (1.2) and (1.3) to convergence with the considered grid points as in [11].

Example 2. Consider $T : U(0,1) \subset C[0,1] \longrightarrow C[0,1]$ as

$$T(z)(s) \quad = \quad z(s) - 1 - \frac{1}{300}\int_0^1 \frac{s}{t+s} z^3(t)dt.$$

Note that $T'(z)u_1(s) = u_1(s) - \frac{1}{100}\int_0^1 \frac{s}{t+s} z^2(t)u_1(t)dt$. By taking $z_0(s) = 1$, we get

$$\left\|I - T'(z_0)\right\| \leq \frac{\log_e(2)}{100}, \quad T'(z_0)^{-1} \in \mathcal{B}(Y, X),$$
$$\left\|T'(z_0)^{-1}\right\| \leq \frac{100}{100 - \log_e(2)}.$$

By applying the conditions (I)–(III), we get

$$Q_1(s) = \frac{2\log_e(2)}{100 - \log_e(2)} s, \; \vartheta = \frac{600 + \log_e(2)}{3(100 - \log_e(2))}$$

and $Q_2(s) = \frac{2\log_e(2)}{100 - \log_e(2)} s$. The majorizing sequences of (1.3) and (1.2) are converged to 2.1938 and 2.1125, respectively for $a_0 = b_1 = b_2 = c_1 = 1$ and $c_2 = c_3 = 0$. In Table 5, the first few terms of majorizing sequences are given.

Table 5. Majorizing sequences for Example 2 when $a_0 = b_1 = b_2 = c_1 = 1$ and $c_2 = c_3 = 0$.

n	Method (1.2)		Method (1.3)	
	α_n	β_n	α_n	β_n
0	0	2.01628632	0	2.01628632
1	2.04506733	2.10474386	2.07506926	2.16655700
2	2.10827846	2.11205293	2.17794204	2.19010040
3	2.11228195	2.11252499	2.19167574	2.19335157
4	2.11253975	2.11255542	2.19356986	2.19380196
5	2.11255637	2.11255738	2.19383221	2.19386438

Example 3. Suppose

$$D^{\nu}_{*0}y(t) \quad = \quad f(t,y), \quad y(0) = 0, \tag{6.2}$$

where $f(t,y) = \frac{40320}{\Gamma(9-\nu)}t^{8-\nu} - 3\frac{\Gamma(5+\frac{\nu}{2})}{\Gamma(5-\frac{\nu}{2})}t^{4-\frac{\nu}{2}} + \frac{9}{4}\Gamma(\nu+1) + \left(\frac{3}{2}t^{\frac{\nu}{2}} - t^4\right)^3 - [y(t)]^{\frac{3}{2}}$, $\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}dt,\ x > 0$ and D^{ν}_{*0} is Caputo fractional differential operator of order $0 < \nu < 1$ (see [7]). Note that $y(t) = t^8 + \frac{9}{4}t^{\nu} - 3t^{4+\frac{\nu}{2}}$ is an exact solution of (6.2).

Using the trapezoidal-based correcter formula given in [7] for FDE (Fractional differential equation), we obtained the discretized system of (6.2) as

$$y_{k+1} \quad = \quad \frac{1}{\Gamma(\nu)}\left[\sum_{j=0}^{k} a_{j,k+1}f\left(t_j, y_j\right) + a_{k+1,k+1}f\left(t_{k+1}, y_{k+1}\right)\right], \tag{6.3}$$

where $h = \frac{1}{64}$, $t_k = kh,\ k = 0, 1, 2, \ldots, 64$ and $a_{j,k+1}$ is as defined in [7]. Since $y(t_0) = y_0$ is known, the Equation (6.3) is a 64×64 system. By solving the obtained system with (1.2) and (1.3) using $a_0 = b_1 = b_2 = c_3 = 1$, $c_1 = c_2 = 0$ and the initial point $2 \cdot (1, 1, \ldots, 1)$, we get the solution approximation of (6.2), shown in Figure 1. Note that the same solution can be obtained for any choices listed in Table 1.

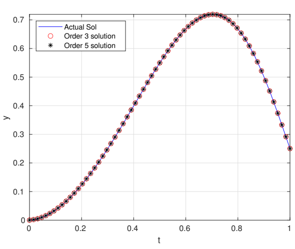

Figure 1. Solution of Example 3.

7 Basin of attraction

There are different types of definitions for the basin of attraction available in the literature to study the dynamics (See [5, 14]). We illustrated this concept with the following examples.

Example 4. Consider the polynomial in complex variable as

$$f(z) = (z-1)^2 z^2 + \frac{16}{25}(z-1)^2 + \frac{1}{4}z^2 + \frac{4}{25},$$

whose zeros are $-\frac{4}{5}i$, $\frac{4}{5}i$, $1-\frac{1}{2}i$ and $1+\frac{1}{2}i$.

Let $\mathcal{D} = \{z \mid -2 \leq Im(z), Re(z) \leq 2\}$ which has all roots of $f(z)$. By dividing the region $\mathcal{D}$ with equally spaced 601×601 grid points, we apply (1.2) and (1.3) with ($a_0 = b_1 = 1 = b_2 = c_2$ and $c_3 = c_1 = 0$) to get the BA as shown in Figure 2. BA Figure for other parameter choices in Table 2 looks almost same as Figure 2. Note that the colors yellow, green, red and blue in Figure 2

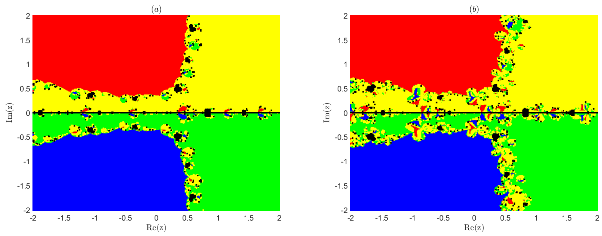

Figure 2. BA using (1.2) and (1.3) with any parameter choices in Table 2.

represent the points that converges to $1+\frac{1}{2}i$, $1-\frac{1}{2}i$, $\frac{4}{5}i$ and $-\frac{4}{5}i$, respectively with 50 iterations and at least 10^{-8} accuracy. Few black marks in the figures means that only a few points of those grid points are not converged.

8 Conclusions

Our convergence analysis of the methods (1.2) and (1.3) unifies the convergence analysis of some existing third and fifth order methods. Moreover, one can obtain the new iterative methods by choosing the parameters appropriately. Various numerical examples are given to validate our theoretical results. It is envisaged to use similar kind of analysis in other class of iterative methods to obtain the convergence order.

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