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Dynamic Complexity of a Predator-Prey Model for IPM with Nonlinear Impulsive Control Incorporating a Regulatory Factor for Predator Releases

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Abstract. The success of integrated pest management depends on spraying the correct amount of pesticides at an appropriate time and releases of natural enemies or pathogens of the pest in appropriate proportions at critical times, with little cost and minimal effects on the environment. Therefore, control decisions require information on instantaneous killing rates of pesticides and numbers of natural enemies to be released, variables that should depend on the densities of both pest and natural enemy population densities in the field. To describe such a control strategy we have proposed a mathematical model of IPM involving releases of natural enemies in relation to a regulatory factor. The threshold condition for the existence and stability of the pest free periodic solution is provided using a cobweb model, the comparison principle and Floquet theory, which reveals the effects of nonlinear control action on pest outbreaks. Bifurcation analyses show that the dynamics of the proposed model can be very complex, including multiple attractors and switch-like transition patterns following small random perturbations. Moreover, the random perturbations and nonlinear impulsive control measures could generate complex switching patterns, which show that the pest population could have outbreaks in complex ways due to environmental noise.

Keywords: IPM, nonlinear control action, pest free periodic solution, switch-like transition, random perturbation.

AMS Subject Classification: 92D05; 92D25; 92D40.

1 Introduction

It is well known that spraying pesticides can kill beneficial organisms as well as the target pests, and can result in outbreaks of secondary pests or rapid resurgence of pests that were initially suppressed [25]. Therefore, it is very important to apply non-chemical control methods or pesticides which kill only the target pest, while protecting its natural enemies. The releasing of natural enemies such as predators, parasites and pathogens to control pests is a type of biological control known as augmentation. This approach uses commercially available species that are applied in a timely manner to prevent population increases, or to suppress a pest population [3,11,13].

Pest management strategies include integrated pest management (IPM), which is defined as the careful consideration of all available pest control techniques and subsequent integration of appropriate measures that discourage the development of pest populations and keep pesticides and other interventions to levels that are economically justified and reduce or minimize risks to human health and the environment. In many cases, the timing and method of biological control applications are more significant influences on the efficacy of biological control than the release rate of the control agent. Additional factors that may limit the relative impact of release rates include natural enemy fecundity, establishment rates, prey availability, dispersal, and cannibalism [2,4,10,17].

Mathematical models can be used to evaluate the effectiveness of multiple biological control factors including the timing of releases, release ratios and density dependent regulatory factors affecting the natural enemies to be augmented [7,14,15,20,21]. Recently, many mathematical models concerning IPM have been proposed and analysed, which have mainly focused on modeling pesticide sprays and natural enemy releases [6,22,23,26,27,29]. Usually, the main assumption is that a proportion of the pest population will be killed instantly after spraying pesticide once, while simultaneously releasing a constant number of natural enemies [18,22,23,24].

However, the release methods and ratios of numbers of natural enemies to be released to their current density in the field (henceforth termed releasing ratios) could significantly affect the outcome of pest control actions. In practice, the densities of the pest and natural enemy populations should be carefully monitored before IPM measures are applied, with the lower the number of natural enemies in the field, the higher the number of them to be released and vice versa. Therefore, taking more factors into account, to address the effects of the density regulatory factor for the natural enemies (predators and not parasitoids or pathogens in this case) on the releasing ratios and pest control, we propose the following model

$$\begin{aligned} \frac{dx(t)}{dt} &= ax(t)\left(1-\frac{x(t)}{K}\right) - bx(t)y(t), \quad t \neq nT, \\ \frac{dy(t)}{dt} &= y(t)[cx(t)-d], \quad t \neq nT, \end{aligned} \tag{1.1}$$

$$\begin{aligned} x(nT^+) &= q_1 x(nT), \quad t = nT, \\ y(nT^+) &= q_2 y(nT) + \frac{\tau\theta}{y(nT)+\theta}, \quad t = nT, \end{aligned}$$

where $x(t)$, $y(t)$ are the densities of the prey and predator populations at time t, respectively, a represents the intrinsic growth rate of the prey, K denotes the carrying capacity of the prey and b is the predation rate of the predator on the prey. The prey's contribution to the predator's growth rate is cxy, and $d > 0$ is the death rate of the predator.

In the model (1.1), the control actions including timing (control at regular intervals), efficacy of the pesticides (survival rate $0 < q_1 \leq 1$ after spraying the pesticides) and releasing methods have been considered. In particular, sometimes the predator may also be killed by the pesticide ($0 < q_2 < 1$), or the pesticides only affect the pest and an impulsive increase of the predator population density is induced by releases of laboratory-bred predators ($q_2 \geq 1, \tau > 0$). Moreover, the nonlinear releasing factor $\frac{\tau\theta}{y(nT)+\theta}$, which is varied by means of the regulatory factor $\theta > 0$ and it is a decreasing function of the density $y(nT)$, is proposed and formulated, i.e. the amount of predators released at $t = nT$ is related to its density $y(nT)$, here n is nonnegative integer with $n = 0, 1, 2, 3, \ldots$. In particular, if the density of the natural enemy at $t = nT$ is large enough to kill the pests, the amount of natural enemy released at time $t = nT$ can be reduced, and τ is the maximal release amount of the predator. As mentioned before, only a constant releasing number (constant τ here) was considered in previous studies, irrespective of how many natural enemies and pests remained in the field [6, 18, 22, 23, 24].

The main purpose of this study is to investigate the global dynamical behavior of system (1.1) and address how the nonlinear impulsive control actions affect the global dynamics and the resulting pest control. Firstly, we analyze the global stability of the so-called pest free periodic solution, and the threshold condition which guarantees the existence and global stability of this pest free periodic solution has been obtained in section 2 and some important biological implications are discussed. Further, we reveal the complexity of the dynamical behavior of system (1.1) by using extensive numerical investigations, bifurcation diagrams, multiple attractors, coexistence and switch-like transitions to reflect the influences on the inherent oscillations of the nonlinear impulsive perturbations. Finally, related biological implications are discussed.

2 Existence and stability of the pest free periodic solution

The common objective of pest control efforts is to eradicate a pest from a designated area, which can be revealed by the stability of the pest free periodic solution of model (1.1). Thus, the existence and stability of the pest free periodic solution of model (1.1) plays a key role in analysing the dynamical behavior and revealing the biological implications. To address this in more detail, we first give some basic properties of the following subsystem:

$$y'(t) = -dy(t), \quad t \neq nT, \quad d > 0, \tag{2.1}$$

$$y(nT^+) = q_2 y(nT) + \tau\theta/\big(y(nT)+\theta\big), \quad t = nT.$$

Solving the first equation in the interval $(nT,(n+1)T]$ yields the following analytical solution:

$$y(t) = y(nT^+)e^{-d(t-nT)}, \;\; t \in (nT,(n+1)T],$$

and an impulsive control strategy related to the natural enemy has been applied at time $t=(n+1)T$, i.e. we have

$$\begin{aligned} y((n+1)T^+) &= q_2 y((n+1)T) + \tau\theta/\big(y((n+1)T)+\theta\big) \\ &= q_2 y(nT^+)e^{-dT} + \tau\theta/\big(y(nT^+)e^{-dT}+\theta\big). \end{aligned}$$

Further, denote $Y_n = y(nT^+)$, and we get the following nonlinear difference equation:

$$Y_{n+1} = q_2 e^{-dT} Y_n + \frac{\tau\theta}{Y_n e^{-dT}+\theta} = f(Y_n). \tag{2.2}$$

Note that the existence and stability of the fixed point of model (2.2) indicate that for subsystem (2.1) there exists a stable periodic solution. In fact, if $1-q_2e^{-dT}>0$, i.e. $q_2e^{-dT}<1$, there exists a unique positive equilibrium Y^* of difference equation (2.2)

$$Y^* = f(Y^*) = 0.5\left(-\theta + \sqrt{\theta^2 + \frac{4\tau\theta e^{-dT}}{1-q_2e^{-dT}}}\right)e^{dT}.$$

Now we show the local stability of Y^*, which can be determined by the inequality $|f'(Y^*)|<1$. In fact, according to the inequality $q_2e^{-dT}<1$ we have $0<1-q_2e^{-dT}<1$, which indicates that the following inequality

$$\sqrt{\theta^2 + 4\tau\theta e^{-dT}/\big(1-q_2e^{-dT}\big)} > \sqrt{\theta^2+4\tau\theta e^{-dT}}$$

holds true, and further

$$\begin{aligned} \frac{4\tau\theta}{\left(\theta+\sqrt{\theta^2+\frac{4\tau\theta e^{-dT}}{1-q_2e^{-dT}}}\right)^2} &< \frac{4\tau\theta}{\left(\theta+\sqrt{\theta^2+4\tau\theta e^{-dT}}\right)^2} \\ &= \frac{4\tau\theta}{2\theta^2+2\theta\sqrt{\theta^2+4\tau\theta e^{-dT}}+4\tau\theta e^{-dT}} < e^{dT}. \end{aligned}$$

Hence

$$q_2 - \frac{4\tau\theta}{\left(\theta+\sqrt{\theta^2+\frac{4\tau\theta e^{-dT}}{1-q_2e^{-dT}}}\right)^2} > q_2 - e^{dT},$$

and all these arguments confirm that

$$f'(Y^*) > e^{-dT}(q_2-e^{dT}) = q_2e^{-dT}-1 > -1.$$

Further, it is easy to see that $q_2e^{-dT}<1$ indicates that

$$f'(Y^*) = q_2e^{-dT} - \frac{4\tau\theta e^{-dT}}{\left(\theta+\sqrt{\theta^2+\frac{4\tau\theta e^{-dT}}{1-q_2e^{-dT}}}\right)^2} < 1.$$

Thus, if $q_2 e^{-dT} < 1$ then $|f'(Y^*)| < 1$, and consequently the unique positive equilibrium Y^* of the difference equation (2.2) is locally asymptotically stable if it exists. Moreover, the stability of Y^* means that model (2.1) has a local stable periodic solution $y^T(t)$, and

$$y^T(t) = Y^* e^{-d(t-nT)}, \quad t \in (nT, (n+1)T]. \tag{2.3}$$

In the following we prove the global attractivity of Y^*. Note that there is a discontinuity point $Y_k = -\theta e^{dT} < 0$ for the function $f(Y) = q_2 e^{-dT} Y + \frac{\tau\theta}{Ye^{-dT}+\theta}$. Solving $f'(Y) = 0$ with respect to Y, we have two stationary points

$$Y_1 = \left(-\sqrt{\tau\theta/q_2} - \theta\right) e^{dT} < 0, \quad Y_2 = \left(\sqrt{\tau\theta/q_2} - \theta\right) e^{dT},$$

where Y_1 is the local maximum and Y_2 is the local minimum of the function $f(Y)$ with $Y_2 > Y_1$ and $f(Y_1) < 0$. Thus, both the points $(Y^*, f(Y^*))$ and $(Y_2, f(Y_2))$ must be on the same curve. Moreover, based on the sign of Y_2 and the positional relations between point Y_2 and Y^*, we consider the following three possible cases, as described in Figure. 1.

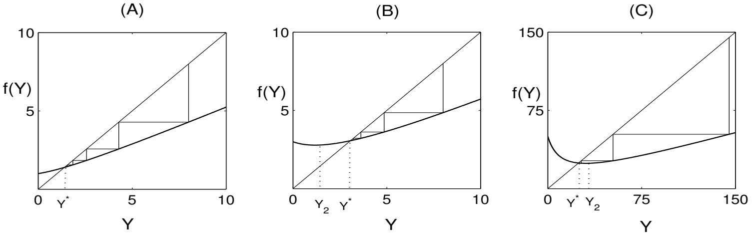

Figure 1. Illustrations of the global attractivity of Y^*. (A) $Y_2 < 0 < Y^*$; (B) $0 < Y_2 < Y^*$; (C) $0 < Y^* < Y_2$.

Case (A) $Y_2 < 0 < Y^*$. For this case the function $f(Y)$ is a monotonically increasing function for $Y > Y_2$, as shown in Figure. 1(A). For any $Y \in [0, Y^*)$, according to $Y < f(Y) < Y^*$ we know that $f^k(Y)$ is monotonically increasing as k increases, and $f^k(Y) \to Y^*$ $(k \to \infty)$. For any $Y > Y^*$, we have $f^k(Y) > Y^*$ for all k, and then according to $f(Y) < Y$ we conclude that $f^k(Y)$ is monotonically decreasing as k increases, which means that $f^k(Y) \to Y^*$ $(k \to \infty)$.

Case (B) $0 < Y_2 < Y^*$, as shown in Figure 1(B). For this case, if $Y \in [Y_2, Y^*)$ or $Y > Y^*$, then the conclusions corresponding to the Case (A) are true, i.e. we have $f^k(Y) \to Y^*$ $(k \to \infty)$. Moreover, for any $Y \in [0, Y_2)$, there must be $f(Y) \in [Y_2, Y^*)$ or $f(Y) > Y^*$, and according to the conclusion of the above we obtain $f^{1+k}(Y) \to Y^*$ $(k \to \infty)$.

Case (C) $Y_2 > Y^* > 0$. For this case, it is easy to know that the function $f(Y)$ is a monotonically decreasing function for $Y \in [0, Y_2]$, and is an increasing function for $Y \in (Y_2, \infty)$, as shown in Figure 1(C).

Next, we show that there exists a positive integer l such that $f^l(Y) \in [Y^*, Y_2]$ for $Y \in [0, Y^*)$ or $Y \in (Y_2, \infty)$.

(1) If $f(0) < Y_2$, there must exist a $Y^1, Y^2 \in (Y_2, \infty)$ such that $f(Y^1) = Y^*$, $f(Y^2) = Y_2$. For any $Y \in [0, Y^*) \cup [Y^1, Y^2]$, there must be $f(Y) \in [Y^*, Y_2]$; For any $Y \in [Y_2, Y^1)$, there must be $f(Y) \in (0, Y^*)$, it means that $f^2(Y) \in (Y^*, Y_2)$; For any $Y \in (Y^2, \infty)$, there exists a positive integer m_1 such that $f^{m_1}(Y) \in (Y_2, Y^2)$, based on the previous conclusions, we know $f^{m_1+m_2}(Y) \in (Y^*, Y_2)$, m_2 is 1 or 2.

(2) If $f(0) > Y_2$, there must exist a $Y^1, Y^2 \in (Y_2, \infty)$, $Y^3 \in (0, Y^*)$ such that $f(Y^1) = Y^*$, $f(Y^2) = f(Y^3) = Y_2$. For any $Y \in [Y^1, Y^2] \cup [Y^3, Y^*)$, there must be $f(Y) \in (Y^*, Y_2)$; For any $Y \in [Y_2, Y^1)$, there must be $f(Y) \in (Y^3, Y^*)$, it means that $f^2(Y) \in (Y^*, Y_2)$; For any $Y \in (Y^2, \infty)$, there exists a positive integer m_3 such that $f^{m_3}(Y) \in (Y_2, Y^2)$, based on the previous conclusions, we know $f^{m_3+m_4}(Y) \in (Y^*, Y_2)$, m_4 is 1 or 2; For any $Y \in [0, Y^3)$, there must be $f(Y) > Y_2$, it means that $f^{m_5}(Y) \in (Y_2, Y^2)$, m_5 is a positive integer, and we have $f^{m_5+m_4}(Y) \in (Y^*, Y_2)$, m_4 is 1 or 2.

In conclusion, there exists a positive integer l such that $f^l(Y) \in [Y^*, Y_2]$ for $Y \in [0, Y^*)$ or $Y \in (Y_2, \infty)$ is true. Thus if $f^k(Y) \to Y^*$ $(k \to \infty)$ for any $Y \in (Y^*, Y_2]$, then $f^n(Y) \to Y^*$ $(n \to \infty)$ for any $Y \in [0, \infty)$ holds.

Hence in the following we will focus on $f^k(Y) \to Y^*$ $(k \to \infty)$ for any $Y \in (Y^*, Y_2]$. According to the function defined by (2.2), we can easily obtain the following functions

$$f(Y) = q_2 e^{-dT} Y + \frac{\tau\theta}{Ye^{-dT} + \theta}, \quad f(f(Y)) = q_2 e^{-dT} f(Y) + \frac{\tau\theta}{f(Y)e^{-dT} + \theta},$$

$$f'(Y) = q_2 e^{-dT} - \frac{\tau\theta e^{-dT}}{(Ye^{-dT} + \theta)^2}, \quad f'(f(Y)) = q_2 e^{-dT} - \frac{\tau\theta e^{-dT}}{(f(Y)e^{-dT} + \theta)^2}.$$

Now we first show that $f'(Y) > -\frac{f(Y)}{Y}$ for any $Y > 0$, i.e.

$$f'(Y)Y > -f(Y), \qquad (Y > 0). \tag{2.4}$$

Since

$$f'(Y)Y = [q_2 e^{-dT} - \frac{\tau\theta e^{-dT}}{(Ye^{-dT} + \theta)^2}]Y = f(Y) - \frac{2\tau\theta Y e^{-dT} + \tau\theta^2}{(Ye^{-dT} + \theta)^2},$$

the inequality (2.4) is equivalent to

$$2f(Y) > \frac{2\tau\theta Y e^{-dT} + \tau\theta^2}{(Ye^{-dT} + \theta)^2} \;\Rightarrow\; 2q_2 e^{-dT} Y > \frac{-\tau\theta^2}{(Ye^{-dT} + \theta)^2}. \tag{2.5}$$

Since $2q_2 e^{-dT} Y > 0$, $\frac{-\tau\theta^2}{(Ye^{-dT}+\theta)^2} < 0$, it is easy to see that the inequality (2.5) is true, which means the inequality (2.4) holds. Note that $Y > 0$ implies $f(Y) > 0$, replacing Y with $f(Y)$ in the inequality (2.4) we obtain

$$f'(f(Y))f(Y) > -f(f(Y)), \qquad (Y > 0). \tag{2.6}$$

Furthermore, $f(Y)$ is decreasing on $(Y^*, Y_2]$, i.e. $f'(Y) < 0$ for any $Y \in (Y^*, Y_2]$. It follows from the inequalities (2.4) and (2.6) that we have

$$f'(f(Y))f'(Y)Y < -\frac{f'(Y)Y}{f(Y)}f(f(Y)) < f(f(Y)), \quad Y \in (Y^*, Y_2],$$

i.e.,

$$f'(f(Y))f'(Y)Y < f(f(Y)), \quad Y \in (Y^*, Y_2]. \tag{2.7}$$

We denote $g(Y) = \frac{f(f(Y))}{Y}$, and then we have

$$g'(Y) = \big(f'(f(Y))f'(Y)Y - f(f(Y))\big)/Y^2.$$

From (2.7) we know $g'(Y) < 0$ and $g(Y)$ is a monotonically decreasing function on $(Y^*, Y_2]$. Moreover, the function $g(Y)$ is continuous on $(0, \infty)$, and

$$g(Y^*) = \frac{f(f(Y^*))}{Y^*} = \frac{f(Y^*)}{Y^*} = \frac{Y^*}{Y^*} = 1, \;\Rightarrow\; g(Y) < 1, \;\; Y \in (Y^*, Y_2],$$

i.e.

$$f^2(Y) = f(f(Y)) < Y, \qquad Y \in (Y^*, Y_2]. \tag{2.8}$$

Since the function f is a monotonically decreasing function on $(Y^*, Y_2]$, combined with the equality $f(Y^*) = Y^*$ and the continuity of f, it follows from the inequality (2.8) that we have $f^3(Y) > f(Y)$, $f^4(Y) < f^2(Y), \ldots$, by induction, we conclude that

$$\begin{aligned} &Y^* < f^{2j}(Y) < f^{2(j-1)}(Y) < \cdots < f^2(Y) < Y, \\ &f(Y) < f^3(Y) < f^5(Y) < \cdots < f^{2j-1}(Y) < f^{2j+1}(Y) < Y^*. \end{aligned}$$

It is easy to see that $f^{2j}(Y)$ is monotonically decreasing and $f^{2j+1}(Y)$ is monotonically increasing as j increases, it means that $\lim\limits_{k\to+\infty} f^k(Y) = Y^*$, $Y \in (Y^*, Y_2]$, which indicates $\lim\limits_{n\to+\infty} f^n(Y) = Y^*$, $Y \in [0, \infty)$.

Based on the above discussions, we know that the unique equilibrium Y^* is globally stable if it exists. Thus, for the existence and stability of the periodic solution of model (2.1) we have the following Lemma.

Lemma 1. *System* (2.1) *has a positive periodic solution* $y^T(t)$ *and for every solution* $y(t)$ *of model* (2.1) *we have* $\left|y(t) - y^T(t)\right| \to 0$ *as* $t \to \infty$, *where* $y^T(t)$ *is defined by* (2.3).

Therefore, we obtain the complete expression for the 'pest-free' periodic solution of system (1.1) over the n-th time interval $nT < t \le (n+1)T$,

$$(0, y^T(t)) = \left(0, \frac{-\theta + \sqrt{\theta^2 + \frac{4\tau\theta e^{-dT}}{1-q_2 e^{-dT}}}}{2e^{-dT}} e^{-d(t-nT)}\right), \;\; t \in (nt, (n+1)t]$$

and we have the following main theorem of this section for system (1.1).

Theorem 1. *The pest free periodic solution* $(0, y^T(t))$ *of model* (1.1) *is globally asymptotically stable provided that*

$$a < \frac{1}{T}\left[\ln\frac{1}{q_1} + \frac{bY^*}{d}\left(1 - e^{-dT}\right)\right].$$

Proof. The local stability of the periodic solution $(0, y^T(t))$ could be determined by considering the behavior of small amplitude perturbations of the solution. Define $x(t) = u(t)$ and $y(t) = y^T(t) + v(t)$, then we have

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \Phi(t)\begin{bmatrix} u(0) \\ v(0) \end{bmatrix}, \ t \in [0, T),$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{bmatrix} a - by^T(t) & 0 \\ cy^T(t) & -d \end{bmatrix}\Phi(t)$$

and $\Phi(0) = E$, the identity matrix, and we have

$$\Phi(T) = \begin{bmatrix} \exp\left[\int_0^T \left(a - by^T(t)\right) dt\right] & 0 \\ * & e^{-dT} \end{bmatrix}.$$

Note that the $*$ shown in the above is not needed for determining the local stability. After re-setting the third and fourth equations of (1.1) become

$$\begin{bmatrix} u(nT^+) \\ v(nT^+) \end{bmatrix} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 - \frac{\tau\theta}{(y^T(nT)+\theta)^2} \end{bmatrix}\begin{bmatrix} u(nT) \\ v(nT) \end{bmatrix} \triangleq B(nT)\begin{bmatrix} u(nT) \\ v(nT) \end{bmatrix}.$$

Hence, if both eigenvalues of matrix

$$M = B(T)\Phi(T) = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 - \frac{\tau\theta}{(y^T(T)+\theta)^2} \end{bmatrix}\begin{bmatrix} \exp\left[\int_0^T \left(a - by^T(t)\right) dt\right] & 0 \\ * & e^{-dT} \end{bmatrix}$$

have absolute values less than one, then the periodic solution $(0, y^T(t))$ is locally stable. In fact, the two Floquet multipliers are as follows:

$$\mu_1 = q_1 \exp\left[\int_0^T \left(a - by^T(t)\right) dt\right], \ \mu_2 = \left(q_2 - \frac{\tau\theta}{(y^T(T)+\theta)^2}\right)e^{-dT}.$$

It is easy to verify that $-1 < \mu_2 < 1$ due to the existence of Y^*, according to the Floquet theory [1], if $|\mu_1| < 1$, then the pest free periodic solution is locally stable. Obviously, $\mu_1 > 0$ holds, and $\mu_1 < 1 \Leftrightarrow \exp\left[\int_0^T \left(a - by^T(t)\right) dt\right] < 1/q_1$, i.e. $\mu_1 < 1$, which is equivalent to

$$\int_0^T \left(a - bY^* e^{-dt})\right) dt < \ln\frac{1}{q_1} \ \Rightarrow \mu_1 < 1 \Leftrightarrow a < \frac{1}{T}\left[\ln\frac{1}{q_1} + \frac{bY^*}{d}(1 - e^{-dT})\right].$$

Therefore, the periodic solution $(0, y^T(t))$ is locally stable provided that $a < \frac{1}{T}[\ln(1/q_1) + \frac{bY^*}{d}(1 - e^{-dT})]$.

Next, we will show the global attractivity of the pest free periodic solution $(0, y^T(t))$ of model (1.1). Since $\mu_1 < 1$, we can choose $\varepsilon_1 > 0$ sufficiently small such that

$$\delta \triangleq q_1 \exp\left[\int_0^T \big(a - b(y^T(t) - \varepsilon_1)\big)\, dt\right] < 1. \tag{2.9}$$

It follows from model (1.1) that

$$\begin{cases} y'(t) & \geq -dy(t),\ t \neq nT,\ d > 0, \\ y(nT^+) & = q_2 y(nT) + \dfrac{\tau\theta}{y(nT) + \theta},\ t = nT. \end{cases}$$

According to the theory of differential inequalities, we consider the comparison equation

$$\begin{cases} z'(t) & = -dz(t),\ t \neq nT,\ d > 0, \\ z(nT^+) & = q_2 z(nT) + \dfrac{\tau\theta}{z(nT) + \theta},\ t = nT. \end{cases}$$

It follows from Lemma 1 and comparison theorem [12] and that we have $y(t) \geq z(t)$ and $z(t) \to y^T(t)$ as $t \to \infty$. Hence, there exists a $t_1 > 0$ such that

$$y(t) \geq z(t) > y^T(t) - \varepsilon_1 \tag{2.10}$$

for all $t > t_1$.

From the first equation of system (1.1) and (2.10) we get

$$\frac{dx(t)}{dt} = ax(t)\left(1 - \frac{x(t)}{K}\right) - bx(t)y(t) \leq x(t)[a - b(y^T(t) - \varepsilon_1)]$$

for $t > t_1$. Thus, we consider the following comparison equation with pulses

$$\begin{cases} \dfrac{dz_1(t)}{dt} & = z_1(t)[a - b(y^T(t) - \varepsilon_1)],\ t \neq nT, \\ z_1(nT^+) & = q_1 z_1(nT),\ t = nT. \end{cases} \tag{2.11}$$

Integrating model (2.11) between pulses $(nT, (n+1)T]$, yields

$$\begin{aligned} z_1((n+1)T) &= q_1 z_1(nT) \exp\left[\textstyle\int_{nT}^{(n+1)T} \big(a - b(y^T(t) - \varepsilon_1)\big)\, dt\right] \\ &= q_1 z_1(nT) \exp\left[\textstyle\int_0^T \big(a - b(y^T(t) - \varepsilon_1)\big)\, dt\right]. \end{aligned}$$

Then by using step by step iterations

$$\begin{aligned} z_1(nT) = q_1 z_1((n-1)T)) \exp\left[\int_0^T \big(a - b(y^T(t) - \varepsilon_1)\big)\, dt\right] \\ = q_1^2 z_1((n-2)T) \exp\left[2\left(\int_0^T \big(a - b(y^T(t) - \varepsilon_1)\big)\, dt\right)\right] \\ = \cdots = q_1^n z_1(0) \exp\left[n\left(\int_0^T \big(a - b(y^T(t) - \varepsilon_1)\big)\, dt\right)\right], \end{aligned}$$

where $z_1(0^+) = q_1 z_1(0) > 0$, and $\lim\limits_{n\to\infty} z_1(nT) = \lim\limits_{n\to\infty} z_1(0)\delta^n = 0$ due to (2.9).

Let $(x(t), y(t))$ be any solution of model (1.1) with initial value (x_0, y_0), and $x(0^+) = q_1 x(0) > 0, y(0^+) = q_2 y(0) + \frac{\tau\theta}{y(0)+\theta} > 0$. According to the comparison theorem we know that $x(nT) \le x(0)\delta^n$, and by incorporating this into the positivity of $x(t)$ we have $0 < x(t) \le x(nT) q_1 \exp(aT) \le x(0)\delta^n q_1 \exp(aT)$ as $nT < t \le (n+1)T$, i.e. we obtain $\lim\limits_{t\to\infty} x(t) = 0$.

Correspondingly, there exists a $t_2 > t_1 > 0$ such that $0 < x(t) \le \varepsilon_2$ for $t \ge t_2$, where $\varepsilon_2 > 0$ is small enough. Further, we have

$$-dy(t) \le y'(t) \le y(t)(c\varepsilon_2 - d)$$

for all $t > t_2$, from which we can obtain the following equation

$$\begin{cases} \dfrac{dz_2(t)}{dt} & = z_2(t)(c\varepsilon_2 - d), \ t \ne nT, \\ z_2(nT^+) & = q_2 z_2(nT) + \dfrac{\tau\theta}{z_2(nT)+\theta}, \ t = nT. \end{cases} \tag{2.12}$$

By employing the same methods as used for the proof of Lemma 1 we get that model (2.12) has a positive periodic solution $z_2^T(t)$, which is globally attractive, where

$$z_2^T(t) = z_2^* \exp[-(d - c\varepsilon_2)(t - nT)], \ t \in [nT, (n+1)T]$$

with

$$z_2^* = \frac{-\theta + \sqrt{\theta^2 + \frac{4\tau\theta}{e^{(d-c\varepsilon_2)T} - q_2}}}{2} e^{(d-c\varepsilon_2)T}.$$

It follows from the comparison theorem on impulsive differential equations that

$$z(t) \le y(t) \le z_2(t).$$

Moreover, $z_2(t) \to z_2^T(t)$ and $z(t) \to y^T(t)$ as $t \to \infty$. Consequently, there exists a t_3 for ε_3 small enough such that $t_3 \ge t_2 > 0$ and

$$y^T(t) - \varepsilon_3 < y(t) < z_2^T(t) + \varepsilon_3$$

for $t > t_3$. Let $\varepsilon_2 \to 0$, then

$$y^T(t) - \varepsilon_3 < y(t) < y^T(t) + \varepsilon_3.$$

Therefore, $y(t) \to y^T(t)$ as $t \to \infty$, which indicates that the pest free periodic solution $(0, y^T(t))$ of model (1.1) is globally asymptotically stable. This completes the proof. □

3 Numerical investigations and biological implications

All numerical simulations were run in Matlab R2012a, based on the solver function ODE45, and consequently the solutions and bifurcation diagrams of corresponding impulsive differential equations were obtained on each impulsive interval $(nT, (n+1)T]$.

3.1 Threshold condition for the pest free periodic solution

The global stability of the pest free periodic solution clarifies that the pest population can be completely eradicated if the integrated control strategies are properly designed. Note that the threshold condition

$$a < \frac{1}{T}\left[\ln\frac{1}{q_1} + \frac{bY^*}{d}(1 - e^{-dT})\right]$$

indicates that if the intrinsic growth rate of the pest is smaller than a threshold depending on the controlling efforts, then the pest will go extinct. However, we can see that it is difficult to solve the inequality with respect to control period T. To get around this, the inequality can be converted into:

$$aT + \frac{b}{2d}\left(-\theta + \sqrt{\theta^2 + \frac{4\tau\theta}{e^{dT} - q_2}}\right)(1 - e^{dT}) < \ln\frac{1}{q_1}. \tag{3.1}$$

It is easy to see that all key parameters including the period T, survival rate $q_i, (i = 1, 2)$, maximum releasing rate τ and nonlinear effect parameter θ are involved in the above threshold condition. This allows us to address the effects of the control tactics on the pest eradication and outbreaks.

In the following we address how the threshold condition shown in (3.1) changes as the parameter T increases, by taking the period T as a bifurcation parameter. Obviously, solving for T from the inequality analytically is impossible. Thus, we let $F(T) = aT + \frac{b}{2d}\left(-\theta + \sqrt{\theta^2 + \frac{4\tau\theta}{e^{dT}-q_2}}\right)(1 - e^{dT})$ and $G(T) = \ln\frac{1}{q_1}$, and the inequality is equivalent to $F(T) < G(T)$, as shown in Figure 2.

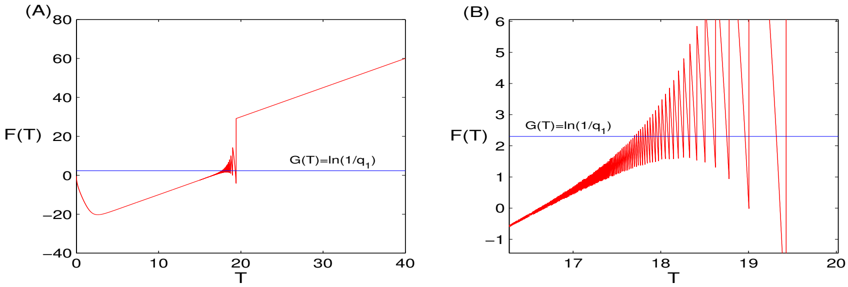

Figure 2. (A) Illustrations of the threshold condition for the stability of the pest free periodic solution, and (B) is an enlarged part of (A) which reveals the complex patterns related to the stability of $(0, y^T(t))$. The parameter values are fixed as follows: $a = 1.5, b = 2, d = 2, \theta = 4, \tau = 25, q_1 = 0.1, q_2 = 1$.

It is interesting to note that the function $F(T)$ could oscillate for a wide range of parameters, for example $T \in (17.70, 19.43)$, which reveals that the stability of the pest free periodic solution $(0, y^T(t))$ could suddenly switch, a change which could be generated by the nonlinear control actions. That is to say, the stable pest free periodic solution could lose its stability frequently as the parameter T traverses some threshold values, as shown in Figure 2, and then becomes unstable eventually as the impulsive period T becomes large enough.

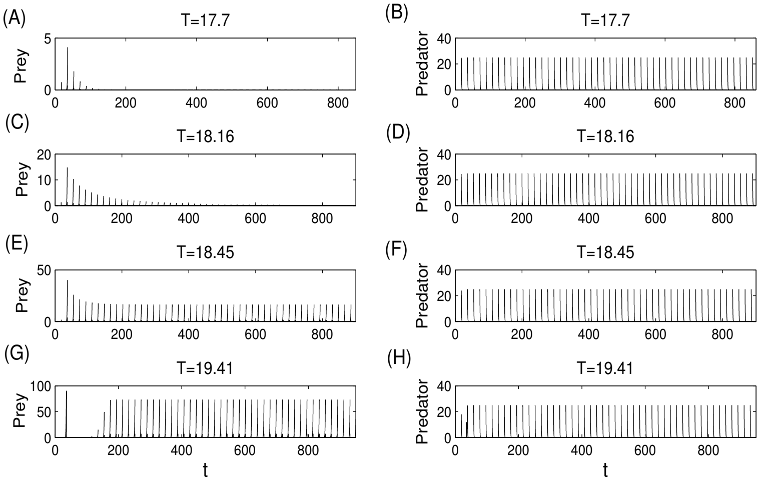

Figure 3. Typical solutions for the prey and predator populations with slight changes to the impulsive period $T = 18.2$. The parameters are fixed as follows: $a = 1.5, b = 2, c = 0.3, d = 2, \theta = 4, \tau = 25, q_1 = 0.1, q_2 = 1, K = 100$.

Furthermore, some typical solutions of model (1.1) are shown in Figure 3, where we choose different values for the impulsive period T around the roots of the equation $F(T) = G(T)$. Note that, as the parameter T increases, the pest population can be eradicated quickly for a relatively small T (i.e. $T = 17.7$ here), as shown in Figure 3(A), and goes to extinction more slowly for $T = 18.16$. After that, the stability of the pest free solution switches on and off as T increases, and we emphasize here that the pest population goes to extinction very slowly if the threshold condition is satisfied once T lies in the oscillation region and close to the threshold value (here T is around 18.2), while the dynamics could be much more complex if the period T is chosen such that the pest free periodic solution becomes unstable. All these results confirm that it is quite difficult to eradicate the pest and that the complex pattern for the pest population is generated if the period of application of the IPM strategy is not chosen properly.

3.2 Bifurcation analysis for the complex dynamics

Based on the discussion from the above subsection, we can see that the dynamics of model (1.1) could be very complex once the threshold condition (3.1) is no longer satisfied. To address this, we carried out one dimensional bifurcation analysis, which is a traditional approach to gain preliminary insight into the properties of a dynamic system and reveals the dynamics given a certain range of parameter variation [6, 18, 27, 28, 30].

It follows from Figure 4 that if the impulsive period exceeds some threshold levels, then both prey and predator populations can oscillate periodically with quite different amplitudes as T varies. Firstly, the pest-free periodic solution becomes unstable and the prey population begins to oscillate with large amplitudes that correspond to its periodic outbreaks. If the pulse period is further increased, a sequence of period adding bifurcations interchanging with regions of chaos is observed.

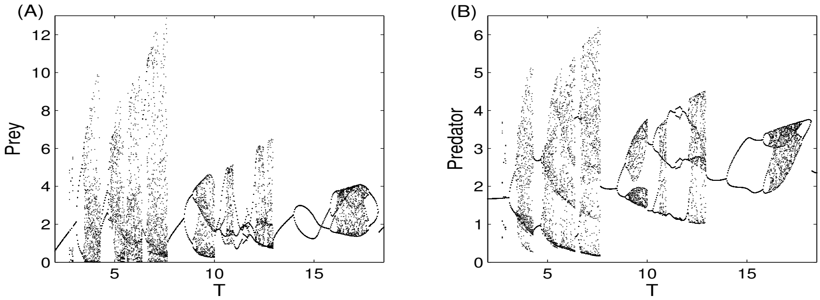

Figure 4. Bifurcation diagrams of model (1.1) with respect to bifurcation parameter T. For each value of the parameter T, system (1.1) is integrated over 500 pulsing cycles, the last 30 stroboscopic measurements of prey or predator population are plotted. The parameter values are fixed as follows:
$a = 3, b = 1, c = 0.3, d = 0.6, K = 10, \tau = 0.5, q_1 = 0.8, q_2 = 2.5, \theta = 4.$

The results shown in Figure 5 reveal how the maximal release amount τ affects the dynamics of system (1.1). It is observed that model (1.1) presents sharp transitions from a periodic solution with period T to a periodic solution with period $2T$ at $\tau \approx 0.5$, i.e. a period-doubling bifurcation occurs; As τ further increases, the period-doubling bifurcations lead system (1.1) to chaotic dynamics. After that, period-halving bifurcations result in various periodic solutions with different periods as τ increases. All these results confirm that varying τ could dramatically change the dynamics of model (1.1).

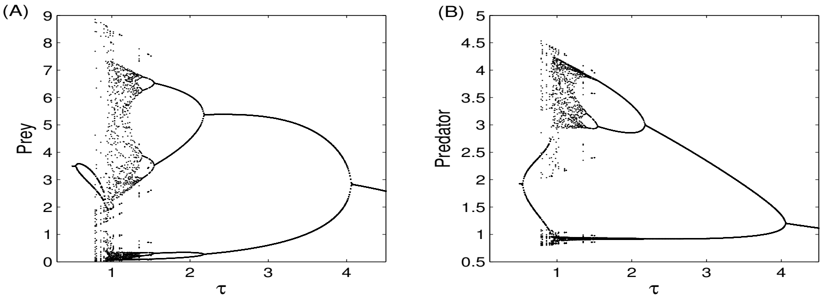

Figure 5. Bifurcation diagrams of model (1.1) with respect to bifurcation parameter τ. The other parameters are identical to those in Figure 4 and $a = 2.85, q_2 = 2, T = 4$.

Meanwhile, bifurcation analyses also indicate that multiple attractors can coexist for a wide range of parameters. In Figure 6 the bifurcation diagram with respect to the bifurcation parameter q_2 shows that three attractors with quite different pest amplitudes can coexist, for example three attractors with different amplitudes could coexist at $q_2 = 1.55$, as shown in Figure 7.

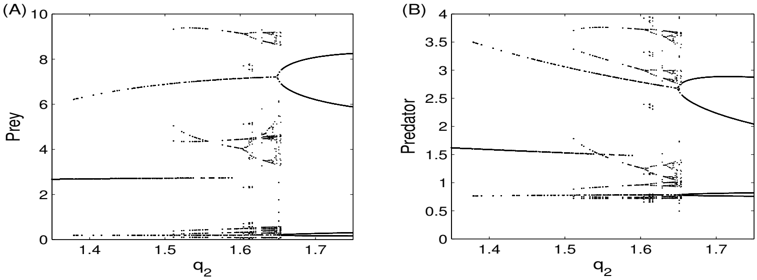

Figure 6. Bifurcation diagrams of model (1.1) with respect to bifurcation parameter q_2. (A) for the prey and (B) for the predator, where all parameter values are fixed as follows: $a = 2, b = 1, c = 0.3, d = 0.6, K = 100, \tau = 0.5, q_1 = 0.8, \theta = 4, T = 4$.

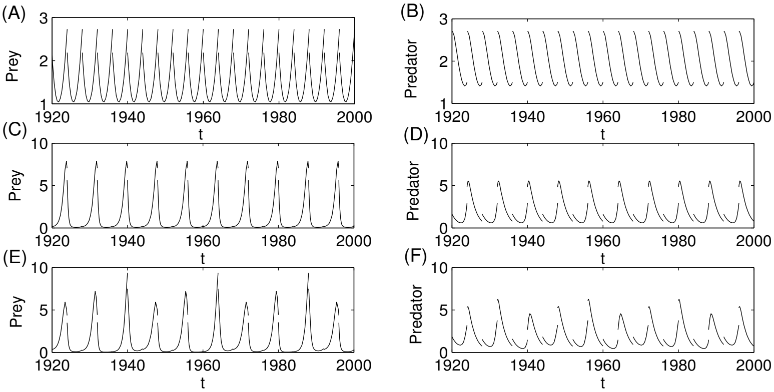

Figure 7. Three coexisting attractors of system (1.1) with parameters as follows: $a = 2, b = 1, c = 0.3, d = 0.6, K = 100, \tau = 0.5, q_1 = 0.8, q_2 = 1.55, \theta = 4, T = 4$. The initial conditions are: $(A - B)(1.3011, 1.6663)$; $(C - D)(1.1391, 1.6981)$; $(E - F)(0.7206, 1.5895)$, respectively.

Similarly, if we choose θ as a bifurcation parameter, then the bifurcation diagrams shown in Figure 8 also reveal that multiple attractors could coexist in model (1.1) for a wide range of parameters, for example two attractors can coexist at $\theta = 0.5$, as shown in Figure 9.

The results shown in Figure 7 and Figure 9 indicate that the attractor is non-unique, i.e. the final stable states of the pest and natural enemy populations depend on their initial densities. The initial densities of the pest and natural enemy populations can affect the outcome of classical biological control, and it can help us to design control strategies and to make management decisions [5, 6, 9, 27]. These results are further confirmed by basins of attraction of initial densities, as shown in Figure 10 and Figure 11. All these results

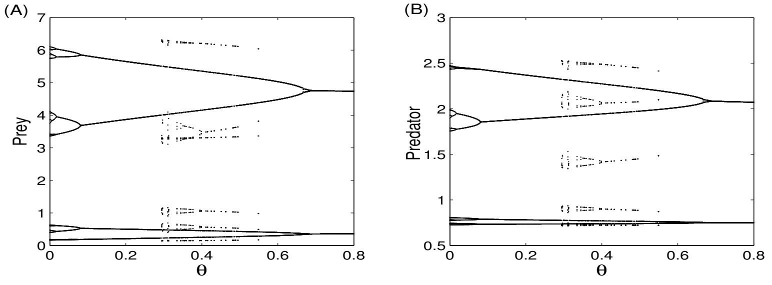

Figure 8. Bifurcation diagrams of model (1.1) with respect to bifurcation parameter θ. The other parameters are identical to those in Figure 6 and $a = 2.15, K = 10, q_2 = 2.5$.

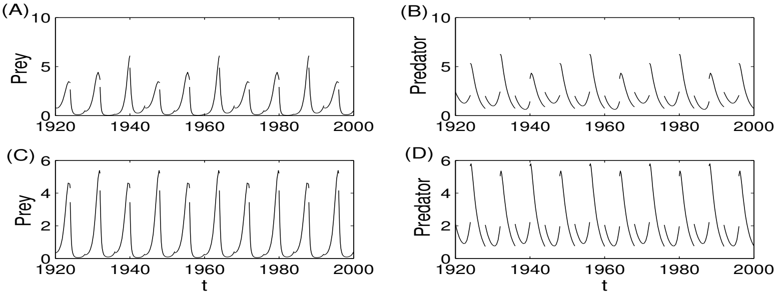

Figure 9. Two coexisting attractors of system (1.1) with parameters as follows: $a = 2.15, b = 1, c = 0.3, d = 0.6, K = 10, \tau = 0.5, q_1 = 0.8, q_2 = 2.5, \theta = 0.5, T = 4$. The initial conditions are: $(A - B)(1.4147, 1.9058); (C - D)(0.7270, 1.9134)$, respectively.

further show that when starting from different initial values the solutions will approach different attractors, and these attractors have quite different amplitudes for the pest population. Obviously, solutions with small amplitudes are biologically desirable.

3.3 Switch-like transitions among multiple attractors

The above studies showed that even small random perturbations can generate switch-like transitions among different attractors [23, 25, 26, 27]. In particular, the attractor with large amplitudes can switch to an attractor with small amplitudes at random times, while extensive numerical simulations indicate that the attractors with a smaller amplitude are robust and are not affected by small random perturbations as proposed in [27]. Therefore, based on the coexistence of multiple attractors discussed in the above and random perturbation methods proposed by Tang et al. [27], we want to show how the nonlinear impulsive control strategies affect the type of switch-like transitions and thus the pest control.

Similarly, various dosages of pesticide applications or different frequencies of pesticide applications and different numbers of natural enemies released in system (1.1) can be mathematically expressed in terms of the key parameters,

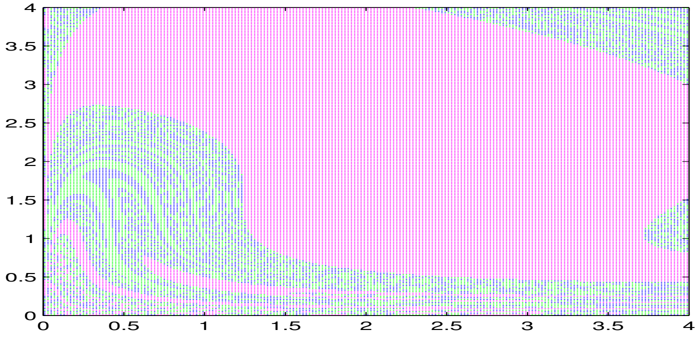

Figure 10. Basin of attraction of the three attractors shown in Figure 7 with the parameters identical to those in Figure 7. The magenta, green and blue points are attracted to the attractors shown in Figure 7 from top to bottom, respectively.

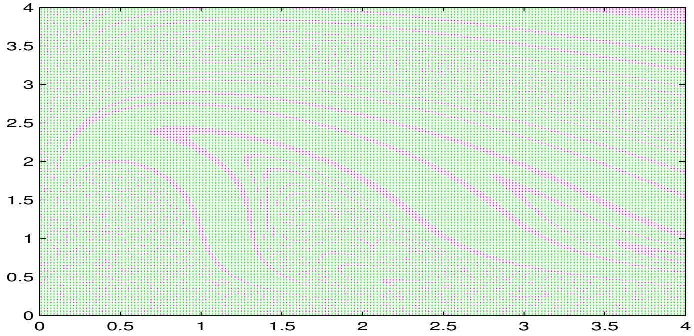

Figure 11. Basin of attraction of the two attractors shown in Figure 9 with the parameters identical to those in Figure 9. The magenta and green points are attracted to the attractors shown in Figure 9 from top to bottom, respectively.

such as q_1 and q_2, i.e. the two parameters can be redefined as $q_{1\eta} = q_1 + \eta_1 u$, $q_{2\eta} = q_2 + \eta_2 u$, where u is a random variable uniformly distributed on $[-1, 1]$, and $\eta_i > 0$ for $i = 1, 2$ represent the intensity of noise. In order to explore this question, we numerically investigated model (1.1) with respect to the switch-like transitions among the attractors under the above stochastic perturbations for the three attractors shown in Figure 7.

In Figure 12, we fix all other parameter values as those in Figure 10, so that there are three stable attractors which can coexist. Further, if we choose the initial values $(x_0, y_0) = (1.1391, 1.6981)$ (or $(0.7206, 1.5895)$), then the stable attractor without random perturbation is an attractor at which the pest population oscillates with a large amplitude (see Figure 7). When we take into account the small random perturbations, numerical simulations imply that the attractors switch from one to another, as shown in Figure 12 for different simulations. By comparison with the numerical simulations obtained in linear impulsive control models [6, 27], we see that an attractor with a larger amplitude can switch to another attractor with smaller amplitude at a random time, and extensive numerical simulations indicate that the attractors with a smaller amplitude are robust and are not affected by these types of small random perturbations. However, once the nonlinear impulsive control is involved in model (1.1), the robustness of the attractors with a smaller amplitude is lost. That is, the switch-like transitions among multiple attractors occur when the small random perturbations are considered in the control parameters, as shown in Figure 12. These numerical results confirm that different dosages of pesticide application and numbers of natural enemies released can influence the dynamics of the classical pest-natural enemy system significantly, and nonlinear impulsive control actions could result in a complex pest outbreak pattern and complex dynamics.

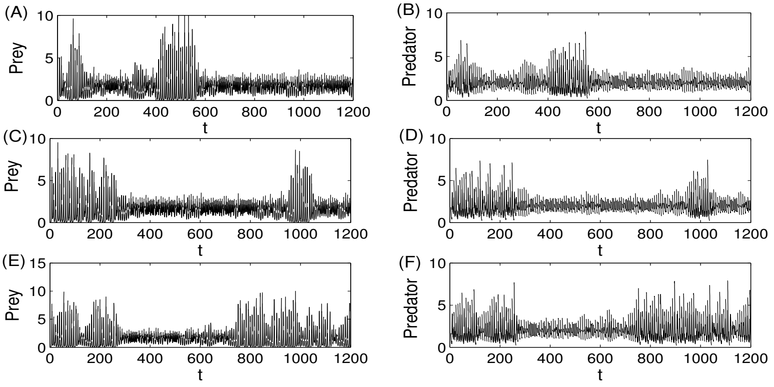

Figure 12. Attractors' switch-like behavior of system (1.1) with small random perturbations on parameters q_1 and q_2, i.e. $\eta_1 = 0$, $\eta_2 = 0.3$. The other parameters are identical to those in Figure 7.

4 Conclusions

In the present work, we have extended a model with linear impulsive control tactics to a model with nonlinear impulsive control measures, which revealed more realistic situations for pest control. That is because before the IPM strategy is applied, it was assumed that the densities of the pest and natural enemy populations had been carefully monitored, and in this case the total number of the natural enemies released must be dependent on their numbers in the field at that time. This indicates that the lower the number of natural enemies in the field, the higher the number of natural enemies that should be released, and vice versa. Therefore, in order to describe this type of releasing strategy, we have proposed a mathematical model of IPM and releases of natural enemies with a regulatory factor at fixed moments.

The threshold condition for the global stability of the pest free periodic solution $(0, y^T(t))$ has been investigated and discussed in detail, and it follows from the inequality (3.1) that we can see how the nonlinear impulsive control actions influence the threshold condition, and consequently affect the success or failure of pest control strategies. Our results indicate that the nonlinear impulsive control parameters including θ can significantly influence the threshold condition and the stability of the pest free periodic solution. In particular, the stability of the pest free solution could switch from stable to unstable, and from unstable to stable as the parameters (such as T or θ) traverse some threshold values, as shown in Figure 2, confirming that the effects of nonlinear impulsive control on pest control should be carefully investigated in more detail from both the mathematical and biological points of view.

Note that the nonlinear impulsive control in model (1.1) could result in a nonlinear difference equation (2.2) determined by the impulsive point series, which is quite useful for addressing the existence and global stability of the pest free periodic solution. In the present paper, we developed the analytical techniques for this nonlinear difference equation to address the existence and global stability of the fixed point, which have improved and extended previously published methods for linear impulsive control [6, 16, 24, 27].

Furthermore, the numerical studies showed that nonlinear control can produce more complex dynamics than those from models with linear control actions. For instance, for the linear impulsive control models, numerical simulations indicate that the attractors with a smaller amplitude are robust and are not affected by some small random perturbations [6, 27]. However, in system (1.1), the attractors with a smaller amplitude no longer have the robustness, i.e. the switch-like transitions could occur among multiple attractors once the nonlinear impulsive control measures are considered. Note that the nonlinear releasing factor turns into a constant τ as the regulatory factor θ tends to infinity, and consequently the condition for the local stability of the pest free periodic solution is identical with the one in a linear control model [6]. Moreover, the bifurcation diagram with respect to θ shown in Figure 8 indicates that the multiple attractors can coexist for a wide range of the parameter θ.

Bifurcation diagrams of the nonlinear control system (1.1) reveal that the control of insect pests depends on the initial densities of pest and natural enemy populations, varying dosages and frequencies of insecticide applications and that the numbers of natural enemies released are crucial for pest control. Most importantly, the random perturbations and nonlinear impulsive control measures could generate complex switching patterns, which shows that the pest population could have outbreaks in a quite complex way due to environmental noise, which further confirms that nonlinear regulatory factors should be taken into account when considering IPM.

Based on the present study we can see that nonlinear impulsive control actions are not only more realistic, but also can generate more interesting results including global stability of the pest free periodic solution and complex switch-like transition behavior. Note that the nonlinear impulsive function (i.e. the nonlinear releasing measure for the natural enemy) considered in the present paper only depends on the density of the natural enemy population, but it is more realistic to consider it as a function of both the pest and natural enemy populations. For example, the nonlinear function could be formulated as $\frac{\tau x(nT)}{\theta+\theta_1 x(nT)+\theta_2 y(nT)}$ (θ_1, θ_2 are positive constants), which is a monotonically increasing function of $x(nT)$ and a decreasing function of $y(nT)$. This will undoubtedly result in more difficulty for analyzing the global dynamics, a subject that we leave for future research. Moreover, in reality, an IPM strategy aims to maintain the density of the pest population below a certain level, such as an economic threshold, rather than eradicate it [8, 19]. Therefore, future research should address how nonlinear impulsive control actions affect the dynamics of models with state-dependent feedback control [22, 26].

Acknowledgements

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