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Fixed point approximation of contractive-like mappings using a stable iterative family and its dynamics via quadratic polynomials

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Abstract. This study aims at presenting a novel bi-parametric family of iterative methods for computing the fixed points of a contractive-like mapping. We thoroughly analyze the strong and stable convergence of the proposed technique and explore its applicability across various problem domains. Regarding convergence, it is proven that for several operators, the Mann iteration is analogous to the proposed multi-step class, and vice-versa. Moreover, numerical tests demonstrate the superior performance of the new procedures compared to existing three-step schemes. We further examine the dynamic behavior of several fixed-point iterative techniques when applied to quadratic polynomials. Based on the outcomes of these experiments, it can be concluded that the proposed family demonstrates both validity and effectiveness.

Keywords: fixed point iteration; Banach space; Mann iteration; Φ-stability; contractive-like operator.

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1 Introduction

The field of fixed point theory holds significant importance and has undergone remarkable development over the last century. It serves as a powerful tool for solving diverse problems in both pure and applied mathematics. Fixed point theory encompasses elements of analysis, topology, and geometry, acting as a bridge among these mathematical disciplines and facilitating fruitful intersections between theory and applications. Due to its wide applicability, fixed point theory finds utility in fields such as differential and integral equations [22], variational inequalities [4, 23], approximation theory [2, 12], and many others. Its versatility and practicality make it a valuable tool for tackling mathematical challenges and advancing various areas of study. Applications of fixed point results extend far beyond the realm of mathematics and find relevance in various disciplines such as statistics, computer science, chemical sciences, physical sciences, economics, engineering, and many others that utilize fixed point theory to address mathematical models representing phenomena arising in diverse areas.

The focus of this research paper revolves around metric fixed-point theory, which concerns theoretical results where geometric conditions are imposed on the function domain, on the underlying space, or sometimes on both. Its origins can be traced back to the Banach Contraction Principle, introduced by the renowned mathematician Stefan Banach [5]. Consider a real Banach space $(B, \|\cdot\|)$ along with a nonempty, convex, and closed subset $H \subseteq B$. Let $\varPhi : H \to H$ be a self-map on H, and let $\varrho \in H$ denote a fixed point of $\varPhi$.. Denote the set of fixed points of $\varPhi$ in H as $\mathcal{F}_{\varPhi} = \{\varrho \in H | \varPhi\varrho = \varrho\}$.

In the literature, Liouville introduced the procedure of successive approximations in 1837, and later, Picard [20] provided a classical proof in the context of ordinary differential equations, demonstrating the existence and uniqueness of solutions to initial value problems in ordinary differential equations. The iterative process proposed by Picard, known as the Picard iteration or sequence of successive approximations, is given by

$$x_{t+1} = \varPhi x_t, \quad t = 0, 1, 2, \dots. \tag{1.1}$$

For the approximation of fixed points in non-expansive mappings, Krasnoselskiĭ [18] proposed an averaged sequence that utilizes two consecutive terms from the Picard iterative process. This sequence, represented as $\{x_t\}_{t=0}^{\infty}$, is generated by initializing with $x_0 \in H$ and is defined by

$$x_{t+1} = 0.5(x_t + \varPhi x_t). \tag{1.2}$$

In 1957, Schaefer [21] extended the concept of the averaged sequence (1.2) by introducing a parameter $\lambda \in (0, 1)$ in place of the constant $1/2$. The Mann procedure is a formal generalization of the Krasnoselskiĭ procedure, where the parameter λ is replaced by a real sequence $\{a_t\}_{t=0}^{\infty}$ in its normal form given by

$$v_{t+1} = (1 - a_t)v_t + a_t\varPhi v_t, \tag{1.3}$$

where $\{a_t\}_{t=0}^{\infty} \in [0, 1]$ and $v_0 \in H$ is a starting value. When $a_t = 1$, the iterative process (1.3) reduces to the Picard scheme (1.1). Furthermore, Kanwar et al. [16] proposed a geometrically constructed fixed point iteration given by

$$x_{t+1} = \frac{mx_t + \varPhi x_t}{m + 1}, \tag{1.4}$$

where the real number $m \geq 0$.

Ishikawa [15] introduced a two-step generalization of the Mann iteration given by

$$\begin{aligned} y_t &= (1 - b_t)x_t + b_t\varPhi x_t, \\ x_{t+1} &= (1 - a_t)x_t + a_t\varPhi y_t. \end{aligned} \tag{1.5}$$

Here, $\{a_t\}_{t=0}^{\infty}$ and $\{b_t\}_{t=0}^{\infty}$ are real sequences in the interval $[0,1]$. The Ishikawa procedure can be seen as a two-step extension of the Mann process, where two different parametric sequences are used. When $b_t = 0$, the Ishikawa iteration reduces to (1.3). Over the past few decades, numerous researchers have developed two-step iterative schemes (see [3, 16]) which aimed at approximating fixed points of diverse nonlinear operators and solving operator equations in appropriate normed spaces.

A three-step iterative procedure was proposed by Karakaya et al. [17] in 2017 for a self-map $\Phi : H \to H$ in a convex and closed subset H of a normed space B. Given an initial point $x_0 \in H$, the sequence $\{x_t\}_{t=0}^{\infty}$ in H is defined by

$$\begin{aligned} y_t &= \Phi x_t, \\ z_t &= (1-\alpha_t)y_t + \alpha_t \Phi y_t, \\ x_{t+1} &= \Phi z_t, \end{aligned} \tag{1.6}$$

where $\{\alpha_t\}_{t=0}^{\infty}$ is a real sequence in $[0,1]$. Subsequently, Ullah and Arshad [24] proposed an iteration process given by

$$\begin{aligned} y_t &= (1-\alpha_t)x_t + \alpha_t \Phi x_t, \\ z_t &= \Phi y_t, \\ x_{t+1} &= \Phi z_t, \end{aligned} \tag{1.7}$$

where $\{\alpha_t\}_{t=0}^{\infty}$ is a real sequence in $[0,1]$. Furthermore, in 2018 Abbas et al. [1] introduced the following three-step iterative process

$$\begin{aligned} y_t &= \Phi x_t, \\ z_t &= \Phi y_t, \\ x_{t+1} &= (1-\alpha_t)z_t + \alpha_t \Phi z_t, \end{aligned} \tag{1.8}$$

where $\{\alpha_t\}_{t=0}^{\infty}$ is a real sequence in $(0,1)$. Very recently, Sharma et al. [22] have presented a three-step iterative approach to calculate the fixed points of a contraction operator

$$\begin{aligned} y_t &= \frac{mx_t + \Phi x_t}{1+m}, \\ z_t &= \Phi y_t, \\ x_{t+1} &= \Phi z_t, \end{aligned} \tag{1.9}$$

where m is a non-negative real number.

There are only a few three-step iterative approaches that have been proposed in the current literature. We propose a three-step bi-parametric generalized scheme given as follows

$$\begin{cases} y_t & = (\alpha x_t + \Phi x_t)/(1+\alpha), \\ z_t & = (\beta y_t + \Phi y_t)/(1+\beta), \\ x_{t+1} & = \Phi z_t, \end{cases} \tag{1.10}$$

where $\alpha, \beta \geq 0$ are real numbers. This scheme presents enhanced characteristics compared to conventional fixed point iterations in practical scenarios.

Furthermore, by assigning specific values to the parameters within our proposed family, many existing schemes can be derived as special cases. This versatility allows for the extraction of known schemes from our broader framework, further enhancing the applicability and compatibility of our proposed technique.

Recently, the study of complex dynamics has proven to be a valuable tool for gaining deeper insights into the behavior of rational maps arising from iterative procedures when applied to equations of the form $P(\zeta) = 0$, where $P : \mathbb{C} \to \mathbb{C}$ is a selected function. By examining the features of this dynamic over rational functions, we can gather crucial details regarding the numerical characteristics of the approach, such as its reliability and stability. The study of iterative techniques applied to rational mappings of complex variables has been extensively explored in the literature, with notable references including [9, 10]. In the present study, we restrict our attention to quadratic polynomials. It is well known that applying an affine transformation to the roots of a polynomial does not affect the qualitative dynamics; for a deeper insight into this behavior, see [11].

The paper is structured as follows: An overview of preliminary results is provided in Section 2 while Section 3 provides a theoretical analysis of strong convergence and stability of the new bi-parametric fixed point iterative family. The importance and role of this family are discussed in Section 4. To validate the theoretical findings, several numerical examples are presented in Section 5, and its complex dynamics are addressed in Section 6. Finally, we conclude with a few remarks in Section 7.

2 Preliminaries

This section contains some significant results and definitions that will aid in supporting the primary findings in the following sections.
Zamfirescu [26] presented this result in 1972:

Theorem 1. *Consider a Banach space* $(B, \|\cdot\|)$ *and a mapping* $\Phi : H(\subseteq B) \to H$ *and let* $\sigma_1, \sigma_2, \sigma_3$*, be real numbers satisfying* $0 \le \sigma_1 < 1$*,* $0 \le \sigma_2, \sigma_3 < 0.5$*, such that for any* $u, v \in H$*, at least one of the following conditions holds:*

($\mathcal{A}_1$*)* $\|\Phi u - \Phi v\| \le \sigma_1 \|u - v\|$;

($\mathcal{A}_2$*)* $\|\Phi u - \Phi v\| \le \sigma_2 \left[\|u - \Phi u\| + \|v - \Phi v\|\right]$;

($\mathcal{A}_3$*)* $\|\Phi u - \Phi v\| \le \sigma_3 \left[\|u - \Phi v\| + \|v - \Phi u\|\right]$.

Then, the mapping Φ *possesses a fixed point* ϱ*, which is unique, and the Picard iteration process* (1.1) *converges to* ϱ*.*

An operator Φ is referred to as a Zamfirescu operator if it fulfills the contractive conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$), and ($\mathcal{A}_3$). A mapping satisfying the condition ($\mathcal{A}_3$) is known as a Chatterjea mapping, while one satisfying ($\mathcal{A}_2$) is known as a Kannan mapping.

In the work of Berinde [7], a new class of operators was introduced for arbitrary Banach spaces. These operators satisfy the following inequality

$$\|\Phi u - \Phi v\| \le 2\omega \|u - \Phi u\| + \omega \|u - v\|, \ \forall\ u, v \in H, \tag{2.1}$$

where $\omega = \max\left\{\sigma_1, \frac{\sigma_2}{1-\sigma_2}, \frac{\sigma_3}{1-\sigma_3}\right\}$, $0 \le \sigma_1 < 1$, $0 \le \sigma_2, \sigma_3 < 0.5$. It was demonstrated that this operator class is broader than the class of Zamfirescu operators. Moreover, he utilized the Ishikawa procedure (1.5) to compute the fixed points of an operator satisfying (2.1) within an arbitrary Banach space (B). Further, a better comprehensive definition of contractivity for (2.1) was considered by Osilike [19]. It is defined by the existence of $P \ge 0$ and $\omega \in [0,1)$ such that for any u and v in H, the following condition holds

$$\|\Phi u - \Phi v\| \le P\|u - \Phi u\| + \omega\|u - v\|.$$

Furthermore, the stability of iterative procedures based on the Picard and Mann iterations was established by Imoru and Olatinwo [14]. They extended the results of Osilike [19] by introducing a contractive condition. Specifically, they assumed the existence of $\omega \in [0,1)$, and a continuous, monotone increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$, such that for any u and v in H, the condition is expressed as follows

$$\|\Phi u - \Phi v\| \le \psi(\|u - \Phi u\|) + \omega\|u - v\|. \tag{2.2}$$

Now, we present several lemmas and definitions whose relevance will be shown in the subsequent section.

Lemma 1. *[6] If $\{\epsilon_t\}_{t=0}^{\infty}$ denotes a positive real sequence such that $\lim_{t\to\infty} \epsilon_t = 0$, and ω is a real number with $0 \le \omega < 1$, then for any positive real sequence $\{s_t\}_{t=0}^{\infty}$ satisfying $s_{t+1} \le \omega s_t + \epsilon_t$, $t \ge 0$, it follows that $\lim_{t\to\infty} s_t = 0$.*

Lemma 2. *[25] Consider two sequences $\{m_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$ of non-negative real numbers satisfying*

$$m_{t+1} \le (1-\mu_t)m_t + k_t, \quad \mu_t \in (0,1) \;\; \forall \;\; t \in \hat{\mathbb{N}} = \mathbb{N} \cup \{0\},$$

where $\sum_{t=0}^{\infty} \mu_t = \infty$ and $\dfrac{k_t}{\mu_t} \to 0$ as $t \to \infty$. Then, $\lim_{t\to\infty} m_t = 0$.

Definition 1. [7] Consider a Banach space B. A sequence $\{x_t\}_{t=0}^{\infty} \subset B$ is said to converge strongly to $l \in B$ if and only if $\|x_t - l\| \to 0$ as $t \to \infty$.

Definition 2. [13] Consider an arbitrary sequence $\{\phi_t\}_{t=0}^{\infty}$ in $H \subseteq B$. An iteration procedure $x_{t+1} = f(\Phi, x_t)$ that converges to a fixed point ϱ is said to be Φ-stable, if for $\epsilon_t = \|\phi_{t+1} - f(\Phi, \phi_t)\|$, where $t \in \hat{\mathbb{N}}$, we have $\lim_{t\to\infty} \epsilon_t = 0$ if and only if $\lim_{t\to\infty} \phi_t = \varrho$.

3 Extensive convergence analysis

This section demonstrates the stability results and strong convergence for the proposed family (1.10).

Theorem 2. *Consider a convex and closed subset $H (\neq \emptyset)$ of a Banach space B. Suppose $\Phi : H \to H$ is a mapping that satisfies condition (2.2) and has a fixed point ϱ. Let the sequence $\{x_t\}_{t=0}^{\infty}$ be obtained from the iterative procedure (1.10) with $x_0 \in H$, where $\alpha, \beta \ge 0$ are real numbers. Then, the sequence $\{x_t\}_{t=0}^{\infty}$ has a strong convergence to the unique fixed point ϱ of Φ.*

Proof. Our aim is to demonstrate that $\lim_{t\to\infty} x_t = \varrho$. By employing the iterative process (1.10), one obtains

$$\|y_t - \varrho\| \le \frac{\omega+\alpha}{1+\alpha}\left|\left|x_t - \varrho\right|\right|, \quad \|z_t - \varrho\| \le \frac{\omega+\beta}{1+\beta}\left|\left|y_t - \varrho\right|\right|. \tag{3.1}$$

Also,

$$\|x_{t+1} - \varrho\| \le \psi(\|\varrho - \Phi\varrho\|) + \omega\|\varrho - z_t\| = \omega\|\varrho - z_t\|. \tag{3.2}$$

Substituting (3.1) in (3.2), one has

$$\|x_{t+1} - \varrho\| \le \omega\left(\frac{\omega+\alpha}{1+\alpha}\right)\left(\frac{\omega+\beta}{1+\beta}\right)\left|\left|x_t - \varrho\right|\right|. \tag{3.3}$$

By repeating the aforementioned process at $1, 2, \ldots, t$ iterative step, we arrive at

$$\|x_{t+1} - \varrho\| \le \omega^{t+1}\left(\frac{\omega+\alpha}{1+\alpha}\right)^{t+1}\left(\frac{\omega+\beta}{1+\beta}\right)^{t+1}\left|\left|x_0 - \varrho\right|\right|. \tag{3.4}$$

Since $\omega < 1$, it is $0 < \left(\frac{\omega+\alpha}{1+\alpha}\right) < 1$, $0 < \left(\frac{\omega+\beta}{1+\beta}\right) < 1$. From (3.4), we have $\lim_{t\to\infty} \|x_{t+1} - \varrho\| = 0$, that implies $\{x_t\}_{t=0}^{\infty}$ converges strongly to ϱ.

Here, we illustrate the uniqueness of the fixed point ϱ. Let us assume that there exist two fixed points $\varrho^*, \varrho \in \mathcal{F}_\Phi$, such that $\Phi\varrho^* = \varrho^*$, and $\Phi\varrho = \varrho$ with $\varrho^* \ne \varrho$. We have,

$$\|\varrho^* - \varrho\| = \|\Phi\varrho - \Phi\varrho^*\| \le \psi(\|\varrho - \Phi\varrho\|) + \omega\|\varrho - \varrho^*\| < \|\varrho - \varrho^*\|,$$

which is a contradiction. Therefore, $\varrho^* = \varrho$. □

Theorem 3. *Under the same assumptions as in Theorem 2, the iterative procedure* (1.10) *is Φ-stable.*

Proof. Consider an arbitrary sequence $\{\phi_t\}_{t=0}^{\infty}$ in H and let the sequence obtained from (1.10) be given by $x_{t+1} = f(\Phi, x_t)$, having convergence towards the unique fixed point ϱ, and $\epsilon_t = \|\phi_{t+1} - f(\Phi, \phi_t)\|$. To prove the Φ-stability of the proposed iteration, we show that $\lim_{t\to\infty} \epsilon_t = 0$ if and only if $\lim_{t\to\infty} \phi_t = \varrho$.

Suppose that $\lim_{t\to\infty} \epsilon_t = 0$. We have that

$$\|\phi_{t+1} - \varrho\| \le \|\phi_{t+1} - f(\Phi, \phi_t)\| + \|f(\Phi, \phi_t) - \varrho\| = \epsilon_t + \|f(\Phi, \phi_t) - \varrho\|.$$

Using a similar result as in (3.3), one can write

$$\|\phi_{t+1} - \varrho\| \le \epsilon_t + \omega\left(\frac{\omega+\alpha}{1+\alpha}\right)\left(\frac{\omega+\beta}{1+\beta}\right)\left|\left|\phi_t - \varrho\right|\right|.$$

Since it is, $\omega < 1, \dfrac{\omega+\alpha}{1+\alpha} < 1$, $\frac{\omega+\beta}{1+\beta} < 1$, this implies $\omega\left(\frac{\omega+\alpha}{1+\alpha}\right)\left(\frac{\omega+\beta}{1+\beta}\right) < 1$. Using Lemma 1 and $\lim_{t\to\infty} \epsilon_t = 0$, we arrive at $\lim_{t\to\infty} \phi_t = \varrho$.

Conversely, let us suppose that $\lim_{t\to\infty} \phi_t = \varrho$. Proceeding similarly as before, one gets

$$\epsilon_t = \|\phi_{t+1} - f(\Phi, \phi_t)\| \leq \|\phi_{t+1} - \varrho\| + \omega \left(\frac{\omega+\alpha}{1+\alpha}\right)\left(\frac{\omega+\beta}{1+\beta}\right) \|\phi_t - \varrho\|.$$

Using the hypothesis, $\lim_{t\to\infty} \phi_t = \varrho$, one gets $\lim_{t\to\infty} \epsilon_t = 0$. This proves that the iterative process (1.10) is Φ-stable. □

Theorem 4. *Assume the same hypotheses as in Theorem 2. Let the sequence $\{v_t\}_{t=0}^{\infty}$ be obtained from the Mann iteration* (1.3) *with $v_0 \in H$, where the real sequence $\{a_t\}_{t=0}^{\infty} \in (0,1)$ satisfies $\sum_{t=0}^{\infty} a_t = \infty$. Then, the following conditions are equivalent:*

(i) the Mann iterative procedure (1.3) *converges to the fixed point ϱ.*

(ii) the new iterative procedure (1.10) *converges to the fixed point ϱ.*

Proof. First, we prove that if the Mann iterative procedure (1.3) converges to ϱ, then the new iterative technique also converges to the same fixed point. Let us assume that the sequence obtained from the Mann procedure converges to ϱ, i.e., $\lim_{t\to\infty} \|v_t - \varrho\| = 0$. Therefore, from Equations (1.3) and (1.10) we obtain

$$\|v_{t+1} - x_{t+1}\| \leq \omega\|v_t - z_t\| + \psi(\|v_t - \Phi v_t\|) + (1-a_t)\|v_t - \Phi v_t\|. \tag{3.5}$$

Moreover,

$$\|v_t - z_t\| \leq \frac{\omega+\beta}{1+\beta}\|v_t - y_t\| + \frac{1}{1+\beta}\Big(\|v_t - \Phi v_t\| + \big(\psi(\|v_t - \Phi v_t\|)\big)\Big). \tag{3.6}$$

Now, substituting (3.6) into (3.5), we get

$$\begin{aligned}\|v_{t+1} - x_{t+1}\| \leq & \frac{\omega(\omega+\beta)}{1+\beta}\|v_t - y_t\| + \frac{\omega+1-a_t}{1+\beta}\|v_t - \Phi v_t\| \\ & + \frac{\omega+1}{1+\beta}\psi(\|v_t - \Phi v_t\|).\end{aligned} \tag{3.7}$$

Further, using a similar result as in (3.6), we can obtain

$$\|v_t - y_t\| \leq \frac{\omega+\alpha}{1+\alpha}\|v_t - x_t\| + \frac{1}{1+\alpha}\left(\|v_t - \Phi v_t\| + (\psi(\|v_t - \Phi v_t\|))\right). \tag{3.8}$$

Substituting (3.8) into (3.7) and using $\omega < 1$, one has

$$\begin{aligned}\|v_{t+1} - x_{t+1}\| \leq & \left(\frac{\omega+\alpha}{1+\alpha}\right)\left(\frac{\omega+\beta}{1+\beta}\right)\|v_t - x_t\| + \left(\frac{\omega}{1+\alpha}\frac{\omega+\beta}{1+\beta} + \frac{\omega+1-a_t}{1+\beta}\right) \\ & \times \|v_t - \Phi v_t\| + \left(\frac{\omega(\omega+\beta)}{(1+\alpha)(1+\beta)} + \frac{\omega+1}{1+\beta}\right)\psi(\|v_t - \Phi v_t\|).\end{aligned}$$

Let us denote $1-\mu = \left(\frac{\omega+\alpha}{1+\alpha}\right)\left(\frac{\omega+\beta}{1+\beta}\right) \in (0,1)$, which implies that $\mu \in (0,1)$ and consider $m_t = \|v_t - x_t\|$, and $k_t = \left(\frac{\omega}{1+\alpha}\frac{\omega+\beta}{1+\beta} + \frac{\omega+1-a_t}{1+\beta}\right)\|v_t - \Phi v_t\| + \left(\frac{\omega(\omega+\beta)}{(1+\alpha)(1+\beta)} + \frac{\omega+1}{1+\beta}\right)\psi(\|v_t - \Phi v_t\|)$. Also, we can write

$$\|v_t - \Phi v_t\| \leq \|v_t - \varrho\| + \omega\|\varrho - v_t\| + \psi(\|\varrho - \Phi\varrho\|) \to 0 \text{ as } t \to \infty, \tag{3.9}$$

from which we get $\lim_{t\to\infty} \psi(\|v_t - \Phi v_t\|) = \psi(\lim_{t\to\infty} \|v_t - \Phi v_t\|) = 0$. Therefore, we get $k_t \to 0$.

From Lemma 2, we have $\|v_t - x_t\| \to 0$ as $t \to \infty$, and therefore, one can obtain $\|x_t - \varrho\| \leq \|x_t - v_t\| + \|v_t - \varrho\| \to 0$ as $t \to \infty$. As a result of this inequality, we obtain that the iterative procedure (1.10) converges to the fixed point ϱ.

Now, we prove the converse. Assume that the iterative procedure (1.10) converges to the fixed point ϱ, i.e., $\lim_{t\to\infty} \|x_t - \varrho\| = 0$. From (1.10) and (1.3), we can write

$$\begin{aligned}\|x_{t+1} - v_{t+1}\| \leq & (1-(1-\omega)a_t)\|z_t - v_t\| + (1-a_t)\|\Phi z_t - z_t\| \\ & + a_t(\psi(\|z_t - \Phi z_t\|)).\end{aligned} \tag{3.10}$$

Moreover, we have

$$\|z_t - v_t\| \leq \|y_t - v_t\| + \frac{1}{1+\beta}\|\Phi y_t - y_t\|, \tag{3.11}$$

and similarly,

$$\|y_t - v_t\| \leq \|x_t - v_t\| + \frac{1}{1+\alpha}\|\Phi x_t - x_t\|. \tag{3.12}$$

Using (3.12) and (3.11), one obtains

$$\|z_t - v_t\| \leq \|x_t - v_t\| + \frac{1}{1+\alpha}\|\Phi x_t - x_t\| + \frac{1}{1+\beta}\|\Phi y_t - y_t\|. \tag{3.13}$$

Substituting (3.13) into (3.10), we arrive at

$$\begin{aligned}\|x_{t+1} - v_{t+1}\| &\leq (1-(1-\omega)a_t)\|x_t - v_t\| + \frac{1-(1-\omega)a_t}{1+\alpha}\|\Phi x_t - x_t\| \\ &+ \frac{1-(1-\omega)a_t}{1+\beta}\|\Phi y_t - y_t\| + (1-a_t)\|\Phi z_t - z_t\| + a_t\psi(\|\Phi z_t - z_t\|).\end{aligned} \tag{3.14}$$

Consider $\mu = (1-\omega)a_t \in (0,1)$, $m_t = \|x_t - v_t\|$, and $\bar{\kappa}_t = \frac{1-(1-\omega)a_t}{1+\alpha}\|\Phi x_t - x_t\| + \frac{1-(1-\omega)a_t}{1+\beta}\|\Phi y_t - y_t\| + (1-a_t)\|\Phi z_t - z_t\| + a_t\psi(\|\Phi z_t - z_t\|)$. Now, using (3.9), we can write

$$\|\Phi x_t - x_t\| \leq (\omega+1)\|x_t - \varrho\| \to 0 \ \text{ as } \ t \to \infty.$$

In a similar way, we obtain

$$\|\Phi y_t - y_t\| \to 0 \text{ and } \|\Phi z_t - z_t\| \to 0 \ \text{ as } \ t \to \infty.$$

Furthermore, $\lim_{t\to\infty} \psi(\|\Phi z_t - z_t\|) = 0$, and thus we obtain that $\bar{\kappa}_t \to 0$.

Now, by using (3.14) and Lemma (2), we have $\|x_t - v_t\| \to 0$ as $t \to \infty$, and therefore, one obtains

$$\|v_t - \varrho\| \leq \|v_t - x_t\| + \|x_t - \varrho\| \to 0 \text{ as } t \to \infty.$$

Therefore, the Mann iteration (1.3) converges to the fixed point of Φ. □

Remark 1. There are several fixed point iterations that share the same equivalence as the proposed family, as mentioned in Theorem 4. For instance, the methods given in Equations (1.2), and (1.4) are analogous to the proposed algorithm, which can be proved by taking $a_t = \frac{1}{2}$ and $\frac{1}{m+1}$, respectively, in the above theorem.

Theorem 5. *Assume the same hypotheses as in Theorem 2. Then, the iteration* (1.10) *converges at least linearly with error equation*

$$\xi_{t+1} = \frac{\Phi'\varrho(\Phi'\varrho + \alpha)(\Phi'\varrho + \beta)}{(1+\alpha)(1+\beta)}\xi_t + O(\xi_t^2).$$

When $\alpha = -\Phi'(\varrho)$, the iteration (1.10) *converges at least quadratically and the error equation becomes*

$$\xi_{t+1} = -\frac{\Phi'\varrho(\Phi'\varrho + \beta)\Phi''\varrho}{2(1+\beta(-1+\Phi'\varrho))}\xi_t^2 - \frac{\Phi'\varrho(\Phi'\varrho + \beta)\Phi'''\varrho}{6(1+\beta(-1+\Phi'\varrho))}\xi_t^3 + O(\xi_t^4).$$

In addition, if $\beta = -\Phi'(\varrho)$, the proposed iteration has at least fourth-order convergence that can seen from the error equation given as follows

$$\xi_{t+1} = -\frac{\Phi'\varrho\Phi''\varrho^3}{8(-1+\Phi'\varrho)^3}\xi_t^4 + O(\xi_t^5),$$

where $\xi_t = x_t - \varrho$ is the error at t^{th} iterative step.

Proof. The proof of convergence for the iterative family (1.10) follows straightforward steps and is therefore omitted. □

Remark 2. (i) The proposed iterative process has been specially constructed for contractive-like mappings. In the case of non-expansive mappings, it may not be practical or applicable.

(ii) As is well known, the convergence of the proposed iterative process depends on the choice of the initial guess. Poor initial guesses can lead to slow convergence or even divergence.

3.1 Special cases

1. It is noteworthy that the well-known iteration proposed by Picard [20] can be regarded as a particular case within our framework. This connection arises when the parameters are set to $\alpha = 0$ and $\beta = 0$.
2. It is worth noting that the iteration introduced by Kumar et al. [22] can be considered as a specific instance of our family (1.10) when we set the parameter values to $\alpha = m \geq 0$ and $\beta = 0$.

3. The iterative process proposed by Ullah and Arshad [24] can be seen as a special case within our iterative family obtained by setting $\alpha = \frac{1-\alpha_t}{\alpha_t}$ and $\beta = 0$, where $\{\alpha_t\}_{t=0}^{\infty}$ is a real sequence in $[0,1]$.
4. It is important to highlight that the three-step iteration proposed by Karakaya et al. [17] arises as a particular case when we assign specific values to the parameters, namely $\alpha = 0$ and $\beta = \frac{1-\alpha_t}{\alpha_t}$, where $\{\alpha_t\}_{t=0}^{\infty}$ is a real sequence in $[0,1]$.

4 Parameter role and its significance

The inclusion of the arbitrary parameters α and β exhibits the following characteristics:

(i). According to the findings of Kanwar et al. [16], it was established that for the convergence of the iterative approach (1.4), the sufficient condition is $-(1+2\alpha) < \Phi'(x_t) < 1$. Similarly, for the proposed iteration, the sufficient conditions for convergence become

$$-(1+2\alpha) < \Phi'(x_t) < 1, \ -(1+2\beta) < \Phi'(y_t) < 1. \tag{4.1}$$

These conditions provide a more extensive criterion for the convergence of the proposed procedure (1.10) compared to the Picard iteration. Considering that for sufficiently large t, the values $x_t \approx \varrho$ and $y_t \approx \varrho$ we arrive at

$$\alpha > -\left(\frac{1+\Phi'(\varrho)}{2}\right), \ \beta > -\left(\frac{1+\Phi'(\varrho)}{2}\right).$$

(ii). Let us assume the three-step Picard iterative procedure (denoted by PM) as

$$\begin{aligned} y_t &= \Phi x_t, \\ z_t &= \Phi y_t, \\ x_{t+1} &= \Phi z_t. \end{aligned} \tag{4.2}$$

The scheme (1.10) has a wider convergence interval due to the presence of parameters in condition (4.1). This implies that various choices of $\Phi(x)$ can be made for the convergence of the iterative procedure (1.10) towards the fixed point ϱ, although the process (4.2) may not converge.

Example 1. To demonstrate the comparison between the simple fixed-point method and the proposed iteration (1.10), we consider $g(x) = x^2 - 5$ and choose the fixed-point counterpart function $\Phi(x) = \frac{5}{x}$. In this case, the desired fixed point is $\varrho = -2.236067977$, and we take an initial guess $x_0 = -2.1$. When the three-step Picard process (4.2) is used, we observe that it diverges. The approximate values alternate between -2.1 and -2.38095, resulting in an infinite loop. However, the proposed iteration (1.10) converges to the fixed point due to its wider interval of convergence, specifically the interval $[-3, 1]$. This highlights the significance of the proposed iteration over the three-step Picard iteration in terms of convergence behavior.

5 Numerical implementation and results

The variants of the proposed family (1.10) have been tested on several numerical problems through which the theoretical results previously obtained can be confirmed. For experimentation, we have considered the following iterative schemes:

- (PM) The three-step Picard iterative procedure in (4.2).
- (KM) The three-step iteration in (1.6) for $\alpha = \frac{2}{3}$.
- (UM) The three-step scheme in (1.7) for $\alpha = \frac{3}{4}$.
- (AM) The three-step iteration in (1.8) for $\alpha = \frac{1}{2}$.
- (VM_1) The three-step procedure in (1.9) for $m = \frac{1}{10}$.
- (NM_1) The proposed three-step iteration in (1.10) for $\alpha = \beta = \frac{1}{2}$.
- (NM_2) The proposed three-step method in (1.10) for $\alpha = \beta = \frac{2}{5}$.
- (NM_3) The proposed three-step scheme in (1.10) for $\alpha = \frac{2}{5}$ and $\beta = 1$.

Furthermore, the second-order procedures considered here are (1.9) (denoted by VM_2) and (1.10) for m and $\alpha = -\Phi'(\varrho) \approx -\Phi'(x_t)$. We have given particular values to the parameter β in our β-dependent family and denote NM_4 (for $\beta = 1$), NM_5 (for $\beta = \frac{2}{5}$), and NM_6 (for $\beta = \frac{1}{2}$). Lastly, we display the results of our new fourth-order method (1.10) (denoted by NM_7) for the parameter values $\alpha = -\Phi'(\varrho) \approx -\Phi'(x_t)$ and $\beta = -\Phi'(\varrho) \approx -\Phi'(y_t)$. In addition, we compute the order of convergence computationally (denoted by p_{COC}) by adopting the following formula

$$p_{COC} \approx \left(\ln \frac{\|x_t - x_{t+1}\|}{\|x_{t-1} - x_t\|} \right) / \left(\ln \frac{\|x_{t-1} - x_t\|}{\|x_{t-2} - x_{t-1}\|} \right), \text{ for } t = 2, 3, 4, \ldots.$$

and the computational time (denoted by e-Time) for each iterative algorithm by using the command `TimeUsed[]`. Numerical computations for this study are performed using Mathematica 11.1 programming package with a setting of $15,000,000$ digits of accuracy. The notation $k(\pm e)$ represents a number k multiplied by $10^{\pm e}$.

5.1 Scalar nonlinear equations

Example 2. Consider the first test problem given by the following nonlinear equation

$$g(x) = \cos(x) - x\exp(x) = 0. \tag{5.1}$$

Based on Equation (5.1), the fixed-point operator selected here is

$$\Phi(x) = \exp(-x)\cos(x). \tag{5.2}$$

The desired solution of Equation (5.1), which is the fixed point of Equation (5.2) is $\varrho = 0.517757363682458298322787 5$. Here, we have compared all the methods taking as initial guess $x_0 = 0.52$.

Example 3. We have tested our procedures on another nonlinear equation,

$$g(x) = \exp(-x) - x = 0. \tag{5.3}$$

Based on Equation (5.3), we have selected the following fixed-point operator

$$\Phi(x) = \exp(-x),$$

to obtain the fixed point $\varrho = 0.5671432904097838729999687$ taking as initial guess $x_0 = 0.6$.

Table 1. Numerical results of several linear-order methods on Examples 2–3.

Ex.	Itr R.E. F.E. p_{COC} e-Time	PM	KM	UM	AM	VM_1	NM_1
2	t	84	27	36	19	62	16
	$\lvert x_{t+1} - x_t \rvert$	6.8(-26)	1.4(-26)	9.2(-26)	2.3(-26)	4.4(-26)	1.4(-26)
	$\lvert g(x_t) \rvert$	1.4(-25)	3.8(-26)	2.3(-25)	7.4(-26)	9.4(-26)	4.0(-26)
	p_{COC}	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	e-Time	0.109	0.062	0.046	0.469	0.735	0.391
3	t	32	14	20	21	28	8
	$\lvert x_{t+1} - x_t \rvert$	8.7(-26)	5.8(-28)	3.6(-27)	1.5(-26)	2.3(-26)	7.9(-26)
	$\lvert g(x_t) \rvert$	1.2(-25)	9.0(-28)	5.3(-27)	2.6(-26)	3.1(-26)	1.2(-25)
	p_{COC}	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	e-Time	0.344	0.188	0.343	0.235	0.313	0.093
Ex.	Itr R.E. F.E. p_{COC} e-Time	NM_2	NM_3				
2	t	20	14				
	$\lvert x_{t+1} - x_t \rvert$	2.3(-26)	1.8(-26)				
	$\lvert g(x_t) \rvert$	6.5(-26)	5.6(-26)				
	p_{COC}	1.0000	1.0000				
	e-Time	0.484	0.296				
3	t	12	13				
	$\lvert x_{t+1} - x_t \rvert$	2.5(-27)	4.4(-26)				
	$\lvert g(x_t) \rvert$	3.8(-27)	7.0(-26)				
	p_{COC}	1.0000	1.0000				
	e-Time	0.093	0.141				

Based on the linearly convergent fixed-point techniques, the computational results of Examples 2–3 are presented in Table 1 with a precision of 25 digits. It is evident from these results that the proposed fixed-point iterations require a few iterations compared to existing methods. We observe that the iterative process PM requires a significantly larger number of iterations to approximate the fixed point than the proposed solvers, highlighting the need for longer execution time of existing algorithms.

In addition, the experimental findings support the theoretical results by demonstrating linear convergence of the proposed techniques, namely NM_1, NM_2, and NM_3. Furthermore, we can utilize second- and fourth-order iterative fixed-point methods to get higher accuracy since they require fewer iterations than linear-order methods. For instance, in the case of the second- and higher-order methods, we obtained results for $t = 11$ and $t = 7$, respectively, as shown in Table 2, indicating that these methods converge faster and provide more accurate solutions. The proposed solvers outperform VM_2 in terms of accuracy, reinforcing their superiority.

Table 2. Numerical results of several second and fourth-order methods on Examples 2–3.

Ex.	Itr R.E. F.E. p_{COC} e-Time	VM_2	NM_4	NM_5	NM_6	NM_7
	t	11	11	11	11	7
2	$\lvert x_{t+1} - x_t\rvert$	1.1(-7408)	1.9(-9329)	2.8(-8310)	2.6(-8618)	2.1(-56811)
	$\lvert g(x_t)\rvert$	3.3(-7408)	5.8(-9329)	8.5(-8310)	7.9(-8618)	6.4(-56811)
	p_{COC}	2.0000	2.0000	2.0000	2.0000	4.0000
	e-Time	14.594	14.844	14.265	13.577	3.484
	t	11	11	11	11	7
3	$\lvert x_{t+1} - x_t\rvert$	2.5(-5571)	8.2(-6428)	1.8(-6956)	2.2(-7828)	8.7(-37849)
	$\lvert g(x_t)\rvert$	4.0(-5571)	1.3(-6427)	2.8(-6956)	3.5(-7828)	1.4(-37848)
	p_{COC}	2.0000	2.0000	2.0000	2.0000	4.0000
	e-Time	4.656	4.563	4.453	4.468	3.781

5.2 Systems of nonlinear equations

Here, we have considered an example of nonlinear systems in two unknowns, represented by $\mathcal{G}(X) = 0$, where $\mathcal{G}(X) = (g_1(X), g_2(X))^t$ and $X = (x_1, x_2)^t$. The numerical results are presented in Table 3 for several linearly convergent techniques. It is observed that the methods UM, NM_1, NM_2, and NM_3 converge to the required fixed point, whereas other procedures, such as PM, UM, and AM, fail to converge to the desired solution.

Furthermore, in Table 4, the higher accuracy is achieved using second- and fourth-order methods by taking $t = 11$ and $t = 7$ iterations, respectively. It is observed that the proposed fixed-point iterations outperform the existing VM_2 technique.

Example 4. Consider the following system of two nonlinear equations

$$\mathcal{G}(X) = \begin{cases} g_1(X) & = x_1^2 - 2x_1 - x_2 + 0.5 = 0, \\ g_2(X) & = x_1^2 + 4x_2^2 - 4 = 0. \end{cases}$$

Here, the fixed-point operators are taken as

$$\begin{cases} x_1 &= \dfrac{x_2 - 0.5 + x_1^2}{-2 + 2x_1} = \Phi_1(X), \\ x_2 &= \dfrac{4 + 4x_2^2 - x_1^2}{8x_2} = \Phi_2(X). \end{cases}$$

Moreover, we assume $B = [-1, 2] \times [-1, 1]$ taking as initial guess $X_0 = (1.6, 0.1)^t$ to reach the approximate solution $(1.900676726367066, 0.3112185654192943)^t$.

Table 3. Numerical results of several linear-order methods on Example 4.

Itr R.E. F.E. p_{COC} e-Time	PM	KM	UM	AM	VM_1	NM_1	NM_2	NM_3
t	–	–	72	–	–	45	50	50
$\|X_{t+1} - X_t\|$	NC	NC	1.1(-15)	NC	NC	1.2(-15)	9.9(-16)	2.2(-16)
$\|\mathcal{G}(X_t)\|$	–	–	1.5(-15)	–	–	1.8(-15)	1.4(-15)	4.2(-16)
p_{COC}	–	–	1.0000	–	–	1.0000	1.0000	1.0000
e-Time	–	–	0.343	–	–	0.203	0.234	0.171

'NC' denotes non-convergence of the fixed point method

Table 4. Numerical results of several second and fourth-order methods on Example 4.

Itr R.E. F.E. p_{COC} e-Time	VM_2	NM_4	NM_5	NM_6	NM_7
t	11	11	11	11	7
$\|X_{t+1} - X_t\|$	1.1(-243)	5.3(-884)	2.3(-593)	3.4(-651)	2.3(-5374)
$\|\mathcal{G}(X_t)\|$	4.6(-243)	1.3(-883)	7.4(-593)	1.0(-651)	6.7(-5375)
p_{COC}	2.0000	2.0000	2.0000	2.0000	4.0000
e-Time	0.124	0.328	0.437	0.421	0.344

5.3 Application in differential systems

Example 5. Let us consider the Van der Pol equation, which is given by

$$x'' - \mu(x^2 - 1)x' + x = 0, \ \mu > 0, \tag{5.4}$$

which governs the flow of current in a vaccum tube, with boundary conditions $x(0) = 0$, and $x(2) = 1$. Further, we consider a uniform partition of the interval $[0, 2]$

$$t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 2, \text{ where } t_i = t_0 + ih, \ h = \frac{2}{n}.$$

In addition, we denote

$$x_0 = x(t_0) = 0,\ x_1 = x(t_1), \ldots, x_{n-1} = x(t_{n-1}),\ x_n = x(t_n) = 1.$$

If we discretize the mathematical model (5.4) using second-order divided differences for the first and second derivatives, then we obtain a system of $(n-1)$ nonlinear equations

$$2h^2x_i - h\mu(x_i^2-1)(x_{i+1}-x_{i-1}) + 2(x_{i+1}+x_{i-1}-2x_i) = 0,\ i=1,2,\ldots,n-1, \quad (5.5)$$

in $(n-1)$ unknowns. Here, we take $\mu = \frac{1}{2}$ and the initial approximation $X_0 = (0.9, 0.9 \cdots, 0.9)^t$. In this problem, the fixed point operator is

$$x_i = \frac{h\mu(x_{i+1}-x_{i-1})(x_i^2+1) + 2(x_{i-1}+x_{i+1})}{2(2-h^2+h\mu(x_{i+1}-x_{i-1})x_i)} = \Phi_i(X),\ i = 1, 2, \ldots (n-1), \quad (5.6)$$

which is used to compute the fixed point (0.4393143837185229, 0.7775271698572037,1.010524232275386,1.131865618324792,1.130248905694175)^t of an operator (5.6) or solution of nonlinear system (5.5). The results are shown in Figure 1, which clearly indicates that linear-order methods behave similarly and require a higher number of iterations in order to obtain eight digits of accuracy. However, the proposed second- and fourth-order require significantly fewer iterations to achieve at least sixteen digits of accuracy.

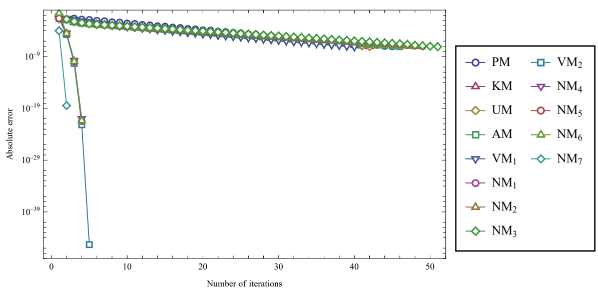

Figure 1. Log plots of iteration number versus residual norms of different methods, for operator $\Phi(x)$ in Example 5.

As a result, the proposed family of fixed-point solvers demonstrates superiority over existing iterative techniques, making it a promising alternative.

6 Complex dynamics of fixed point techniques

The fixed-point operator ($\mathcal{FPO}$) related to the proposed family (1.10) for the quadratic polynomial $P(\zeta) = \zeta^2 + c$ is given by

$$\mathcal{FPO}_P(\zeta) = \frac{\left(\dfrac{\beta(c+\zeta(\alpha+\zeta))}{\alpha+1} + \dfrac{(c+\zeta(\alpha+\zeta))^2}{(\alpha+1)^2} + c\right)^2}{(\beta+1)^2} + c,$$

where the above rational function depends on the parameters α, β and c.

In 1994, Blanchard [9] considered some properties (i) $\mathcal{C}(-i\sqrt{c}) = \infty$, (ii) $\mathcal{C}(i\sqrt{c}) = 0$, (iii) $\mathcal{C}(\infty) = 1$ with the conjugacy map $\mathcal{C}(\zeta) = \frac{\zeta - i\sqrt{c}}{\zeta + i\sqrt{c}}$, and proved that the operator obtained from Newton's iteration always has conjugation to the rational operator ζ^2. Analogously, via the same conjugacy map, the operator $\mathcal{FPO}_P(\zeta)$ has conjugation to the operator $\mathcal{O}_P(\zeta) = (\mathcal{C} \circ \mathcal{FPO}_P \circ \mathcal{C}^{-1})(\zeta)$, and so we arrive at

$$\mathcal{O}_P(\zeta) = \frac{d + c - i\sqrt{c}}{d + c + i\sqrt{c}}, \quad d = \frac{\left(-\frac{\beta\left(4c\zeta + i\alpha\sqrt{c}(\zeta^2 - 1)\right)}{(\alpha+1)(\zeta-1)^2} + \frac{\left(4c\zeta + i\alpha\sqrt{c}(\zeta^2-1)\right)^2}{(\alpha+1)^2(\zeta-1)^4} + c \right)^2}{(\beta+1)^2}.$$

Now, we recall some essential concepts from complex dynamics (refer to [8]) used in this study. For a given rational map $\mathcal{R} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the orbit of a point $\zeta_0 \in \hat{\mathbb{C}}$ is defined as $\mathrm{Orb}(\zeta_0) = \{\zeta_0, \mathcal{R}(\zeta_0), \mathcal{R}^2(\zeta_0), \cdots, \mathcal{R}^n(\zeta_0), \cdots\}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere. The objective of this study is to analyze the asymptotic nature of the orbit based on the starting point ζ_0. Specifically, the efficient fixed-point procedures are designed to investigate the dynamic plane of the function $\mathcal{R}$. We obtain these phase spaces by categorizing initial points according to the asymptotic nature of their orbits.

Table 5. Fixed point operators of the iterative methods PM, KM, UM, AM, VM_1, NM_2, and NM_3.

Method	$\mathcal{O}_P(\zeta)$
PM	$-\frac{(\zeta+1)^4\left(\zeta^2-6\zeta+1\right)^2}{\left(\zeta^8-8\zeta^7+60\zeta^6-184\zeta^5+6\zeta^4-184\zeta^3+60\zeta^2-8\zeta+1\right)}$
KM	$-\frac{\left(-3\zeta^4+16\zeta^3+22\zeta^2+16\zeta-3\right)^2}{\left(23\zeta^8-160\zeta^7+772\zeta^6-2400\zeta^5+1226\zeta^4-2400\zeta^3+772\zeta^2-160\zeta+23\right)}$
UM	$-\frac{\left(-15\zeta^4+88\zeta^3+46\zeta^2+40\zeta-15\right)^2}{\left(e_0\zeta^8-e_1\zeta^7+e_2\zeta^6-e_3\zeta^5+e_4\zeta^4-e_5\zeta^3+e_6\zeta^2-e_7\zeta+e_8\right)}$
AM	$-\frac{\left(3\zeta^8-24\zeta^7+4\zeta^6+152\zeta^5+498\zeta^4+152\zeta^3+4\zeta^2-24\zeta+3\right)}{\left(5\zeta^8-40\zeta^7+220\zeta^6-600\zeta^5+62\zeta^4-600\zeta^3+220\zeta^2-40\zeta+5\right)}$
VM_1	$-\frac{8\left(-30\zeta^4+141\zeta^3+218\zeta^2+101\zeta-30\right)^2}{\left(w_0\zeta^8-w_1\zeta^7+w_2\zeta^6-w_3\zeta^5+w_4\zeta^4-w_5\zeta^3+w_6\zeta^2-w_7\zeta+w_8\right)}$
NM_2	$-\frac{\left(197\zeta^4-1604\zeta^3+70\zeta^2-916\zeta+253\right)^2}{\left(s_0\zeta^8-s_1\zeta^7+s_2\zeta^6-s_3\zeta^5+s_4\zeta^4-s_5\zeta^3+s_6\zeta^2-s_7\zeta+s_8\right)}$
NM_3	$-\frac{\left(31\zeta^4-388\zeta^3+182\zeta^2-284\zeta+59\right)^2}{\left(r_0\zeta^8-r_1\zeta^7+r_2\zeta^6-r_3\zeta^5+r_4\zeta^4-r_5\zeta^3+r_6\zeta^2-r_7\zeta+r_8\right)}$

A point ζ_0 in the complex plane $\mathbb{C}$ is termed as fixed point (FP) of $\mathcal{R}$ if $\mathcal{R}(\zeta_0) = \zeta_0$. A periodic point ζ_0 with period $\mathfrak{p} > 1$ is defined by $\mathcal{R}^{\mathfrak{p}}(\zeta_0) = \zeta_0$ such that $\mathcal{R}^k(\zeta_0) \neq \zeta_0$ for $1 < k < \mathfrak{p}$. Moreover, a pre-periodic point refers to a non-periodic point ζ_0 if we can find some positive integer k such that $\mathcal{R}^k(\zeta_0)$ is periodic. Furthermore, the point ζ_0 is critical if $\mathcal{R}'(\zeta_0) = 0$. In terms of classification of FPs ζ_0 is considered to be attractor (AFP) if $|\mathcal{R}'(\zeta_0)| < 1$, a superattractor (SFP) if $|\mathcal{R}'(\zeta_0)| = 0$, a repulsor (RFP) if $|\mathcal{R}'(\zeta_0)| > 1$, and a parabolic point if $|\mathcal{R}'(\zeta_0)| = 1$.

The attraction basin of γ is defined as the set containing all pre-images of arbitrary order, i.e., $\mathcal{B}(\gamma) = \{\zeta_0 \in \hat{\mathbb{C}} : \mathcal{R}^n(\zeta_0) \to \gamma,\ n \to \infty\}$. The set in which the orbits converge to an attractor is referred to as the Fatou set (denoted as $\mathfrak{F}(\mathcal{R})$). Its complement in the complex plane is known as the Julia set, denoted as $\mathcal{J}(\mathcal{R})$. Consequently, the $\mathcal{J}(\mathcal{R})$ encompasses all RFPs, periodic orbits, and their pre-images. This implies that the attraction basins of any FP are part of $\mathfrak{F}(\mathcal{R})$, while their boundaries belong to $\mathcal{J}(\mathcal{R})$.

6.1 Stability analysis via dynamics

This section considers the study of the dynamical analysis of the operator $\mathcal{O}_P(\zeta)$ to determine its fixed points. First, we discuss the dynamics of the proposed fixed point iteration NM_1 for $c = -1$. The FPs of $\mathcal{O}_P(\zeta)$ are $\zeta = 1,\ \infty,\ 2-\sqrt{5}$, $2 \pm \sqrt{5},\ 0.2956 \pm 0.0754i,\ 0.2544 \pm 0.7027i$ and $0.1785 \pm 2.3143i$, which are considered as roots of the equation $\mathcal{O}_P(\zeta) = \zeta$ with multiplicity one. Further, to study the fixed points stability, we first compute the derivative of $\mathcal{O}_P(\zeta)$,

$$\mathcal{O}'_P(\zeta)=\frac{-5832(\zeta-1)^7\left(35\zeta^3+151\zeta^2+73\zeta-3\right)\left(13\zeta^4-122\zeta^3+32\zeta^2-70\zeta+19\right)}{\eta^2}, \tag{6.1}$$

where $\eta = (1289\zeta^8 - 8492\zeta^7 + 25108\zeta^6 - 72020\zeta^5 + 83462\zeta^4 - 72532\zeta^3 + 34708\zeta^2 - 9004z + 1097)$. From (6.1), we obtain that $\zeta = 1$ and $\zeta = \infty$ are SFPs, since $|\mathcal{O}'_P(\zeta)| = 0$. However, at other fixed points, we have

$$\begin{aligned}
&|\mathcal{O}'_P(2-\sqrt{5})| = |-0.2976| < 1, \quad |\mathcal{O}'_P(2+\sqrt{5})| = |20.0754| > 1,\\
&|\mathcal{O}'_P(0.2956 \pm 0.0754i)|=|6.6207|>1,\ |\mathcal{O}'_P(0.2544 \pm 0.7027i)|=|9.6299|>1,\\
&|\mathcal{O}'_P(0.1785 \pm 2.3143i)|=|6.214| > 1,
\end{aligned}$$

which implies that $2-\sqrt{5}$ is an attractive fixed point (AFP), whereas $0.2956 \pm 0.0754i$, $0.2544 \pm 0.7027i$ and $0.1785 \pm 2.3143i$ are repulsive fixed points (RFP). In association with the polynomial, the dynamical planes obtained from the fixed-point operator of NM_1, which depicts the attraction basins where the orbit of each point converges to a root.

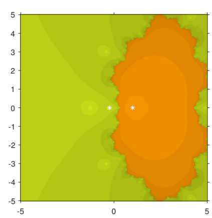

Figure 2. Dynamical plane of the proposed iterative method NM_1.

In Figure 2, each basin of attraction indicates the attracting or superattracting fixed points marked by white stars ('*'). We have assigned different colors to each initial point based on the iterations required to reach a root. Points that converge to the first attracting fixed point 1 are colored in orange,

Table 6. Dynamical analysis of the fixed point iterations $PM, KM, UM, AM, VM_1, NM_2$, and NM_3.

Method	Fixed points	Attracting FP	Repelling FP	Superattracting FP
PM	$1,\ \infty$		$\surd$	
	$2 \pm \sqrt{5}$		$\surd$	
	$0.1761 \pm 0.0388i$		$\surd$	
	$-0.3493 \pm 0.8135i$		$\surd$	
	$1.1732 \pm 6.1535i$		$\surd$	
KM	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$		$\surd$	
	$0.7520 \pm 4.1439i$		$\surd$	
	$-0.1281 \pm 0.8789i$		$\surd$	
	$0.1587 \pm 0.0527i$		$\surd$	
UM	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$		$\surd$	
	$-0.6078 \pm 4.3331i$		$\surd$	
	$-0.0175 \pm 0.7380i$		$\surd$	
	$0.2700 \pm 0.0475i$		$\surd$	
AM	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$		$\surd$	
	$1.3929 \pm 4.7804i$		$\surd$	
	$-0.3686 \pm 0.7557i$		$\surd$	
	$0.1757 \pm 0.0581i$		$\surd$	
VM_1	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$		$\surd$	
	$0.3501 \pm 5.6247i$		$\surd$	
	$-0.2175 \pm 0.8003i$		$\surd$	
	$0.2062 \pm 0.0422i$		$\surd$	
NM_2	$1,\ \infty$			$\surd$
	$2 - \sqrt{5}$	$\surd$		
	$2 + \sqrt{5}$		$\surd$	
	$0.1261 + 2.6735i$		$\surd$	
	$0.1842 + 0.7349i$		$\surd$	
	$0.2728 + 0.0691i$		$\surd$	
NM_3	$1,\ \infty$			$\surd$
	$2 - \sqrt{5}$	$\surd$		
	$2 + \sqrt{5}$		$\surd$	
	$0.4222 + 1.9404i$		$\surd$	
	$0.3468 + 0.7235i$		$\surd$	
	$0.2561 - 0.0976i$		$\surd$	

$\surd$ denotes to the type of fixed point exhibited by the corresponding operator.

while points converging to the second fixed point $2 - \sqrt{5}$ are colored in yellow. On the other hand, the points that show divergence are marked in black color. In addition, the brightness of the colored points indicates the number of iterations required to reach the polynomial root.

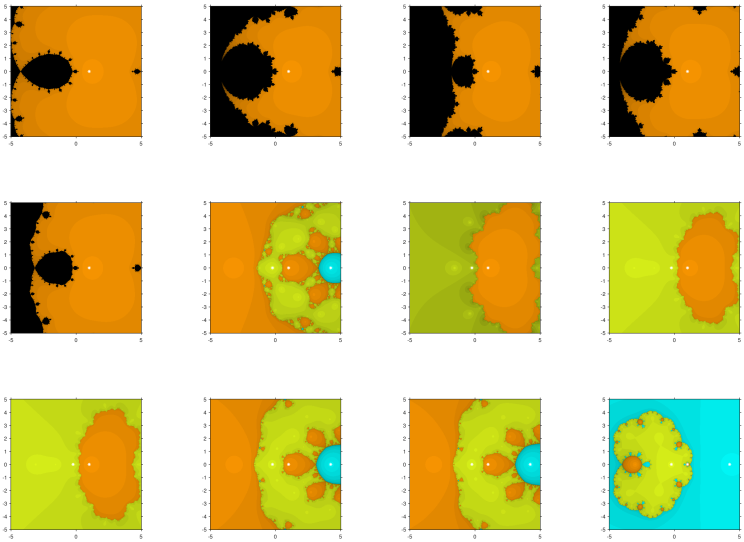

Figure 3. Dynamical planes of the iterative methods PM, KM, UM, AM, VM_1, VM_2, NM_2, NM_3, NM_4, NM_5, NM_6, and NM_7, respectively.

Table 7. Fixed point operators of the iterations VM_2, NM_4, NM_5, NM_6, and NM_7.

Method	$\mathcal{O}_P(\zeta)$
VM_2	$\dfrac{\left(3\zeta^4-4\zeta^3+10\zeta^2+12\zeta-5\right)^2}{\left(7\zeta^8-40\zeta^7+52\zeta^6+72\zeta^5+122\zeta^4+40\zeta^3-172\zeta^2+312\zeta-137\right)}$
NM_4	$-\dfrac{\left(31\zeta^4-388\zeta^3+182\zeta^2-284\zeta+59\right)^2}{\left(m_0\zeta^8-m_1\zeta^7+m_2\zeta^6-m_3\zeta^5+m_4\zeta^4-m_5\zeta^3+m_6\zeta^2-m_7\zeta+m_8\right)}$
NM_5	$\dfrac{\left(19\zeta^4-12\zeta^3+42\zeta^2+68\zeta-37\right)^2}{\left(u_0\zeta^8-u_1\zeta^7+u_2\zeta^6+u_3\zeta^5+u_4\zeta^4-u_5\zeta^3-u_6\zeta^2+u_7\zeta-u_8\right)}$
NM_6	$\dfrac{8\left(-2\zeta^4+\zeta^3-4\zeta^2-7\zeta+4\right)^2}{\left(23\zeta^8-104\zeta^7+28\zeta^6+520\zeta^5+554\zeta^4-568\zeta^3-836\zeta^2+1496\zeta-601\right)}$
NM_7	$\dfrac{\left(5\zeta^4+4\zeta^3+6\zeta^2-12\zeta+13\right)^2}{\left(7\zeta^8-8\zeta^7+116\zeta^6-216\zeta^5-70\zeta^4+776\zeta^3-1004\zeta^2+1368z\zeta-713\right)}$

Similarly, we have computed the fixed points of various methods considered in this paper, and Tables 5–8 summarize their corresponding dynamics, with values $e_0 = 287$, $e_1 = 1456$, $e_2 = 7972$, $e_3 = 35568$, $e_4 = 26234$, $e_5 = 29712$, $e_6 = 14116$, $e_7 = 2896$, $e_8 = 287$, $w_0 = 7441$, $w_1 = 49448$, $w_2 = 355540$, $w_3 = 1263224$, $w_4 = 402422$, $w_5 = 1104504$, $w_6 = 432980$, $w_7 = 68648$, $w_8 = 7441$, $s_0 = 196489$, $s_1 = 1250408$, $s_2 = 3987948$, $s_3 = 12591224$, $s_4 = 13427750$, $s_5 = 12236824$, $s_6 = 5713868$, $s_7 = 1418888$, $s_8 = 171289$, $r_0 = 18247$, $r_1 = 129608$, $r_2 = 375996$, $r_3 = 916808$, $r_4 = 1087394$, $r_5 = 926488$, $r_6 = 435692$, $r_7 = 120152$, $r_8 = 15727$, $m_0 = 18247$, $m_1 = 129608$, $m_2 = 375996$, $m_3 = 916808$, $m_4 = 1087394$, $m_5 = 926488$, $m_6 = 435692$,

$m_7 = 120152$, $m_8 = 15727$, $u_0 = 263$, $u_1 = 1240$, $u_2 = 564$, $u_3 = 5496$, $u_4 = 5978$, $u_5 = 5160$, $u_6 = 9068$, $u_7 = 16136$, and $u_8 = 6569$. In addition, the dynamical planes are used to check the stability of the procedures (see Figure 3). We have assigned a cyan color to those initial points, which converge to the attracting or superattracting fixed point $2+\sqrt{5}$. Based on the dynamical analysis, it is evident that the existing linearly convergent methods exhibit unstable behavior, as evidenced by the presence of black-shaded regions. However, the newly proposed procedures demonstrate stability, since they do not present divergent regions. Furthermore, all higher-order methods lack regions of divergence, implying greater stability. Therefore, from a dynamic perspective, it can be concluded that the proposed methods outperform the compared solvers in terms of stability.

Table 8. Dynamical analysis of the fixed point iterations VM_2, NM_4, NM_5, NM_6, and NM_7.

Method	Fixed points	Attracting FP	Repelling FP	Superattracting FP
VM_2	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$			$\surd$
	$-0.4407 \pm 1.5825i$		$\surd$	
	$0.4640 + 0.2916i$		$\surd$	
NM_4	$1,\ \infty$			$\surd$
	$2 - \sqrt{5}$	$\surd$		
	$2 + \sqrt{5}$		$\surd$	
	$0.4222 \pm 1.9404i$		$\surd$	
	$0.3468 \pm 0.7235i$		$\surd$	
	$0.2561 \pm 0.0976i$		$\surd$	
NM_5	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$			$\surd$
	$-0.8455 \pm 1.3691i$		$\surd$	
	$0.6029 \pm 0.2410i$		$\surd$	
NM_6	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$			$\surd$
	$-0.9154 \pm 1.3246i$		$\surd$	
	$0.6256 \pm 0.2289i$		$\surd$	
NM_7	$1,\ \infty$			$\surd$
	$2 \pm \sqrt{5}$			$\surd$
	$-0.0696 \pm 3.2194i$		$\surd$	
	$0.2084 \pm 1.0528i$		$\surd$	

$\surd$ denotes to the type of the fixed point exhibited by the corresponding operator.

7 Concluding remarks

We have developed a robust class of fixed-point iterative techniques specifically designed for solving nonlinear problems. Our methods have been rigorously proven to exhibit strong convergence and to outperform several well-known iterative procedures in terms of convergence rate. Additionally, the proposed family allows for the recovery of pre-existing fixed-point iterations by assigning specific parameter values. Furthermore, the numerical results demonstrate that the proposed iterations provide more accurate approximations of fixed points compared to existing three-step fixed-point procedures of the same order. The dynamical planes further illustrate the stability of these fixed-point iterations. By utilizing the stable three-step proposed class of fixed-point techniques, we can effectively address nonlinear problems with enhanced accuracy and efficiency.

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