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A numerical scheme to simulate the distributed-order time 2D Benjamin Bona Mahony Burgers equation with fractional-order space

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Abstract. In this study, a new class of the Benjamin Bona Mahony Burgers equation is introduced, which considers the distributed-order in the time variable and fractional-order space in the Caputo form in the 2D case. The 2D-modified orthonormal normalized shifted Ultraspherical polynomials are derived from 1D-modified orthonormal normalized shifted Ultraspherical polynomials and 2D-modified orthonormal normalized shifted Ultraspherical polynomials and the orthonormal normalized shifted Ultraspherical polynomials are applied to approximate of the space and time variables, respectively. Moreover, the convergence analysis of these basis functions is investigated. Due to the time variable being in the distributed-order mode and the space variable being in the fractional-order case, to apply the desired numerical algorithm for this type of equation, operational matrices of ordinary, fractional and distributed-order derivatives are computed. In the proposed method, once the unknown function is approximated using the mentioned polynomial, the matrix form of the residual function is derived and then a system of algebraic equations is adopted by applying the collocation approach. An approximate solution is extracted for the original problem by solving constructed equation system. Several examples are examined to demonstrate the accuracy and capability of the method.

Keywords: distributed-order time; fractional-order space; 2D Benjamin Bona Mahony Burgers equation; ultraspherical polynomials; Caputo fractional derivative.

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1 Introduction

Although the fractional calculation was proposed a few decades ago, it is an attractive topic for researchers because it is still developing and expanding. At first, fractional calculus was discussed only from the point of view of mathematical theory. Over time, the application of this branch of mathematics came to the notice and a new window was opened in research activities. As far as today, it can be claimed that it covers many scientific fields such as dynamics [7], electronics [5], mechanics [18] and signal processing [10]. The main foundation of using fractional calculus in various sciences is fractional equations. Solving these types of equations does not mean obtaining an analytical solution. Therefore, the efficiency of numerical methods in solving fractional equations (fractional differential or integral equations) is very important. For instance, see [6, 17, 19] and references therein.

According to the types of the concept of fractional derivative (Caputo, Riemann, Atangana-Baleanu and etc.) and the generalization of fractional equations to fractional variable order and distributed-order equations, various problems are also widely modeled. The fractional derivative of the distributed order is obtained by integrating from product of the weight function and the desired fractional derivative with respect to its fractional order. The equations that have been modeled and studied in this way include Schrödinger [20], diffusion-wave [28], inhomogeneous parabolic [26] and telegraph equation [23].

Some authors have investigated different numerical methods for the ordinary and fractional forms of this equation in various dimensional. For example, the finite difference [21], radial basis functions [11], B-spline collocation [27], Legendre spectral element [13], and adomain decomposition method [14].

The Benjamin Bona Mahony Burgers equation is a partial differential equation used to model wave phenomena in physical systems, particularly in fluid dynamics. The Benjamin Bona Mahony Burgers equation combines advection and diffusion effects. It describes the evolution of nonlinear waves and can exhibit a variety of interesting phenomena, including shock wave and soliton formations and wave breaking. In summary, the Benjamin Bona Mahony Burgers equation is a mathematical tool to characterize the model of the long wavelength surface waves in liquids, acoustic, acoustic-gravity, and hydromagnetic waves in harmonic crystals, compressible fluids, and cold plasma, respectively [1,8,12]. The importance of Benjamin Bona Mahony Burgers equation prompted us to generalize it to distributed-order time and fractional-order space in the sense of Caputo. So, we are interested in the equation

$$\begin{aligned} f(\mathbf{z},t) = &\int_0^1 \theta(\varrho)\,{}_0^cD_t^\varrho \upsilon(\mathbf{z},t)d\varrho \\ &- \mathfrak{D}^{1+\kappa}\upsilon_t(\mathbf{z},t) - \mathfrak{D}^{1+\kappa}\upsilon(\mathbf{z},t) + \mathfrak{D}^{\kappa}\upsilon(\mathbf{z},t)\big(1+\upsilon(\mathbf{z},t)\big), \end{aligned} \tag{1.1}$$

on $(\mathbf{z},t) = (x,y,t) \in [0,\xi_1]\times[0,\xi_2]\times[0,1]$ with the initial condition $\upsilon(x,y,0) = \varphi_0(x,y)$ and the boundary conditions

$$\begin{aligned} &\upsilon(0,y,t) = \varphi_1(y,t),\ \upsilon(\xi_1,y,t) = \varphi_2(y,t),\\ &\upsilon(x,0,t) = \varphi_3(x,t),\ \upsilon(x,\xi_2,t) = \varphi_4(x,t), \end{aligned}$$

where $\mathfrak{D}^\kappa = {}_0^cD_x^\kappa + {}_0^cD_y^\kappa$ with $0<\kappa\le 1$, φ_i are known continuous functions for $i=1,2,3,4$ and $\theta(\varrho)$, the distribution weight function has the properties: $\theta(\varrho)>0$ and $\int_0^1\theta(\varrho)d\varrho<\infty$.

In numerical schemes for solving fractional and ordinary equations, the application of polynomials as basis functions can be considered as an efficient method. One of these polynomials, which has been used in the recent decade to solve kinds of problems, is orthogonal Ultraspherical polynomials because these polynomials are related to the famous Chebyshev and Legendre polynomials. The Ultraspherical polynomials are utilized for solving second-order equations [15], fractal-fractional Riccati equation [24], Lane-Emden equation [25], some types of ordinary fractional problems [3] and $(2n+1)$-order linear differential equations [16]. See references [2, 4] for more details.

In this work, we use the technique of approximating the unknown function of distributed-order time 2D Benjamin Bona Mahony Burgers equation with fractional-order space by two types of Ultraspherical polynomials. Since the space variable is in two-dimensional case, these polynomials are also extended to 2D which are introduced for the first time. Then, by using the calculated derivative operator matrices and Gauss-Legendre quadrature integration formula, the derivatives of unknown function are also approximated. The high accuracy, flexibility, fast convergence and ease of implementation are the main advantages of Gauss-Legendre quadrature. The method is generally stable and less prone to numerical errors compared to other approximation techniques. By replacing all these approximations in the main equation and using collocation points, an algebraic equation system is obtained. Solving this system is equivalent to calculating the numerical solution of problem (1.1). Finally, some examples are considered for the correctness and accuracy of the proposed method.

The operational matrix method is a numerical technique used to solve differential and integro-differential equations. It involves representing the equations in matrix form, allowing for efficient computation and solution. The method offers several advantages, including: simplicity, efficiency, accuracy, versatility, flexibility and numerical stability. On the other hand, the cons of the operational matrix method are discretization error, limited applicability to complex geometries, computational requirements, convergence issues and limited support for discontinuous solutions.

The choice of Ultraspherical polynomials has been motivated by two primary factors. Firstly, emphasis is placed on converting these polynomials into a two-dimensional structure while concurrently performing an error analysis, a task that has not been previously undertaken. Secondly, the association between these polynomials and Chebyshev and Legendre polynomials ensures spectral accuracy, thereby establishing one of the method's stability prospects.

So, the novelty of the article is contingent upon two key aspects. Firstly, the introduction of a novel equation type, which expands the existing knowledge base. Secondly, the proposal of innovative basis functions, contributing to the advancement of the field.

The rest of this paper is as follows: Essential prerequisites are given in Section 2. The shifted Ultraspherical and modified shifted Ultraspherical polynomials with their properties are introduced in Sections 3 and 4, respectively. The proposed approach is explained in Section 5. The results of numerical are provided in Section 6. The conclusion of the study is rendered in Section 7.

2 Preliminaries

Here, we review some prerequisites used in this work.

DEFINITION 1. [22] The Caputo fractional differentiation of order κ of the differentiable function $g(x)$ over $[a, b]$ is defined with $g^{(p)}(x)$ for $\kappa = p$ and

$$ {}_a^cD_x^{\kappa}g(x) = \frac{1}{\Gamma(p-\kappa)} \int_a^x (x-t)^{p-\kappa-1} g^{(p)}(t)dt, \quad p-1 < \kappa < p \in \mathbb{N}. \quad (2.1) $$

Property 1. For $0 < \kappa \leq 1$, it can be concluded ${}_0^cD_x^{1+\kappa}g(x) = {}_0^cD_x^{\kappa}g'(x)$.

Property 2. [22] Suppose that $s \geq 1$ is a given constant. Then, we get

$$ {}_0^cD_x^{\kappa}x^s = \frac{\Gamma(s+1)}{\Gamma(s-\kappa+1)} x^{s-\kappa}, \quad 0 < \kappa \leq 1. $$

Theorem 1. *[9] An M-point Gauss-Legendre quadrature integration on $[a, b]$ is given by*

$$ \int_a^b g(x)dx \simeq \frac{b-a}{2} \sum_{\iota=1}^{M} \varpi_\iota g\left(\frac{b-a}{2} z_\iota + \frac{a+b}{2}\right), $$

where $\varpi_\iota = \frac{2}{(1-z_\iota^2)\left(P'_M(z_\iota)\right)^2}$ and z_ι represents the zeros of $P_M(x)$ (Legendre polynomial of M degree) which are called the Gauss-Legendre integration weights and nodes, respectively.

3 Orthonormal normalized shifted Ultraspherical polynomials

In this part, the orthonormal normalized shifted Ultraspherical polynomials with some properties is introduced on the interval $[0, 1]$.

DEFINITION 2. [24] The shifted Ultraspherical polynomials of degree $\tilde{i}$, $\mathcal{U}_{\tilde{i},\mathfrak{d}}$ on $[0, 1]$ for $\mathfrak{d} \geq 0$ are given by

$$ \mathcal{U}_{\tilde{i},\mathfrak{d}}(t) = \sum_{k=0}^{\tilde{i}} \lambda_{\tilde{i},k} t^k, \qquad \tilde{i} = 0, 1, \ldots, $$

$$ \lambda_{\tilde{i},k} = (-1)^{\tilde{i}-k} \binom{\tilde{i}}{k} \sqrt{\frac{2(\tilde{i}+\mathfrak{d})}{\tilde{i}!\Gamma(\tilde{i}+2\mathfrak{d})}} \frac{\Gamma(\tilde{i}+k+2\mathfrak{d})}{\Gamma(k+\mathfrak{d}+0.5)}. $$

These polynomials are equivalent to Legendre and Chebyshev polynomials for some values of $\mathfrak{d}$ as

$$ \mathcal{U}_{\tilde{i},\frac{1}{2}}(t) = \mathcal{P}_{\tilde{i}}(t), \ \mathcal{U}_{\tilde{i},0}(t) = \mathcal{T}_{\tilde{i}}^{*}(t), \ \mathcal{U}_{\tilde{i},1}(t) = \mathcal{T}_{\tilde{i}}^{**}(t), $$

where $\mathcal{P}_{\tilde{i}}(t)$ are shifted Legendre polynomials and $\mathcal{T}_{\tilde{i}}^{*}(t)$ and $\mathcal{T}_{\tilde{i}}^{**}(t)$ indicate the first and second kind of shifted Chebyshev polynomials on $[0, 1]$, respectively. So, the spectral accuracy of the approach is guaranteed for values

$\mathfrak{d} = 0, \frac{1}{2}, 1$. The set of orthonormal normalized shifted Ultraspherical polynomials $\{\mathcal{U}_{\tilde{i},\mathfrak{d}}(t)\}_{\tilde{i}\geq 0}$ forms an orthonormal system with respect to the weight function $w_{\mathfrak{d}}(t) = (t - t^2)^{\mathfrak{d}-0.5}$,

$$\int_0^1 \mathcal{U}_{\tilde{i},\mathfrak{d}}(t)\mathcal{U}_{\tilde{j},\mathfrak{d}}(t)w_{\mathfrak{d}}(t)dt = \delta_{\tilde{i}\tilde{j}},$$

where $\delta_{\tilde{i}\tilde{j}}$ is the well-known Kronecker delta function. Therefore, this property them allows us to expand any function $u(t) \in L^2_{w_{\mathfrak{d}}}([0,1])$ as follows

$$u(t) \simeq \sum_{\tilde{i}=0}^{\mathbf{m}} \bar{u}_{\tilde{i}}\mathcal{U}_{\tilde{i},\mathfrak{d}}(t) \triangleq \bar{U}^T\Phi_{\mathbf{m},\mathfrak{d}}(t), \tag{3.1}$$

in which $\bar{u}_{\tilde{i}} = \int_0^1 u(t)\mathcal{U}_{\tilde{i},\mathfrak{d}}(t)w_{\mathfrak{d}}(t)dt$ as

$$\bar{U} = [\bar{u}_0\ \bar{u}_1\ \dots\ \bar{u}_{\mathbf{m}}]^T,\ \ \Phi_{\mathbf{m},\mathfrak{d}}(t) = [\mathcal{U}_{0,\mathfrak{d}}(t)\quad \mathcal{U}_{1,\mathfrak{d}}(t)\quad \dots\quad \mathcal{U}_{\mathbf{m},\mathfrak{d}}(t)]^T. \tag{3.2}$$

In this work, three types of operational matrices of the orthonormal normalized shifted Ultraspherical polynomials are needed, including first-order ordinary and fractional derivatives, and distributed-order differentiation which are statemented and proved in the following theorems.

Theorem 2. *The first-order classical derivative of the vector $\Phi_{\mathbf{m},\mathfrak{d}}(t)$ expressed in relation* (3.2) *can be represented as*

$$\frac{d\Phi_{\mathbf{m},\mathfrak{d}}(t)}{dt} = \mathbf{P}^{(1)}\Phi_{\mathbf{m},\mathfrak{d}}(t),$$

where $\mathbf{P}^{(1)}$ is called the derivative matrix and obtained as

$$\mathbf{P}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mathbf{p}^{(1)}_{21} & \mathbf{p}^{(1)}_{22} & \mathbf{p}^{(1)}_{23} & \dots & \mathbf{p}^{(1)}_{2\mathbf{m}} & \mathbf{p}^{(1)}_{2(\mathbf{m}+1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \mathbf{p}^{(1)}_{(\mathbf{m}+1)1} & \mathbf{p}^{(1)}_{(\mathbf{m}+1)2} & \mathbf{p}^{(1)}_{(\mathbf{m}+1)3} & \dots & \mathbf{p}^{(1)}_{(\mathbf{m}+1)\mathbf{m}} & \mathbf{p}^{(1)}_{(\mathbf{m}+1)(\mathbf{m}+1)} \end{bmatrix},$$

which

$$\mathbf{p}^{(1)}_{ij} = \sum_{r=1}^{i-1}\sum_{l=0}^{j-1} \lambda_{i-1,r}\lambda_{j-1,l}\frac{r\Gamma(r+l+\mathfrak{d}-0.5)\Gamma(\mathfrak{d}+0.5)}{\Gamma(r+l+2\mathfrak{d})}. \tag{3.3}$$

Also, the κ-fractional derivative of the vector $\Phi_{\mathbf{m},\mathfrak{d}}(t)$ can be expanded as

$${}_0^cD_t^{\kappa}\Phi_{\mathbf{m},\mathfrak{d}}(t) \simeq \mathbf{P}^{(\kappa)}\Phi_{\mathbf{m},\mathfrak{d}}(t),$$

where

$$\mathbf{P}^{(\kappa)} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mathbf{p}^{(\kappa)}_{21} & \mathbf{p}^{(\kappa)}_{22} & \mathbf{p}^{(\kappa)}_{23} & \dots & \mathbf{p}^{(\kappa)}_{2\mathbf{m}} & \mathbf{p}^{(\kappa)}_{2(\mathbf{m}+1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \mathbf{p}^{(\kappa)}_{(\mathbf{m}+1)1} & \mathbf{p}^{(\kappa)}_{(\mathbf{m}+1)2} & \mathbf{p}^{(\kappa)}_{(\mathbf{m}+1)3} & \dots & \mathbf{p}^{(\kappa)}_{(\mathbf{m}+1)\mathbf{m}} & \mathbf{p}^{(\kappa)}_{(\mathbf{m}+1)(\mathbf{m}+1)} \end{bmatrix},$$

with

$$\mathbf{p}_{ij}^{(\kappa)}=\sum_{r=1}^{i-1}\sum_{l=0}^{j-1}\lambda_{i-1,r}\lambda_{j-1,l}\frac{\Gamma(r+1)}{\Gamma(r-\kappa+1)}\frac{\Gamma(r+l-\kappa+\mathfrak{d}+0.5)\Gamma(\mathfrak{d}+0.5)}{\Gamma(r+l-\kappa+2\mathfrak{d}+1)}. \quad (3.4)$$

Proof. Since $\frac{d\mathcal{U}_{0,\mathfrak{d}}(t)}{dt}=0$, the first row of $\mathbf{P}^{(1)}$ is also equal to zero. So,

$$\frac{d\mathcal{U}_{i-1,\mathfrak{d}}(t)}{dt}=\sum_{r=1}^{i-1}\lambda_{i-1,r}rt^{r-1}, \qquad i=2,\dots,\mathbf{m}+1.$$

On the other hand, it can be shown that

$$\mathbf{p}_{ij}^{(1)}=\sum_{r=1}^{i-1}\sum_{l=0}^{j-1}\lambda_{i-1,r}\lambda_{j-1,l}r\int_0^1 t^{r+l-1}(t-t^2)^{\mathfrak{d}-0.5}dt, \quad j=1,\dots,\mathbf{m}+1. \quad (3.5)$$

By solving the last integral, we get

$$\int_0^1 t^{r+l-1}(t-t^2)^{\mathfrak{d}-0.5}dt=\frac{\Gamma(r+l+\mathfrak{d}-0.5)\Gamma(\mathfrak{d}+0.5)}{\Gamma(r+l+2\mathfrak{d})}. \quad (3.6)$$

Thus, relation (3.3) is obtained via replacing (3.6) in (3.5). The same manner is used to prove relation (3.4). Assume $0<\kappa<1$, then ${}_0^cD_t^\kappa\mathcal{U}_{0,\mathfrak{d}}(t)=0$ and

$${}_0^cD_t^\kappa\mathcal{U}_{i-1,\mathfrak{d}}(t)=\sum_{r=1}^{i-1}\lambda_{i-1,r}\frac{\Gamma(r+1)}{\Gamma(r-\kappa+1)}t^{r-\kappa}, \quad i=2,\dots,\mathbf{m}+1. \quad (3.7)$$

By extending the relation (3.7) by the orthonormal normalized shifted Ultraspherical polynomials for $1\le j\le \mathbf{m}+1$, the expressed assertion is proved as

$$\begin{aligned}\mathbf{p}_{ij}^{(\kappa)}&=\sum_{r=1}^{i-1}\sum_{l=0}^{j-1}\lambda_{i-1,r}\lambda_{j-1,l}\frac{\Gamma(r+1)}{\Gamma(r-\kappa+1)}\int_0^1 t^{r+l-\kappa}(t-t^2)^{\mathfrak{d}-0.5}dt\\&=\sum_{r=1}^{i-1}\sum_{l=0}^{j-1}\lambda_{i-1,r}\lambda_{j-1,l}\frac{\Gamma(r+1)}{\Gamma(r-\kappa+1)}\frac{\Gamma(r+l-\kappa+\mathfrak{d}+0.5)\Gamma(\mathfrak{d}+0.5)}{\Gamma(r+l-\kappa+2\mathfrak{d}+1)}.\end{aligned}$$

□

Theorem 3. *The distributed-order differentiation of the vector $\Phi_{\mathbf{m},\mathfrak{d}}(t)$ introduced in* (3.2) *can be represented as follows*

$$\int_0^1\theta(\varrho){}_0^cD_t^\varrho\Phi_{\mathbf{m},\mathfrak{d}}(t)d\varrho\simeq\mathbf{G}^{(\varrho)}\Phi_{\mathbf{m},\mathfrak{d}}(t),$$

where $\mathbf{G}^{(\varrho)}$ is an $(\mathbf{m}+1)$ *matrix in the form*

$$\left[\mathbf{G}^{(\varrho)}\right]_{ij}=\frac{1}{2}\sum_{\iota=1}^{M}\varpi_\iota\theta\big(\frac{1}{2}(z_\iota+1)\big)\mathbf{p}_{ij}^{(\frac{1}{2}(z_\iota+1))},$$

in which $\mathbf{p}_{ij}^{(\frac{1}{2}(z_\iota+1))}$, ϖ_ι and z_ι are defined in Theorem 2 and Theorem 1.

Proof. From Theorem 2, one obtains

$$\int_0^1 \theta(\varrho)\, {}_0^c D_t^{\varrho} \varPhi_{\mathbf{m},\mathfrak{d}}(t) d\varrho \simeq \left(\int_0^1 \theta(\varrho) \mathbf{P}^{(\varrho)} d\varrho \right) \varPhi_{\mathbf{m},\mathfrak{d}}(t) \triangleq \mathbf{G}^{(\varrho)} \varPhi_{\mathbf{m},\mathfrak{d}}(t).$$

So, the proof is completed by applying an M-point Legendre quadrature as follows

$$\left[\int_0^1 \theta(\varrho) \mathbf{P}^{(\varrho)} d\varrho \right]_{ij} = \int_0^1 \theta(\varrho) \mathbf{p}_{ij}^{(\varrho)} d\varrho \simeq \frac{1}{2} \sum_{\iota=1}^{M} \varpi_\iota \theta\big(\frac{1}{2}(z_\iota + 1)\big) \mathbf{p}_{ij}^{(\frac{1}{2}(z_\iota+1))}.$$

Note that, we set $M = 25$ in this study for all examples. □

4 Modified orthonormal normalized shifted Ultraspherical polynomials

In this section, we define the modified orthonormal normalized shifted Ultraspherical polynomials in cases of one and two dimensional. Also, we introduce some essential properties of these polynomials which are applied to obtain an approximate solution of the problem (1.1).

4.1 1D-modified orthonormal normalized shifted Ultraspherical polynomials

By taking the change of variable $\frac{1}{\xi}x$ in the orthonormal normalized shifted Ultraspherical polynomials, we define the modified orthonormal normalized shifted Ultraspherical polynomials on $[0, \xi]$.

Definition 3. The modified orthonormal normalized shifted Ultraspherical polynomials $\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)$ in the arbitrary interval $[0, \xi]$ with $\mu_{\tilde{i},k}{=}\frac{1}{\xi^k}\lambda_{\tilde{i},k}$ is defined as

$$\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x) = \sum_{k=0}^{\tilde{i}} \mu_{\tilde{i},k} x^k. \tag{4.1}$$

The modified orthonormal normalized shifted Ultraspherical polynomials $\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)$ generate an orthogonal system with weight function $w_{\mathfrak{d},\xi}(x){=}(\frac{x}{\xi} - x^2/\xi^2)^{\mathfrak{d}-0.5}$ over $[0, \xi]$. Namely, they satisfy the following relation

$$\int_0^{\xi} \mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x) \mathcal{G}_{\tilde{j},\mathfrak{d},\xi}(x) w_{\mathfrak{d},\xi}(x) dx = \xi \delta_{\tilde{i}\tilde{j}}.$$

Any square integrable function $u(x) \in L^2_{w_{\mathfrak{d},\xi}}([0, \xi])$ can be approximated via the modified shifted Ultraspherical polynomials as follows

$$u(x) \simeq \sum_{\tilde{i}=0}^{\mathbf{n}} c_{\tilde{i}} \mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x) = \mathbf{C}^T \varOmega_{\mathbf{n},\xi}(x), \tag{4.2}$$

in which $c_{\tilde{i}}=\frac{1}{\xi}\int_0^{\xi} u(x)\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)w_{\mathfrak{d},\xi}(x)dx$ and

$$\mathbf{C}=[c_0\ c_1\ \dots\ c_{\mathbf{n}}]^T,\ \ \Omega_{\mathbf{n},\xi}(x)=[\mathcal{G}_{0,\mathfrak{d},\xi}(x)\ \mathcal{G}_{1,\mathfrak{d},\xi}(x)\ \dots\ \ \mathcal{G}_{\mathbf{n},\mathfrak{d},\xi}(x)]^T. \tag{4.3}$$

In the following, relationships the classic and fractional derivatives operational matrices of 1D-modified orthonormal normalized shifted Ultraspherical polynomials are calculated.

Theorem 4. *Assume $\Omega_{\mathbf{n},\xi}(x)$ is the vector defined in relation* (4.3)*. Then, the first order derivative of the vector $\Omega_{\mathbf{n},\xi}(x)$ can be expressed in the following form*

$$\frac{d\Omega_{\mathbf{n},\xi}(x)}{dx}=\mathbf{D}^{(1)}_{(\mathbf{n}+1)}\Omega_{\mathbf{n},\xi}(x),$$

where $\mathbf{D}^{(1)}_{(\mathbf{n}+1)}$ is an $(\mathbf{n}+1)\times(\mathbf{n}+1)$ matrix with the following elements

$$\big[\mathbf{D}^{(1)}_{(\mathbf{n}+1)}\big]_{ij}=\begin{cases}0, & i=1,\\ \displaystyle\sum_{r=1}^{i-1}\sum_{l=0}^{j-1}\mu_{i-1,r}\mu_{j-1,l}r\xi^{r+l-1}\frac{\Gamma(r+l+\mathfrak{d}-0.5)\Gamma(\mathfrak{d}+0.5)}{\Gamma(r+l+2\mathfrak{d})}, & j\neq 1.\end{cases} \tag{4.4}$$

Proof. For $\tilde{i}=0$, from relation (4.1) clearly it can be achieved $\frac{d\mathcal{G}_{0,\mathfrak{d},\xi}(x)}{dx}=0$ and for $\tilde{i}=1,2,\dots,\mathbf{n}$, we have

$$\frac{d\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)}{dx}=\sum_{r=1}^{\tilde{i}}\mu_{\tilde{i},r}rx^{r-1}.$$

Now, we expand $\frac{d\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)}{dx}$ in terms of the modified orthonormal normalized shifted Ultraspherical polynomials. More precisely, using the change of variable $\tau=x/\xi$ with $d\tau=1/\xi dx$, one obtains

$$\frac{d\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)}{dx}=\sum_{l=0}^{\tilde{j}}\big[\mathbf{D}^{(1)}_{(\mathbf{n}+1)}\big]_{\tilde{i}\tilde{j}}\mathcal{G}_{\tilde{j},\mathfrak{d},\xi}(x),$$

where

$$\begin{aligned}\big[\mathbf{D}^{(1)}_{(\mathbf{n}+1)}\big]_{\tilde{i}\tilde{j}}&=\frac{1}{\xi}\int_0^{\xi}\frac{d\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)}{dx}\mathcal{G}_{\tilde{i},\mathfrak{d},\xi}(x)w_{\mathfrak{d},\xi}(x)dx\\&=\frac{1}{\xi}\sum_{r=1}^{\tilde{i}}\sum_{l=0}^{\tilde{j}}\mu_{\tilde{i},r}\mu_{\tilde{j},l}r\int_0^{\xi}x^{r+l-1}\Big(\frac{x}{\xi}-\frac{x^2}{\xi^2}\Big)^{\mathfrak{d}-0.5}dx\\&=\frac{1}{\xi}\sum_{r=1}^{\tilde{i}}\sum_{l=0}^{\tilde{j}}\mu_{\tilde{i},r}\mu_{\tilde{j},l}r\xi^{r+l}\int_0^{\xi}\tau^{r+l-1}(\tau-\tau^2)^{\mathfrak{d}-0.5}d\tau\\&=\sum_{r=1}^{\tilde{i}}\sum_{l=0}^{\tilde{j}}\mu_{\tilde{i},r}\mu_{\tilde{j},l}r\frac{\xi^{r+l-1}\Gamma(r+l+\mathfrak{d}-0.5)\Gamma(\mathfrak{d}+0.5)}{\Gamma(r+l+2\mathfrak{d})}.\end{aligned} \tag{4.5}$$

Finally, replacing $\tilde{i} = i-1$, and $\tilde{j} = j-1$, the proof of relation (4.4) is resulted. □

Theorem 5. *The κ-order fractional derivative* $(0 < \kappa < 1)$ *of the vector* $\Omega_{\mathbf{n},\xi}(x)$ *given in* (4.3) *can be approximated as*

$$ {}_0^cD_x^{\kappa}\Omega_{\mathbf{n},\xi}(x) \simeq \mathbf{D}^{(\kappa)}_{(\mathbf{n}+1)}\Omega_{\mathbf{n},\xi}(x), $$

where $\mathbf{D}^{(\kappa)}_{(\mathbf{n}+1)}$ *is an* $(\mathbf{n}+1)$*-order matrix as*

$$ \mathbf{D}^{(\kappa)}_{(\mathbf{n}+1)} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \rho_{21} & \rho_{22} & \rho_{23} & \cdots & \rho_{2\mathbf{n}} & \rho_{2(\mathbf{n}+1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \rho_{(\mathbf{n}+1)1} & \rho_{(\mathbf{n}+1)2} & \rho_{(\mathbf{n}+1)3} & \cdots & \rho_{(\mathbf{n}+1)\mathbf{n}} & \rho_{(\mathbf{n}+1)(\mathbf{n}+1)} \end{bmatrix}, \quad (4.6) $$

in which

$$ \rho_{ij} = \sum_{r=1}^{i-1}\sum_{l=0}^{j-1} \mu_{i-1,r}\mu_{j-1,l}\frac{\xi^{r+l-1}\Gamma(r+1)\Gamma(r+l-\kappa+\mathfrak{d}+0.5)\Gamma(\mathfrak{d}+0.5)}{\Gamma(r-\kappa+1)\Gamma(r+l-\kappa+2\mathfrak{d}+1)}. $$

Proof. Similar to proof of Theorem 4 and applying Lemma 2, the proof is archived. So, we omit the details of proof. □

4.2 2D-modified orthonormal normalized shifted Ultraspherical polynomials

Herein, we generalize the modified orthonormal normalized shifted Ultraspherical polynomials of two-dimensional using the modified orthonormal normalized shifted Ultraspherical polynomials of one-dimensional. In order to facilitate the approximation of the solution to the main problem, it becomes imperative to establish a comprehensive definition for the modified orthonormal normalized shifted Ultraspherical polynomials in two dimensions. The utilization of these specific polynomials is predicated on the fundamental principle of the separability exhibited by the exact solution.

Definition 4. The 2D-modified orthonormal normalized shifted Ultraspherical polynomials $\Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y)$ is defined over $[0,\xi_1]\times[0,\xi_2]$ as follows

$$ \Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y) = \mathcal{G}_{\tilde{i},\mathfrak{d}_1,\xi_1}(x)\mathcal{G}_{\tilde{j},\mathfrak{d}_2,\xi_2}(y), \qquad \tilde{i}=0,1,\dots,\mathbf{n}_1, \quad \tilde{j}=0,1,\dots,\mathbf{n}_2. $$

The family $\{\Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y)\}_{\tilde{i},\tilde{j}=0}^{\infty}$ satisfies the orthogonal condition with weight function $w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}(x,y) = w_{\mathfrak{d}_1,\xi_1}(x)w_{\mathfrak{d}_2,\xi_2}(y)$, i.e.,

$$ \int_0^{\xi_1}\int_0^{\xi_2} \Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y)\Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{k},\tilde{l}}(x,y)w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}(x,y)dydx = \xi_1\xi_2\delta_{\tilde{i}\tilde{k}}\delta_{\tilde{j}\tilde{l}}. $$

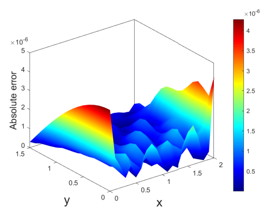
So, every function $u(x,y) \in L^2_{w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}}([0,\xi_1]\times[0,\xi_2])$ can be expanded by 2D-modified shifted Ultraspherical polynomials as follows:

$$u(x,y) \simeq \sum_{\tilde{i}=0}^{\mathbf{n}_1}\sum_{\tilde{j}=0}^{\mathbf{n}_2} w_{\tilde{i}\tilde{j}}\Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y) \triangleq \mathbf{W}^T\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y), \tag{4.7}$$

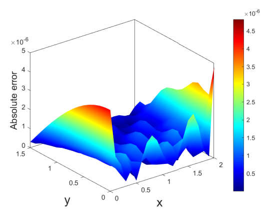
where $w_{\tilde{i}\tilde{j}} = \frac{1}{\xi_1\xi_2}\int_0^{\xi_1}\int_0^{\xi_2} u(x,y)\Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y)w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}(x,y)dydx$ with

$$\begin{aligned}
&\mathbf{W} = [w_{00}\ w_{01}\ \dots\ w_{0\mathbf{n}_2}\ w_{10}\ w_{11}\ \dots\ w_{1\mathbf{n}_2}\ \dots\ w_{\mathbf{n}_10}\ w_{\mathbf{n}_11}\ \dots\ w_{\mathbf{n}_1\mathbf{n}_2}]^T,\\
&\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) = [\Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{0,0}(x,y)\ \dots\ \Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{0,\mathbf{n}_2}(x,y)\ \Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{1,0}(x,y)\\
&\qquad \dots\ \Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)]^T.
\end{aligned} \tag{4.8}$$

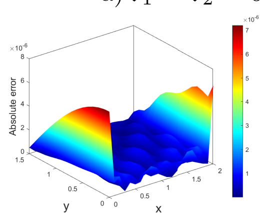
For example, $u(x,y) = \sin(x)\cos(y)$ on $[0,2]\times[0,\frac{\pi}{2}]$ is approximated by $\Psi_{6,6}(x,y)$. The absolute errors are shown in Figure 1 for different values of $\mathfrak{d}_1$ and $\mathfrak{d}_2$.



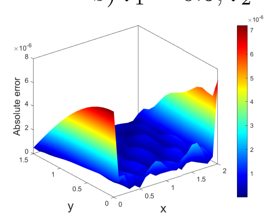
a) $\mathfrak{d}_1 = \mathfrak{d}_2 = 0.5$



b) $\mathfrak{d}_1 = 0.5, \mathfrak{d}_2 = 1$



c) $\mathfrak{d}_1 = 1, \mathfrak{d}_2 = 0.5$



d) $\mathfrak{d}_1 = \mathfrak{d}_2 = 1$

Figure 1. The absolute error of $\sin(x)\cos(y)$ by 2D-modified orthonormal normalized shifted Ultraspherical polynomials for some $\mathfrak{d}_1$ and $\mathfrak{d}_2$.

Theorem 6. *The κ-order fractional derivative with respect to x for $0 < \kappa < 1$ of the vector $\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)$ which is expressed by (4.8), is approximated as*

$${}^c_0D^\kappa_x\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) \simeq \mathbf{Q}^{(\kappa)}\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y),$$

where $\mathbf{Q}^{(\kappa)}$ is an $(\mathbf{n}_1+1)(\mathbf{n}_2+1)$ square matrix in the form of

$$\mathbf{Q}^{(\kappa)} = \mathbf{D}^{(\kappa)}_{(\mathbf{n}_1+1)} \otimes \mathbf{I}_{(\mathbf{n}_2+1)},$$

in which $\mathbf{D}^{(\kappa)}_{(\mathbf{n}_1+1)}$ *is defined by* (4.6) *and* $\mathbf{I}_{(\mathbf{n}_2+1)}$ *is an* $(\mathbf{n}_2+1)$ *identity matrix. Also, the* κ*-order fractional derivative with respect to* y *of the vector* $\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)$ *can be given as follows*

$$ {}^{c}_{0}D^{\kappa}_{y}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) \simeq \mathbf{R}^{(\kappa)}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y), $$

where $\mathbf{R}^{(\kappa)}$ *is* ${}_{(\mathbf{n}_1+1)(\mathbf{n}_2+1)\times(\mathbf{n}_1+1)(\mathbf{n}_2+1)}$*-matrix as*

$$ \mathbf{R}^{(\kappa)} = \begin{bmatrix} \mathbf{D}^{(\kappa)}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{O}_{(\mathbf{n}_2+1)} \\ \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{D}^{(\kappa)}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{O}_{(\mathbf{n}_2+1)} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{D}^{(\kappa)}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} \\ \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{D}^{(\kappa)}_{(\mathbf{n}_2+1)} \end{bmatrix}, $$

in which $\mathbf{D}^{(\kappa)}_{(\mathbf{n}_2+1)}$ *is the matrix defined by Theorem 5 with* $(\mathbf{n}_2+1)$*-order and* $\mathbf{O}_{(\mathbf{n}_2+1)}$ *is an* $(\mathbf{n}_2+1)$*-zero matrix.*

Proof. Since the proof of both parts of the Theorem is straightforward, therefore, it is left to the reader. □

Similarly, the first-order derivative operational matrix for the modified orthonormal normalized shifted Ultraspherical polynomials is calculated in two-dimensional mode.

Theorem 7. *The first-order derivatives of vector* $\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)$ *satisfy the below relations*

$$ \frac{\partial\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)}{\partial x} = \mathbf{Q}^{(1)}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y), \qquad \frac{\partial\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)}{\partial y} = \mathbf{R}^{(1)}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y), $$

where $\mathbf{Q}^{(1)} = \mathbf{D}^{(\kappa)}_{(\mathbf{n}_1+1)} \otimes \mathbf{I}_{(\mathbf{n}_2+1)}$ *and*

$$ \mathbf{R}^{(1)} = \begin{bmatrix} \mathbf{D}^{(1)}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{O}_{(\mathbf{n}_2+1)} \\ \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{D}^{(1)}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{O}_{(\mathbf{n}_2+1)} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{D}^{(1)}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} \\ \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \mathbf{O}_{(\mathbf{n}_2+1)} & \cdots & \mathbf{D}^{(1)}_{(\mathbf{n}_2+1)} \end{bmatrix}, $$

in which $\mathbf{D}^{(1)}_{(\mathbf{n}_2+1)}$ *is introduced in Theorem 4,* $\mathbf{I}_{(\mathbf{n}_2+1)}$ *and* $\mathbf{O}_{(\mathbf{n}_2+1)}$ *are an* $(\mathbf{n}_2+1)$ *identity and* $(\mathbf{n}_2+1)$ *zero matrices, respectively.*

Remark 1. The following relations can be concluded for vector $\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)$ from Property 1 for $0<\kappa<1$, by

$$ {}^{c}_{0}D^{1+\kappa}_{x}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) \simeq \mathbf{Q}^{(\kappa)}\mathbf{Q}^{(1)}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) \triangleq \mathbf{S}^{(1+\kappa)}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y), $$
$$ {}^{c}_{0}D^{1+\kappa}_{y}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) \simeq \mathbf{R}^{(\kappa)}\mathbf{R}^{(1)}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) \triangleq \mathbf{T}^{(1+\kappa)}\Psi^{(\eth_1,\eth_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y), $$

where $\mathbf{Q}^{(1)}, \mathbf{Q}^{(\kappa)}, \mathbf{R}^{(1)}$ and $\mathbf{R}^{(\kappa)}$ are introduced in Theorems 6 and 7.

4.3 Analysis of convergence

Herein, the convergence analysis of the modified orthonormal normalized shifted Ultraspherical polynomials in one and two dimensions is investigated.

Theorem 8. *Let* $u(x)$ *and* ${}_0^c D_x^{\mathfrak{j}\kappa} u(x)$ *be in* $C([0,\xi])$ *for* $\mathfrak{j} = 1, 2, \ldots, \mathbf{n}+1$ *with* $0 \leq \kappa < 1$ *and* $u_{\mathbf{n}}(x)$ *be the approximation of* $u(x)$ *by 1D-modified orthonormal normalized shifted Ultraspherical polynomials as the relation* (4.2) *that is the best approximation of* $u(x)$ *in space of* $\Xi_{\mathbf{n},\mathfrak{d}} = span\{\mathcal{G}_{0,\mathfrak{d},\xi}(x), \mathcal{G}_{1,\mathfrak{d},\xi}(x), \ldots, \mathcal{G}_{\mathbf{n},\mathfrak{d},\xi}(x)\}$. *Then, with* $\mathcal{M}_{\mathbf{n}} = \max_{0\leq x\leq\xi} \left| {}_0^c D_x^{(\mathbf{n}+1)\kappa} u(x) \right|$, *we have*

$$\|u(x) - u_{\mathbf{n}}(x)\|_{L^2_{w_{\mathfrak{d},\xi}}} \leq \frac{\mathcal{M}_{\mathbf{n}} \xi^{(\mathbf{n}+1)\kappa+1}}{\Gamma\big((\mathbf{n}+1)\kappa+1\big)} \sqrt{\frac{\Gamma\big(2(\mathbf{n}+1)\kappa+\mathfrak{d}+0.5\big)\Gamma\big(\mathfrak{d}+0.5\big)}{\Gamma\big(2(\mathbf{n}+1)\kappa+2\mathfrak{d}+1\big)}}. \quad (4.9)$$

Proof.

$$\big|u(x) - \overline{u}(x)\big| \leq \frac{x^{(\mathbf{n}+1)\kappa}}{\Gamma\big((\mathbf{n}+1)\kappa+1\big)} \max_{0\leq x\leq\xi} \left| {}_0^c D_x^{(\mathbf{n}+1)\kappa} u(x) \right|.$$

By the best approximation in $\Xi_{\mathbf{n},\mathfrak{d}}$ and the change of variable $\mathfrak{w} = \frac{x}{\xi}$, we get

$$\begin{aligned}
\big\|u(x) - u_{\mathbf{n}}(x)\big\|^2_{L^2_{w_{\mathfrak{d},\xi}}} &\leq \big\|u(x) - \overline{u}(x)\big\|^2_{L^2_{w_{\mathfrak{d},\xi}}} \\
&\leq \int_0^\xi \left(\frac{x^{(\mathbf{n}+1)\kappa}}{\Gamma\big((\mathbf{n}+1)\kappa+1\big)} \max_{0\leq x\leq\xi} \left| {}_0^c D_x^{(\mathbf{n}+1)\kappa} u(x) \right| \right)^2 w_{\mathfrak{d},\xi}(x) dx \\
&= \frac{\mathcal{M}_{\mathbf{n}}^2}{\Big(\Gamma\big((\mathbf{n}+1)\kappa+1\big)\Big)^2} \int_0^\xi x^{2(\mathbf{n}+1)\kappa} (\frac{x}{\xi} - \frac{x^2}{\xi^2})^{\mathfrak{d}-0.5} dx \\
&= \frac{\mathcal{M}_{\mathbf{n}}^2 \xi^{2(\mathbf{n}+1)\kappa+1}}{\Big(\Gamma\big((\mathbf{n}+1)\kappa+1\big)\Big)^2} \int_0^1 \mathfrak{w}^{2(\mathbf{n}+1)\kappa} (\mathfrak{w} - \mathfrak{w}^2)^{\mathfrak{d}-0.5} d\mathfrak{w} \\
&= \frac{\mathcal{M}_{\mathbf{n}}^2 \xi^{2(\mathbf{n}+1)\kappa+1}}{\Big(\Gamma\big((\mathbf{n}+1)\kappa+1\big)\Big)^2} \int_0^1 \mathfrak{w}^{2(\mathbf{n}+1)\kappa+\mathfrak{d}-0.5} (1 - \mathfrak{w})^{\mathfrak{d}-0.5} d\mathfrak{w} \\
&= \frac{\mathcal{M}_{\mathbf{n}}^2 \xi^{2(\mathbf{n}+1)\kappa+1}}{\Big(\Gamma\big((\mathbf{n}+1)\kappa+1\big)\Big)^2} \mathbf{B}\big(2(\mathbf{n}+1)\kappa+\mathfrak{d}+0.5, \mathfrak{d}+0.5\big),
\end{aligned}$$

where $\mathbf{B}$ is the Beta function. From the relationship between Beta and Gamma functions, completes the proof as

$$\big\|u(x) - u_{\mathbf{n}}(x)\big\|^2_{L^2_{w_{\mathfrak{d},\xi}}} \leq \frac{\mathcal{M}_{\mathbf{n}}^2 \xi^{2(\mathbf{n}+1)\kappa+1}}{\Big(\Gamma\big((\mathbf{n}+1)\kappa+1\big)\Big)^2} \frac{\Gamma\big(2(\mathbf{n}+1)\kappa+\mathfrak{d}+0.5\big)\Gamma\big(\mathfrak{d}+0.5\big)}{\Gamma\big(2(\mathbf{n}+1)\kappa+2\mathfrak{d}+1\big)}.$$

□

Remark 2. The expansion of function $u(x,y) \in L^2_{w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}}([0,\xi_1]\times[0,\xi_2])$ via 2D-modified orthonormal normalized shifted Ultraspherical polynomials according to (4.7), can be rewritten as

$$u(x,y) \simeq \sum_{\tilde{i}=0}^{\mathbf{n}_1}\sum_{\tilde{j}=0}^{\mathbf{n}_2} w_{\tilde{i}\tilde{j}}\Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y) = \sum_{\tilde{k}=1}^{\mathbf{r}} \overline{w}_{\tilde{k}}\overline{\Theta}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{k}}(x,y) \triangleq \overline{\mathbf{W}}^T\overline{\Psi}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y),$$

where $\mathbf{r} = (\mathbf{n}_1+1)(\mathbf{n}_2+1)$ and

$$\begin{aligned}\overline{\mathbf{W}} =& [\overline{w}_1 \quad \overline{w_2} \quad \dots \quad \overline{w}_{(\mathbf{n}_1+1)(\mathbf{n}_2+1)}]^T,\\ \overline{\Psi}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y) =& [\overline{\Theta}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_1(x,y) \quad \overline{\Theta}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_2(x,y) \quad \dots \quad \overline{\Theta}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{(\mathbf{n}_1+1),(\mathbf{n}_2+1)}(x,y)]^T,\end{aligned}$$

in which $\overline{w}_{\tilde{k}} = w_{\tilde{i}\tilde{j}}$ and $\overline{\Theta}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{k}}(x,y) = \Theta^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{i},\tilde{j}}(x,y)$ with $\tilde{k} = \tilde{i}(\mathbf{n}_2+1)+\tilde{j}+1$.

Theorem 9. *If $u(x,y) \in C^{\mathbf{r}+1}([0,\xi_1]\times[0,\xi_2])$ and $u_{\mathbf{n}_1,\mathbf{n}_2}(x,y) = \overline{\mathbf{W}}^T\overline{\Psi}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)$ is the best approximation of $u(x,y)$ by 2D-modified shifted Ultraspherical polynomials in space of $\Xi^{\mathfrak{d}_1,\mathfrak{d}_2}_{\mathbf{n}_1,\mathbf{n}_2} = span\{\overline{\Theta}^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\tilde{k}}(x,y)\}$ for $1 \le \tilde{k} \le \mathbf{r}$, then*

$$\begin{aligned}\left\|u(x,y) - u_{\mathbf{n}_1,\mathbf{n}_2}(x,y)\right\|_{L^2_{w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}}} \le& \sqrt{8}\dot{\mathcal{C}}\ddot{\mathcal{C}}\xi_1^{\mathbf{r}+1}\xi_2^{\mathbf{r}+3/2}\sqrt{\Gamma\big(\mathfrak{d}_1+0.5\big)\Gamma\big(\mathfrak{d}_2+0.5\big)}\\ &\times\sqrt{4^{\mathbf{r}+1}\big(\mathbf{r}+0.5\big)!/\sqrt{\pi}\big(\big(\mathbf{r}+0.5\big)!\big)^3},\end{aligned}$$

where $\dot{\mathcal{C}}_i = \max_i \dot{\mathcal{N}}_i$, $\quad \ddot{\mathcal{C}}_i = \max_i \ddot{\mathcal{N}}_i$, in which

$$\dot{\mathcal{N}}_i = \max_{0\le x\le\xi_1,\, 0\le y\le\xi_2}\left|\frac{\partial^i u(x,y)}{\partial x^i}\right|, \quad \ddot{\mathcal{N}}_i = \max_{0\le x\le\xi_1,\, 0\le y\le\xi_2}\left|\frac{\partial^i u(x,y)}{\partial y^i}\right|.$$

Proof. The truncation error Taylor series in two variables with $\mathbf{r}$ term yields

$$\big|u(x,y) - \overline{u}(x,y)\big| \le \frac{1}{(\mathbf{r}+1)!}\left|\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{\mathbf{r}+1} u(\bar{x},\bar{y})\right|,$$

where $(\bar{x},\bar{y}) \in (0,\xi_1)\times(0,\xi_2)$ and $\overline{u}(x,y)$ is Taylor expansion of $u(x,y)$. So, we have

$$\begin{aligned}\|\mathcal{E}\| :=& \left\|u(x,y) - u_{\mathbf{n}_1,\mathbf{n}_2}(x,y)\right\|_{L^2_{w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}}} \le \left\|u(x,y) - \overline{u}(x,y)\right\|_{L^2_{w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}}}\\ &= \int_0^{\xi_1}\int_0^{\xi_2}\big[u(x,y) - \overline{u}(x,y)\big]^2 w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}(x,y)dydx\\ &\le \frac{1}{(\mathbf{r}+1)!^2}\int_0^{\xi_1}\int_0^{\xi_2}\big[\big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\big)^{\mathbf{r}+1}u(\bar{x},\bar{y})\big]^2 w^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\xi_1,\xi_2}(x,y)dydx.\end{aligned}$$

By the binomial expansion and the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\|\mathcal{E}\| \leq 2\dot{\mathcal{N}}_i^2 \ddot{\mathcal{N}}_{\mathbf{r}+1-i}^2 \sum_{i=0}^{\mathbf{r}+1} \frac{1}{\big(i!(\mathbf{r}+1-i)!\big)^2} \int_0^{\xi_1}\int_0^{\xi_2} x^{2i} y^{2\mathbf{r}+2-2i} w_{\mathfrak{d}_1,\xi_1}(x) w_{\mathfrak{d}_2,\xi_2}(y)\,dy\,dx.$$

Similar to the calculation of integral in Theorem 8, one obtains

$$\|\mathcal{E}\| \leq 2\dot{\mathcal{N}}_i^2 \ddot{\mathcal{N}}_{\mathbf{r}+1-i}^2 \sum_{i=0}^{\mathbf{r}+1} \frac{\xi_1^{2i+1}\xi_2^{2\mathbf{r}+3-2i}}{\big(i!(\mathbf{r}+1-i)!\big)^2} \frac{\Gamma\big(2i+\mathfrak{d}_1+0.5\big)\Gamma\big(\mathfrak{d}_1+0.5\big)}{\Gamma\big(2i+2\mathfrak{d}_1+1\big)} \times \frac{\Gamma\big(2\mathbf{r}+2-2i+\mathfrak{d}_2+0.5\big)\Gamma\big(\mathfrak{d}_2+0.5\big)}{\Gamma\big(2\mathbf{r}+2-2i+2\mathfrak{d}_2+1\big)}. \tag{4.10}$$

Hence, the relation (4.10) and $\max_{\mathfrak{p},\mathfrak{q}\geq 0} \frac{\Gamma(2\mathfrak{p}+\mathfrak{q}+0.5)}{\Gamma(2\mathfrak{p}+2\mathfrak{q}+1)} \leq 2$ yield

$$\begin{aligned}\|\mathcal{E}\| &\leq 8\dot{\mathcal{N}}_i^2 \ddot{\mathcal{N}}_{\mathbf{r}+1-i}^2 \xi_1^{2\mathbf{r}+2}\xi_2^{2\mathbf{r}+3}\Gamma\big(\mathfrak{d}_1+0.5\big)\Gamma\big(\mathfrak{d}_2+0.5\big)\sum_{i=0}^{\mathbf{r}+1}\frac{1}{\big(i!(\mathbf{r}+1-i)!\big)^2}\\ &\leq 8\dot{\mathcal{C}}^2\ddot{\mathcal{C}}^2\xi_1^{2\mathbf{r}+2}\xi_2^{2\mathbf{r}+3}\Gamma\big(\mathfrak{d}_1+0.5\big)\Gamma\big(\mathfrak{d}_2+0.5\big)\frac{4^{\mathbf{r}+1}\big(\mathbf{r}+0.5\big)!}{\sqrt{\pi}\big(\big(\mathbf{r}+0.5\big)!\big)^3}.\end{aligned}$$

By taking the square root from the inequality, the desired result is obtained. □

5 The proposed method

To solve the problem (1.1), the unknown function $v(x,y,t)$ is approximated by 2D-modified orthonormal normalized shifted Ultraspherical and the orthonormal normalized shifted Ultraspherical polynomials as follows

$$v(x,y,t) \simeq \big(\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y)\big)^T \mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t), \tag{5.1}$$

where $\mathbf{V}$ is an unknown $\mathbf{r}\times(\mathbf{m}+1)$ matrix. So, from Theorems 2, 3 and 6 with Remark 1, one can show that

$$\int_0^1 \theta(\varrho)\,{}_0^cD_t^{\varrho} v(x,y,t)\,d\varrho \simeq \big(\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y)\big)^T \mathbf{V}\mathbf{G}^{(\varrho)}\Phi_{\mathbf{m},\mathfrak{d}}(t), \tag{5.2}$$

$$\begin{aligned}&\mathfrak{D}^{\kappa}v(x,y,t)\simeq(\mathbf{Q}^{(\kappa)}\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y))^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t)+\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y)^T(\mathbf{R}^{(\kappa)})^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t),\\ &\mathfrak{D}^{1+\kappa}v(x,y,t)\simeq(\mathbf{S}^{(1+\kappa)}\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y))^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t)+(\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y)\mathbf{T}^{(1+\kappa)})^T\\ &\quad\times\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t),\quad \mathfrak{D}^{1+\kappa}v_t(x,y,t)\simeq(\mathbf{S}^{(1+\kappa)}\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y))^T\mathbf{V}\mathbf{P}^{(1)}\Phi_{\mathbf{m},\mathfrak{d}}(t)\\ &\quad+(\mathbf{T}^{(1+\kappa)}\Psi_{\mathbf{n}_1,\mathbf{n}_2}^{(\mathfrak{d}_1,\mathfrak{d}_2)}(x,y))^T\mathbf{V}\mathbf{P}^{(1)}\Phi_{\mathbf{m},\mathfrak{d}}(t).\end{aligned} \tag{5.3}$$

Substituting relations (5.1)–(5.3) into (1.1), the residual function is obtained as

$$\begin{aligned}\mathcal{R}(x,y,t) \triangleq \big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)\big)^T \Big[\mathbf{V}\mathbf{G}^{(\varrho)} - (\mathbf{S}^{(1+\kappa)})^T\mathbf{V}\mathbf{P}^{(1)} - (\mathbf{T}^{(1+\kappa)})^T\mathbf{V}\mathbf{P}^{(1)} \\ - (\mathbf{S}^{(1+\kappa)})^T\mathbf{V} - (\mathbf{T}^{(1+\kappa)})^T\mathbf{V} + (\mathbf{Q}^{(\kappa)})^T\mathbf{V} + (\mathbf{R}^{(\kappa)})^T\mathbf{V}\Big]\Phi_{\mathbf{m},\mathfrak{d}}(t) \\ + \Big(\big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)\big)^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t)\Big)\Big(\big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)\big)^T(\mathbf{Q}^{(\kappa)})^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t) \\ + \big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)\big)^T(\mathbf{R}^{(\kappa)})^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t)\Big) - f(x,y,t). \qquad (5.4)\end{aligned}$$

Also, replacing (1.1) into initial-boundary conditions yields

$$\begin{aligned}\Pi_1(y,t) &\triangleq \big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(0,y)\big)^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t) - \varphi_1(y,t),\\ \Pi_2(y,t) &\triangleq \big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(\xi_1,y)\big)^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t) - \varphi_2(y,t),\\ \Pi_3(x,t) &\triangleq \big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,0)\big)^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t) - \varphi_3(x,t),\\ \Pi_4(x,t) &\triangleq \big(\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,\xi_2)\big)^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(t) - \varphi_4(x,t),\end{aligned} \qquad (5.5)$$

and $\Pi_0 \triangleq \Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)^T\mathbf{V}\Phi_{\mathbf{m},\mathfrak{d}}(0) - \varphi_0(x,y)$. From (5.4)–(5.5) and the shifted Chebyshev collocation points

$$x_i = \frac{\xi_1}{2}\Big(1-\cos\big(\frac{(2i-1)\pi}{2(\mathbf{n}_1+1)}\big)\Big),\quad y_j = \frac{\xi_2}{2}\Big(1-\cos\big(\frac{(2j-1)\pi}{2(\mathbf{n}_2+1)}\big)\Big), \qquad (5.6)$$

and $t_k = \frac{1}{2}\Big(1-\cos\big(\frac{(2k-1)\pi}{2(\mathbf{m}+1)}\big)\Big)$, the $\mathbf{r}\times(\mathbf{m}+1)$ system is obtained as follows

$$\begin{cases}\mathcal{R}(x_i,y_j,t_k)=0, & i=2,3,\dots,\mathbf{n}_1,\ j=2,3,\dots,\mathbf{n}_2,\ k=2,3,\dots,\mathbf{m}+1,\\ \Pi_0(x_i,y_j)=0, & i=1,2,\dots,\mathbf{n}_1+1,\ j=1,2,\dots,\mathbf{n}_2+1,\\ \Pi_l(y_j,t_k)=0, & j=1,2,\dots,\mathbf{n}_2+1,\ k=2,3,\dots,\mathbf{m}+1,\ l=1,2,\\ \Pi_r(x_i,t_k)=0, & i=2,3,\dots,\mathbf{n}_1,\ k=2,3,\dots,\mathbf{m}+1,\ r=3,4.\end{cases} \qquad (5.7)$$

Eventually, the unknown vector $\mathbf{V}$ is computed by solving the above equation system and the approximate solution is extracted for the problem (1.1) by placing $\mathbf{V}$ in (5.1). The algorithm of the proposed numerical method in pseudo code format is designed in Algorithm 1.

Remark 3. Considering the establishment of spectral accuracy for the constant parameter in the orthonormal normalized shifted Ultraspherical polynomials with three values $0, 1/2$ and 1, in all examples these parameters are considered as $\mathfrak{d} = \mathfrak{d}_1 = 1/2$ and $\mathfrak{d}_2 = 1$. Note that $\mathcal{U}_{0,0}(t) = 0$ and it should be omitted from the polynomials due to the violation of the orthogonality condition.

6 Numerical results

In this section, three examples are evaluated to show the correctness of the established method. To this end, the validity of the results is calculated by the

Algorithm 1 Algorithm in pseudo code format

1: **Inputs:** κ, ξ_1, ξ_2, $\mathbf{m}$, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathfrak{d}_1$, $\mathfrak{d}_2$, $\mathfrak{d}$ and the functions $\theta(\varrho)$, $\varphi_0(x,y)$, $\varphi_1(y,t)$, $\varphi_2(y,t)$, $\varphi_3(x,t)$, $\varphi_4(x,t)$ and $f(x,y,t)$.
2: Define the vectors $\Phi_{\mathbf{m},\mathfrak{d}}(t)$, $\Psi^{(\mathfrak{d}_1,\mathfrak{d}_2)}_{\mathbf{n}_1,\mathbf{n}_2}(x,y)$ and operator matrices $\mathbf{P}^{(1)}$, $\mathbf{P}^{(\kappa)}$, $\mathbf{G}^{(\varrho)}$, $\mathbf{Q}^{(\kappa)}$, $\mathbf{R}^{(\kappa)}$, $\mathbf{Q}^{(1)}$, $\mathbf{R}^{(1)}$, $\mathbf{S}^{(1+\kappa)}$, $\mathbf{T}^{(1+\kappa)}$.
3: Define the unknown matrix $\mathbf{V}$ and unknown function $\upsilon(x,y,t)$ from the relation (5.1).
4: Compute the relations (5.2)–(5.3) and the residual function $\mathcal{R}(x,y,t)$ from Equation (5.4).
5: Compute the shifted Chebyshev collocation points x_i, y_j and t_k from Equation(5.6).
6: Compute the system of algebraic equations from Equation (5.7).
7: Solve the obtained system of algebraic equations in (5.7) with command "fsolve" to get the unknown matrix $\mathbf{V}$.
8: **Outputs:** Compute the approximate function $\upsilon(x,y,t)$ by inserting the obtained matrix $\mathbf{V}$ in Equation (5.1).

below well-known errors formulas:

$$\mathcal{L}^{\infty}=\max_{x,y}|\upsilon(x,y,1)-\bar{\upsilon}(x,y,1)|,\ \mathcal{L}^{2}=\Big(\int_0^{\xi_2}\int_0^{\xi_1}\big(\upsilon(x,y,1)-\bar{\upsilon}(x,y,1)\big)^2dxdy\Big)^{1/2},$$

where $\bar{\upsilon}(x,y,1)$ is the approximation solution obtained from the mentioned method and $\upsilon(x,y,1)$ is the exact solution in the final time.

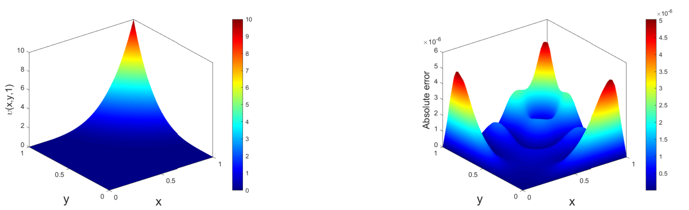
Example 1. Consider the problem (1.1) with $\theta(\varrho)=\Gamma(5-\varrho)$ over $(x,y)\in[0,1]^2$ and

$$f(x,y,t)=240t^3\Bigg[\frac{(t-1)}{\ln(t)}x^4y^4-\frac{4x^{3-\kappa}y^4}{\Gamma(4-\kappa)}-\frac{4x^4y^{3-\kappa}}{\Gamma(4-\kappa)}-\frac{tx^{3-\kappa}y^4}{\Gamma(4-\kappa)}-\frac{tx^4y^{3-\kappa}}{\Gamma(4-\kappa)}$$
$$+\frac{tx^{4-\kappa}y^4}{\Gamma(5-\kappa)}+\frac{tx^4y^{4-\kappa}}{\Gamma(5-\kappa)}+\frac{10t^5x^{8-\kappa}y^8}{\Gamma(5-\kappa)}+\frac{10t^5x^8y^{8-\kappa}}{\Gamma(5-\kappa)}\Bigg],$$

where the true solution is $\upsilon(x,y,t)=10t^4x^4y^4$. The initial and boundary conditions can be derived from the expressed exact solution. For solving this problem, we utilize our method for different values of $(\mathbf{n}_1,\mathbf{n}_2,\mathbf{m})$ and κ. The obtained numerical results are presented in Table 1 and Figure 2. In Table 1, we provide $\mathcal{L}^2$ and $\mathcal{L}^\infty$ errors for four values $\kappa=0.5,0.7,0.9,1$ and some values of $\mathbf{n}_1,\mathbf{n}_2$, and $\mathbf{m}$. One can observe that increasing the number of polynomials bases decrease the error for each κ. The convergence of the results in this table is easily visible. Also, Figure 2 is presented to demonstrate the accuracy of our scheme. This figure illustrates the approximate solution and its related absolute error for $\mathbf{n}_1=\mathbf{n}_2=\mathbf{m}=7$ and $\kappa=0.9$. The results of Table 1 and Figure 2 reveal the reliability of our approach in obtaining the approximate solution of this example.

Table 1. The outcome errors obtained by some values of $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$ and κ in Example 1.

$\mathcal{L}^2$ $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	$\kappa = 1$
$(4,4,4)$	$5.1787E-04$	$2.8426E-04$	$7.9270E-05$	$2.8558E-07$
$(5,5,5)$	$8.6815E-05$	$4.9044E-05$	$1.4350E-05$	$1.2712E-07$
$(6,6,6)$	$2.3319E-05$	$1.3778E-05$	$4.2149E-06$	$1.9634E-08$
$(7,7,7)$	$7.1672E-06$	$4.1390E-06$	$1.2705E-06$	$1.1319E-08$
$\mathcal{L}^\infty$ $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	$\kappa = 1$
$(4,4,4)$	$1.8220E-03$	$9.3574E-04$	$2.3541E-04$	$5.9512E-07$
$(5,5,5)$	$3.6992E-04$	$2.0556E-04$	$5.2214E-05$	$4.4241E-07$
$(6,6,6)$	$9.7408E-05$	$5.4558E-05$	$1.4763E-05$	$7.8959E-08$
$(7,7,7)$	$3.2195E-05$	$9.2204E-06$	$5.0375E-06$	$4.0906E-08$



a) b)

Figure 2. The outcomes obtained for $\upsilon(x, y, 1)$ (a) and related absolute error (b) with $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{m} = 7$ and $\kappa = 0.9$ in Example 1.

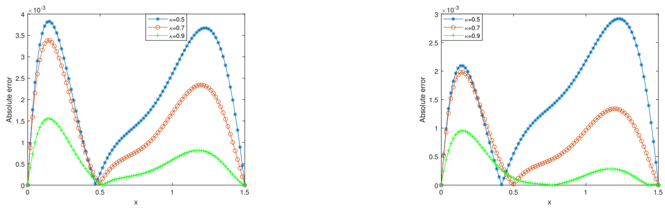
Example 2. Consider the problem (1.1) in $[0, \frac{3}{2}] \times [0, 2]$, $\theta(\varrho) = \Gamma(7 - \varrho)$ and

$$f(x,y,t) = \frac{5!t^4(6t\ln(t) - 5\ln(t) - t + 1)}{\ln(t)^2}\sin(x)e^{-y} - \left(t^5 + 5t^4\right)$$
$$\times \Big(- x^{2-\kappa}\mathbf{M}_{2,3-\kappa}(-x^2)e^{-y} + y^{1-\kappa}\mathbf{M}_{1,2-\kappa}(-y)\sin(x)\Big)$$
$$+ \left(t^{10}\sin(x)e^{-y} + t^5\right)\Big(x^{1-\kappa}\mathbf{M}_{2,2-\kappa}(-x^2)e^{-y} - y^{1-\kappa}\mathbf{M}_{1,2-\kappa}(-y)\sin(x)\Big),$$

where $\mathbf{M}_{\varepsilon,\varsigma}$ is the Mittag-Leffler function as $\mathbf{M}_{\varepsilon,\varsigma}(x) = \sum_{\hat{i}=0}^{\infty} \dfrac{x^{\hat{i}}}{\Gamma(\varepsilon\hat{i} + \varsigma)}$ and we apply the first 20 terms of the Mittag-Leffler series in our computational. Also, the analytical solution is $\upsilon(x, y, t) = t^5 \sin(x)e^{-y}$. We have employed the offered technique to solve this problem with several values of κ and $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m}) = (6, 6, 7)$. The outcomes extracted by the stated method for $y = 0.5, 1, 1.5$ and $t = 1$ are shown in Table 2. The behavior of the obtained results in $y = 1, 1.5$ and $t = 1$ with several values of fractional κ for this problem is shown in Figure 3. Furthermore, the graph of absolute error between the exact and approximate solutions with $\kappa = 1$ is depicted in Figure 4. According to the results of Table 2 and Figures 3, 4, it can be seen that the reported absolute error is suitable in some selected points x and the established method has a good ability to calculate the numerical solution for this example.

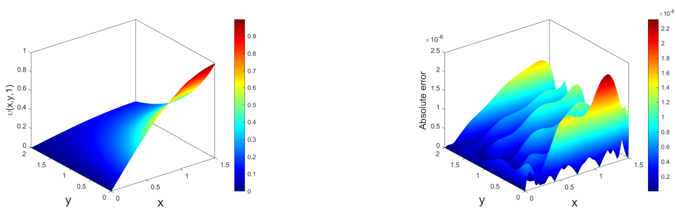
Table 2. The outcome errors obtained by some values of κ with $t = 1$ in Example 2.

x	y	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$
0.3	0.5	$4.1959E-03$	$3.4113E-03$	$1.5243E-03$
0.6		$5.4598E-04$	$4.7505E-04$	$2.0642E-04$
0.9		$2.0693E-03$	$1.5913E-03$	$6.5988E-04$
1.2		$4.4774E-03$	$3.3493E-03$	$1.3199E-03$
0.3	1	$2.2376E-03$	$1.9642E-03$	$9.1111E-04$
0.6		$8.9928E-04$	$4.7628E-04$	$1.3006E-04$
0.9		$2.0049E-03$	$1.2241E-03$	$4.0865E-04$
1.2		$3.6541E-03$	$2.3432E-03$	$8.0943E-04$
0.3	1.5	$1.0620E-03$	$1.1554E-03$	$6.0191E-04$
0.6		$9.8986E-04$	$2.6147E-04$	$4.4065E-05$
0.9		$1.7780E-03$	$6.9227E-04$	$8.0733E-05$
1.2		$2.9002E-03$	$1.3388E-03$	$2.8168E-04$



a) b)

Figure 3. The absolute errors for $\upsilon(x, y, 1)$ with $y = 1$ (a) and $y = 1.5$ (b) for $\kappa = 0.5, 0.7, 0.9$ in Example 2.



a) b)

Figure 4. The outcomes obtained for $\upsilon(x, y, 1)$ (a) and related absolute error (b) with $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m}) = (6, 6, 7)$ and $\kappa = 1$ in Example 2.

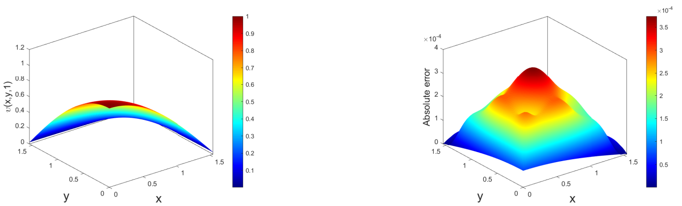
Example 3. Consider the problem (1.1) on $[0, \frac{\pi}{2}]^2$ with $\theta(\varrho) = \Gamma(\frac{5}{2} - \varrho)$ and

$$\begin{aligned} f(x,y,t) =& \frac{\Gamma(\frac{5}{2})\sqrt{t}(t-1)}{\ln(t)} \cos(x)\cos(y)(t^{\frac{3}{2}} + \frac{3}{2}t^{\frac{1}{2}})\Big(x^{1-\kappa}\mathbf{M}_{2,2-\kappa}(-x^2)\cos(y) \\ &+ y^{1-\kappa}\mathbf{M}_{2,2-\kappa}(-y^2)\cos(x)\Big) - \Big(t^{\frac{3}{2}} + t^3\cos(x)\cos(y)\Big) \\ &\times \Big(x^{2-\kappa}\mathbf{M}_{2,3-\kappa}(-x^2)\cos(y) + y^{2-\kappa}\mathbf{M}_{2,3-\kappa}(-y^2)\cos(x)\Big), \end{aligned}$$

where the exact solution is $\upsilon(x,y,t) = t^{\frac{3}{2}}\cos(x)\cos(y)$. This problem is solved by the suggested method for some values of $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$ and κ. The obtained results are collected in Table 3, and it can be shown that the outcomes improve by increasing the values $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$. These results confirm that the numerical solutions converges to the analytic solution. Furthermore, the behavior of the obtained results with $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m}) = (7, 7, 6)$ and $\kappa = 0.8$ for this problem is shown in Figure 5. From acquired outcomes, it can be observed that the explained technique has reliability for solving this example.

Table 3. The outcome errors obtained by some values of $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$ and κ in Example 3.

$\mathcal{L}^2$ $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	$\kappa = 1$
$(4, 4, 3)$	$2.1178E-03$	$1.7230E-03$	$1.1671E-03$	$8.8077E-04$
$(5, 5, 4)$	$1.0134E-03$	$1.1018E-03$	$9.2819E-04$	$7.9740E-04$
$(6, 6, 5)$	$6.7290E-04$	$5.4595E-04$	$3.8025E-04$	$2.7888E-04$
$(7, 7, 6)$	$3.1447E-04$	$3.1423E-04$	$2.8788E-04$	$2.4439E-04$
$\mathcal{L}^\infty$ $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m})$	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	$\kappa = 1$
$(4, 4, 3)$	$1.9988E-03$	$1.5795E-03$	$1.2968E-03$	$1.1457E-03$
$(5, 5, 4)$	$1.2779E-03$	$1.2005E-03$	$9.7397E-04$	$8.8454E-04$
$(6, 6, 5)$	$7.2253E-04$	$6.2712E-04$	$4.6526E-04$	$3.5316E-04$
$(7, 7, 6)$	$4.0341E-04$	$3.9244E-04$	$3.3940E-04$	$2.8042E-04$



a) b)

Figure 5. The outcomes obtained for $\upsilon(x, y, 1)$ (a) and related absolute error (b) with $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m}) = (7, 7, 6)$ and $\kappa = 0.8$ in Example 3.

7 Conclusions

In this paper, a new family of basic functions named 2D-modified orthonormal normalized shifted Ultraspherical polynomials and the novel form Benjamin Bona Mahony Burgers equation in the case of distributed-order time with fractional-order space were introduced. A direct algorithm based on operational matrices of basic functions was proposed to solve this equation. In the presented method, the main equation is converted to solve an algebraic system of equations in which the accuracy of the mentioned scheme was investigated via the several numerical examples. The report of the results shows the good ability of the approach in obtaining suitable numerical solutions. Considering the remarkable accuracy exhibited by 2D-modified orthonormal normalized shifted Ultraspherical polynomials in numerical solution for the distributed-order time 2D Benjamin Bona Mahony Burgers equation, future investigations must concentrate on multiple aspects encompassing including the expansion of this derivative to diverse equations and the replacement of the operator matrix approach with other numerical techniques.

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