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A boundary value problem with an integral condition for a certain fractional differential equation

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Abstract. The aim of this work is to prove the existence and the uniqueness of the solution of one dimensional initial boundary value problem for a parabolic equation with a Caputo time fractional differential operator supplemented by periodic nonlocal boundary condition and integral condition. First, an a priori estimate is established for the associated problem. Secondly, the density of the operator range generated by the considered problem is proved by using the functional analysis method.

Keywords: energy inequality; fractional differential equation; Caputo derivatives; integral boundary conditions; strong solution.

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1 Introduction

Some problems related to physical and technical issues can be effectively described in terms of nonlocal problems with integral conditions in partial differential equations. These nonlocal conditions arise mainly when the values on the boundary cannot be measured directly, while their average values are known. This type of problem can be found in various physics problems such as heat conduction [8, 9, 12, 17, 18], plasma physics [28], thermoelasticity [30], electrochemistry [11], chemical diffusion [14] and underground water flow [15, 25, 31].

In recent years, fractional differential equations are playing a major role in various fields such as physics, biology, engineering, signal processing, control theory, finance, fractal dynamics and many other physical processes and diverse applications [10, 21, 26]. Studying or finding approximate and exact solutions to the partial fractional differential equations is regarded as a very important task. Many powerful and efficient methods have been proposed to obtain the numerical solution [3] and the exact solution of partial fractional differential equations such as optimal homotopy analysis method [5], optimal homotopy asymptotic method [29], variational iteration method [7], Adomian decomposition method [6], and many others. As a result, fractional boundary value problems have gained increasing attention in research, as a large number of physical phenomena and many problems in modern physics and technology can be described in terms of nonlocal problems for example problems in partial differential equations with integral conditions.

Several methods have been used to investigate the existence and uniqueness of solution for fractional partial differential equations, including the Lax-Milgram theorem, fixed point theorem and numerical method such as finite element methods or spectral methods [13, 16, 20]. For our problem (2.1)–(2.3), we use the functional analysis method, the so-called energy inequality method, because it's the most powerful tool to prove the existence and uniqueness of the solution for fractional differential equation with integral condition. In the literature, there are many papers using the functional analysis method such as [1, 2, 4, 19, 22, 23, 24].

The motivation of this paper lies in developing the used method for a fractional order partial differential equation with periodic and nonlocal condition of integral type. First, a priori estimate is established for the strong solution of the problem. Subsequently, the existence and uniqueness of the strong solution of the problems is established. This paper is organized as follows: In Section 2, the problem is stated. Section 3 deals with the proof of the uniqueness of the solution using an a priori estimate. For the existence of the solution, the density of the range of the operator generated by the considered problem is proved in Section 4. Finally, Section 5 presents the conclusion.

2 Statement of the problem

In this section, The problem of fractional partial differential equation with integral condition is stated as follows: Let us consider the rectangular domain $Q = [0, 1] \times [0, T]$ such that $0 < T < +\infty$. In the domain Q, we consider the following equation.

$$\mathcal{L}u = {}_0^c\partial_t^\alpha u - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), \qquad \forall (x,t) \in (0,1) \times (0,T), \tag{2.1}$$

with the initial condition

$$\ell u = u(x,0) = \varphi(x), \qquad \forall x \in (0,1),$$

the periodic boundary condition

$$u(0,t) = u(1,t), \qquad \forall t \in (0,T), \tag{2.2}$$

and the integral condition

$$\int_0^1 u(x,t)dx = 0, \qquad \forall t \in (0,T). \tag{2.3}$$

In addition, we assume that the function $a(x,t)$ satisfies the condition:

$$0 < a_0 \le a(x,t) \le a_1, \quad \forall (x,t) \in Q, \tag{2.4}$$

and the function $\varphi(x)$ satisfies the compatibilities conditions:

$$\varphi(0) = \varphi(1), \quad \int_0^1 \varphi(x)dx = 0.$$

The symbol ${}_0^c\partial_t^\alpha$ denotes the time fractional derivative operator in the Caputo sense of order $0 < \alpha < 1$. It is defined by

$${}_0^c\partial_t^\alpha v(t) = I^{1-\alpha} \frac{\partial v(\tau)}{\partial \tau},$$

where

$$I^{1-\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} v(\tau) d\tau,$$

for a certain function v [27], where $\Gamma(\cdot)$ is the Gamma function. For $\alpha = 1$ the Caputo derivative becomes a conventional first derivative of the function $v(t)$.

The given problem (2.1)–(2.3) can be considered as finding a solution of the operator equation $Lu = (\mathcal{L}u, \ell u) = \mathcal{F} = (f, \varphi)$, where the operator L has as a domain of definition $D(L)$ consisting of functions $u \in L^2(Q)$ such that $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \in L^2(Q)$ and satisfying the conditions (2.2) and (2.3).

The operator L is an operator acting on E into $\mathbb{F}$, where E is Banach space of functions $u \in L^2(Q)$, with a finite norm

$$\|u\|_E^2 = \left\|x(1-x){}_0^c\partial_t^\alpha u\right\|_{L^2(Q)}^2 + \sup_{0\le t\le T} I^{1-\alpha}\left(\left\|x(1-x)\frac{\partial u}{\partial x}\right\|_{L^2(0,1)}^2 + \|u\|_{L^2(0,1)}^2\right).$$

$\mathbb{F}$ is Hilbert space of functions $\mathcal{F} = (f, \varphi)$, with the finite norm

$$\|\mathcal{F}\|_{\mathbb{F}}^2 = \left\|x(1-x)f\right\|_{L^2(Q)}^2 + \left\|x(1-x)\frac{\partial \varphi}{\partial x}\right\|_{L^2(0,1)}^2 + \|\varphi\|_{L^2(0,1)}^2.$$

Then, we show that the operator L has a closure $\overline{L}$ and later on, in Section 3, we establish an energy inequality of the following type (see Theorem 1):

$$\|u\|_E \le C\,\|Lu\|_{\mathbb{F}}, \qquad \forall u \in D(L). \tag{2.5}$$

Definition 1. A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of problem (2.1)–(2.3).

Since the points of the graph of the operator $\overline{L}$ are limits of sequences of points of the graph of L, we can extend the a priori estimate (2.5) to be applied to strong solutions by taking the limits, that is, we have the inequality

$$\|u\|_E \leq C\left\|\overline{L}u\right\|_{\mathbb{F}}, \quad \forall u \in D(\overline{L}).$$

From this inequality, we deduce the uniqueness of a strong solution, if it exists, and that the range of the operator $\overline{L}$ coincides with the closure of the range of L.

Proposition 1. *The operator $L : E \longrightarrow \mathbb{F}$ admits a closure $\overline{L}$.*

The following a priori estimate gives the uniqueness of the solution of the formulated linear problem.

3 Uniqueness of the solution

In this section, the uniqueness of the solution will be proved using the energy inequality method.

Theorem 1. *There exists a positive constant C, such that for each function $u \in D(L)$, we have*

$$\|u\|_E \leq C\,\|Lu\|_F\,, \quad \forall x,t \in Q. \tag{3.1}$$

Proof. Let

$$Mu = x^2(1-x)^2 {}_0^c\partial_t^\alpha u + \int_x^1 \frac{d\zeta}{a(\zeta,t)} \int_0^x \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,t)}}{\int_0^1 \frac{d\zeta}{a(\zeta,t)}} {}_0^c\partial_t^\alpha u d\zeta$$
$$+\int_0^x \frac{d\zeta}{a(\zeta,t)} \int_x^1 \left(\lambda - \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,t)}}{\int_0^1 \frac{d\zeta}{a(\zeta,t)}}\right) {}_0^c\partial_t^\alpha u d\zeta,$$

where λ is a scalar parameter such that $\lambda > 2a_1$.

We denote by $\Im_x u = \int_0^x u(\zeta,t)d\zeta$, taking the scalar product in $L^2(Q_t)$, where $Q_t = [0,1]\times[0,t]$ of (2.1) and the operator Mu, with $0 < t \leq T$, we have

$$\int_{Q_t} {}_0^c\partial_\tau^\alpha u M u dx d\tau - \int_{Q_t} \frac{\partial}{\partial x}\left(a(x,\tau)\frac{\partial u}{\partial x}\right) M u dx d\tau = \int_{Q_t} f M u dx d\tau. \tag{3.2}$$

Substituting Mu by its expression in the first term in the left-hand side of (3.2), integrating by parts with respect to x, we obtain

$$\int_0^1 x^2(1-x)^2 {}_0^c\partial_\tau^\alpha u {}_0^c\partial_\tau^\alpha u dx = \int_0^1 x^2(1-x)^2 ({}_0^c\partial_\tau^\alpha u)^2 dx,$$

$$\int_0^1 {}_0^c\partial_\tau^\alpha u \int_x^1 \frac{d\zeta}{a(\zeta,\tau)} \int_0^x \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}} {}_0^c\partial_\tau^\alpha u d\zeta dx$$
$$= \int_0^1 \frac{{}_0^c\partial_\tau^\alpha \Im_x u}{a(x,\tau)} \int_0^x \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}} {}_0^c\partial_\tau^\alpha u d\zeta dx$$
$$- \int_0^1 \frac{2(2x-1)(x-x^2)+\lambda\int_0^x \frac{d\zeta}{a(\zeta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}} {}_0^c\partial_\tau^\alpha u {}_0^c\partial_\tau^\alpha \Im_x u \int_x^1 \frac{d\zeta}{a(\zeta,\tau)} dx,$$

$$\int_0^1 {}_0^c\partial_\tau^\alpha u \int_0^x \frac{d\zeta}{a(\zeta,\tau)} \int_x^1 \Big(\lambda - \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}}\Big) {}_0^c\partial_\tau^\alpha u d\zeta dx$$
$$= \frac{\lambda}{2}\int_0^1 \frac{({}_0^c\partial_\tau^\alpha \Im_x u)^2}{a(x,\tau)} dx + \int_0^1 \frac{{}_0^c\partial_\tau^\alpha \Im_x u}{a(x,\tau)} \int_x^1 \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}} {}_0^c\partial_\tau^\alpha u d\zeta dx$$
$$- \int_0^1 \frac{2(2x-1)(x-x^2)+\lambda\int_0^x \frac{d\zeta}{a(\zeta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}} {}_0^c\partial_\tau^\alpha u {}_0^c\partial_\tau^\alpha \Im_x u \int_0^x \frac{d\zeta}{a(\zeta,\tau)} dx.$$

From the last equalities, we get

$$\int_0^1 {}_0^c\partial_\tau^\alpha u M u dx = \int_0^1 x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2 dx + \frac{\lambda}{2}\int_0^1 \frac{({}_0^c\partial_\tau^\alpha \Im_x u)^2}{a(x,\tau)} dx$$
$$- \int_0^1 \left(2(2x-1)(x-x^2) + \lambda\int_0^x \frac{d\zeta}{a(\zeta,\tau)}\right) {}_0^c\partial_\tau^\alpha u {}_0^c\partial_\tau^\alpha \Im_x u dx$$
$$+ \int_0^1 \frac{{}_0^c\partial_\tau^\alpha \Im_x u}{a(x,\tau)} dx \int_0^1 \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}} {}_0^c\partial_\tau^\alpha u d\zeta. \quad (3.3)$$

Integrating by parts the last two terms in the right-hand side of (3.3), with respect to x, we get

$$\int_0^1 {}_0^c\partial_\tau^\alpha u M u dx = \int_0^1 x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2 dx + \frac{\lambda}{2}\int_0^1 \frac{({}_0^c\partial_\tau^\alpha \Im_x u)^2}{a(x,\tau)} dx$$
$$+ \frac{1}{2}\int_0^1 \left(12(x-x^2) - 2 + \frac{\lambda}{a(x,\tau)}\right) ({}_0^c\partial_\tau^\alpha \Im_x u)^2 dx$$
$$- \int_0^1 \frac{{}_0^c\partial_\tau^\alpha \Im_x u}{a(x,\tau)} dx \int_0^1 \frac{12(\zeta-\zeta^2) - 2 + \frac{\lambda}{a(\zeta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}} {}_0^c\partial_\tau^\alpha \Im_\zeta u d\zeta. \quad (3.4)$$

Similarly, substituting Mu by its expression in the second term of the left-hand side of Equation (3.2), integrating by parts with respect to x and using the condition (2.3), we obtain

$$
\begin{aligned}
&-\int_0^1 x^2(1-x)^2\frac{\partial}{\partial x}\left(a(x,\tau)\frac{\partial u}{\partial x}\right){}_0^c\partial_\tau^\alpha u dx=\int_0^1 2a(x,\tau)(1-2x)\\
&\quad\times(x-x^2)\frac{\partial u}{\partial x}{}_0^c\partial_\tau^\alpha u dx+\int_0^1 x^2(1-x)^2a(x,\tau)\frac{\partial u}{\partial x}{}_0^c\partial_\tau^\alpha\frac{\partial u}{\partial x}dx,\\
&-\int_0^1\frac{\partial}{\partial x}\left(a(x,\tau)\frac{\partial u}{\partial x}\right)\int_x^1\frac{d\zeta}{a(\zeta,\tau)}\int_0^x\frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta\frac{d\eta}{a(\eta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}{}_0^c\partial_\tau^\alpha u d\zeta dx\\
&\quad=-u(1,\tau)\int_0^1\frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta\frac{d\eta}{a(\eta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}{}_0^c\partial_\tau^\alpha u d\zeta\\
&\quad+\int_0^1\frac{2(2x-1)(x-x^2)+\lambda\int_0^x\frac{d\zeta}{a(\zeta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}u{}_0^c\partial_\tau^\alpha u dx\\
&\quad+\int_0^1\frac{2(2x-1)(x-x^2)+\lambda\int_0^x\frac{d\zeta}{a(\zeta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}a(x,\tau)\frac{\partial u}{\partial x}{}_0^c\partial_\tau^\alpha u\int_x^1\frac{d\zeta}{a(\zeta,\tau)}dx,\\
&-\int_0^1\frac{\partial}{\partial x}\left(a(x,\tau)\frac{\partial u}{\partial x}\right)\int_0^x\frac{d\zeta}{a(\zeta,\tau)}\int_x^1\left(\lambda-\frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta\frac{d\eta}{a(\eta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}\right)\\
&\quad\times{}_0^c\partial_\tau^\alpha u d\zeta dx=u(0,\tau)\int_0^1\frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta\frac{d\eta}{a(\eta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}{}_0^c\partial_\tau^\alpha u d\zeta\\
&\quad+\int_0^1\left(\lambda-\frac{2(2x-1)(x-x^2)+\lambda\int_0^x\frac{d\zeta}{a(\zeta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}\right)u{}_0^c\partial_\tau^\alpha u dx\\
&-\int_0^1\left(\lambda-\frac{2(2x-1)(x-x^2)+\lambda\int_0^x\frac{d\zeta}{a(\zeta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}\right)a(x,\tau)\frac{\partial u}{\partial x}{}_0^c\partial_\tau^\alpha u\int_0^x\frac{d\zeta}{a(\zeta,\tau)}dx.
\end{aligned}
$$

From the last equalities and using the condition (2.2), we get

$$
\begin{aligned}
&-\int_0^1\frac{\partial}{\partial x}\left(a(x,\tau)\frac{\partial u}{\partial x}\right)Mudx\\
&\quad=\int_0^1 x^2(1-x)^2a(x,\tau)\frac{\partial u}{\partial x}{}_0^c\partial_\tau^\alpha\frac{\partial u}{\partial x}dx+\lambda\int_0^1 u{}_0^c\partial_\tau^\alpha u dx. \qquad (3.5)
\end{aligned}
$$

From (3.4) and (3.5), equality (3.2) becomes

$$
\begin{aligned}
&\int_{Q_t}x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2dxd\tau+\frac{\lambda}{2}\int_{Q_t}\frac{({}_0^c\partial_\tau^\alpha\Im_x u)^2}{a(x,\tau)}dxd\tau\\
&+\frac12\int_{Q_t}\left(12(x-x^2)-2+\frac{\lambda}{a(x,\tau)}\right)({}_0^c\partial_\tau^\alpha\Im_x u)^2dxd\tau\\
&-\int_{Q_t}\frac{{}_0^c\partial_\tau^\alpha\Im_x u}{a(x,\tau)}dx\int_0^1\frac{12(\zeta-\zeta^2)-2+\frac{\lambda}{a(\zeta,\tau)}}{\int_0^1\frac{d\zeta}{a(\zeta,\tau)}}{}_0^c\partial_\tau^\alpha\Im_\zeta u d\zeta d\tau
\end{aligned}
$$

$$+\int_{Q_t} x^2(1-x)^2 a(x,\tau)\frac{\partial u}{\partial x}{}_0^c\partial_\tau^\alpha \frac{\partial u}{\partial x}dxd\tau+\lambda\int_{Q_t} u{}_0^c\partial_\tau^\alpha u dxd\tau=\int_{Q_t} fMudxd\tau. \quad (3.6)$$

By using Holder's inequality in the fourth term in the left-hand side of (3.6), we get

$$\int_{Q_t} \frac{{}_0^c\partial_\tau^\alpha \Im_x u}{a(x,\tau)}dx\int_0^1 \frac{12(\zeta-\zeta^2)-2+\frac{\lambda}{a(\zeta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}}{}_0^c\partial_\tau^\alpha \Im_\zeta u d\zeta d\tau \quad (3.7)$$

$$\leq \int_0^t \sqrt{\lambda\int_0^1 \frac{({}_0^c\partial_\tau^\alpha \Im_x u)^2}{a(x,\tau)}dx}\sqrt{\int_0^1 \left(12(\zeta-\zeta^2)-2+\frac{\lambda}{a(\zeta,\tau)}\right)({}_0^c\partial_\tau^\alpha \Im_\zeta u)^2 d\zeta d\tau}.$$

Using the fact that

$$2u(t){}_0^c\partial_t^\alpha u(t) \geq {}_0^c\partial_t^\alpha (u(t))^2, \quad 0<\alpha<1, \quad (3.8)$$

and (2.4), the last two terms in the left-hand side of (3.6) are controlled by

$$\int_{Q_t} x^2(1-x)^2 a(x,\tau)\frac{\partial u}{\partial x}{}_0^c\partial_\tau^\alpha \frac{\partial u}{\partial x}dxd\tau \geq \frac{a_0}{2}\int_{Q_t} x^2(1-x)^2{}_0^c\partial_\tau^\alpha \left(\frac{\partial u}{\partial x}\right)^2 dxd\tau, \quad (3.9)$$

$$\lambda\int_{Q_t} u{}_0^c\partial_\tau^\alpha u dxd\tau \geq \frac{\lambda}{2}\int_{Q_t} {}_0^c\partial_\tau^\alpha (u)^2 dxd\tau. \quad (3.10)$$

Using the Dirichlet formula and integrating by parts the right-hand side of (3.9) and (3.10), with respect to τ, we get

$$\frac{a_0}{2}\int_{Q_t} x^2(1-x)^2{}_0^c\partial_\tau^\alpha \left(\frac{\partial u}{\partial x}\right)^2 dxd\tau$$

$$=\frac{a_0}{2\Gamma(1-\alpha)}\int_0^1 x^2(1-x)^2\int_0^t \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)^2\int_s^t (\tau-s)^{-\alpha}d\tau ds dx$$

$$=\frac{a_0}{2(1-\alpha)\Gamma(1-\alpha)}\int_0^1 x^2(1-x)^2\int_0^t (t-s)^{1-\alpha}\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)^2 ds dx$$

$$=\frac{a_0}{2}I^{1-\alpha}\left\|x(1-x)\frac{\partial u}{\partial x}\right\|^2_{L^2(0,1)}-\frac{a_0}{2\Gamma(2-\alpha)}t^{1-\alpha}\left\|x(1-x)\frac{\partial \varphi}{\partial x}\right\|^2_{L^2(0,1)}, \quad (3.11)$$

and

$$\frac{\lambda}{2}\int_{Q_t} {}_0^c\partial_\tau^\alpha (u)^2 dxd\tau = \frac{\lambda}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{\partial (u)^2}{\partial s}\int_s^t (\tau-s)^{-\alpha}d\tau ds dx$$

$$=\frac{\lambda}{2(1-\alpha)\Gamma(1-\alpha)}\int_0^1\int_0^t (t-s)^{1-\alpha}\frac{\partial (u)^2}{\partial s}ds dx$$

$$=\frac{\lambda}{2}I^{1-\alpha}\left\|u\right\|^2_{L^2(0,1)}-\frac{\lambda}{2\Gamma(2-\alpha)}t^{1-\alpha}\left\|\varphi\right\|^2_{L^2(0,1)}. \quad (3.12)$$

Therefore, by combining (3.7), (3.11) and (3.12) with (3.6), we get the following expression:

$$\int_{Q_t} x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2 dxd\tau + \frac{a_0}{2} I^{1-\alpha}\left\|x(1-x)\frac{\partial u}{\partial x}\right\|^2_{L^2(0,1)} + \frac{\lambda}{2} I^{1-\alpha}\|u\|^2_{L^2(0,1)}$$
$$+\frac{1}{2}\int_0^t \left(\sqrt{\lambda\int_0^1 \frac{({}_0^c\partial_\tau^\alpha \Im_x u)^2}{a(x,\tau)}dx} - \sqrt{\int_0^1 \left(12(x-x^2)-2+\frac{\lambda}{a(x,\tau)}\right)({}_0^c\partial_\tau^\alpha \Im_x u)^2 dx}\right)^2 d\tau$$
$$\leq \int_{Q_t} fMudxd\tau + \frac{\lambda}{2\Gamma(2-\alpha)} t^{1-\alpha}\|\varphi\|^2_{L^2(0,1)} + \frac{a_0}{2\Gamma(2-\alpha)} t^{1-\alpha}\left\|x(1-x)\frac{\partial \varphi}{\partial x}\right\|^2_{L^2(0,1)}. \tag{3.13}$$

Substituting Mu by its expression in the first term in the right-hand side of (3.13), we obtain

$$\int_{Q_t} fMudxd\tau = \int_{Q_t} x^2(1-x)^2 f\,{}_0^c\partial_\tau^\alpha u dxd\tau$$
$$+\int_{Q_t} f\int_x^1 \frac{d\zeta}{a(\zeta,\tau)}\int_0^x \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}}\,{}_0^c\partial_\tau^\alpha u d\zeta dxd\tau$$
$$+\int_{Q_t} f\int_0^x \frac{d\zeta}{a(\zeta,\tau)}\int_x^1 \left(\lambda - \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}}\right){}_0^c\partial_\tau^\alpha u d\zeta dxd\tau. \tag{3.14}$$

Using (2.4) and ϵ-Cauchy inequality, each term in the right-hand side of (3.14), can be, respectively, controlled by

$$\int_{Q_t} x^2(1-x)^2 f\,{}_0^c\partial_\tau^\alpha u dxd\tau \leq \frac{1}{4}\int_{Q_t} x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2 dxd\tau + \int_{Q_t} x^2(1-x)^2 f^2 dxd\tau,$$

$$\int_{Q_t} f\int_x^1 \frac{d\zeta}{a(\zeta,\tau)}\int_0^x \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}}\,{}_0^c\partial_\tau^\alpha u d\zeta dxd\tau$$
$$\leq \frac{1}{8}\int_{Q_t} x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2 dxd\tau + \left(\frac{64a_1^2}{a_0^2}+\frac{16\lambda^2 a_1^2}{a_0^4}\right)\int_{Q_t} x^2(1-x)^2 f^2 dxd\tau,$$

and

$$\int_{Q_t} f\int_0^x \frac{d\zeta}{a(\zeta,\tau)}\int_x^1 \left(\lambda - \frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta \frac{d\eta}{a(\eta,\tau)}}{\int_0^1 \frac{d\zeta}{a(\zeta,\tau)}}\right){}_0^c\partial_\tau^\alpha u d\zeta dxd\tau$$
$$\leq \frac{1}{8}\int_{Q_t} x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2 dxd\tau + \left(\frac{64a_1^2}{a_0^2}+\frac{16\lambda^2 a_1^2}{a_0^4}\right)\int_{Q_t} x^2(1-x)^2 f^2 dxd\tau.$$

By combining the previous inequalities with (3.13), we arrive at

$$\frac{1}{2}\int_{Q_t} x^2(1-x)^2({}_0^c\partial_\tau^\alpha u)^2 dxd\tau + \frac{a_0}{2} I^{1-\alpha}\left\|x(1-x)\frac{\partial u}{\partial x}\right\|^2_{L^2(0,1)}$$
$$+\frac{\lambda}{2} I^{1-\alpha}\|u\|^2_{L^2(0,1)} + \frac{1}{2}\int_0^t \left(\left(\lambda\int_0^1 \frac{({}_0^c\partial_\tau^\alpha \Im_x u)^2}{a(x,\tau)}dx\right)^{1/2}\right.$$
$$\left.-\left(\int_0^1 \left(12(x-x^2)-2+\frac{\lambda}{a(x,\tau)}\right)({}_0^c\partial_\tau^\alpha \Im_x u)^2 dx\right)^{1/2}\right)^2 d\tau$$
$$\leq \left(2\left(64a_1^2/a_0^2+16\lambda^2 a_1^2/a_0^4\right)+1\right)\int_{Q_t} x^2(1-x)^2 f^2 dxd\tau$$
$$+\frac{a_0}{2\Gamma(2-\alpha)}T^{1-\alpha}\left\|x(1-x)\frac{\partial\varphi}{\partial x}\right\|^2_{L^2(0,1)} + \frac{\lambda}{2\Gamma(2-\alpha)}T^{1-\alpha}\|\varphi\|^2_{L^2(0,1)}.$$

If we drop the fourth term in the last inequality and by taking the least upper bound of the left side with respect to t from 0 to T, we get the desired estimate (3.1) with

$$C^2=\max\left(\frac{128a_1^2}{a_0^2}+\frac{32\lambda^2 a_1^2}{a_0^4}+1, \frac{a_0 T^{1-\alpha}}{2\Gamma(2-\alpha)}, \frac{\lambda}{2\Gamma(2-\alpha)}T^{1-\alpha}\right)/\min\left(\frac{1}{2},\frac{a_0}{2},\frac{\lambda}{2}\right).$$

□

Consequently, the a priori estimate (3.1) can be extended to strong solutions, then we have the inequality

$$\|u\|_E \leq C\left\|\overline{L}u\right\|_F, \qquad \forall u\in D(\overline{L}).$$

The last inequality implies the following corollaries:

Corollary 1. A strong solution of (2.1)–(2.3) if it exists, it is unique and depends continuously on $\mathcal{F}$.

Corollary 2. The rang $R(\overline{L})$ of $\overline{L}$ is closed in $\mathbb{F}$ and $\overline{R(L)} = R(\overline{L})$.

Corollary (2) shows that, to prove that problem (2.1)–(2.3) has a strong solution for arbitrary $\mathcal{F}$, it suffices to prove that the set R(L) is dense in $\mathbb{F}$.

4 Existence of the solution

To prove the existence of the solution of problem (2.1)–(2.3), it is sufficient to show that $R(L)$ is dense in $\mathbb{F}$, that is $\overline{R(L)} = \mathbb{F}$.

The proof is based on the following lemma.

Lemma 1. *Suppose that $a(x,t)$ and its derivatives are bounded. Let $D_0(L) = \{u\in D(L), u(x,0)=0\}$. If, for $u\in D_0(L)$ and for some function $w\in L^2(\Omega)$, we have*

$$\int_Q \frac{x^2(1-x)^2}{a(x,t)}\mathcal{L}u\overline{w}dxdt = 0, \tag{4.1}$$

then $w=0$.

Proof. From (4.1) we obtain

$$\int_Q \frac{x^2(1-x)^2}{a(x,t)} {}_0^c\partial_t^\alpha u\overline{w}dxdt = \int_Q \frac{x^2(1-x)^2}{a(x,t)} \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\overline{w}dxdt. \quad (4.2)$$

For a given $w(x,t) \in L^2(Q)$, we introduce the function

$$v(x,t) = (x-x^2)w + \int_0^x a(\zeta,t)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)w(\zeta,t)d\zeta,$$

then,

$$\int_0^x \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta = \frac{x-x^2}{a(x,t)}\int_0^x a(\zeta,t)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)w(\zeta,t)d\zeta,$$

this implies

$$\int_0^1 \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta = 0.$$

Equality (4.2), can be written as follows:

$$\int_Q {}_0^c\partial_t^\alpha u\overline{Nv}dxdt = \int_Q A(t)u\overline{v}dxdt, \quad (4.3)$$

where

$$\begin{cases} A(t)u = \frac{\partial}{\partial x}\left((x-x^2)\frac{\partial u}{\partial x}\right), \\ Nv = \frac{x-x^2}{a(x,t)}v - \int_0^x \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta = \frac{(x-x^2)^2}{a(x,t)}w. \end{cases} \quad (4.4)$$

We introduce the smoothing operators:

$$J_\epsilon^{-1} = (I+\epsilon{}_0^c\partial_t^\alpha)^{-1} \text{ and } (J_\epsilon^{-1})^* = \left(I+\epsilon{}_t^c\partial_T^\alpha\right)^{-1},$$

with respect to t, then these operators provide the solution of the problems:

$$\begin{cases} u_\epsilon(t) + \epsilon{}_0^c\partial_t^\alpha u_\epsilon(t) = u(t), & u_\epsilon(0) = 0, \\ v_\epsilon^*(t) + \epsilon\int_t^T \frac{(s-t)^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial v_\epsilon^*(s)}{\partial s}ds = v(t), & v_\epsilon^*(T) = 0. \end{cases}$$

We also have the following properties:
for any $g \in L^2(0,T)$ the function $J_\epsilon^{-1}g$, $(J_\epsilon^{-1})^*g \in W_2^1(0,T)$. If $g \in D(L)$, then $J_\epsilon^{-1}g \in D(L)$ and we have

$$\begin{cases} \lim \left\|J_\epsilon^{-1}g - g\right\|_{L^2(0,T)} = 0, & \text{for } \epsilon \to 0, \\ \lim \left\|(J_\epsilon^{-1})^*g - g\right\|_{L^2(0,T)} = 0, & \text{for } \epsilon \to 0. \end{cases}$$

Substituting the function u into (4.3) by the smoothing function u_ϵ and using the relation

$$A(t)u_\epsilon = J_\epsilon^{-1}A(t)u,$$

we obtain

$$\int_Q uN\overline{{}_0^c\partial_\tau^\alpha v_\epsilon^*}dxdt = \int_Q A(t)u\overline{v_\epsilon^*}dxdt. \qquad (4.5)$$

The left-hand side of (4.5) is a continuous linear functional of u, hence the function v_ϵ^* has the derivatives $(x-x^2)\frac{\partial v_\epsilon^*}{\partial x} \in L^2(Q)$, $\frac{\partial}{\partial x}\left((x-x^2)\frac{\partial v_\epsilon^*}{\partial x}\right) \in L^2(Q)$, and the following conditions are satisfied:

$$\begin{cases} xv_\epsilon^*|_{x=0} = (1-x)v_\epsilon^*|_{x=1} = 0, \\ x\frac{\partial v_\epsilon^*}{\partial x}|_{x=0} = (1-x)\frac{\partial v_\epsilon^*}{\partial x}|_{x=1}. \end{cases}$$

Substituting the function ${}_0^c\partial_t^\alpha u$ in (4.3), such that

$$\begin{aligned} {}_0^c\partial_t^\alpha u =& \frac{x-x^2}{a(x,t)}v_\epsilon^* - \int_0^x\left(\zeta\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) + \frac{\zeta-\zeta^2}{a(\zeta,t)}\right)v_\epsilon^*(\zeta,t)d\zeta \\ &- \int_x^1\left(\frac{\zeta-\zeta^2}{a(\zeta,t)} - (1-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\right)v_\epsilon^* d\zeta, \end{aligned} \qquad (4.6)$$

and using the properties of the smoothing operators, we have

$$\int_Q {}_0^c\partial_t^\alpha u\overline{Nv}dxdt = \int_Q A(t)u\overline{v_\epsilon^*}dxdt + \epsilon\int_Q A(t)u\int_t^T \frac{(s-t)^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial v_\epsilon^*(s)}{\partial s}dsdxdt. \qquad (4.7)$$

Integrating by parts each term in the right-hand side of (4.7), using (4.6) with respect to x and t, we obtain

$$\begin{aligned} &\int_Q A(t)u\overline{v_\epsilon^*}dxdt + \epsilon\int_Q A(t)u\int_t^T \frac{(s-t)^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial v_\epsilon^*(s)}{\partial s}dsdxdt \\ &= -\int_Q a(x,t)\frac{\partial u}{\partial x}{}_0^c\partial_t^\alpha\frac{\partial u}{\partial x}dxdt + \epsilon\int_Q a(x,t)\left({}_0^c\partial_t^\alpha\frac{\partial u}{\partial x}\right)^2 dxdt. \end{aligned}$$

Using (3.8) and (2.4), Equation (4.7) becomes

$$\int_Q {}_0^c\partial_t^\alpha u\overline{Nv}dxdt \le -a_0 I^{1-\alpha}\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(0,1)} + \epsilon\int_Q a(x,t)\left({}_0^c\partial_t^\alpha\frac{\partial u}{\partial x}\right)^2 dxdt. \qquad (4.8)$$

We replace ${}_0^c\partial_t^\alpha u$ by its representation (4.6) in the left-hand side of (4.8), we obtain

$$\begin{aligned} \int_Q {}_0^c\partial_t^\alpha u\overline{Nv}dxdt =& \int_Q \frac{x-x^2}{a(x,t)}v_\epsilon^*(\zeta,t)d\zeta\overline{Nv}dxdt \\ &- \int_Q\int_0^x\left(\zeta\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) + \frac{\zeta-\zeta^2}{a(\zeta,t)}\right)v_\epsilon^*(\zeta,t)d\zeta\overline{Nv}dxdt \\ &- \int_Q\int_x^1\left(\frac{\zeta-\zeta^2}{a(\zeta,t)} - (1-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\right)v_\epsilon^* d\zeta\overline{Nv}dxdt. \end{aligned} \qquad (4.9)$$

Substituting Nv by its expression in each term in the right-hand side of (4.9), integrating with respect to x, we obtain

$$\int_Q \frac{x-x^2}{a(x,t)} v_\epsilon^* d\zeta \overline{Nv} dxdt = \int_Q \left(\frac{x-x^2}{a(x,t)}\right)^2 |v|^2 dxdt$$
$$- \int_Q \frac{x-x^2}{a(x,t)} v \int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta dxdt + \int_Q \frac{x-x^2}{a(x,t)} Nv\overline{(v_\epsilon^* - v)} dxdt, \tag{4.10}$$
$$- \int_Q \int_0^x \left(\zeta \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) + \frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v_\epsilon^* d\zeta \overline{Nv} dxdt$$
$$= -\frac{1}{2}\int_0^T \left(\int_0^1 \frac{\zeta-\zeta^2}{a(\zeta,t)} v d\zeta\right)^2 dt - \frac{1}{2}\int_0^T \left(\int_0^1 dx \int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta\right)^2 dt$$
$$+ \int_Q x \left(\int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta\right)^2 dxdt - \int_Q \frac{x-x^2}{a(x,t)} v \int_0^x \zeta \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta dxdt$$
$$+ \int_Q \int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta \int_0^x \frac{\zeta-\zeta^2}{a(\zeta,t)} v d\zeta dxdt$$
$$- \int_Q \int_0^x \left(\zeta \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) + \frac{\zeta-\zeta^2}{a(\zeta,t)}\right) (v_\epsilon^* - v) d\zeta \overline{Nv} dxdt, \tag{4.11}$$

and

$$- \int_Q \int_x^1 \left(\frac{\zeta-\zeta^2}{a(\zeta,t)} - (1-\zeta)\frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\right) v_\epsilon^* d\zeta \overline{Nv} dxdt$$
$$= -\frac{1}{2}\int_0^T \left(\int_0^1 \frac{\zeta-\zeta^2}{a(\zeta,t)} v d\zeta\right)^2 dt - \frac{1}{2}\int_0^T \left(\int_0^1 dx \int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta\right)^2 dt$$
$$+ \int_Q (1-x) \left(\int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta\right)^2 dxdt + \int_Q \frac{x-x^2}{a(x,t)} v \int_x^1 (1-\zeta)$$
$$\times \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta dxdt + \int_Q \int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) v d\zeta \int_x^1 \frac{\zeta-\zeta^2}{a(\zeta,t)} v d\zeta dxdt$$
$$- \int_Q \int_x^1 \left(\frac{\zeta-\zeta^2}{a(\zeta,t)} - (1-\zeta)\frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\right) (v_\epsilon^* - v) d\zeta \overline{Nv} dxdt. \tag{4.12}$$

By combining (4.10)–(4.12), we get

$$\int_Q {}_0^c\partial_t^\alpha u \overline{Nv} dxdt = \int_Q (Nv)^2 dxdt - \int_0^T \left(\int_0^1 Nv dx\right)^2 dt$$
$$+ \int_Q \left[\frac{x-x^2}{a(x,t)} - \int_0^x \left(\zeta \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) + \frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\right.$$
$$\left. - \int_x^1 \left(\frac{\zeta-\zeta^2}{a(\zeta,t)} - (1-\zeta)\frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\right)\right] (v_\epsilon^* - v) d\zeta \overline{Nv} dxdt.$$

Since

$$\int_Q (Nv)^2 - \int_0^T \left(\int_0^1 Nv\right)^2 = \frac{1}{2}\int_0^T \int_0^1 \int_0^1 \left|(Nv)(x,t) - (Nv)(y,t)\right|^2 dxdydt,$$

using (4.8), for sufficiently small ϵ, we conclude that

$$Nv(x,t) = Nv(y,t) \quad \forall x, y \in [0,1], \quad t \in [0,T].$$

then $Nv = 0$, so from (4.4), the assertion of the Lemma 1 is established. □

Theorem 2. *The range $R(L)$ of the operator L is dense in $\mathbb{F}$.*

Proof. Since $\mathbb{F}$ is a Hilbert space, we have $\overline{R(L)} = \mathbb{F}$ if and only if the relation

$$\int_Q x^2(1-x)^2 f\overline{g}dxdt + \int_0^1 x^2(1-x)^2 \frac{d\ell u}{dx}\frac{\overline{d\varphi_1}}{dx}dx + \int_0^1 \ell u\overline{\varphi_1} = 0, \qquad (4.13)$$

for an arbitrary $u \in D(L)$ and $(g, \varphi_1) \in \mathbb{F}$, implies that $g = 0$ and $\varphi_1 = 0$.

Putting $u \in D_0(L)$ in (4.13), we conclude from Lemma 1 that $g = \frac{w}{a(x,t)} = 0$, then $g = 0$. a.e. Taking $u \in D(L)$ in (4.13) yields

$$\int_0^1 x^2(1-x)^2 \frac{d\ell u}{dx}\frac{\overline{d\varphi_1}}{dx}dx + \int_0^1 \ell u\overline{\varphi_1} = 0,$$

Since the range of the trace operator is everywhere dense in Hilbert space with the norm

$$\int_0^1 x^2(1-x)^2 \left|\frac{d\ell u}{dx}\right|^2 dx + \int_0^1 |\ell u|^2 = 0.$$

Hence, $\varphi_1 = 0$. This completes the proof. □

5 Conclusions

In this work, we studied the existence and uniqueness of a strong solution for a parabolic equation with a Caputo time fractional differential operator supplemented by periodic nonlocal boundary condition and integral condition. The used method is one of the most efficient functional analysis methods for solving fractional differential equations with boundary integral conditions, the so-called energy-integral method or a priori estimates method. We constructed suitable multiplicators for each problem, which provide the a priori estimate. From there, it was possible to establish the solution's uniqueness and continuous dependence on the initial data. Subsequently, we established the main result concerning the existence of the solution for the considered problem. Our approach primarily relies on operator theory.

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