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Robust computing technique for reaction diffusion 2D parabolic problems with shift

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Abstract. Higher dimensional singularly perturbed problems frequently appears in many mathematical modelling. Solving such higher dimensional problem is not as much easy as possible. So order reducing technique namely alternating direction method is one such good choice for solving them. Further singular perturbation problem has its own complexities like boundary and/or interior layers, hence it requires a fitted method on special mesh. Reaction diffusion type singular perturbation problem with space shift is considered in this article. The presence of space shift leads strong interior layer in the solution. To take care of interior and boundary layers, a special mesh is constructed. Hence the problem considered in this article is solved by alternating direction method and fitted difference method with bilinear interpolation. Further the convergence analysis also carried out with rate one in both time and space. Computational validation is also done.

Keywords: ADI method; 2-D parabolic problems; bilinear interpolation; reaction diffusion equation.

AMS Subject Classification: 34K26; 35B25; 65M22; 65M50; 65N22.

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1 Introduction

Higher dimensional differential equations are source of study for analysing the solution of some special equations like Navier-Stokes (N-S) equation. But solving such higher dimensional equations are not that much easy in theoretical and computational perspective. Reducing the higher dimensional problem into a sequence of lower dimensional problems resolves the higher dimensional complexity. One such tech is available in the literature called Alternating Direction Implicit (ADI) method [4,5]. This method reduces the higher dimensional complexity. Further, whether the problem is higher dimensional or not, the singular perturbation problem has its own complexity, that is the solutions has steep gradient near some points in the domain of definition. Small neighbourhoods of these points are called boundary layers regions. In these small neighbourhoods the standard finite difference methods will not produce a satisfactory numerical solution, hence the convergence of the numerical solutions is not guaranteed. So it requires some special type of computational method on some special mesh domain. In this course of study, two dimensional singularly perturbed space shift delay differential equations of parabolic type is considered. Further, ADI method is employed to reduce the higher dimension and fitted numerical method with bilinear interpolation for singular perturbation nature and space shift/delay arguments. Some of the nontrivial results revealed for singularly perturbed delay differential equations (SPDDEs) in the series of articles by Lange and Miura. To cite a few, [12, 13]. These are the bench marking results for SPDDEs. Much later, numerical aspects of SPDDEs have been addressed by many researchers, here it is mentioned a few of them by Kadalbajoo and Sharma [9, 10], Amiraliyev et al [1, 2], Subburayan and Ramanujam [20, 21], for one dimensional parabolic problems [11, 18] and for higher dimension many of them contributed, here it is mentioned some of the recent articles by Das and Natesan [6, 7] and Priyadarshana and Mohapatra [16], Priyadarshana et. al [17]. Our interest in this article is to study reaction diffusion type 2 dimensional SPDDEs. We proposed a fitted method with interpolation on non uniform mesh and proved its convergence.

There are some notations and terminologies which are used in the rest of the article, they are as follows: C is generic positive constant. For computational purpose, the number of grids in spatial domains are Ω_{ξ_1} and Ω_{ξ_2} same and it is N. Let the index set be $I_K = \{r\}_{r=1}^{K}$, K is any positive integer. It is assumed that ε satisfies $\sqrt{\varepsilon} \leq CN^{-1}$ for definite purpose. The norm for studying the analysis is $\|\chi\|_D = \sup_{\mathbf{z}\in D} \|\chi(\mathbf{z})\|$, where $\mathbf{z} = (\xi_1, \xi_2)$.

The section descriptions of the article are as follows: the problem under consideration is stated in the Section 2. The dimension reduction ADI method is applied in Section 3, further stability of solution of dimension reduced problems also addressed here. Bilinear interpolation along with a fitted numerical techniques is elucidated in Section 4. Error bounds of numerical solution and convergence are shown in Section 5. In Section 6, numerical validation of theoretical claims via test problems are given. Section 7 offers final remarks of the article.

Remark 1. If $\sqrt{\varepsilon} > CN^{-1}$, that is, the mesh size is smaller than $\sqrt{\varepsilon}$ then the classical analysis for standard difference method on uniform mesh hold true [14].

2 Problem definition

Numerous applications have been modeled by higher dimensional problems, one such two dimensional problem with shift argument is the celebrated Black-Scholes equation [3]. It motivates us to choose the following type of PDEs: we look for φ such that, it is solution of the problem

$$\mathfrak{Q}\varphi:=\varphi_t-\varepsilon\Delta\varphi+P(\mathbf{z})\varphi+Q(\mathbf{z})\varphi(\mathbf{z}-\mathbf{d},t)=g(\mathbf{z},t),\ (\mathbf{z},t)\in\Omega\times(0,T], \tag{2.1}$$

$$\varphi(\mathbf{z},0)=\varphi_0(\mathbf{z}),\ \mathbf{z}\in\Omega, \tag{2.2}$$

$$\varphi(\mathbf{z},t)=0,\ \text{on}\ \partial\Omega\times[0,T], \tag{2.3}$$

$$\varphi(\mathbf{z},t)=0,\ \text{on}\ [-d_{\xi_1},0]\times[-d_{\xi_2},1]\times[0,T]\cup[-d_{\xi_1},1]\times[-d_{\xi_2},0]\times[0,T], \tag{2.4}$$

where $\mathbf{d}=(d_{\xi_1},d_{\xi_2})$, $\Omega_{\xi_1}=(0,1)=\Omega_{\xi_2}$, $\Omega=\Omega_{\xi_1}\times\Omega_{\xi_2}$, $\Omega^*=\Omega^*_{\xi_1}\times\Omega^*_{\xi_2}$, $\Omega^*_\nu=\Omega^-_\nu\cup\Omega^+_\nu$, $\Omega^-_\nu=(0,d_\nu)$, $\Omega^+_\nu=(d_\nu,1),\nu=\xi_1,\xi_2$, the coefficients φ_0,P,Q,g are assumed to have sufficient differentiability and boundedness on $\bar{\Omega}$. Also we assume that, $P(\mathbf{z})=P_1(\mathbf{z})+P_2(\mathbf{z})$, $P_1,P_2\geq P_0>0$, $Q(\mathbf{z})=Q_1(\mathbf{z})+Q_2(\mathbf{z})$, $0\geq Q_1,Q_2\geq l$, $g=g_1+g_2$, $g_1(\xi_1,0,t)=g_1(\xi_1,1,t)=0$, $g_2(0,\xi_2,t)=g_2(1,\xi_2,t)=0$.

Let $\mathfrak{Q}_{\xi_1}:=-\varepsilon\frac{\partial^2}{\partial\xi_1^2}+P_1(\mathbf{z})I+Q_1(\mathbf{z})I_d$ and $\mathfrak{Q}_{\xi_2}:=-\varepsilon\frac{\partial^2}{\partial\xi_2^2}+P_2(\mathbf{z})I+Q_2(\mathbf{z})I_d$ be two differential operators, $I_d\varphi(\mathbf{z},t)=\varphi(\mathbf{z}-\mathbf{d},t)$, then by combining properly the operators gives the $\mathfrak{Q}$, it is $\mathfrak{Q}:=\frac{\partial}{\partial t}+\mathfrak{Q}_{\xi_1}+\mathfrak{Q}_{\xi_2}$.

Remark 2. The above problem has boundary layer regions at all the boundaries and interior layer regions around the internal lines $\xi_1=d_{\xi_1}$ and $\xi_2=d_{\xi_2}$.

3 Dimension reduced problems and stability

We begin with a remark.

Remark 3. We use the same definition of $\Delta t=\frac{T}{M}$ and $\bar{\Omega}^M_t=\{t_k=k\Delta t\}^M_{k=0}$ as defined in [19].

3.1 ADI method

After the discretization with respect to time, we assume that the solution of the problem at $t=t_n$ is $\hat{\varphi}^n$. We begin with the initial condition $\hat{\varphi}^0(\mathbf{z})=\varphi_0(\mathbf{z})$, $\mathbf{z}\in\bar{\Omega}_{\xi_1}\times\bar{\Omega}_{\xi_2}$. Now, we discretize (2.1)-(2.3) using the ADI method and obtain the lower dimensional problems: let $\xi_2\in\Omega_{\xi_2}$, then

$$\begin{cases}(I+\Delta t\mathfrak{Q}_{\xi_1})\hat{\varphi}^{n+\frac12}=\hat{\varphi}^n+\Delta tg_1(t_{n+1}),\ \xi_1\in\Omega_{\xi_1},\\ \hat{\varphi}^{n+\frac12}(0,\xi_2)=\hat{\varphi}^{n+\frac12}(1,\xi_2)=0,\end{cases} \tag{3.1}$$

let $\xi_1\in\Omega_{\xi_1}$, then

$$\begin{cases}(I+\Delta t\mathfrak{Q}_{\xi_2})\hat{\varphi}^{n+1}=\hat{\varphi}^{n+\frac12}+\Delta tg_2(t_{n+1}),\ \xi_2\in\Omega_{\xi_2},\\ \hat{\varphi}^{n+1}(\xi_1,0)=\hat{\varphi}^{n+1}(\xi_1,1)=0,\end{cases} \tag{3.2}$$

where at $t=t_n$, we have the exact solution $\hat{\varphi}^n(\xi_1,\xi_2)$. For convenience, we denote $\mathcal{G}_{\xi_1}:=I+\Delta t\mathfrak{Q}_{\xi_1}$ and $\mathcal{G}_{\xi_2}:=I+\Delta t\mathfrak{Q}_{\xi_2}$ and then Equations (3.1) and (3.2) can be written as $\mathcal{G}_{\xi_1}\hat{\varphi}^{n+\frac12}=\hat{\varphi}^n+\Delta tg_1(t_{n+1})$ and $\mathcal{G}_{\xi_2}\hat{\varphi}^{n+1}=\hat{\varphi}^{n+\frac12}+\Delta tg_2(t_{n+1})$.

Suppose that, (2.1) has the exact solution at $t = t_n$, then the semi-discrete problems can be defined: let $\xi_2 \in \Omega_{\xi_2}$, then

$$\begin{cases} \mathcal{G}_{\xi_1}\bar{\varphi}^{n+\frac{1}{2}} = \varphi(\xi_1, \xi_2, t_n) + \Delta t g_1(t_{n+1}),\ \xi_1 \in \Omega_{\xi_1}, \\ \bar{\varphi}^{n+\frac{1}{2}}(0, \xi_2) = \bar{\varphi}^{n+\frac{1}{2}}(1, \xi_2) = 0, \end{cases}$$

let $\xi_1 \in \Omega_{\xi_1}$, then

$$\begin{cases} \mathcal{G}_{\xi_2}\bar{\varphi}^{n+1} = \Delta t g_2(t_{n+1}) + \bar{\varphi}^{n+\frac{1}{2}},\ \xi_2 \in \Omega_{\xi_2}, \\ \bar{\varphi}^{n+1}(\xi_1, 0) = \bar{\varphi}^{n+1}(\xi_1, 1) = 0, \end{cases}$$

Let us call the test functions $\Gamma(\xi_1) = 1+\xi_1,\ \xi_1 \in \overline{\Omega}_{\xi_1}$, $\Gamma(\xi_2) = 1+\xi_2,\ \xi_2 \in \overline{\Omega}_{\xi_2}$ and which are used adequately in the rest.

Lemma 1. *Suppose* $\chi \in C^0(\bar{\Omega}_{\xi_1}) \cap C^2(\Omega^*_{\xi_1})$ *possesses non negativeness of* $\chi(0)$, $\chi(1)$, $\mathcal{G}_{\xi_1}\chi(\xi_1)$, $\xi_1 \in \Omega^*_{\xi_1}$ *and* $\chi'(d_{\xi_1}-) - \chi'(d_{\xi_1}+) \geq 0$, *then* $\chi(\xi_1) \geq 0$, $\xi_1 \in \bar{\Omega}_{\xi_1}$.

Proof. Analogous to [21] we prove the lemma. It is shown that $\mathcal{G}_{\xi_1}\Gamma(\xi_1) > 0$, $x \neq d_{\xi_1}$ and $\Gamma'(d^-_{\xi_1}) - \Gamma'(d^+_{\xi_1}) \geq 0$. By imitating the proof of [21] the claim is achieved. □

Lemma 2. *Let* $\chi \in C^0(\bar{\Omega}_{\xi_1}) \cap C^2(\Omega^*_{\xi_1})$ *be a function then* $|\chi| \leq C \max\{|\chi(0)|,$ $|\chi(1)|,\ \sup_{\xi_1 \in \Omega^*_{\xi_1}} |\mathcal{G}_{\xi_1}\chi(\xi_1)|\}$.

Lemma 3. *Let* $\chi \in C^0(\bar{\Omega}_{\xi_2}) \cap C^2(\Omega^*_{\xi_2})$ *possesses the non negativeness of* $\chi(0)$, $\chi(1)$, $\mathcal{G}_{\xi_2}\chi(\xi_2)$, $\xi_2 \in \Omega^*_{\xi_2}$ *and* $\chi'(d_{\xi_2}-) - \chi'(d_{\xi_2}+) \geq 0$, *then* $\chi(\xi_2) \geq 0$, $\xi_2 \in \bar{\Omega}_{\xi_2}$.

Remark 4. As stated in [19], we state the local and global errors and their estimates are $\| LE_n \| \leq C\Delta t^2$, $\sup_n \|GE_n\|_\infty \leq C\Delta t$, where $LE_n = \varphi(\xi_1, \xi_2, t_n) - \bar{\varphi}^n(\xi_1, \xi_2)$, $GE_n = \varphi(t_n) - \hat{\varphi}^n$.

Let us assume that $\hat{\varphi}^{n+\frac{1}{2}}$ has a decomposition $\hat{\varphi}^{n+\frac{1}{2}} = v^{n+\frac{1}{2}} + w^{n+\frac{1}{2}}$ for calculating the error and the bounds of derivatives. Further, from the asymptotic approximation theory we have $v^{n+\frac{1}{2}} = \sum_{k=0}^{2} \varepsilon^{\frac{k}{2}} v_k^{n+\frac{1}{2}}$ and the singular component $w^{n+\frac{1}{2}} = w_1^{n+\frac{1}{2}} + w_2^{n+\frac{1}{2}}$, which leads the good amount of results on the derivatives of the solution. To find these $v_k^{n+\frac{1}{2}}$, $k = 0, 1, 2$, $w_j^{n+\frac{1}{2}}$, $j = 1, 2$ one has to solve the following problems: first let us fix $\xi_2 \in \Omega_{\xi_2}$.

$$\begin{cases} v_0^{n+\frac{1}{2}} + \Delta t\,(P_1 + Q_1 I_{\mathbf{d}})\, v_0^{n+\frac{1}{2}} = v_0^n + \Delta t g_1(t_{n+1}),\ \xi_1 \in \Omega^*_{\xi_1}, \\ v_0^{n+\frac{1}{2}}(\xi_1) = \hat{\varphi}^{n+\frac{1}{2}}(\xi_1),\ \xi_1 \in [-d_{\xi_1}, 0), \end{cases} \tag{3.3}$$

$$\begin{cases} v_1^{n+\frac{1}{2}} + \Delta t\,(P_1 + Q_1 I_{\mathbf{d}})\, v_1^{n+\frac{1}{2}} = v_1^n + \Delta t\sqrt{\varepsilon}\frac{d^2 v_0^{n+\frac{1}{2}}}{d\xi_1^2},\ \xi_1 \in \Omega^*_{\xi_1}, \\ v_1^{n+\frac{1}{2}}(\xi_1) = 0,\ \xi_1 \in [-d_{\xi_1}, 0), \end{cases} \tag{3.4}$$

$$\begin{cases} \mathcal{G}_{\xi_1} v_2^{n+\frac{1}{2}} = v_2^n + \sqrt{\varepsilon}\Delta t \frac{d^2}{d\xi_1^2}\left(v_1^{n+\frac{1}{2}}\right), \ \xi_1 \in \Omega^*_{\xi_1}, \\ v_2^{n+\frac{1}{2}}(\xi_1) = 0, \ \xi_1 \in [-d_x, 0], v_2^{n+\frac{1}{2}}(1) = 0, \\ \frac{d}{d\xi_1} v_2^{n+\frac{1}{2}}(d_{\xi_1}^-) = \frac{d}{d\xi_1} v_2^{n+\frac{1}{2}}(d_{\xi_1}^+), \end{cases} \tag{3.5}$$

and the functions $v^{n+\frac{1}{2}}$ and $w^{n+\frac{1}{2}}$ can be obtained from the BVPs:

$$\begin{cases} \mathcal{G}_{\xi_1} v^{n+\frac{1}{2}} = v^n + \Delta t g_1(t_{n+1}), \ \xi_1 \in \Omega^*_{\xi_1}, \\ v^{n+\frac{1}{2}}(\xi_1) = \begin{cases} \hat{\varphi}^{n+\frac{1}{2}}(\xi_1), & \xi_1 \in [-d_{\xi_1}, 0), \\ v_0^{n+\frac{1}{2}}(0) + \sqrt{\varepsilon} v_1^{n+\frac{1}{2}}(0), & \xi_1 = 0, \end{cases} \\ v^{n+\frac{1}{2}}(1) = \sum_{k=0}^{1} \varepsilon^{\frac{k}{2}} v_k^{n+\frac{1}{2}}(1), \\ \left[v^{n+\frac{1}{2}}(d_{\xi_1})\right] = \sum_{k=0}^{1} \varepsilon^{\frac{k}{2}} \left[v_k^{n+\frac{1}{2}}(d_{\xi_1})\right], \\ \left[\frac{d}{d\xi_1} v^{n+\frac{1}{2}}(d_{\xi_1})\right] = \sum_{k=0}^{1} \varepsilon^{\frac{k}{2}} \left[\frac{d}{d\xi_1} v_k^{n+\frac{1}{2}}(d_{\xi_1})\right], \end{cases} \tag{3.6}$$

$$\begin{cases} \mathcal{G}_{\xi_1} w^{n+\frac{1}{2}} = w^n, \ \xi_1 \in \Omega^*_{\xi_1}, \\ w^{n+\frac{1}{2}}(\xi_1) = \begin{cases} 0, & \xi_1 \in [-d_{\xi_1}, 0]), \\ \hat{\varphi}^{n+\frac{1}{2}}(0) - v^{n+\frac{1}{2}}(0), & \xi_1 = 0, \end{cases} \\ w^{n+\frac{1}{2}}(1) = \hat{\varphi}^{n+\frac{1}{2}}(1) - \sum_{k=0}^{1} \varepsilon^{k/2} v_k^{n+\frac{1}{2}}(1), \\ \left[w^{n+\frac{1}{2}}(d_{\xi_1})\right] = -\left[v^{n+\frac{1}{2}}(d_{\xi_1})\right], \ \left[\frac{d}{d\xi_1} w^{n+\frac{1}{2}}(d_{\xi_1})\right] = -\left[\frac{d}{d\xi_1} v^{n+\frac{1}{2}}(d_{\xi_1})\right], \end{cases} \tag{3.7}$$

where $\left[\chi(\zeta)\right] = \chi(\zeta^+) - \chi(\zeta^-)$ denote jump discontinuity. Further the initial guesses are $v^0 = \hat{\varphi}^0, \ w^0 = 0$.

Theorem 1. *The derivative bounds of the components $v^{n+\frac{1}{2}}$ and $w^{n+\frac{1}{2}}$ are*

$$\left\|\frac{d^k v^{n+\frac{1}{2}}}{d\xi_1^k}\right\|_{\Omega^*} \le C(1 + \varepsilon^{\frac{2-k}{2}}), \ k = 0, 1, 2, 3,$$

$$\left|\frac{d^k w^{n+\frac{1}{2}}(\xi_1)}{d\xi_1^k}\right| \le C\varepsilon^{-\frac{k}{2}} \begin{cases} \exp\left(\frac{-\sqrt{P_0}\xi_1}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{\sqrt{P_0}(\xi_1 - d_{\xi_1})}{\sqrt{\varepsilon}}\right), \ \xi_1 \in \Omega^-_{\xi_1}, \\ \exp\left(\frac{\sqrt{P_0}(d_{\xi_1} - \xi_1)}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{\sqrt{P_0}(\xi_1 - 1)}{\sqrt{\varepsilon}}\right), \ \xi_1 \in \Omega^+_{\xi_1}, \end{cases}$$

$k = 0, 1, 2, 3.$

Proof. Equations (3.3), (3.4) are algebraic, by integration of (3.5), and using the argument presented in [8, 22], Lemma 1 we have $\|v^{n+\frac{1}{2}}\| \le C$.

Successive differentiation's of the Equations (3.3), (3.4) and (3.5) we have $\|\frac{d^k}{d\xi_1^k} v^{n+\frac{1}{2}}\| \le C(\varepsilon^{\frac{2-k}{2}} + 1)$. The functions $\hat{\varphi}^{n+\frac{1}{2}}$ and $v^{n+\frac{1}{2}}$ are bounded from Lemma 2, hence $w^{n+\frac{1}{2}}$. Let us assume that $|w^{n+\frac{1}{2}}(0)|, |w^{n+\frac{1}{2}}(d_{\xi_1})|, |w^{n+\frac{1}{2}}(1)| \le \gamma$.

Now consider the BVP

$$\begin{cases} \mathcal{G}_{\xi_1} w^{n+\frac{1}{2}} = w^n, \ \xi_1 \in \Omega^-_{\xi_1}, \\ w^{n+\frac{1}{2}}(0) = \hat{\varphi}^{n+\frac{1}{2}}(0) - v^{n+\frac{1}{2}}(0), \ w^{n+\frac{1}{2}}(d_{\xi_1}) = -v^{n+\frac{1}{2}}(d_{\xi_1}-), \end{cases}$$

by the barrier function

$$\theta_1^{\pm}(\xi_1) = C^* \left(\exp\left(\frac{-\sqrt{P_0}\xi_1}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{-\sqrt{P_0}(d_{\xi_1} - \xi_1)}{\sqrt{\varepsilon}}\right)\right) \pm w^{n+\frac{1}{2}}, \ \xi_1 \in \Omega^-_{\xi_1},$$

where C^* is an appropriate constant, and it is $\max\{\gamma, |\hat{\varphi}^{n+\frac{1}{2}}(0) - v^{n+\frac{1}{2}}(0)|, |\hat{\varphi}^{n+\frac{1}{2}}(1) - \sum_{k=0}^{1} \varepsilon^{k/2} v_k^{n+\frac{1}{2}}(1)|, |v^{n+\frac{1}{2}}(d_{\xi_1}-)|, |v^{n+\frac{1}{2}}(d_{\xi_1}+)|, \sup|w^n|\}$ and by [24, Theorem 2.1], we see that

$$\begin{aligned} &\theta_1^{\pm}(0) = C^*(1 + \exp(-\sqrt{P_0/\varepsilon} d_{\xi_1})) \pm (\hat{\varphi}^{n+\frac{1}{2}}(0) - v^{n+\frac{1}{2}}(0)) \geq 0, \\ &\theta_1^{\pm}(d_{\xi_1}) = C^*(1 + \exp(-\sqrt{P_0/\varepsilon} d_{\xi_1})) \mp v^{n+\frac{1}{2}}(d_{\xi_1}-) \geq 0, \quad \mathcal{G}_{\xi_1}\theta_1^{\pm}(\xi_1) = \\ &\quad [1+\Delta t(P_1 - P_0)] \left(\exp\left(\frac{-\sqrt{P_0}\xi_1}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{-\sqrt{P_0}(d_{\xi_1} - \xi_1)}{\sqrt{\varepsilon}}\right)\right) \pm w^n \geq 0, \end{aligned}$$

we arrive that $\mathcal{G}_{\xi_1}\theta_1^{\pm}(\xi_1) \geq 0$, and therefore,

$$| \, w^{n+\frac{1}{2}} \, | \leq C \left(\exp\left(\frac{-\sqrt{P_0}\xi_1}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{-\sqrt{P_0}(d_{\xi_1} - \xi_1)}{\sqrt{\varepsilon}}\right)\right).$$

Analogously, on the domain $\Omega^+_{\xi_1}$, we have the BVP

$$\begin{cases} \mathcal{G}_{\xi_1} w^{n+\frac{1}{2}} = w^n, \ \xi_1 \in \Omega^+_{\xi_1}, \\ w^{n+\frac{1}{2}}(d_{\xi_1}) = -v^{n+\frac{1}{2}}(d_{\xi_1}+), \ w^{n+\frac{1}{2}}(1) = \hat{\varphi}^{n+\frac{1}{2}}(1) - \sum_{k=0}^{1} \varepsilon^{k/2} v_k^{n+\frac{1}{2}}(1). \end{cases}$$

To estimate the bound, we consider the following barrier function $\theta_2^{\pm}(\xi_1) = C^*[\chi(\xi_1) + \exp(-\sqrt{P_0/\varepsilon}(1-\xi_1))] \pm w^{n+\frac{1}{2}}$, where

$$\chi(\xi_1) = \begin{cases} 0, & \xi_1 < d_{\xi_1}, \\ \exp(-\sqrt{P_0/\varepsilon}(\xi_1 - d_{\xi_1})), & \xi_1 \geq d_{\xi_1}, \end{cases}$$

and we see that $\theta_2(d_{\xi_1}) \geq 0, \ \theta_2(1) \geq 0$,

$$\begin{aligned} \mathcal{G}_{\xi_1}\theta_2^{\pm}(\xi_1) =& [1 + \Delta t(P_1 - P_0)](\exp(\frac{-\sqrt{P_0}(d_{\xi_1} - \xi_1)}{\sqrt{\varepsilon}}) + \exp(\frac{-\sqrt{P_0}(1 - \xi_1)}{\sqrt{\varepsilon}})) \\ &+ (\exp(\frac{-\sqrt{P_0}(d_{\xi_1} - \xi_1)}{\sqrt{\varepsilon}}) + \exp(\frac{-\sqrt{P_0}(1 - \xi_1)}{\sqrt{\varepsilon}}) \\ &+ \Delta t Q_1 \exp(\frac{-\sqrt{P_0}(1 - d_{\xi_1} - \xi_1)}{\sqrt{\varepsilon}})) \pm w^n \geq 0, \end{aligned}$$

then, by the result [24, Theorem 2.1] we have the estimate

$$| \, w^{n+\frac{1}{2}} \, | \leq C^*(\exp(\frac{-\sqrt{P_0}(\xi_1 - d_{\xi_1})}{\sqrt{\varepsilon}}) + \exp(\frac{-\sqrt{P_0}(1 - \xi_1)}{\sqrt{\varepsilon}})).$$

Successive differentiation's of the function and applying the above estimate gives the desired result. □

Analogous manner the function φ^{n+1} is decomposed and $\varphi^{n+1} = v^{n+1} + w^{n+1}$. The components in the finite sum $v^{n+1} = \sum_{k=0}^{2} \varepsilon^{k/2} v_k^{n+1}$ are obtained from the following:

$$\begin{cases} v_0^{n+1} + \Delta t\,(P_2 + Q_2 I_{\mathbf{d}})\, v_0^{n+1} = v_0^{n+\frac{1}{2}} + \Delta t g_2(t_{n+1}),\ \xi_2 \in \Omega^*_{\xi_2}, \\ v_0^{n+1}(\xi_2) = 0,\ \xi_2 \in [-d_{\xi_2}, 0), \end{cases} \tag{3.8}$$

$$\begin{cases} v_1^{n+1} + \Delta t\,(P_2 + Q_2 I_{\mathbf{d}})\, v_1^{n+1} = v_1^{n+\frac{1}{2}} + \sqrt{\varepsilon}\Delta t \frac{d^2 v_0^{n+1}}{d\xi_2^2},\ \xi_2 \in \Omega^*_{\xi_2}, \\ v_1^{n+1}(\xi_2) = 0,\ \xi_2 \in [-d_{\xi_2}, 0), \end{cases} \tag{3.9}$$

$$\begin{cases} \mathcal{G}_{\xi_2} v_2^{n+1} = v_2^{n+\frac{1}{2}} + \sqrt{\varepsilon}\Delta t \frac{d^2}{d\xi_2^2}\left(v_1^{n+1}\right),\ \xi_2 \in \Omega^*_{\xi_2}, \\ v_2^{n+1}(\xi_2)=0,\ \xi_2\in[-d_{\xi_2}, 0], v_2^{n+1}(1) = 0,\ \frac{dv_2^{n+1}(d^-_{\xi_2})}{d\xi_2} = \frac{dv_2^{n+1}(d^+_{\xi_2})}{d\xi_2}. \end{cases} \tag{3.10}$$

The components v^{n+1}, w^{n+1} are obtained from the problems :

$$\begin{cases} \mathcal{G}_{\xi_2} v^{n+1} = v^{n+\frac{1}{2}} + \Delta t g_2(t_{n+1}),\ \xi_2 \in \Omega^*_{\xi_2}, \\ v^{n+1}(\xi_2) = \begin{cases} 0, & \xi_2 \in [-d_{\xi_2}, 0), \\ \sum_{k=0}^{1} \varepsilon^k v_k^{n+1}(0), & \xi_2 = 0, \end{cases} \quad v^{n+1}(1) = \sum_{k=0}^{1} \varepsilon^{k/2} v_k^{n+1}(1), \\ \left[v^{n+1}(d_{\xi_2})\right] = \sum_{k=0}^{2} \varepsilon^{k/2} \left[v_k^{n+1}(d_{\xi_2})\right],\ \left[\frac{dv^{n+1}(d_{\xi_2})}{d\xi_2}\right] = \sum_{k=0}^{2} \varepsilon^{k/2} \left[\frac{dv_k^{n+1}(d_{\xi_2})}{d\xi_2}\right], \end{cases} \tag{3.11}$$

$$\begin{cases} \mathcal{G}_{\xi_2} w^{n+1} = w^{n+\frac{1}{2}},\ \xi_2 \in \Omega^*_{\xi_2}, \\ w^{n+1}(\xi_2) = \begin{cases} 0, & \xi_2 \in [-d_{\xi_2}, 0), \\ \hat{\varphi}^{n+1}(0) - v^{n+1}(0), & \xi_2 = 0, \end{cases} \quad w^{n+1}(1) = \hat{\varphi}^{n+1}(1) - v^{n+1}(1), \\ \left[w^{n+1}(d_{\xi_2})\right] = -[v^{n+1}(d_{\xi_2})],\ \left[\frac{d}{d\xi_2} w^{n+1}(d_{\xi_2})\right] = -\left[\frac{d}{d\xi_2} v^{n+1}(d_{\xi_2})\right] \end{cases} \tag{3.12}$$

and we have the following result.

Theorem 2. *The derivatives estimates of the components v^{n+1} and w^{n+1} are*

$$\left\| \frac{d^k v^{n+1}}{d\xi_2^k} \right\|_{\Omega^*} \le C(1 + \varepsilon^{\frac{2-k}{2}}),\ k = 0, 1, 2, 3,$$

$$\left| \frac{d^k w^{n+1}(\xi_1)}{d\xi_2^k} \right| \le C\varepsilon^{-k/2} \begin{cases} \exp\left(\frac{-\sqrt{P_0}\xi_2}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{\sqrt{P_0}(\xi_2 - d_{\xi_2})}{\sqrt{\varepsilon}}\right),\ \xi_2 \in \Omega^-_{\xi_2}, \\ \exp\left(\frac{\sqrt{P_0}(\xi_2 - 1)}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{\sqrt{P_0}(d_{\xi_2} - \xi_2)}{\sqrt{\varepsilon}}\right),\ \xi_2 \in \Omega^+_{\xi_2}. \end{cases}$$

Proof. The Equations (3.8)–(3.9) are algebraic equations. Therefore, $\|v_i^{n+1}\|_{\Omega^*_{\xi_2}} \le C,\ i = 1, 2$. From the Lemma 3, we have $\|v_2^{n+1}\| \le C$. Successive differentiation's leads $\|\frac{d^k}{d\xi_2^k} v^{n+1}\|_{\Omega^*_{\xi_2}} \le C(1 + \varepsilon^{\frac{2-k}{2}}),\ k = 1, 2, 3$. From the Equations

(3.8)–(3.10) we see that $v^{n+1}(d^-_{\xi_2})$, $v^{n+1}(d^+_{\xi_2})$ are bounded. So let us assume that $|w^{n+1}(0)|, |w^{n+1}(1)|, |w^{n+1}(d_{\xi_2}-)|, |w^{n+1}(d_{\xi_2}+)| \leq \gamma$. Consider the problem

$$\begin{aligned}\mathcal{G}_{\xi_2} w^{n+1} =& w^{n+\frac{1}{2}},\ \xi_2 \in \Omega^-_{\xi_2},\\ w^{n+1}(0) =& \hat{\varphi}^{n+1}(0) - v^{n+1}(0),\ w^{n+1}(d^-_{\xi_2}) = -v^{n+1}(d^-_{\xi_2}),\end{aligned}$$

and the barrier function

$$\theta^{\pm}_1(\xi_2) = C\gamma^* \exp(\frac{-\sqrt{P_0}\xi_2}{\sqrt{\varepsilon}}) + \exp(\frac{-\sqrt{P_0}(d_{\xi_2} - \xi_2)}{\sqrt{\varepsilon}})) \pm w^{n+1},\ \xi_2 \in \bar{\Omega}^-_{\xi_2},$$

where $\gamma^* = \max\{\gamma, |\hat{\varphi}^{n+1}(0) - v^{n+1}(0)|, |v^{n+1}(d^-_{\xi_2})|, |v^{n+1}(d^+_{\xi_2})|, |\hat{\varphi}^{n+1}(1) - v^{n+1}(1)|, \sup|w^{n+\frac{1}{2}}|\}$. It is shown that $\theta^{\pm}_1(0) \geq 0$ and $\theta^{\pm}_1(d_{\xi_2}) \geq 0$ and for a suitable choice of C, and similar to the proof of above theorem we see that

$$|w^{n+1}| \leq C\Big(\exp\Big(\frac{-\sqrt{P_0}\xi_2}{\sqrt{\varepsilon}}\Big) + \exp\Big(\frac{-\sqrt{P_0}(d_{\xi_2} - \xi_2)}{\sqrt{\varepsilon}}\Big)\Big),\ \xi_2 \in \Omega^-_{\xi_2}.$$

To prove the result in $\Omega^+_{\xi_2}$, we consider the problem

$$\begin{aligned}\mathcal{G}_{\xi_2} w^{n+1} =& w^{n+\frac{1}{2}},\ \xi_2 \in \Omega^+_{\xi_2},\\ w^{n+1}(d^+_{\xi_2}) =& - v^{n+1}(d^+_{\xi_2}),\ w^{n+1}(1) = \hat{\varphi}^{n+1}(1) - v^{n+1}(1),\end{aligned}$$

and by a suitable barrier function

$$\theta^{\pm}_2(\xi_2) = C\gamma^* \Big(\mu(\xi_2) + \exp\Big(\frac{-\sqrt{P_0}(1 - \xi_2)}{\sqrt{\varepsilon}}\Big)\Big) \pm w^{n+1},\ \ \xi_2 \in \bar{\Omega}^+_{\xi_2},$$

where

$$\mu(\xi_2) = \begin{cases} 0, & \xi_2 < d_{\xi_2},\\ \exp\Big(\frac{-\sqrt{P_0}(d_{\xi_2} - \xi_2)}{\sqrt{\varepsilon}}\Big), & \xi_2 \geq d_{\xi_2},\end{cases}$$

we prove that $\theta^{\pm}_2 \geq 0$ and
$|w^{n+1}(\xi_2)| \leq C\Big(\exp\Big(\frac{-\sqrt{P_0}(d_{\xi_2}-\xi_2)}{\sqrt{\varepsilon}}\Big) + \exp\Big(\frac{-\sqrt{P_0}(1-\xi_2)}{\sqrt{\varepsilon}}\Big)\Big),\ \xi_2 \in \Omega^+_{\xi_2}$.

Successive differentiation's leads the desired results in $\Omega^*_{\xi_2}$. □

4 Discrete problem

4.1 Grid points

The problem (2.1)–(2.3) exhibits boundary layers at $\mu = 0, d_\mu, 1,\ \mu = \xi_1, \xi_2$ from Theorems 1 and 2. Accordingly, we subdivide the domain into small subdomains, for that define $\tau_{1,\mu} = \min\{\frac{d_\mu}{4}, \frac{2}{\sqrt{P_0}}\sqrt{\varepsilon}\ln N\}$, $\tau_{2,\mu} = \min\{\frac{1-d_\mu}{4}, \frac{2}{\sqrt{P_0}}\sqrt{\varepsilon}\ln N\}$, $\mu = \xi_1, \xi_2$. Using these parameters $\tau_{i,\mu},\ i = 1, 2,\ \mu = \xi_1, \xi_2$, the domains $\bar{\Omega}_{\xi_1}$ and $\bar{\Omega}_{\xi_2}$ are subdivided as follows: $\bar{\Omega}_\mu = \cup^6_{i=1}\Omega_{i,\mu}$, where $\Omega_{1,\mu} =$

$[0, \tau_{1,\mu}]$, $\Omega_{2,\mu} = [\tau_{1,\mu}, d_\mu - \tau_{1,\mu}]$, $\Omega_{3,\mu} = [d_\mu - \tau_{1,\mu}, d_\mu]$, $\Omega_{4,\mu} = [d_\mu, d_\mu + \tau_{2,\mu}]$, $\Omega_{5,\mu} = [d_\mu + \tau_{2,\mu}, 1 - \tau_{2,\mu}]$, $\Omega_{6,\mu} = [1 - \tau_{2,\mu}, 1]$.

On each sub-domain $\Omega_{i,\mu}$, $i = 1, \dots, 6$ the following $\frac{N}{8}, \frac{N}{4}, \frac{N}{8}, \frac{N}{8}, \frac{N}{4}, \frac{N}{8}$, number of grid points placed, respectively. Calling the mesh sizes as $h(i) = \xi_{1,i} - \xi_{1,i-1}$ and $k(j) = \xi_{2,j} - \xi_{2,j-1}$ and hence define the meshes $\bar{\Omega}^N_{\xi_1} = \{\xi_{1,r}\}_{r=0}^{r=N}$, $\bar{\Omega}^N_{\xi_2} = \{\xi_{2,r}\}_{r=0}^{r=N}$, where $\xi_{1,0} = 0$, $\xi_{1,r} = \xi_{1,r-1} + h(r)$ and $\xi_{2,0} = 0$, $\xi_{2,r} = \xi_{2,r-1} + k(r)$, $r = 1, 2, \dots, N$.

The partition of 2D domain is presented in the Figure 1.

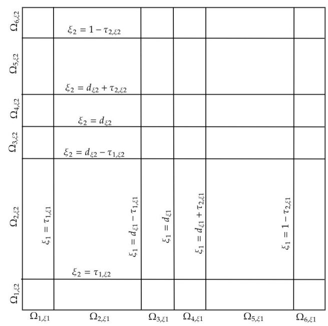

Figure 1. 2D domain partition.

4.2 Fitted difference method

On the discrete sets $\bar{\Omega}^N_{\xi_1}$ and $\bar{\Omega}^N_{\xi_2}$, fitted difference schemes with interpolation are defined, fix $\xi_2 = \xi_{2,j}$,

$$\mathcal{G}^N_{\xi_1} \Phi^{n+\frac{1}{2}}_{i,j} := \begin{cases} \Phi^{n+\frac{1}{2}}_{i,j} + \Delta t(-\varepsilon \delta^2_{\xi_1} \Phi^{n+\frac{1}{2}}_{i,j} + P_{1_{i,j}} \Phi^{n+\frac{1}{2}}_{i,j} + Q_{1_{i,j}} I^N_d \Phi^{n+\frac{1}{2}}_{i,j}) & \\ = \Phi^n_{i,j} + \Delta t g_1(\xi_{1,i}, \xi_{2,j}, t_{n+1}), & i \neq \frac{N}{2}, \\ D^-_{\xi_1} \Phi^{n+\frac{1}{2}}_{N/2,j} = D^+_{\xi_1} \Phi^{n+\frac{1}{2}}_{N/2,j}, & i = \frac{N}{2}, \\ \Phi^{n+\frac{1}{2}}_{0,j} = \hat{\varphi}^{n+\frac{1}{2}}(0, \xi_{2,j}); \ \Phi^{n+\frac{1}{2}}_{N,j} = \hat{\varphi}^{n+\frac{1}{2}}(1, \xi_{2,j}), & \end{cases} \tag{4.1}$$

fix $\xi_1 = \xi_{1,i}$,

$$\mathcal{G}^N_{\xi_2} \Phi^{n+1}_{i,j} := \begin{cases} \Phi^{n+1}_{i,j} + \Delta t(-\varepsilon \delta^2_{\xi_2} \Phi^{n+1}_{i,j} + P_{2_{i,j}} \Phi^{n+1}_{i,j} + Q_{2_{i,j}} I^N_d \Phi^{n+1}_{i,j}) & \\ = \Phi^{n+\frac{1}{2}}_{i,j} + \Delta t g_2(\xi_{1,i}, \xi_{2,j}, t_{n+1}), & j \neq \frac{N}{2}, \\ D^-_{\xi_2} \Phi^{n+1}_{i,N/2} = D^+_{\xi_2} \Phi^{n+1}_{i,N/2}, & j = \frac{N}{2}, \\ \Phi^{n+1}_{i,0} = \hat{\varphi}^{n+1}(\xi_{1,i}, 0); \ \Phi^{n+1}_{i,N} = \hat{\varphi}^{n+1}(\xi_{1,i}, 1), & \end{cases} \tag{4.2}$$

where

$$I_{\mathbf{d}}^N \Phi_{i,j}^{n+\frac{1}{2}} = \begin{cases} 0, \ i < \frac{N}{2}, \\ \Phi_{p,q}^{n+\frac{1}{2}} l_{p,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q,\xi_2}(\xi_{2,j} - d_{\xi_2}) \\ +\Phi_{p+1,q}^{n+\frac{1}{2}} l_{p+1,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q,\xi_2}(\xi_{2,j} - d_{\xi_2}) \\ +\Phi_{p,q+1}^{n+\frac{1}{2}} l_{p,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q+1,\xi_2}(\xi_{2,j} - d_{\xi_2}) \\ +\Phi_{p+1,q+1}^{n+\frac{1}{2}} l_{p+1,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q+1,\xi_2}(\xi_{2,j} - d_{\xi_2}), i > \frac{N}{2}, \end{cases}$$

$$I_{\mathbf{d}}^N \Phi_{i,j}^{n+1} = \begin{cases} 0, \ j < \frac{N}{2}, \\ \Phi_{p,q}^{n+1} l_{p,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q,\xi_2}(\xi_{2,j} - d_{\xi_2}) \\ +\Phi_{p+1,q}^{n+1} l_{p+1,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q,\xi_2}(\xi_{2,j} - d_{\xi_2}) \\ +\Phi_{p,q+1}^{n+1} l_{p,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q+1,\xi_2}(\xi_{2,j} - d_{\xi_2}) \\ +\Phi_{p+1,q+1}^{n+1} l_{p+1,\xi_1}(\xi_{1,i} - d_{\xi_1}) l_{q+1,\xi_2}(\xi_{2,j} - d_{\xi_2}), j > \frac{N}{2}, \end{cases}$$

$l_{p,\xi_1}(\xi_{1,i}-d_{\xi_1}) = \frac{\xi_{1,p+1}-(\xi_{1,i}-d_{\xi_1})}{h(p+1)}$, $l_{p+1,\xi_1}(\xi_{1,i}-d_{\xi_1}) = \frac{(\xi_{1,i}-d_{\xi_1})-\xi_{1,p}}{h(p+1)}$, $l_{q,\xi_2}(\xi_{2,j}-d_{\xi_2}) = \frac{\xi_{2,q+1}-(\xi_{2,j}-d_{\xi_2})}{k(q+1)}$, $l_{q+1,\xi_2}(\xi_{2,j}-d_{\xi_2}) = \frac{(\xi_{2,j}-d_{\xi_2})-\xi_{2,q}}{k(q+1)}$, $\xi_{1,p}, \xi_{1,p+1}, \xi_{2,q}, \xi_{2,q+1}$ are the nodes, $\xi_{1,i} - d_{\xi_1} \in [\xi_{1,p}, \xi_{1,p+1}]$ and $\xi_{2,j} - d_{\xi_2} \in [\xi_{2,q}, \xi_{2,q+1}]$. The operators $\mathcal{G}_{\xi_1}^N$ and $\mathcal{G}_{\xi_2}^N$ produces M-matrices by [15]. Note: we call

$$s(\xi_{1,i}) = \begin{cases} 1 + \xi_{1,i}, & i \le \frac{N}{2}, \\ 1 + \frac{d_{\xi_1}+\xi_{1,i}}{2}, & i \ge \frac{N}{2}, \end{cases} \qquad s(\xi_{2,j}) = \begin{cases} 1 + \xi_{2,j}, & j \le \frac{N}{2}, \\ 1 + \frac{d_{\xi_2}+\xi_{2,j}}{2}, & j \ge \frac{N}{2} \end{cases}$$

and which are utilized in the coming lemmas.

5 Error analysis

Lemma 4. *Assume that $\chi_{o,j}^{n+\frac{1}{2}}$, $\chi_{N,j}^{n+\frac{1}{2}}$, $\mathcal{G}_{\xi_1}^N \chi_{i,j}^{n+\frac{1}{2}}$ are non negatives and $[D_{\xi_1}^+ - D_{\xi_1}^-]\chi_{N/2,j}^{n+\frac{1}{2}} \le 0$, then $\chi_{i,j}^{n+\frac{1}{2}}$ is also non negative for all i.*

Lemma 5. *Assume that $\chi_{i,0}^{n+1}$, $\chi_{i,N}^{n+1}$, $\mathcal{G}_{\xi_2}^N \chi_{i,j}^{n+1}$ are non negatives and $[D_{\xi_2}^+ - D_{\xi_2}^-]\chi_{i,N/2}^{n+1} \le 0$, then $\chi_{i,j}^{n+1} \ge 0$ is also non negative for all j.*

Lemma 6. *Let $\Phi_{i,j}^{n+\frac{1}{2}}$ be a solution of (4.1), then for all i, $|\Phi_{i,j}^{n+\frac{1}{2}}| \le C \max\{|\Phi_{0,j}^{n+\frac{1}{2}}|, |\Phi_{N,j}^{n+\frac{1}{2}}|, \sup_i |\mathcal{G}_{\xi_1}^N \Phi_{i,j}^{n+\frac{1}{2}}|\}$.*

Lemma 7. *Let $\Phi_{i,j}^{n+1}$ be a solution of (4.2), then for all j, $|\Phi_{i,j}^{n+1}| \le C \max\{|\Phi_{i,0}^{n+1}|, \sup_j |\mathcal{G}_{\xi_2}^N \Phi_{i,j}^{n+1}|, |\Phi_{i,N}^{n+1}|\}$.*

Numerical stability's of (4.1) and (4.2) are guaranteed by Lemmas 6 and 7.

To estimate absolute error bound $\Phi^{n+\frac{1}{2}}$ is decomposed as $\Phi^{n+\frac{1}{2}} = V^{n+\frac{1}{2}} + W^{n+\frac{1}{2}}$, they satisfy the following difference equations:

$$\begin{cases} \mathcal{G}^N_{\xi_1} V^{n+\frac{1}{2}}_{k,j} = \Delta t g_1(\xi_{1,k}, \xi_{2,j}, t_{n+1}) + V^n_{k,j}, \ k \neq 0, N/2, N, \\ D^+_{\xi_1} V^{n+\frac{1}{2}}_{N/2,j} - D^-_{\xi_1} V^{n+\frac{1}{2}}_{N/2,j} = \left[\frac{d}{d\xi_1} v^{n+\frac{1}{2}}(d_{\xi_1})\right], \\ V^{n+\frac{1}{2}}_{0,j} = v^{n+\frac{1}{2}}(\xi_{1,0}, \xi_{2,j}), \ V^{n+\frac{1}{2}}_{N,j} = v^{n+\frac{1}{2}}(1), \end{cases} \tag{5.1}$$

$$\begin{cases} \mathcal{G}^N_{\xi_1} W^{n+\frac{1}{2}}_{k,j} = W^n_{k,j}, \ k \neq N, N/2, 0, \\ D^+_{\xi_1} W^{n+\frac{1}{2}}_{N/2,j} - D^-_{\xi_1} W^{n+\frac{1}{2}}_{N/2,j} = -\left[\frac{d}{d\xi_1} v^{n+\frac{1}{2}}(d_{\xi_1})\right], \\ W^{n+\frac{1}{2}}_{0,j} = w^{n+\frac{1}{2}}(0), \ W^{n+\frac{1}{2}}_{N,j} = w^{n+\frac{1}{2}}(1). \end{cases} \tag{5.2}$$

Analogously, $\Phi^{n+1} = V^{n+1} + W^{n+1}$ and V^{n+1}, W^{n+1} are solutions of the difference problems:

$$\begin{cases} \mathcal{G}^N_{\xi_2} V^{n+1}_{i,k} = V^{n+\frac{1}{2}}_{i,k} + g_2(\xi_{1,i}, \xi_{2,k}, t_{n+1})\Delta t, \ k \neq 0, N/2, N, \\ V^{n+1}_{i,0} = v^{n+1}(0), \ V^{n+1}_{i,N} = v^{n+1}(1), \\ D^+_{\xi_2} V^{n+1}_{i,N/2} - D^-_{\xi_2} V^{n+1}_{i,N/2} = \left[\frac{d}{d\xi_2} v^{n+1}(d_{\xi_2})\right], \end{cases} \tag{5.3}$$

$$\begin{cases} \mathcal{G}^N_{\xi_2} W^{n+1}_{i,k} = W^{n+\frac{1}{2}}_{i,k}, \ k \neq 0, N/2, N, \\ W^{n+1}_{i,0} = w^{n+1}(0), \ W^{n+1}_{i,N} = w^{n+1}(1), \\ D^+_{\xi_2} W^{n+1}_{i,N/2} - D^-_{\xi_2} W^{n+1}_{i,N/2} = -\left[\frac{d}{d\xi_2} v^{n+1}(d_{\xi_2})\right]. \end{cases} \tag{5.4}$$

Lemma 8. *Let the mesh functions $\Phi^{\frac{1}{2}}_{i,j}$ and $V^{\frac{1}{2}}_{i,j}$ be defined by (4.1) and (5.1), respectively, then* $|\Phi^{\frac{1}{2}}_{i,j} - V^{\frac{1}{2}}_{i,j}| \leq C \begin{cases} N^{-1}, \ i = \frac{N}{8}, \dots, \frac{3N}{8}, \frac{5N}{8}, \dots, \frac{7N}{8}, \\ N^{-1} + \beta, \quad \text{otherwise}, \end{cases}$ *where the constant* $\beta = \max_{i,j} |\Phi^{\frac{1}{2}}_{i,j} - V^{\frac{1}{2}}_{i,j}|$.

Proof. For a fixed j we assume that $\chi^{\pm}(\xi_{1,i}) = C[N^{-1}s(\xi_{1,i}) + \chi(\xi_{1,i})\beta] \pm [\Phi^{\frac{1}{2}}_{i,j} - V^{\frac{1}{2}}_{i,j}]$, $\forall i$, where $\beta = \max_{i,j} |\Phi^{\frac{1}{2}}_{i,j} - V^{\frac{1}{2}}_{i,j}|$,

$$\chi(\xi_{1,i}) = \begin{cases} 1 - \frac{\xi_{1,i}}{\tau_{1,\xi_1}}, & i = 0, 1, \dots, \frac{N}{8}, \\ 0, & i = \frac{N}{8} + 1, \dots, \frac{3N}{8} - 1, \\ \frac{\xi_{1,i} - (d_{\xi_1} - \tau_{1,\xi_1})}{\tau_{1,\xi_1}}, & i = \frac{3N}{8}, \dots, \frac{N}{2}, \\ 1 - \frac{\xi_{1,i} - d_{\xi_1}}{\tau_{2,\xi_1}}, & i = \frac{N}{2}, \dots, \frac{5N}{8}, \\ 0, & i = \frac{5N}{8} + 1, \dots, \frac{7N}{8} - 1, \\ \frac{\xi_{1,i} - (1 - \tau_{2,\xi_1})}{\tau_{2,\xi_1}}, & i = \frac{7N}{8}, \dots, N. \end{cases}$$

Obvious that $\chi^{\pm}(\xi_{1,k}) \geq 0, \ k = 0, N$ and by the arguments of [22], we have $\mathcal{G}^N_{\xi_1} \chi^{\pm}(\xi_{1,i}) = \mathcal{G}^N_{\xi_1}(CN^{-1}s(\xi_{1,i}) + \beta\chi(\xi_{1,i})) \pm \mathcal{G}^N_{\xi_1}(\Phi^{\frac{1}{2}}_{i,j} - V^{\frac{1}{2}}_{i,j}) \geq 0, \ i \neq \frac{N}{2}$, $(D^+_{\xi_1} - D^-_{\xi_1})\chi^{\pm}(\xi_{1,\frac{N}{2}}) \leq 0$. Applying Lemma 4, we have $\chi^{\pm}(\xi_{1,i}) \geq 0$. $\square$

Lemma 9. *Let $v^{\frac{1}{2}}$ and $V^{\frac{1}{2}}$ be two solutions of (3.6) and (5.1), respectively, then $|v^{\frac{1}{2}}(\xi_{1,i},\xi_2) - V^{\frac{1}{2}}_{i,\xi_2}| \le CN^{-1},\ \forall i.$*

Proof. To derive the estimate,

$$\mathcal{G}^N_{\xi_1}(v^{\frac{1}{2}}(\xi_{1,i},\xi_2) - V^{\frac{1}{2}}_{i,\xi_2}) = \Delta t[-\varepsilon(\delta^2_{\xi_1} - \frac{d^2}{d\xi_1^2}) + Q_{1_{i,j}}[I^N_{\mathbf{d}} - I_{\mathbf{d}}]]v^{\frac{1}{2}}(\xi_{1,i},\xi_2),$$

by the results from [14, 23, 25], we have $|\mathcal{G}^N_{\xi_1}(v^{\frac{1}{2}}(\xi_{1,i},\xi_2) - V^{\frac{1}{2}}_{i,\xi_2})| \le C\frac{\Delta t}{N}$. By considering the mesh function $\chi^{\pm}(\xi_{1,i}) = CN^{-1}s(\xi_{1,i}) \pm (v^{\frac{1}{2}}(\xi_{1,i},\xi_2) - V^{\frac{1}{2}}_{i,\xi_2})$, we prove that $\chi^{\pm}(\xi_{1,k}) \ge 0,\ k = 0, N,\ \mathcal{G}^N_{\xi_1}\chi^{\pm}(\xi_{1,k}) \ge 0$ and $(D^+_{\xi_1} - D^-_{\xi_1})\chi^{\pm}(\xi_{1,\frac{N}{2}}) \le 0$. As an application of the Lemma 4, we get the result. □

Lemma 10. *Let $w^{\frac{1}{2}}$ and $W^{\frac{1}{2}}$ be two solutions of (3.7) and (5.2), respectively, then $|w^{\frac{1}{2}}(\xi_{1,i},\xi_2) - W^{\frac{1}{2}}_{i,\xi_2}| \le CN^{-1}\ln N,\ \forall i.$*

Proof. As an application of Lemmas 8, 9 and Theorem 1, we have

$$\begin{aligned}
&|\hat{\varphi}^{\frac{1}{2}}(\xi_{1,i},\xi_2) - \Phi^{\frac{1}{2}}_{i,\xi_2}| \le |\Phi^{\frac{1}{2}}_{i,\xi_2} - V^{\frac{1}{2}}_{i,\xi_2}| + |v^{\frac{1}{2}}(\xi_{1,i},\xi_2) - V^{\frac{1}{2}}_{i,\xi_2}| \\
&\quad + |\hat{\varphi}^{\frac{1}{2}}(\xi_{1,i},\xi_2) - v^{\frac{1}{2}}(\xi_{1,i},\xi_2)| \le CN^{-1} \\
&+ C\begin{cases} \exp\left(\frac{-\sqrt{P_0}\xi_{1,i}}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{(\xi_{1,i}-d_{\xi_1})\sqrt{P_0}}{\sqrt{\varepsilon}}\right), & i = 0,1,\ldots,\frac{N}{2}, \\ \exp\left(\frac{(\xi_{1,i}-d_{\xi_1})\sqrt{P_0}}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{\sqrt{P_0}(\xi_{1,i}-1)}{\sqrt{\varepsilon}}\right), & i = \frac{N}{2}+1,\ldots,N, \end{cases} \\
&+ C\begin{cases} N^{-1}, & i{=}\frac{N}{8},\ldots,\frac{3N}{8},\frac{5N}{8},\ldots,\frac{7N}{8}, \\ N^{-1}+\beta, & \text{otherwise}, \end{cases} \le C\begin{cases} \frac{1}{N}, & i{=}\frac{N}{8},\ldots,\frac{3N}{8},\frac{5N}{8},\ldots,\frac{7N}{8}, \\ \frac{1}{N}+\beta+1, & \text{otherwise}, \end{cases}
\end{aligned}$$

where $\beta = \max_{i,j}|\Phi^{\frac{1}{2}}_{i,j} - V^{\frac{1}{2}}_{i,j}|$. Hence, $|\hat{\varphi}^{\frac{1}{2}}(\xi_{1,i},\xi_2) - \Phi^{\frac{1}{2}}_{i,\xi_2}| \le CN^{-1},\ i = \frac{N}{8},\ldots,\frac{3N}{8},\frac{5N}{8},\ldots,\frac{7N}{8}$. Therefore, $|w^{\frac{1}{2}}(\xi_{1,i},\xi_2) - W^{\frac{1}{2}}_{i,\xi_2}| \le CN^{-1}, i = \frac{N}{8},\ldots,\frac{3N}{8}, \frac{5N}{8},\ldots,\frac{7N}{8}$. In the region $[0,\tau_{1,\xi_1}]$, we define the mesh function

$$\chi_1^{\pm}(\xi_{1,i}) = CN^{-1}\Big(s(\xi_{1,i}) + \frac{\tau_{\xi_1}}{\varepsilon^2}\xi_{1,i}\Big) \pm (w^{\frac{1}{2}} - W^{\frac{1}{2}}),\ \xi_{1,i} \in [0,\tau_{1,\xi_1}] \cap \overline{\Omega}^N_{\xi_1},$$

where $\tau_{\xi_1} = \min\{\tau_{1,\xi_1},\tau_{2,\xi_1}\}$. Then for $i = 0, \frac{N}{8}$ we have, $\chi_1^{\pm}(\xi_{1,i}) \ge 0$. Further, $|\mathcal{G}^N_{\xi_1}(w^{\frac{1}{2}} - W^{\frac{1}{2}})| \le C\Delta t\varepsilon^{-2}N^{-1}$. Now, let $\xi_{1,i} \in (0,\tau_{1,\xi_1}) \cap \overline{\Omega}^N_{\xi_1}$

$$\begin{aligned}
\mathcal{G}^N_{\xi_1}\chi_1^{\pm}(\xi_{1,i}) &= CN^{-1}\mathcal{G}^N_{\xi_1}s(\xi_{1,i}) + CN^{-1}\frac{\tau_{\xi_1}}{\varepsilon^2}\mathcal{G}^N_{\xi_1}\xi_{1,i} \pm \mathcal{G}^N_{\xi_1}(w^{\frac{1}{2}} - W^{\frac{1}{2}}) \\
&\ge CN^{-1}(1 + \Delta t + \frac{\Delta t}{\varepsilon^2}) \mp C\frac{\Delta t}{\varepsilon^2}N^{-1} \ge 0
\end{aligned}$$

for a properly assumed $C > 0$. Application of Lemma 4, we obtain that

$$|w^{\frac{1}{2}} - W^{\frac{1}{2}}| \le CN^{-1}\ln N,\ i = 0,1,\ldots,N/8.$$

Let $\xi_{1,i} \in [d_{\xi_1} - \tau_{1,\xi_1}, d_{\xi_1} + \tau_{2,\xi_1}] \cap \overline{\Omega}^N_{\xi_1}$, then $\chi_2^{\pm}(\xi_{1,i}) = CN^{-1}\big(s(\xi_{1,i}) + \frac{\tau_{\xi_1}}{\varepsilon^2}\alpha_1(\xi_{1,i})\big) \pm (w^{\frac{1}{2}} - W^{\frac{1}{2}})$, where

$$\alpha_1(\xi_{1,i}) = \begin{cases} 0, & \xi_{1,i} \in [0, d_{\xi_1} - \tau_{1,\xi_1}], \\ \xi_{1,i} - d_{\xi_1} + \tau_{1,\xi_1}, & \xi_{1,i} \in [d_{\xi_1} - \tau_{1,\xi_1}, d_{\xi_1}], \\ d_{\xi_1} + \tau_{2,\xi_1} - \xi_{1,i}, & \xi_{1,i} \in [d_{\xi_1}, d_{\xi_1} + \tau_{2,\xi_1}]. \end{cases}$$

We see that $\chi_2^{\pm}(\xi_{1,i}) \geq 0,\ i = \frac{3N}{8}, \frac{5N}{8}$, $\mathcal{G}^N_{\xi_1}\chi_2^{\pm}(\xi_{1,i}) \geq 0,\ i = \frac{3N}{8}+1, \ldots, \frac{5N}{8} - 1,\ i \neq \frac{N}{2}$, hence by Lemma 4, $|w^{\frac{1}{2}} - W^{\frac{1}{2}}| \leq CN^{-1}\ln N$, $i = \frac{3N}{8}+1, \ldots, \frac{5N}{8}-1$. Similarly, we prove the result in the domain $[1 - \tau_{2,\xi_1}, 1]$ by using the mesh function

$$\chi_3^{\pm}(\xi_{1,i}) = CN^{-1}\Big(s(\xi_{1,i}) + \frac{\tau_{\xi_1}}{\varepsilon^2}\alpha_2(\xi_{1,i})\Big) \pm (w^{\frac{1}{2}} - W^{\frac{1}{2}}),\ \xi_{1,i} \in [1 - \tau_{2,\xi_1}, 1] \cap \overline{\Omega}^N_{\xi_1},$$

where $\alpha_2(\xi_{1,i}) = \begin{cases} 0, & \xi_{1,i} \in [0, 1 - \tau_{2,\xi_1}], \\ \xi_{1,i} + \tau_{2,\xi_1} - 1, & \xi_{1,i} \in [1 - \tau_{2,\xi_1}, 1]. \end{cases}$

Therefore, $|w^{\frac{1}{2}}(\xi_{1,i}, \xi_2) - W^{\frac{1}{2}}_{i,\xi_2}| \leq CN^{-1}\ln N,\ \forall i.$ □

Lemma 11. *$\|\hat{\varphi}^{\frac{1}{2}} - \Phi^{\frac{1}{2}}\| \leq CN^{-1}\ln N$, where $\hat{\varphi}^{\frac{1}{2}}$ and $\Phi^{\frac{1}{2}}$ are solutions of (3.1) and (4.1), respectively.*

Proof. From the above two lemmas, we get

$$|\hat{\varphi}^{\frac{1}{2}}(\xi_{1,i}, \xi_2) - \Phi^{\frac{1}{2}}_{i,\xi_2}| \leq |v^{\frac{1}{2}}(\xi_{1,i}, \xi_2) - V^{\frac{1}{2}}_{i,\xi_2}| + |w^{\frac{1}{2}}(\xi_{1,i}, \xi_2) - W^{\frac{1}{2}}_{i,\xi_2}| \leq CN^{-1}\ln N.$$

Hence the proof. □

Lemma 12. *Let v^1, w^1, $\hat{\varphi}^1$, V^1, W^1, and Φ^1 be the solutions of (3.11), (3.12), (3.2), (5.3), (5.4) and (4.2), respectively, then*

$$\|v^1 - V^1\| \leq CN^{-1},\ \|w^1 - W^1\| \leq C\frac{\ln N}{N},\ \|\hat{\varphi}^1 - \Phi^1\| \leq C\frac{\ln N}{N}.$$

Proof. We know that $\hat{\varphi}^1(\xi_1, 0) = \Phi^1_{\xi_1,0}$ and $\hat{\varphi}^1(\xi_1, 1) = \Phi^1_{\xi_1,N}$. Analogous to Lemma 8, this lemma is proved. For that assume

$$\theta_1^{\pm}(\xi_{2,j}) = C(N^{-1}s(\xi_{2,j}) + \zeta\chi(\xi_{2,j})) \pm (\Phi^1 - V^1),\ \zeta = \max_{i,j}|\Phi^1_{i,j} - V^1_{i,j}|,$$

where

$$\chi(\xi_{2,j}) = \begin{cases} 1 - \frac{\xi_{2,j}}{\tau_{1,\xi_2}}, & j = 0, 1, \ldots, \frac{N}{8}, \\ 0, & j = \frac{N}{8}+1, \ldots, \frac{3N}{8} - 1, \\ \frac{\xi_{2,j} - (d_{\xi_2} - \tau_{1,\xi_2})}{\tau_{1,\xi_2}}, & j = \frac{3N}{8}, \ldots, \frac{N}{4}, \\ 1 - \frac{\xi_{2,j} - d_{\xi_2}}{\tau_{2,\xi_2}}, & j = \frac{N}{4}, \ldots, \frac{5N}{8}, \\ 0, & j = \frac{5N}{8}+1, \ldots, \frac{7N}{8} - 1, \\ \frac{\xi_{2,j} - (1 - \tau_{2,\xi_2})}{\tau_{2,\xi_2}}, & j = \frac{7N}{8}, \ldots, N, \end{cases}$$

we prove the following,

$$\theta_1^{\pm}(\xi_{2,0}) \geq 0, \theta_1^{\pm}(\xi_{2,N}) \geq 0, \mathcal{G}_{\xi_2}^N \theta_1^{\pm}(\xi_{2,j}) \geq 0,\ j \neq \frac{N}{2},$$
$$[D_{\xi_2}^- - D_{\xi_2}^+]\theta_1^{\pm}(\xi_{2,j}) \geq 0,\ j = \frac{N}{2}$$

and hence by the Lemma 5,

$$|\Phi^1 - V^1| \leq C \begin{cases} N^{-1},\ j = \frac{N}{8}, \dots, \frac{3N}{8},\ \frac{5N}{8}, \dots, \frac{7N}{8},\ i = 1, \dots, \frac{3N}{4}, \\ N^{-1} + \zeta, \quad \text{otherwise.} \end{cases}$$

The solutions v^1 and V^1 of (3.11) and (5.3) satisfy

$$\mathcal{G}_{\xi_2}^N (v^1(\xi_1, \xi_{2,j}) - V_{\xi_1,j}^1) = \Delta t[-\varepsilon(\delta_{\xi_2}^2 - \frac{d^2}{d\xi_2^2}) + Q_{2_{i,j}}[I_{\mathbf{d}}^N - I_{\mathbf{d}}]]v^1(\xi_1, \xi_{2,j}),$$

and $|\mathcal{G}_{\xi_2}^N (v^1(\xi_1, \xi_{2,j}) - V_{\xi_1,j}^1)| \leq C\Delta t N^{-1}$. Suitably assuming a mesh function we prove that $\|v^1 - V^1\| \leq CN^{-1}$. To estimate $\|w^1 - W^1\|$, we write

$$|\hat{\varphi}^1(\xi_1, \xi_{2,j}) - \Phi_{\xi_1,j}^1| \leq |\Phi_{\xi_1,j}^1 - V_{\xi_1,j}^1| + |v^1(\xi_1, \xi_{2,j}) - V_{\xi_1,j}^1| + |\hat{\varphi}^1(\xi_1, \xi_{2,j})$$
$$- v^1(\xi_1, \xi_{2,j})| \leq C \begin{cases} N^{-1},\ j = \frac{N}{8}, \dots, \frac{3N}{8},\ \frac{5N}{8}, \dots, \frac{7N}{8}, \\ 1 + N^{-1} + \zeta, \quad \text{otherwise.} \end{cases}$$

Hence, $|\hat{\varphi}^1(\xi_1, \xi_{2,j}) - \Phi_{\xi_1,j}^1| \leq CN^{-1},\ j = \frac{N}{8}, \dots, \frac{3N}{8},\ \frac{5N}{8}, \dots, \frac{7N}{8}$ and $|w^1(\xi_1, \xi_{2,j}) - W_{\xi_1,j}^1| \leq CN^{-1}, j = \frac{N}{8}, \dots, \frac{3N}{8},\ \frac{5N}{8}, \dots, \frac{7N}{8}$. Using the barrier function

$$\theta_2^{\pm}(\xi_{2,j}) = CN^{-1} \frac{\tau_{\xi_2}}{\varepsilon^2} \xi_{2,j} \pm (w^1 - W^1),\ \xi_{2,j} \in [0, \tau_{1,\xi_2}] \cap \overline{\Omega}_{\xi_2}^N,$$

where $\tau_{\xi_2} = \min\{\tau_{1,\xi_2}, \tau_{2,\xi_2}\}$. For $j = 0, \frac{N}{8}$ we proved that $\theta_2^{\pm}(\xi_{2,j}) \geq 0$, and $\mathcal{G}_{\xi_2}^N \theta_2^{\pm}(\xi_{2,j}) \geq 0,\ j = 1, \dots, \frac{N}{8} - 1$. Therefore, for $j = 0, 1 \dots, \frac{N}{8}$, we have $\|w^1 - W^1\| \leq CN^{-1} \ln N$. Using $\theta_3^{\pm}$ and $\theta_4^{\pm}$

$$\theta_3^{\pm}(\xi_{2,j}) = CN^{-1} \begin{cases} 0, & \xi_{2,j} \in [0, d_{\xi_2} - \tau_{1,\xi_2}], \\ \frac{\tau_{\xi_2}}{\varepsilon^2} \left(\frac{\xi_{2,j} - d_{\xi_2} + \tau_{1,\xi_2}}{\tau_{1,\xi_2}} \right), & \xi_{2,j} \in [d_{\xi_2} - \tau_{1,\xi_2}, d_{\xi_2}], \\ \frac{\tau_{\xi_2}}{\varepsilon^2} \left(\frac{d_{\xi_2} + \tau_{2,\xi_2} - \xi_{2,j}}{\tau_{2,\xi_2}} \right), & \xi_{2,j} \in [d_{\xi_2}, d_{\xi_2} + \tau_{2,\xi_2}] \end{cases}$$
$$\pm (w^1 - W^1), \xi_{2,j} \in [d_{\xi_2} - \tau_{1,\xi_2}, d_{\xi_2} + \tau_{2,\xi_2}] \cap \overline{\Omega}_{\xi_2}^N,$$
$$\theta_4^{\pm}(\xi_{2,j}) = CN^{-1} \begin{cases} 0, & \xi_{2,j} \in [0, 1 - \tau_{2,\xi_2}], \\ \frac{\tau_{\xi_2}}{\varepsilon^2} \left(\frac{\xi_{2,j} - 1 + \tau_{2,\xi_2}}{\tau_{2,\xi_2}} \right), & \xi_{2,j} \in [1 - \tau_{2,\xi_2}, 1], \end{cases}$$
$$\pm (w^1 - W^1), \xi_{2,j} \in [1 - \tau_{2,\xi_2}, 1] \cap \overline{\Omega}_{\xi_2}^N,$$

we have shown $\|w^1 - W^1\| \leq CN^{-1} \ln N$. Hence $\|\hat{\varphi}^1 - \Phi^1\| \leq CN^{-1} \ln N$. □

Theorem 3. *Let $\hat{\varphi}^{n+\frac{1}{2}}$, $\hat{\varphi}^{n+1}$, $\Phi_{i,j}^{n+\frac{1}{2}}$ and $\Phi_{i,j}^{n+1}$ be the solutions of (3.1), (3.2), (4.1) and (4.2), respectively, then $\|\hat{\varphi}^{n+\frac{1}{2}} - \Phi^{n+\frac{1}{2}}\| \leq CN^{-1}\ln N$, $\|\hat{\varphi}^{n+1} - \Phi^{n+1}\| \leq CN^{-1}\ln N$.*

Proof. Iterating the integer value n one can have the desired. We know that $\hat{\varphi}^{n+\frac{1}{2}}(0,\xi_2) - \Phi_{0,\xi_2}^{n+\frac{1}{2}} = 0$, $\hat{\varphi}^{n+\frac{1}{2}}(1,\xi_2) - \Phi_{N,\xi_2}^{n+\frac{1}{2}} = 0$, $\hat{\varphi}^{n+1}(\xi_1,0) - \Phi_{\xi_1,0}^{n+1} = 0$ and $\hat{\varphi}^{n+1}(\xi_1,1) - \Phi_{\xi_1,N}^{n+1} = 0$.
$\mathcal{G}_{\xi_1}^N(\Phi^{n+\frac{1}{2}} - V^{n+\frac{1}{2}}) = \mathcal{G}_{\xi_1}^N\Phi^{n+\frac{1}{2}} - \mathcal{G}_{\xi_1}^N V^{n+\frac{1}{2}} = \Phi^n - V^n$,
$\|\mathcal{G}_{\xi_1}^N(\Phi^{n+\frac{1}{2}} - V^{n+\frac{1}{2}})\| \leq C\begin{cases} N^{-1}, \ i,j = \frac{N}{8},\dots,\frac{3N}{8},\ \frac{5N}{8},\dots,\frac{7N}{8}, \\ 1 + N^{-1} + \zeta, \quad \text{otherwise,}\end{cases}$
$\mathcal{G}_{\xi_2}^N(\Phi^{n+1} - V^{n+1}) = \mathcal{G}_{\xi_2}^N\Phi^{n+1} - \mathcal{G}_{\xi_2}^N V^{n+1} = \Phi^{n+\frac{1}{2}} - V^{n+\frac{1}{2}}$,
$\|\mathcal{G}_{\xi_2}^N(\Phi^{n+1} - V^{n+1})\| \leq C\begin{cases} N^{-1}, \ i,j = \frac{N}{8},\dots,\frac{3N}{8},\ \frac{5N}{8},\dots,\frac{7N}{8}, \\ 1 + N^{-1} + \zeta, \quad \text{otherwise,}\end{cases}$ where $\zeta = \max_n \max_{i,j}|\Phi_{i,j}^n - V_{i,j}^n|$ and repeatedly applying Lemmas 6, 7, iterating the value of n, we prove that $\|\Phi^\mu - V^\mu\| \leq C\begin{cases} N^{-1}, \ i,j = \frac{N}{8},\dots,\frac{3N}{8},\ \frac{5N}{8},\dots,\frac{7N}{8}, \\ 1 + N^{-1} + \zeta, \quad \text{otherwise,}\end{cases}$ $\mu = n+\frac{1}{2}, n+1$.

Using the discrete functions $\chi_1^{\pm}(\xi_{1,i}) = CN^{-1}s(\xi_{1,i}) \pm [v^{n+\frac{1}{2}}(\xi_{1,i},\xi_{2,j}) - V_{i,j}^{n+\frac{1}{2}}]$, $\forall i$, $\chi_2^{\pm}(\xi_{2,j}) = CN^{-1}s(\xi_{2,j}) \pm [v^{n+1}(\xi_{1,i},\xi_{2,j}) - V_{i,j}^{n+1}]$, $\forall j$, and Lemmas 6, 7, we prove that

$$\|v^{n+\frac{1}{2}} - V^{n+\frac{1}{2}}\| \leq CN^{-1}, \quad \|v^{n+1} - V^{n+1}\| \leq CN^{-1}.$$

It has been shown that $|w^\mu(\xi_{1,i},\xi_{2,j}) - W_{i,j}^\mu| \leq CN^{-1}$, $i,j = \frac{N}{8},\dots,\frac{3N}{8}, \frac{5N}{8},\dots,\frac{7N}{8}$, $\mu = n+\frac{1}{2},\ n+1$.

When $\xi_{1,i} \in [0,\tau_{1,\xi_1}] \cap \overline{\Omega}_{\xi_1}^N$, by using $\chi_3^{\pm}(\xi_{1,i}) = CN^{-1}(s(\xi_{1,i}) + \frac{\tau_{\xi_1}}{\varepsilon^2}\xi_{1,i}) \pm (w^{n+\frac{1}{2}} - W^{n+\frac{1}{2}})$, where $\tau_{\xi_1} = \min\{\tau_{1,\xi_1},\tau_{2,\xi_1}\}$ and Lemma 6, it is estimated that $|w^{n+\frac{1}{2}} - W^{n+\frac{1}{2}}| \leq CN^{-1}\ln N$, $i = 0,\dots,\frac{N}{8}$. Similarly, $|w^{n+1} - W^{n+1}| \leq CN^{-1}\ln N$, $j = 0,\dots,\frac{N}{8}$. Desired estimate can be obtained by suitable choice barrier functions in $[d_\mu - \tau_{1,\nu}, d_\nu + \tau_{2,\nu}]$ and $[1-\tau_{2,\nu},1]$, $\nu = \xi_1,\xi_2$. □

Theorem 4. *$\|\varphi - \Phi\| \leq C(\Delta t + N^{-1}\ln N)$, where $\varphi(\xi_{1,i},\xi_{2,j},t_n)$ and $\Phi_{i,j}^n$ are solutions of (2.1) and (4.2), respectively.*

Proof. We see that

$$\begin{aligned}\varphi(\xi_{1,i},\xi_{2,j},t_n) - \Phi_{i,j}^n &= \varphi(\xi_{1,i},\xi_{2,j},t_n) - \hat{\varphi}^n(\xi_{1,i},\xi_{2,j}) + \hat{\varphi}^n(\xi_{1,i},\xi_{2,j}) - \Phi_{i,j}^n,\\ \|\varphi(t_n) - \Phi^n\| &\leq \|\hat{\varphi}^n - \Phi^n\| + \|\varphi(t_n) - \hat{\varphi}^n\|.\end{aligned}$$

From the Remark 4 and Theorem 3 we have

$$\begin{aligned}\|\varphi(t_n) - \Phi^n\| &\leq \|\varphi(t_n) - \hat{\varphi}^n\| + \|\hat{\varphi}^n - \Phi^n\| \leq C\Delta t + CN^{-1}\ln N,\\ \sup_n \|\varphi(t_n) - \Phi^n\| &\leq C(\Delta t + N^{-1}\ln N).\end{aligned}$$

□

6 Numerical validation

We present two numerical examples for validating the final result in this section. The exact solutions of examples are not available on hand. So dual mesh / double mesh technique is applied for calculating the error and rate of convergence. We put

$$E_\varepsilon^N = \max_{i,j} \mid \Phi_{i,j}^N(h,k,\Delta t) - \Phi_{i,j}^N(h/2,k/2,\Delta t/2) \mid, \quad D_{\xi_1,\xi_2}^N = \max_\varepsilon E_\varepsilon^N,$$

$$\rho^N = \log_2\left(D_{\xi_1,\xi_2}^N / D_{\xi_1,\xi_2}^{2N}\right),$$

where $\Phi_{i,j}^N(h,k,\Delta t)$ and $\Phi_{i,j}^N(h/2,k/2,\Delta t/2)$ are the calculated solution at the grid $(\xi_{1,i},\xi_{2,j},t_n)$ with step lengths $(h,k,\Delta t)$ and $(h/2,k/2,\frac{\Delta t}{2})$, respectively, D_{ξ_1,ξ_2}^N is calculated for a fixed N and for all ε.

Example 1. Consider reaction diffusion type 2D problem (2.1).

$$\frac{\partial\varphi}{\partial t} - \varepsilon\Delta\varphi + P(\mathbf{z})\varphi(\mathbf{z},t) + Q(\mathbf{z})\varphi(\mathbf{z}-\mathbf{d},t) = g(\mathbf{z},t), \ (\mathbf{z},t) \in \Omega \times (0,T],$$

$P_1(\mathbf{z}) = 1 + \xi_1^2 + \xi_2^2$, $P_2(\mathbf{z}) = 1 + \xi_1\xi_2$, $Q_1(\mathbf{z}) = -0.25 - \xi_1^2(1-\xi_1)$, $Q_2(\mathbf{z}) = -0.25 - \xi_2(1-\xi_2)^2$, $g_1(\mathbf{z},t) = \frac{\xi_2^2(1-\xi_2)}{(1+\xi_1^2+\xi_2^2)}\exp(t^2 - \frac{\xi_1\xi_2}{1+(\xi_1^2+\xi_2^2)})$, $g_2(\mathbf{z},t) = \exp(-t^2) \times \xi_1(1-\xi_1)\exp(-\xi_1^2-\xi_2^2)/2, \varphi_0 = \xi_1(1-\xi_1)\xi_2(1-\xi_2)$, $d_{\xi_1} = 0.5 = 2d_{\xi_2}$. Error and rate of convergence for this example are presented in Table 1 for different values of ε, N and M. The total execution time for this example is 3.8773e+03 seconds. Numerical solution and loglog plot are given in Figures 2 and 3, respectively.

Table 1. Rate of convergence and error for the Example 1.

ε / ROC	N Number spatial grids & M number of temporal grids (N=M)				
$\downarrow$	16	32	64	128	256
10^{-1}	1.710e-03	7.961e-04	3.801e-04	1.851e-04	9.125e-05
ROC	**1.103e+00**	**1.066e+00**	**1.038e+00**	**1.021e+00**	-
10^{-3}	4.5551e-03	2.3588e-03	9.4068e-04	3.1361e-04	9.2535e-5
ROC	**9.4944e-01**	**1.3263e+00**	**1.5847e+00**	**1.7609e+00**	-
10^{-5}	1.044e-02	8.007e-03	4.801e-03	2.451e-03	1.074e-3
ROC	**3.8274e-01**	**7.3800e-01**	**9.6961e-01**	**1.1901e+00**	-
10^{-7}	1.4289e-02	1.3072e-02	1.0035e-02	6.9484e-03	3.972e-3
ROC	**1.2836e-01**	**3.8150e-01**	**5.3024e-01**	**8.0689e-01**	-
10^{-9}	1.5557e-02	1.3767e-02	1.0259e-02	6.8440e-03	4.264e-3
ROC	**1.7636e-01**	**4.2436e-01**	**5.8393e-01**	**6.8252e-01**	-
D_{ξ_1,ξ_2}^N	**1.5557e-02**	**1.3767e-02**	**1.0259e-02**	**6.9484e-03**	4.264e-3
ρ^N	**1.7636e-01**	**4.2436e-01**	**5.6209e-01**	**7.0435e-01**	-

Example 2. Consider reaction diffusion type 2D problem (2.1).

$$\frac{\partial\varphi}{\partial t} - \varepsilon\Delta\varphi + P(\mathbf{z})\varphi(\mathbf{z},t) + Q(\mathbf{z})\varphi(\mathbf{z}-\mathbf{d},t) = g(\mathbf{z},t), \ (\mathbf{z},t) \in \Omega \times (0,T],$$

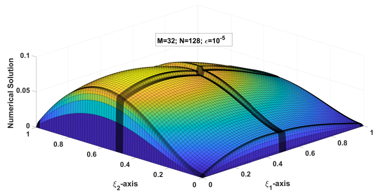

Figure 2. Computational solution graph of Example 1 where the parameters: $d_{\xi_1} = d_{\xi_2} = 0.5$, $N = 2^7$, $M = 2^5$, $\varepsilon = 10^{-5}$.

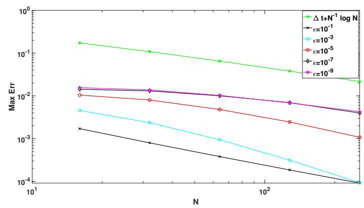

Figure 3. Maximum error loglog plot of Example 1.

$P_1(\mathbf{z})=1+\exp(\frac{-1}{1+\xi_1^2\xi_2^2})$, $P_2(\mathbf{z})=\frac{1+\xi_1\xi_2}{1+\xi_1^4+\xi_2^4}$, $Q_1(\mathbf{z})=\frac{-1}{1+\xi_1^2}$, $Q_2(\mathbf{z})=-\exp(\frac{-1}{1+\xi_1\xi_2})$, $g_1(\mathbf{z},t)=\xi_2(1-\xi_2)\exp(\frac{-t^2}{1+\xi_1\xi_2+\xi_1^2\xi_2^3})$, $g_2(\mathbf{z},t)=\xi_1\sin(1-\xi_1)(t^2+\frac{\exp(-t)}{(1+\xi_1^4+\xi_2^4)})$, $d_{\xi_1} = 0.5 = d_{\xi_2}$.

Error and rate of convergence for this example are presented in Tables 2 and 3 for different values of ε, N and M. The total execution time for the results given in Table 2 is $4.2382e+03$ seconds, for the Table 3, when $N = 2n$, $M = 4n$ the total execution time is $2.7239e+04$ seconds and for the other case $N = 4n$, $M = 2n$, it is $4.7445e+04$ seconds. Numerical solution and loglog plot are given in Figures 4 and 5, respectively.

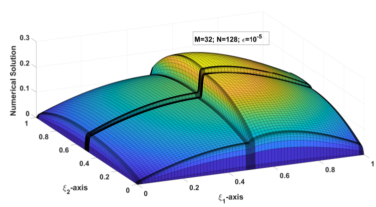

Figure 4. Computational solution graph of Example 2 where the parameters: $d_{\xi_1} = d_{\xi_2} = 0.5$, $N = 2^7$, $M = 2^5$, $\varepsilon = 10^{-5}$.

Table 2. Rate of convergence and error for the Example 2.

ε / ROC	N Number spatial grids & M number of temporal grids (N=M)				
$\downarrow$	16	32	64	128	256
10^{-1}	3.6035e-03	1.6948e-03	8.1479e-04	3.9827e-04	1.967e-4
ROC	**1.0883e+00**	**1.0566e+00**	**1.0327e+00**	**1.0177e+00**	-
10^{-3}	1.8122e-02	9.6294e-03	3.9510e-03	1.3446e-03	4.019e-4
ROC	**9.1218e-01**	**1.2852e+00**	**1.5551e+00**	**1.7423e+00**	-
10^{-5}	4.5311e-02	3.5076e-02	2.1834e-02	1.1477e-02	5.136e-3
ROC	**3.6938e-01**	**6.8393e-01**	**9.2786e-01**	**1.1599e+00**	-
10^{-7}	5.4166e-02	5.2314e-02	4.2835e-02	3.1553e-02	1.875e-2
ROC	**5.0210e-02**	**2.8840e-01**	**4.4100e-01**	**7.5121e-01**	-
10^{-9}	5.5681e-02	5.3360e-02	4.2990e-02	3.0487e-02	1.978e-2
ROC	**6.1421e-02**	**3.1176e-01**	**4.9581e-01**	**6.2394e-01**	-
$D^N_{\xi_1,\xi_2}$	**5.5681e-02**	**5.3360e-02**	**4.2990e-02**	**3.1553e-02**	**1.978e-2**
ρ^N	**6.1421e-02**	**3.1176e-01**	**4.4622e-01**	**6.7353e-01**	-

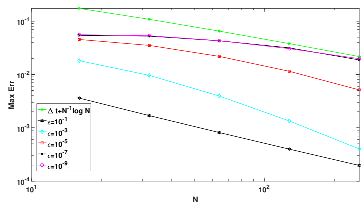

Figure 5. Maximum error loglog plot of Example 2.

7 Concluding remarks

Two dimensional SPDDEs of reaction diffusion type with space shift was examined in this article. This problem has many complexities due to higher dimension, singular perturbation nature, and shift arguments. These complexities resolved by applying ADI method, fitted method with bilinear interpolation on special mesh. Convergence of the numerical solution is established and whose rate of convergence is $O(\Delta t + N^{-1}\ln N)$.

Table 3. Rate of convergence and error for the Example 2.

$n \rightarrow$	16	32	64	128
ε / ROC	$N = 2n$-Number spatial grids & $M = 4n$-number of temporal grids			
10^{-1}	1.7838e-03	8.4119e-04	4.0553e-04	1.9862e-04
ROC	**1.0844e+00**	**1.0526e+00**	**1.0298e+00**	-
10^{-3}	1.3120e-02	5.9508e-03	2.2339e-03	7.2021e-04
ROC	**1.1406e+00**	**1.4135e+00**	**1.6330e+00**	-
10^{-5}	3.6910e-02	2.3134e-02	1.2810e-02	6.3327e-03
ROC	**6.7396e-01**	**8.5280e-01**	**1.0163e+00**	-
10^{-7}	5.4407e-02	4.3650e-02	3.1645e-02	1.8624e-02
ROC	**3.1781e-01**	**4.6400e-01**	**7.6477e-01**	-
10^{-9}	5.5526e-02	4.3830e-02	3.0649e-02	1.9680e-02
ROC	**3.4124e-01**	**5.1609e-01**	**6.3911e-01**	-
$D^N_{\xi_1,\xi_2}$	**5.5526e-02**	**4.3830e-02**	**3.1645e-02**	**1.9680e-02**
ρ^N	**3.4124e-01**	**4.6996e-01**	**6.8524e-01**	-
ε / ROC	$N = 4n$-Number spatial grids & $M = 2n$-number of temporal grids			
10^{-1}	7.9271e-04	3.9231e-04	1.9516e-04	9.7333e-05
ROC	**1.0148e+00**	**1.0073e+00**	**1.0037e+00**	-
10^{-3}	2.3175e-03	7.4168e-04	2.1247e-04	5.7087e-05
ROC	**1.6437e+00**	**1.8035e+00**	**1.8960e+00**	-
10^{-5}	1.8617e-02	9.0271e-03	3.6086e-03	1.2232e-03
ROC	**1.0443e+00**	**1.3228e+00**	**1.5608e+00**	-
10^{-7}	4.1044e-02	3.0868e-02	1.8283e-02	9.4681e-03
ROC	**4.1104e-01**	**7.5563e-01**	**9.4935e-01**	-
10^{-9}	4.1153e-02	2.9808e-02	1.9525e-02	1.1865e-02
ROC	**4.6526e-01**	**6.1040e-01**	**7.1856e-01**	-
$D^N_{\xi_1,\xi_2}$	**4.1153e-02**	**3.0868e-02**	**1.9525e-02**	**1.1865e-02**
ρ^N	**4.1486e-01**	**6.6080e-01**	**7.1856e-01**	-

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