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A class of nonlinear systems with new boundary conditions: existence of solutions, stability and travelling waves

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Abstract. In this work, we begin by introducing a new notion of coupled closed fractional boundary conditions to study a class of nonlinear sequential systems of Caputo fractional differential equations. The existence and uniqueness of solutions for the class of systems is proved by applying Banach contraction principle. The existence of at least one solution is then accomplished by applying Schauder fixed point theorem. The Ulam Hyers stability, with a limiting-case example, is also discussed. In a second part of our work, we use the *tanh* method to obtain a new travelling wave solution for the coupled system of Burgers using time and space Khalil derivatives. By bridging these two aspects, we aim to present an understanding of the system's behaviour.

Keywords: Caputo sequential derivative; coupled system of FDEs; existence and uniqueness; Ulam-Hyers stability; fixed point theorem; travelling wave; Khalil derivative.

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1 Introduction

Coupled systems of second-order ordinary differential equations (CSSODEs) play a crucial role in modeling a variety of physical, biological, and engineering phenomena. These systems consist of two ordinary equations whose solutions are interdependent, often requiring simultaneous solutions to describe the dynamics of the system comprehensively. A common application of the CSSODEs can be found in the study of mechanical vibrations, where the equations govern the motion of interconnected mechanical components. Similarly, in electrical engineering, these systems describe the behavior of circuits with multiple interacting components, such as inductors and capacitors [3, 9, 18].

Coupled systems involving Caputo derivatives (CSCDs) have garnered significant attention (by studying existence of solutions and their Ulam-Hyers stabilities) in recent years due to their applicability in modeling complex phenomena in various scientific and engineering disciplines, see [7, 10, 24, 28] for more details. Other important papers dealing with fixed point theorems, and Ulam–Hyers stability can be found in [20, 21, 22].

One crucial aspect in the analysis of the CSCDs is the consideration of boundary conditions. While the literature extensively covers classical boundary conditions, a novel contribution has emerged in the form of closed boundary conditions, see the work [15, 23]. Building upon the well-established concept of closed boundary conditions, our research seeks to delve deeper into their application in the domain of CSCDs.

Closed boundary conditions, in their traditional form, play a pivotal role in defining the behaviour of a system within a specified domain. These conditions encapsulate the interactions between the system and its surroundings, ensuring a well-posed problem with a clear set of constraints at the boundaries, see [24]. However, as we navigate the landscape of fractional calculus and Caputo derivatives, the traditional closed boundary conditions may require adaptation to address the unique characteristics introduced by fractional order derivatives.

This paper focuses on analyzing a class of coupled differential systems with derivatives close to two in the sense of the Caputo derivative. This focus lies also in introducing a novel concept termed "coupled closed fractional boundary conditions". This innovation aims to capture the intricate interplay between fractional order derivatives and boundary effects in differential systems. The coupled closed fractional boundary conditions provide a more comprehensive framework for understanding and solving differential equations involving Caputo derivatives, enriching the mathematical tools available for researchers and practitioners alike.

By shedding light on this novel concept, our work contributes to the evolving field of fractional derivatives, paving the way for a more nuanced understanding of differential systems with sequential Caputo derivatives and their associated boundary conditions. This nuanced understanding is critical for advancing both theoretical and applied aspects of fractional differential equations. For instance, in [4], the authors introduced an important concept of coupled closed boundary conditions to investigate the existence and uniqueness of solutions for a system of nonlinear sequential fractional differential equations. This work complements our research by providing a foundational approach to handling complex boundary conditions in fractional systems, thereby enhancing the applicability and robustness of such problems in various scientific and engineering disciplines. The sequential system of [4] is the following:

$$\begin{cases} {}^{c}D^{q_1}\varphi(t) = \rho_1(t,\varphi(t),\psi(t)), & t \in J = [0,T]\,, \\ {}^{c}D^{q_2}\psi(t) = \rho_2(t,\varphi(t),\psi(t)), & t \in J = [0,T]\,, \end{cases}$$

complemented with the conditions

$$\begin{cases} \varphi(T) = \alpha_1\psi(0) + \beta_1 T\psi'(0), & T\varphi'(T) = \gamma_1\psi(0) + \delta_1 T\psi'(0), \\ \psi(T) = \alpha_2\varphi(0) + \beta_2 T\varphi'(0), & T\psi'(T) = \gamma_2\varphi(0) + \delta_2 T\varphi'(0), \end{cases}$$

where, ${}^cD^{q_1}, {}^cD^{q_2}$ denote the Caputo fractional derivatives of order q_1, q_2, $1 < q_1, q_2 < 2$ respectively, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \in \mathbb{R}$, $T > 0$, and $\rho_1, \rho_1 \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

In another paper, B. Ahmed et al. [2] proposed new nonlocal variant of closed boundary conditions and studied an integro-differential problem containing a Caputo fractional derivative and mixed Riemann–Liouville type integral nonlinearities supplemented. Specifically, the authors explored the criteria ensuring existence of solutions for their studied problem.

Also, in [6] the authors investigated the existence, uniqueness, stability and approximate solutions of a thermostat fractional differential equation involving ABC derivative. Very recently, A. Lamamri et al. [13] investigated a more general couple Caputo sequential differential system by incorporating fractional derivatives in some of the initial conditions. The authors focused on proving the existence of unique solutions for the system.

In this paper, we have to investigate two parts. In the first part, motivated by the above cited papers on CSSODEs and by the paper of [4], we study the existence of solutions and their Ulam-Hyers stability for the following coupled problem:

$$\begin{cases} D^{\alpha_1} D^{\alpha_2} x(t) = f_1(t, x(t), y(t)), \\ D^{\beta_1} D^{\beta_2} y(t) = f_2(t, x(t), y(t)), \end{cases} \tag{1.1}$$

under the coupled closed fractional boundary conditions:

$$\begin{cases} x(1) = A_1 y(0) + B_1 D^{\beta^*} y(0), \\ D^{\alpha^*} x(1) = \varepsilon_1 y(0) + \zeta_1 D^{\beta^*} y(0), \\ y(1) = A_2 x(0) + B_2 D^{\alpha^*} x(0), \\ D^{\beta^*} y(1) = \varepsilon_2 x(0) + \zeta_2 D^{\alpha^*} x(0). \end{cases} \tag{1.2}$$

We impose the following conditions:

The derivatives $D_i^{\alpha}, D_i^{\beta}, D^{\beta^*}, D^{\alpha^*}$ are in the sense of Caputo.

* The parameters $A_1, B_1, \varepsilon_1, \zeta_1, A_2, B_2, \varepsilon_2, \zeta_2 \in \mathbb{R}$.

** To guarantee the absence of semi-group and commutativity properties on the considered Caputo derivatives and to obtain the above closed conditions of Alsaadi et al. as a particular case, we suppose that the following conditions are valid: $0 < \alpha_1, \beta_1, \alpha_2, \beta_2 \leqslant 1, \alpha^* = \min\{\alpha_1, \alpha_2\}, \beta^* = \min\{\beta_1, \beta_2\}$, $\beta_1 + \beta_2 > 1, \alpha_1 + \alpha_2 > 1$.

*** We also suppose that the two functions f_1, f_2 are continuous over $J \times \mathbb{R}^2$.

The reader can see clearly that our problem is more general than the above two problems of [4]. Another important motivation is in introduction of the coupled closed fractional boundary conditions, which lead to a significant difference between our system and the associated integral representations presented in [4]. Another important novelty of our problem is given by the sequential concept on the derivatives of the left hand side of our problem. The absence of the semi group and the commutativity on the derivatives of our system is also to be noted in this study.

In the second part, we focus on the application of the tanh method; we employ this technique to uncover a new traveling wave solution for a system of Burgers equations, involving time and space conformable derivatives in accordance with Khalil's framework. The tanh method is particularly effective for solving nonlinear differential equations, allowing us to find exact solutions by transforming the original problem into a simpler form.

This approach highlights the versatility and robustness of the tanh method in dealing with complex systems that are influenced by fractional derivatives. In particular, the use of conformable derivatives as defined by Khalil provides a more flexible framework to model physical phenomena, as these derivatives generalize the concept of differentiation to non-integer orders, offering more accurate descriptions of processes with memory and hereditary properties.

By integrating these two distinct yet interconnected aspects, our study aims to present to the reader and to deliver a comprehensive understanding of the system's behavior from a holistic perspective. The theoretical groundwork laid by the Caputo derivatives in the first part is effectively complemented by the practical applications demonstrated through the tanh method, via Khalil approach, in the second part. This synthesis underscores the critical role of fractional calculus in expanding classical mathematical approaches to tackle more intricate and realistic problems. By doing so, we significantly enhance our capability to analyze nonlinear systems with fractional derivatives, providing valuable insights and broader applicability in various scientific and engineering contexts.

The paper is arranged as follows: In Section 2, we recall some background notions that will be used later. Section 3 is concerned with the main existence results for the class of coupled systems equipped with introduced closed conditions. The Ulam-Hyers stability of the solutions for the class of systems is investigated in the same Section. Some detailed examples are discussed in Section 4. In Section 5, we employ the tanh technique to uncover a new travelling wave solution for a coupled system of Burgers equations, incorporating conformable derivatives in the sense of Khalil. At the end a conclusion follows.

2 Preliminaries on fractional calculus

We introduce some definitions and lemmas on fractional calculus. We invite the reader the see [12, 17] for more informations on these concepts.

2.1 Definitions and lemmas

DEFINITION 1. For any $\alpha > 0$ and any continuous function $f : J \longmapsto \mathbb{R}$, the Riemann-Liouville integral is defined by:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

DEFINITION 2. If $f \in C^n(J,\mathbb{R})$, such that $n-1<\alpha\leq n$, then the Caputo derivative is defined by:

$$D^{\alpha}f(t) \quad = \quad I^{n-\alpha}\frac{d^n}{dt^n}(f(t)).$$

The following two lemmas will be used in the present paper.

Lemma 1. *Let $n\in\mathbb{N}^*$, and $n-1<\alpha\leq n$. Then, the homogeneous differential equation $D^{\alpha}y(t)=0; t\in J$ admits as general solution the function y given by*

$$y(t)=\sum_{i=0}^{n-1}c_it^i, c_i\in\mathbb{R}.$$

Lemma 2. *Let $n\in\mathbb{N}^*$ and $n-1<\alpha\leq n$. Then, the property*

$$I^{\alpha}D^{\alpha}y(t)=y(t)+\sum_{i=0}^{n-1}c_it^i, c_i\in\mathbb{R}$$

is valid.

Let us now pass to prove the following equivalence:

Lemma 3. *Let G_1 and G_1 be two continuous functions over J. Suppose also that $0<\alpha_1,\beta_1,\alpha_2,\beta_2\leqslant 1, \alpha^*=\min\{\alpha_1,\alpha_2\}, \beta^*=\min\{\beta_1,\beta_2\}, \beta_1+\beta_2>1, \alpha_1+\alpha_2>1$, and $A_1,B_1,\varepsilon_1,\zeta_1,A_2,B_2,\varepsilon_2,\zeta_2\in\mathbb{R}$. Then, the linear differential problem*

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}x(t)=G_1(t), \quad D^{\beta_1}D^{\beta_2}y(t)=G_2(t),\\ x(1)=A_1y(0)+B_1D^{\beta^*}y(0),\\ D^{\alpha^*}x(1)=\varepsilon_1y(0)+\zeta_1D^{\beta^*}y(0),\\ y(1)=A_2x(0)+B_2D^{\alpha^*}x(0),\\ D^{\beta^*}y(1)=\varepsilon_2x(0)+\zeta_2D^{\alpha^*}x(0), \end{cases}$$

is equivalent to the following integral problem

$$\begin{aligned}
&(x(t),y(t))\\
=&\Bigg(I^{\alpha_1+\alpha_2}G_1(t)+\Bigg[\Gamma(\alpha_2-\alpha^*+1)\left(\frac{\varepsilon_1}{\Delta}+\frac{\varepsilon_1\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2}G_1(1)\\
&-\Gamma(\alpha_2-\alpha^*+1)\left(\varepsilon_1\big[\Delta^{-1}\nu+\frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\big]+1\right)I^{\alpha_1+\alpha_2-\alpha^*}G_1(1)\\
&+\varepsilon_1\Delta^{-1}\Gamma(\alpha_2-\alpha^*+1)\left(\frac{1}{\varepsilon_2}+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}G_2(1)
\end{aligned}$$

$$+\frac{\varepsilon_1\Delta^{-1}\varphi\Gamma(\alpha_2-\alpha^*+1)}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}G_2(1)\Bigg]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}+\left(\frac{\varphi\Delta^{-1}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)$$
$$\times I^{\alpha_1+\alpha_2}G_1(1)-\left(\frac{\varphi\Delta^{-1}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2-\alpha^*}G_1(1)+\Bigg(\frac{1}{\varepsilon_2}$$
$$+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\Bigg)I^{\beta_1+\beta_2-\beta^*}G_2(1)+\frac{\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}G_2(1),$$
$$I^{\beta_1+\beta_2}G_2(t)+\Bigg[\varphi\Delta^{-1}I^{\alpha_1+\alpha_2}G_1(1)-\varphi\Delta^{-1}\nu I^{\alpha_1+\alpha_2-\alpha^*}G_1(1)$$
$$+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2}I^{\beta_1+\beta_2-\beta^*}G_2(1)+\varphi I^{\beta_1+\beta_2}G_2(1)\Bigg]\frac{t^{\beta_2}}{\Gamma(\beta_2+1)}$$
$$+\left(\Delta^{-1}+\frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2}G_1(1)-\left(\Delta^{-1}\nu+\frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)$$
$$\times I^{\alpha_1+\alpha_2-\alpha^*}G_1(1)+\Delta^{-1}\left(\frac{1}{\varepsilon_2}+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}G_2(1)$$
$$+\frac{\Delta^{-1}\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}G_2(1)\Bigg),$$

where

$$\nu:=\frac{\Gamma(\alpha_2-\alpha^*+1)}{\Gamma(\alpha_2+1)},\quad \Delta:=A_1-\varepsilon_1\nu\neq 0,$$
$$\varphi:=\Big(\frac{A_2-\Delta^{-1}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}-\frac{1}{\Gamma(\beta_2+1)}\Big)^{-1}.$$

Proof. Applying Lemma 2 to the above linear problem, we can write

$$(x(t),y(t))$$
$$=(I^{\alpha_1+\alpha_2}G_1(t)+c_0\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}+c_1,I^{\beta_1+\beta_2}G_2(t)+k_0\frac{t^{\beta_2}}{\Gamma(\beta_2+1)}+k_1).$$

Consequently, we obtain

$$D^{\alpha^*}x(t)=I^{\alpha_1+\alpha_2-\alpha^*}G_1(t)+c_0t^{\alpha_2-\alpha^*}/\Gamma(\alpha_2-\alpha^*+1),$$
$$D^{\beta^*}y(t)=I^{\beta_1+\beta_2-\beta^*}G_2(t)+k_0t^{\beta_2-\beta^*}/\Gamma(\beta_2-\beta^*+1).$$

In the above four expressions of $x(t),y(t),D^{\beta^*}y(t),D^{\alpha^*}x(t)$, by taking $t=0$, then $t=1$, we obtain the following quantities with four unknown parameters k_1,c_1,k_0,c_0, that need to be determined:

$$y(0)=k_1,x(0)=c_1,D^{\beta^*}y(0)=0,D^{\alpha^*}x(0)=0,$$
$$x(1)=I^{\alpha_1+\alpha_2}G_1(1)+c_0\frac{1}{\Gamma(\alpha_2+1)}+c_1,$$

$$
\begin{aligned}
y(1) &= I^{\beta_1+\beta_2}G_2(1) + k_0/\Gamma(\beta_2+1) + k_1,\\
D^{\alpha^*}x(1) &= I^{\alpha_1+\alpha_2-\alpha^*}G_1(1) + c_0/\Gamma(\alpha_2-\alpha^*+1),\\
D^{\beta^*}y(1) &= I^{\beta_1+\beta_2-\beta^*}G_2(1) + k_0/\Gamma(\beta_2-\beta^*+1).
\end{aligned}
$$

Thanks to our imposed conditions, we obtain the following four values for the above unknown parameters:

$$
\begin{aligned}
c_0 &= \varepsilon_1\Gamma(\alpha_2-\alpha^*+1)\Big(\Delta^{-1} + \frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Big)I^{\alpha_1+\alpha_2}G_1(1)\\
&\quad - \Gamma(\alpha_2-\alpha^*+1)\left(\varepsilon_1\big[\Delta^{-1}\nu + \frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\big] + 1\right)I^{\alpha_1+\alpha_2-\alpha^*}G_1(1)\\
&\quad + \varepsilon_1\Delta^{-1}\Gamma(\alpha_2-\alpha^*+1)\left(\frac{1}{\varepsilon_2} + \frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}G_2(1)\\
&\quad + \frac{\varepsilon_1\Delta^{-1}\varphi\Gamma(\alpha_2-\alpha^*+1)}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}G_2(1),\\
c_1 &= \left(\frac{\varphi\Delta^{-1}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2}G_1(1) - \left(\frac{\varphi\Delta^{-1}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2-\alpha^*}G_1(1)\\
&\quad + \left(\frac{1}{\varepsilon_2} + \frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}G_2(1) + \frac{\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}G_2(1),\\
k_0 &= \varphi\Delta^{-1}I^{\alpha_1+\alpha_2}G_1(1) - \varphi\Delta^{-1}\nu I^{\alpha_1+\alpha_2-\alpha^*}G_1(1) + \frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2}I^{\beta_1+\beta_2-\beta^*}\\
&\quad \times G_2(1) + \varphi I^{\beta_1+\beta_2}G_2(1),\\
k_1 &= \left(\Delta^{-1} + \frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2}G_1(1) - \left(\Delta^{-1}\nu + \frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)\\
&\quad \times I^{\alpha_1+\alpha_2-\alpha^*}G_1(1) + \Delta^{-1}\left(\frac{1}{\varepsilon_2} + \frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}\\
&\quad \times G_2(1) + \frac{\Delta^{-1}\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}G_2(1).
\end{aligned}
$$

The first implication of the equivalence is thus proved.
The second implication is trivial and hence, we omit it. □

2.2 Fixed points of Banach spaces

Since we use fixed point theory to study the above problem, so to be able to do this, we need to introduce the following notions:

1. We consider the space $S \times S := \{(x,y) : x, y \in C(J,\mathbb{R})\}$.
2. Then, over this space, we take the sum norm $\|(x,y)\|_{S\times S} = \|x\|_\infty + \|y\|_\infty$, where, $\|x\|_\infty = \sup_{t\in J}|x(t)|$, $\|y\|_\infty = \sup_{t\in J}|y(t)|$.

3. Also, we take the nonlinear coupled operator $Q : S \times S \to S \times S$: $Q(x,y) = \big(Q_1(x,y), Q_2(x,y)\big)$, such that, for any $t \in J$, the components of the operator are given by the following two expressions:

$$
\begin{aligned}
Q_1(x,y)(t) &= I^{\alpha_1+\alpha_2} f_1(t,x(t),y(t)) + \Bigg[\varepsilon_1 \Gamma(\alpha_2-\alpha^*+1)\Bigg(\Delta^{-1} \\
&+ \frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Bigg) I^{\alpha_1+\alpha_2} f_1(1,x(1),y(1)) - \Gamma(\alpha_2-\alpha^*+1) \\
&\times \Bigg(\varepsilon_1\Big[\Delta^{-1}\nu + \frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Big] + 1\Bigg) I^{\alpha_1+\alpha_2-\alpha^*} f_1(1,x(1),y(1)) \\
&+ \varepsilon_1\Delta^{-1}\Gamma(\alpha_2-\alpha^*+1)\Bigg(\frac{1}{\varepsilon_2} + \frac{(\Delta^{-1}-A_2)}{\varepsilon_2^2}\frac{\varphi}{\Gamma(\beta_2-\beta^*+1)}\Bigg) \\
&\times I^{\beta_1+\beta_2-\beta^*} f_2(1,x(1),y(1)) + \frac{\varepsilon_1\Delta^{-1}\varphi\Gamma(\alpha_2-\alpha^*+1)}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)} \\
&\times I^{\beta_1+\beta_2} f_2(1,x(1),y(1))\Bigg] \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \Bigg(\frac{\varphi\Delta^{-1}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Bigg) I^{\alpha_1+\alpha_2} \\
&\times f_1(1,x(1),y(1)) - \Bigg(\frac{\varphi\Delta^{-1}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Bigg) I^{\alpha_1+\alpha_2-\alpha^*} f_1(1,x(1),y(1)) \\
&+ \Bigg(\frac{1}{\varepsilon_2} + \frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\Bigg) I^{\beta_1+\beta_2-\beta^*} f_2(1,x(1),y(1)) \\
&+ \frac{\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)} I^{\beta_1+\beta_2} f_2(1,x(1),y(1)),
\end{aligned}
$$

$$
\begin{aligned}
Q_2(x,y)(t) &= I^{\beta_1+\beta_2} f_2(t,x(t),y(t)) + \Bigg[\varphi\Delta^{-1} I^{\alpha_1+\alpha_2} f_1(1,x(1),y(1)) \\
&- \varphi\Delta^{-1}\nu I^{\alpha_1+\alpha_2-\alpha^*} f_1(1,x(1),y(1)) + \frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2} \\
&\times I^{\beta_1+\beta_2-\beta^*} f_2(1,x(1),y(1)) + \varphi I^{\beta_1+\beta_2} f_2(1,x(1),y(1))\Bigg] \frac{t^{\beta_2}}{\Gamma(\beta_2+1)} \\
&+ \Bigg(\Delta^{-1} + \frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Bigg) \times I^{\alpha_1+\alpha_2} f_1(1,x(1),y(1)) \\
&- \Bigg(\Delta^{-1}\nu + \frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Bigg) I^{\alpha_1+\alpha_2-\alpha^*} f_1(1,x(1),y(1)) \\
&+ \Delta^{-1}\Bigg(\frac{1}{\varepsilon_2} + \frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\Bigg) \times I^{\beta_1+\beta_2-\beta^*} f_2(1,x(1),y(1)) \\
&+ \frac{\Delta^{-1}\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)} I^{\beta_1+\beta_2} f_2(1,x(1),y(1)).
\end{aligned}
$$

The reader can remark that the fixed points of the above coupled operator are the vector solutions of (1.1)–(1.2).

2.3 Hypotheses

We consider the following hypotheses:
$(H1)$: There exist positive constants $\iota_{f1,1}, \iota_{f1,2}, \iota_{f2,1}, \iota_{f2,2}$, such that for any $t \in J$ and for any $y_i, y_i^* \in \mathbb{R}$, the inequalities

$$\begin{aligned} |f_1(t, y_1, y_2) - f_1(t, {y_1}^*, {y_2}^*)| &\leq \sum_{i=1}^{2} \iota_{f1,i} |y_i - {y_i}^*|, \\ |f_2(t, y_1, y_2) - f_2(t, {y_1}^*, {y_2}^*)| &\leq \sum_{i=1}^{2} \iota_{f2,i} |y_i - {y_i}^*| \end{aligned}$$

are valid, such that

$$\mu := \max(\iota_{f1,1}, \iota_{f1,2}), \mu^* := \max(\iota_{f2,1}, \iota_{f2,2}).$$

$(H2)$: There exist positive constants $U_i, V_i, W_i; i = 1, 2$, such that

$$|f_i(t, x, y)| \leq U_i |x| + V_i |y| + W_i, \text{ for } t \in J, x, y \in \mathbb{R}.$$

The following quantities will be used in the proof of our results:

$$\begin{aligned} &\delta = \frac{1}{\Gamma(\alpha_1 + \alpha_2 + 1)}, \quad \delta^* = \frac{1}{\Gamma(\alpha_1 + \alpha_2 - \alpha^* + 1)}, \quad \vartheta = \frac{1}{\Gamma(\beta_1 + \beta_2 + 1)}, \\ &\vartheta^* = \frac{1}{\Gamma(\beta_1 + \beta_2 - \beta^* + 1)}, \ \varsigma_1 = \frac{\varphi \Delta^{-2}}{\varepsilon_2 \Gamma(\beta_2 - \beta^* + 1)}, \quad \varsigma_2 = \frac{1}{\varepsilon_2} + \frac{(\Delta^{-1} - A_2)\varphi}{\varepsilon_2^2 \Gamma(\beta_2 - \beta^* + 1)}, \\ &\Upsilon = \mu\Big(\delta + \Big[|\varepsilon_1| \Gamma(\alpha_2 - \alpha^* + 1)|\Delta^{-1} + \varsigma_1|\delta + \Gamma(\alpha_2 - \alpha^* + 1) \\ &\quad \times \big(|\varepsilon_1|[\nu|\Delta^{-1}| + |\varsigma_1|] + 1\big)\delta^* + |\varepsilon_1 \Delta^{-1}| \Gamma(\alpha_2 - \alpha^* + 1)|\varsigma_2|\vartheta^* + \frac{|\varepsilon_1 \Delta \varsigma_1|}{\delta^*}\vartheta\Big] \\ &\quad \times \frac{1}{\Gamma(\alpha_2 + 1)} + |\Delta\varsigma_1|\delta + \nu|\Delta\varsigma_1|\delta^* + |\varsigma_2|\vartheta^* + \frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2 - \beta^* + 1)}\vartheta\Big), \\ &\Upsilon^* = \mu^*\Big(\vartheta + \Big[|\varphi\Delta^{-1}|(\delta + \nu\delta^*) + \frac{|(\Delta^{-1} - A_2)\varphi|}{|\varepsilon_2|}\vartheta^* \\ &\quad + |\varphi|\vartheta\Big] \frac{1}{\Gamma(\beta_2 + 1)} + (|\Delta^{-1}| + |\varsigma_1|)(\delta + \nu\delta^*) + |\Delta^{-1}\varsigma_2|\vartheta^* + |\Delta\varsigma_1|\vartheta\Big). \end{aligned}$$

3 Main results

3.1 Application of Banach contraction principle

Using $(H1)$, we prove the following result as a consequence of Banach fixed point theorem.

Theorem 1. *If the functions f_1 and f_2 satisfy $(H1)$, with $\Omega < 1; \Omega = \Upsilon + \Upsilon^*$, then, system (1.1)–(1.2) has a unique solution.*

Proof. First of all, it trivial to prove the stability of the above Banach space by Q. This point is thus omitted in the proof of this theorem. But, we have to prove the contraction of the operator. To do this, let consider $x, y \in S \times S$. So, some easy calculations and by passing to the maximum over J will allow us to obtain the following two estimates on $Q_i, i = 1, 2$:

$$\begin{aligned}
&\left\|Q_1(y_1,y_2)-Q_1(x_1,x_2)\right\|_\infty \le \mu\Big(\delta+\Big[|\varepsilon_1|\Gamma(\alpha_2-\alpha^*+1)|\Delta^{-1}+\varsigma_1|\delta \\
&+\Gamma(\alpha_2-\alpha^*+1)\big(|\varepsilon_1|[\nu|\Delta^{-1}|+|\varsigma_1|]+1\big)\delta^*+|\varepsilon_1\Delta^{-1}|\Gamma(\alpha_2-\alpha^*+1)|\varsigma_2|\vartheta^* \\
&+\frac{|\varepsilon_1\Delta\varsigma_1|}{\delta^*}\vartheta\Big]\frac{1}{\Gamma(\alpha_2+1)}+|\Delta\varsigma_1|\delta+\nu|\Delta\varsigma_1|\delta^*+|\varsigma_2|\vartheta^* \\
&+\frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2-\beta^*+1)}\vartheta\Big)\left\|(y_1,y_2)-(x_1,x_2)\right\|_{S\times S}, \\
&\left\|Q_2(y_1,y_2)-Q_2(x_1,x_2)\right\|_\infty \le \mu^*\Big(\vartheta+\Big[|\varphi\Delta^{-1}|(\delta+\nu\delta^*)+\frac{|(\Delta^{-1}-A_2)\varphi|}{|\varepsilon_2|}\vartheta^* \\
&+|\varphi|\vartheta\Big]\frac{1}{\Gamma(\beta_2+1)}+(|\Delta^{-1}|+|\varsigma_1|)(\delta+\nu\delta^*)+|\Delta^{-1}\varsigma_2|\vartheta^*+|\Delta\varsigma_1|\vartheta\Big) \\
&\times\left\|(y_1,y_2)-(x_1,x_2)\right\|_{S\times S}.
\end{aligned}$$

Consequently, the following two inequalities

$$\begin{aligned}
\left\|Q_1(y_1,y_2)-Q_1(x_1,x_2)\right\|_\infty &\le \Upsilon\ \left\|(y_1,y_2)-(x_1,x_2)\right\|_{S\times S}, \\
\left\|Q_2(y_1,y_2)-Q_2(x_1,x_2)\right\|_\infty &\le \Upsilon^*\ \left\|(y_1,y_2)-(x_1,x_2)\right\|_{S\times S}
\end{aligned}$$

are valid. Then, we have

$$\left\|Q(y_1,y_2)-Q(x_1,x_2)\right\|_{S\times S} \le \Omega\ \left\|(y_1,y_2)-(x_1,x_2)\right\|_{S\times S}.$$

The contraction of the above coupled operator is thus achieved. □

3.2 Application of Schauder fixed point theorem

Under the weaker hypothesis $(H2)$, we can establish the following existence result as a consequence of Schauder fixed point theorem. The following quantities will be needed:

$$\begin{aligned}
\eth_1 =&\delta(U_1+V_1)+\Big[(U_1+V_1)|\varepsilon_1|\Gamma(\alpha_2-\alpha^*+1)|\Delta^{-1} \\
&+(U_1+V_1)\varsigma_1|\delta+\Gamma(\alpha_2-\alpha^*+1)\big(|\varepsilon_1|[\nu|\Delta^{-1}|+|\varsigma_1|]+1\big)\delta^* \\
&+(U_2+V_2)|\varepsilon_1\Delta^{-1}|\Gamma(\alpha_2-\alpha^*+1)|\varsigma_2|\vartheta^*+(U_2+V_2)\frac{|\varepsilon_1\Delta\varsigma_1|}{\delta^*}\vartheta\Big]\frac{1}{\Gamma(\alpha_2+1)} \\
&+(U_1+V_1)|\Delta\varsigma_1|\delta+(U_1+V_1)\nu|\Delta\varsigma_1|\delta^*+(U_2+V_2)|\varsigma_2|\vartheta^* \\
&+(U_2+V_2)\frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2-\beta^*+1)}\vartheta, \\
\eth_2 =&(U_2+V_2)\vartheta+\Big[(U_1+V_1)|\varphi\Delta^{-1}|(\delta+\nu\delta^*)+(U_2+V_2) \\
&\times\Big(\frac{|(\Delta^{-1}-A_2)\varphi|}{|\varepsilon_2|}\vartheta^*+|\varphi|\vartheta\Big)\Big]\frac{1}{\Gamma(\beta_2+1)}+(U_1+V_1)(|\Delta^{-1}|+|\varsigma_1|)(\delta+\nu\delta^*)
\end{aligned}$$

$$
\begin{aligned}
&+ (U_2 + V_2)\Big(|\Delta^{-1}\varsigma_2|\vartheta^* + |\Delta\varsigma_1|\vartheta\Big),\\
\tau_1 =&\delta[W_1]+\Big[[W_1]|\varepsilon_1|\Gamma(\alpha_2 - \alpha^* + 1)|\Delta^{-1}\\
&+ [W_1]\varsigma_1|\delta + \Gamma(\alpha_2 - \alpha^* + 1)\big(|\varepsilon_1|[\nu|\Delta^{-1}| + |\varsigma_1|] + 1\big), \delta^*\\
&+ [W_2]|\varepsilon_1\Delta^{-1}|\Gamma(\alpha_2 - \alpha^* + 1)|\varsigma_2|\vartheta^* + [W_2]\frac{|\varepsilon_1\Delta\varsigma_1|}{\delta^*}\vartheta\Big]\frac{1}{\Gamma(\alpha_2 + 1)}\\
&+ [W_1]|\Delta\varsigma_1|\delta + [W_1]\nu|\Delta\varsigma_1|\delta^* + [W_2]|\varsigma_2|\vartheta^* + [W_2]\frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2 - \beta^* + 1)}\vartheta,\\
\tau_2 =&[W_2]\vartheta+\Big[[W_1]|\varphi\Delta^{-1}|(\delta+\nu\delta^*)+[W_2]\Big(\frac{|(\Delta^{-1}-A_2)\varphi|}{|\varepsilon_2|}\vartheta^*+|\varphi|\vartheta\Big)\Big]\frac{1}{\Gamma(\beta_2 + 1)}\\
&+ [W_1](|\Delta^{-1}| + |\varsigma_1|)(\delta + \nu\delta^*) + [W_2]\Big(|\Delta^{-1}\varsigma_2|\vartheta^* + |\Delta\varsigma_1|\vartheta\Big).
\end{aligned}
$$

Theorem 2. *Assume that* $(H2)$ *is satisfied and* $\eth_1 + \eth_2 < 1$. *Then, (1.1)–(1.2) has at least one solution defined over* J.

Proof. We need to prove that Q has a fixed point into a subset of the form

$$
B_R := \Big\{(x_1, x_2) \in S \times S; \big\|(x_1, x_2)\big\|_{S\times S} \leq R\Big\},
$$

for a suitable positive R. For $(y_1, y_2) \in B_R$, we can write

$$
\begin{aligned}
&\big\|Q_1(y_1, y_2)\big\|_\infty \leq \delta[(U_1 + V_1)R + W_1] + \Big[[(U_1 + V_1)R + W_1]|\varepsilon_1|\\
&\times \Gamma(\alpha_2-\alpha^*+1)|\Delta^{-1}|+[(U_1 + V_1)R+W_1]\varsigma_1|\delta+\Gamma(\alpha_2-\alpha^* + 1)\big(|\varepsilon_1|[\nu|\Delta^{-1}|\\
&+ |\varsigma_1|]+1\big)\delta^*+[(U_2+V_2)R+W_2]|\varepsilon_1\Delta^{-1}|\Gamma(\alpha_2-\alpha^*+1)|\varsigma_2|\vartheta^* + [(U_2 + V_2)R\\
&+ W_2]\frac{|\varepsilon_1\Delta\varsigma_1|}{\delta^*}\vartheta\Big]\frac{1}{\Gamma(\alpha_2 + 1)} + [(U_1 + V_1)R + W_1]|\Delta\varsigma_1|\delta + [(U_1 + V_1)R\\
&+ W_1]\nu|\Delta\varsigma_1|\delta^* + [(U_2 + V_2)R + W_2]|\varsigma_2|\vartheta^* + [(U_2 + V_2)R + W_2]\\
&\times \frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2 - \beta^* + 1)}\vartheta\\
\leq&\Big(\delta(U_1 + V_1) + \Big[(U_1 + V_1)|\varepsilon_1|\Gamma(\alpha_2 - \alpha^* + 1)|\Delta^{-1}| + (U_1 + V_1)\varsigma_1|\delta\\
&+ \Gamma(\alpha_2-\alpha^*+1)\big(|\varepsilon_1|[\nu|\Delta^{-1}| + |\varsigma_1|] + 1\big)\delta^* + (U_2 + V_2)|\varepsilon_1\Delta^{-1}|\Gamma(\alpha_2 - \alpha^* + 1)\\
&\times |\varsigma_2|\vartheta^*+(U_2+V_2)\frac{|\varepsilon_1\Delta\varsigma_1|}{\delta^*}\vartheta\Big]\frac{1}{\Gamma(\alpha_2+1)}+(U_1+V_1)|\Delta\varsigma_1|\delta + (U_1 + V_1)\nu|\Delta\varsigma_1|\delta^*\\
&+ (U_2 + V_2)|\varsigma_2|\vartheta^* + (U_2 + V_2)\frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2 - \beta^* + 1)}\vartheta\Big)R + \Big(\delta W_1 + \Big[W_1|\varepsilon_1|\\
&\times \Gamma(\alpha_2-\alpha^*+1)|\Delta^{-1}|+[W_1]\varsigma_1|\delta+\Gamma(\alpha_2-\alpha^*+1)\big(|\varepsilon_1|[\nu|\Delta^{-1}| + |\varsigma_1|] + 1\big)\delta^*\\
&+ [W_2]|\varepsilon_1\Delta^{-1}|\Gamma(\alpha_2-\alpha^*+1)|\varsigma_2|\vartheta^*+[W_2]\frac{|\varepsilon_1\Delta\varsigma_1|}{\delta^*}\vartheta\Big]\frac{1}{\Gamma(\alpha_2+1)}+[W_1]|\Delta\varsigma_1|\delta\\
&+ [W_1]\nu|\Delta\varsigma_1|\delta^* + [W_2]|\varsigma_2|\vartheta^* + [W_2]\frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2 - \beta^* + 1)}\vartheta\Big) \leq \eth_1 R + \tau_1,
\end{aligned}
$$

and

$$\begin{aligned}
&\left\|Q_2(y_1,y_2)\right\|_\infty \leq [(U_2+V_2)R+W_2]\vartheta+\Big[[(U_1+V_1)R+W_1]|\varphi\Delta^{-1}|(\delta+\nu\delta^*)\\
&+[(U_2+V_2)R+W_2]\Big(\frac{|(\Delta^{-1}-A_2)\varphi|}{|\varepsilon_2|}\vartheta^*+|\varphi|\vartheta\Big)\Big]\frac{1}{\Gamma(\beta_2+1)}+[(A_1+B_1)R\\
&+C_1](|\Delta^{-1}|+|\varsigma_1|)(\delta+\nu\delta^*)+[(U_2+V_2)R+W_2]\Big(|\Delta^{-1}\varsigma_2|\vartheta^*+|\Delta\varsigma_1|\vartheta\Big)\\
&\quad\leq\Big((U_2+V_2)\vartheta+\Big[(U_1+V_1)|\varphi\Delta^{-1}|(\delta+\nu\delta^*)+(U_2+V_2)\\
&\times\Big(\frac{|(\Delta^{-1}-A_2)\varphi|}{|\varepsilon_2|}\vartheta^*+|\varphi|\vartheta\Big)\Big]\frac{1}{\Gamma(\beta_2+1)}+(U_1+V_1)(|\Delta^{-1}|+|\varsigma_1|)(\delta+\nu\delta^*)\\
&+(U_2+V_2)\Big(|\Delta^{-1}\varsigma_2|\vartheta^*+|\Delta\varsigma_1|\vartheta\Big)\Big)R+\Big([W_2]\vartheta+\Big[[W_1]|\varphi\Delta^{-1}|(\delta+\nu\delta^*)\\
&+[W_2]\Big(\frac{|(\Delta^{-1}-A_2)\varphi|}{|\varepsilon_2|}\vartheta^*+|\varphi|\vartheta\Big)\Big]\frac{1}{\Gamma(\beta_2+1)}+[W_1](|\Delta^{-1}|+|\varsigma_1|)(\delta+\nu\delta^*)\\
&+[W_2]\Big(|\Delta^{-1}\varsigma_2|\vartheta^*+|\Delta\varsigma_1|\vartheta\Big)\Big)\leq\eth_2R+\tau_2.
\end{aligned}$$

If $R>(\tau_1+\tau_2)/(1-(\eth_1+\eth_2))$, then Q maps B_R into itself. A number R that satisfies the above inequality exists in view of the condition $\eth_1+\eth_2<1$.

Furthermore, thanks to Arzela-Ascoli theorem [16], we state that Q is a completely continuous operator in B_R; this can be justified as follows:
* The fact that f_1, f_2 are supposed continuous guarantees that Q is continuous on $S\times S$.
** The operator Q maps any bounded set M of $S\times S$ into a bounded set $Q(M)$ of the same space. (We can take $M:=B_R$).
***The equi continuity of the operator: we can prove it as follows. Let $t_1,t_2\in J$, $t_1<t_2$ and let B_R be the above bounded set of $S\times S$. So by considering $x=(x_1,x_2)\in B_R$, we can state that for each $t\in J$, we have

$$\begin{aligned}
&|Q_1x(t_1)-Q_1x(t_2)|\leq\delta\Big[(U_1+V_1)R+W_1\Big]\Big[|{t_1}^{\alpha_1+\alpha_2}-{t_2}^{\alpha_1+\alpha_2}|+|t_1-t_2|^{\alpha_1+\alpha_2}\Big]\\
&+\Bigg|\Big[\varepsilon_1\Gamma(\alpha_2-\alpha^*+1)\Big(\Delta^{-1}+\frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\Big)I^{\alpha_1+\alpha_2}f_1(1,x(1),y(1))\\
&-\Gamma(\alpha_2-\alpha^*+1)\Big(\varepsilon_1\big[\Delta^{-1}\nu+\frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\big]+1\Big)I^{\alpha_1+\alpha_2-\alpha^*}f_1(1,x(1),y(1))\\
&+\varepsilon_1\Delta^{-1}\Gamma(\alpha_2-\alpha^*+1)\left(\frac{1}{\varepsilon_2}+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}f_2(1,x(1),y(1))\\
&+\frac{\varepsilon_1\Delta^{-1}\varphi\Gamma(\alpha_2-\alpha^*+1)}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}f_2(1,x(1),y(1))\Big]\Bigg|\frac{|t_1^{\alpha_2}-t_2^{\alpha_2}|}{\Gamma(\alpha_2+1)},
\end{aligned}$$

and

$$|Q_2x(t_1)-Q_2x(t_2)|\leq\vartheta\Big[(U_2+V_2)R+W_2\Big]\Big[|{t_1}^{\beta_1+\beta_2}-{t_2}^{\beta_1+\beta_2}|+|t_1-t_2|^{\beta_1+\beta_2}\Big]$$

$$+\left|\left[\varphi\Delta^{-1}I^{\alpha_1+\alpha_2}f_1(1,x(1),y(1))-\varphi\Delta^{-1}\nu I^{\alpha_1+\alpha_2-\alpha^*}f_1(1,x(1),y(1))\right.\right.$$
$$\left.\left.+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2}I^{\beta_1+\beta_2-\beta^*}f_2(1,x(1),y(1))+\varphi I^{\beta_1+\beta_2}f_2(1,x(1),y(1))\right]\right|\frac{|t_1^{\beta_2}-t_2^{\beta_2}|}{\Gamma(\beta_2+1)}.$$

So, according to the Arzela-Ascoli theorem, we state that $Q(B_R)$ is relatively compact. Then, Q is a completely continuous operator. Hence, by Schauder fixed point theorem, Q has at least a fixed point.
The proof of Theorem 2 is thus achieved. □

3.3 Stability in the sense of Ulam-Hyers

In order to study the Ulam-Hyers stability, we have first to adopt the following two definitions to our problem.

Definition 3. The problem (1.1)–(1.2) has the Ulam-Hyers stability if there exists a real number $\Sigma > 0$, such that for each $\gamma_1 > 0, \gamma_2 > 0$ and for each $(x^*, y^*) \in S \times S$ solution of the coupled inequality

$$\begin{cases} |D^{\alpha_1}D^{\alpha_2}x^*(t) - f_1(t,x^*(t),y^*(t))| \le \gamma_1, \\ \left|D^{\beta_1}D^{\beta_2}y^*(t) - f_2(t,x^*(t),y^*(t))\right| \le \gamma_2, \end{cases} \tag{3.1}$$

$$\begin{cases} x^*(1) = A_1y^*(0) + B_1D^{\beta^*}y^*(0), \\ D^{\alpha^*}x^*(1) = \varepsilon_1y^*(0) + \zeta_1D^{\beta^*}y^*(0), \\ y^*(1) = A_2x^*(0) + B_2D^{\alpha^*}x^*(0), \\ D^{\beta^*}y^*(1) = \varepsilon_2x^*(0) + \zeta_2D^{\alpha^*}x^*(0), \end{cases} \tag{3.2}$$

there exists $(x, y) \in S \times S$ satisfying (1.1)–(1.2), such that,

$$\left\|(x - x^*, y - y^*)\right\|_{S\times S} \le \gamma\Sigma,$$

where, $\gamma := \max\{\gamma_1 > 0, \gamma_2 > 0\}$.

Using the same quantity γ, we give the following definition.

Definition 4. The system (1.1)–(1.2) has the Ulam-Hyers stability in the generalized sense if there exists $\Sigma \in C(\mathbb{R}^+, \mathbb{R}^+)$; $\Sigma(0) = 0$, such that for each $\gamma_1 > 0, \gamma_2 > 0$, and for any $(x^*, y^*) \in S \times S$ solution of (3.1)–(3.2), there exists a solution $(x, y) \in S \times S$ of (1.1)–(1.2) , that satisfies

$$\left\|(x - x^*, y - y^*)\right\|_{S\times S} \le \Sigma(\gamma).$$

Now, we can prove the following third main result.

Theorem 3. *The conditions of Theorem* 1 *guarantee the Ulam-Hyers stability of (1.1)–(1.2).*

Proof. Let $(x^*, y^*) \in S\times S$ be a solution of (3.1)–(3.2), and let, by Theorem 1, $(x, y) \in S\times S$ be the unique solution of (1.1)–(1.2). The inequalities (3.1) allow us to write

$$\left|x^*(t) - I^{\alpha_1+\alpha_2}f_1(t,x^*(t),y^*(t)) + \left[\varepsilon_1\Gamma(\alpha_2 - \alpha^* + 1)\right.\right.$$

$$\times\left(\Delta^{-1}+\frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2}f_1(1,x^*(1),y^*(1))-\Gamma(\alpha_2-\alpha^*+1)$$
$$\times\left(\varepsilon_1\left[\Delta^{-1}\nu+\frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right]+1\right)I^{\alpha_1+\alpha_2-\alpha^*}f_1(1,x^*(1),y^*(1))$$
$$+\varepsilon_1\Delta^{-1}\Gamma(\alpha_2-\alpha^*+1)\left(\frac{1}{\varepsilon_2}+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}f_2(1,x^*(1),y^*(1))$$
$$+\frac{\varepsilon_1\Delta^{-1}\varphi\Gamma(\alpha_2-\alpha^*+1)}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}f_2(1,x^*(1),y^*(1))\Bigg]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}$$
$$-\left(\frac{\varphi\Delta^{-1}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2}f_1(1,x^*(1),y^*(1))$$
$$+\left(\frac{\varphi\Delta^{-1}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2-\alpha^*}f_1(1,x^*(1),y^*(1))$$
$$-\left(\frac{1}{\varepsilon_2}+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)I^{\beta_1+\beta_2-\beta^*}f_2(1,x^*(1),y^*(1))$$
$$-\frac{\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}f_2(1,x^*(1),y^*(1))\Bigg|\le\frac{\gamma_1}{\Gamma(\alpha_1+\alpha_2+1)},$$

and

$$\Bigg|y(t)-I^{\beta_1+\beta_2}f_2(t,x^*(t),y^*(t))-\Big[\varphi\Delta^{-1}I^{\alpha_1+\alpha_2}f_1(1,x^*(1),y^*(1))$$
$$-\varphi\Delta^{-1}\nu I^{\alpha_1+\alpha_2-\alpha^*}f_1(1,x^*(1),y^*(1))+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2}I^{\beta_1+\beta_2-\beta^*}$$
$$\times f_2(1,x^*(1),y^*(1))+\varphi I^{\beta_1+\beta_2}f_2(1,x^*(1),y^*(1))\Big]\frac{t^{\beta_2}}{\Gamma(\beta_2+1)}$$
$$-\left(\Delta^{-1}+\frac{\varphi\Delta^{-2}}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)\times I^{\alpha_1+\alpha_2}f_1(1,x^*(1),y^*(1))$$
$$+\left(\Delta^{-1}\nu+\frac{\varphi\Delta^{-2}\nu}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}\right)I^{\alpha_1+\alpha_2-\alpha^*}f_1(1,x^*(1),y^*(1))$$
$$-\Delta^{-1}\left(\frac{1}{\varepsilon_2}+\frac{(\Delta^{-1}-A_2)\varphi}{\varepsilon_2^2\Gamma(\beta_2-\beta^*+1)}\right)\times I^{\beta_1+\beta_2-\beta^*}f_2(1,x^*(1),y^*(1))$$
$$-\frac{\Delta^{-1}\varphi}{\varepsilon_2\Gamma(\beta_2-\beta^*+1)}I^{\beta_1+\beta_2}f_2(1,x^*(1),y^*(1))\Bigg|\le\frac{\gamma_2}{\Gamma(\beta_1+\beta_2+1)}.$$

Combining both (1.1)–(1.2) and (3.1)–(3.2), we get

$$\|x-x^*\|_\infty\le\frac{\gamma_1}{\Gamma(\alpha_1+\alpha_2+1)}$$
$$+\mu\Big(\delta+\Big[|\varepsilon_1|\Gamma(\alpha_2-\alpha^*+1)|\Delta^{-1}+\varsigma_1|\delta+\Gamma(\alpha_2-\alpha^*+1)\big(|\varepsilon_1|[\nu|\Delta^{-1}|$$
$$+|\varsigma_1|]+1\big)\delta^*+|\varepsilon_1\Delta^{-1}|\Gamma(\alpha_2-\alpha^*+1)|\varsigma_2|\vartheta^*+\frac{|\varepsilon_1\Delta\varsigma_1|}{\delta^*}\vartheta\Big]\frac{1}{\Gamma(\alpha_2+1)}$$
$$+|\Delta\varsigma_1|\delta+\nu|\Delta\varsigma_1|\delta^*+|\varsigma_2|\vartheta^*+\frac{|\varphi|}{|\varepsilon_2|\Gamma(\beta_2-\beta^*+1)}\vartheta\Big)\left\|(x-x^*,y-y^*)\right\|_{S\times S},$$

and

$$\|y-y^*\|_\infty \leq \frac{\gamma_2}{\Gamma(\beta_1+\beta_2+1)}\mu^*$$
$$\times\left(\vartheta+\left[|\varphi\Delta^{-1}|(\delta+\nu\delta^*)+\frac{|(\Delta^{-1}-A_2)\varphi|}{|\varepsilon_2|}\vartheta^*+|\varphi|\vartheta\right]\frac{1}{\Gamma(\beta_2+1)}+(|\Delta^{-1}|\right.$$
$$\left.+|\varsigma_1|)(\delta+\nu\delta^*)+|\Delta^{-1}\varsigma_2|\vartheta^*+|\Delta\varsigma_1|\vartheta\right)\left\|(x-x^*,y-y^*)\right\|_{S\times S}.$$

Therefore, we state that the inequalities

$$\|x-x^*\|_\infty \leq \frac{\gamma_1}{\Gamma(\alpha_1+\alpha_2+1)}+\Upsilon\left\|(x-x^*,y-y^*)\right\|_{S\times S},$$
$$\|y-y^*\|_\infty \leq \frac{\gamma_2}{\Gamma(\beta_1+\beta_2+1)}+\Upsilon^*\left\|(x-x^*,y-y^*)\right\|_{S\times S}$$

are valid. Consequently, we have

$$\left\|(x-x^*,y-y^*)\right\|_{S\times S}$$
$$\leq\left(\frac{\gamma_1}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{\gamma_2}{\Gamma(\beta_1+\beta_2+1)}\right)+\Omega\left\|(x-x^*,y-y^*)\right\|_{S\times S},$$
$$\left\|(x-x^*,y-y^*)\right\|_{S\times S}\leq\frac{\frac{\gamma_1}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{\gamma_2}{\Gamma(\beta_1+\beta_2+1)}}{1-\Omega}.$$

Thus,

$$\left\|(x-x^*,y-y^*)\right\|_{S\times S}\leq\Sigma\gamma,$$

where, γ is the maximum between γ_1 and γ_2. Hence, (1.1)–(1.2) has the Ulam Hyers stability. □

Remark 1. In the case $\Sigma(\gamma)=\gamma\Sigma$, we obtain the generalised Ulam Hyers stability for (1.1)–(1.2).

4 Illustrative examples

Example 1. Consider the following sequential coupled system:

$$\begin{cases} D^{0.98}D^{0.99}x(t)=\dfrac{\sin t}{t^2+40}x(t)+\dfrac{\cos t}{t^2+50}y(t),\\ D^{0.99}D^{0.98}y(t)=\dfrac{t}{e^t+50}x(t)+\dfrac{t+1}{e^{2t}+60}y(t),\end{cases}$$

under the fractional closed conditions:

$$\begin{cases} x(1)=\frac{1}{7}y(0)+\frac{1}{9}D^{0.98}y(0),\\ D^{0.98}x(1)=\frac{1}{2}y(0)+\frac{1}{5}D^{0.98}y(0),\\ y(1)=\frac{1}{11}x(0)+\frac{1}{13}D^{0.98}x(0),\\ D^{0.98}y(1)=\frac{1}{3}x(0)+\frac{1}{4}D^{0.98}x(0).\end{cases}$$

We see that

$$f_1(t,x(t),y(t)) = \frac{\sin t}{t^2+40}x(t) + \frac{\cos t}{t^2+50}y(t),\ f_2(t,x(t),y(t)) = \frac{t}{e^t+50}x(t)$$
$$+\frac{t+1}{e^{2t}+60}y(t),\ \ l_{f_{1,1}}=\frac{1}{40},\ l_{f_{1,2}}=\frac{1}{50},\ l_{f_{2,1}}=\frac{1}{50},\ l_{f_{2,2}}=\frac{1}{60},\ \mu=\frac{1}{40},\ \ \mu^*=\frac{1}{50},$$
$$\alpha_1 = 0.98,\ \ \alpha_2 = 0.99,\ \ \beta_1 = 0.99,\ \ \beta_2 = 0.98,\ \ \alpha^* = 0.98,\ \ \beta^* = 0.98,$$
$$\delta = 0.5139,\ \ \delta^* = 1.0042,\ \ \vartheta = 0.5139,\ \ \vartheta^* = 1.0042,\ \ A_1{=}1/7,\ A_2{=}1/11,$$
$$B_1 = \frac{1}{9},\ B_1 = \frac{1}{13},\ \varepsilon_1 = \frac{1}{2},\ \varepsilon_2 = \frac{1}{3},\ \varsigma_1 = 3.07133,\ \varsigma_2 = 0.3902,\ \nu = 0.9984,$$
$$\Delta = -0.3564,\ \ \Delta^{-1} = -2.8058,\ \ \Delta^{-2} = 7.8727,\ \ \varphi = 0.1293.$$

We also have

$$\Upsilon = 0.2462,\ \ \Upsilon^* = 0.2568,\ \ \Omega = \Upsilon + \Upsilon^* = 0.5030 < 1.$$

So, thanks to Theorem 1, this system has a unique solution $(x(t), y(t))$ defined over $[0,1]$.Also, by Theorem 3, this system is Ulam-Hyers stable.

Example 2. Consider the following sequential coupled system:

$$\begin{cases} D^{0.9}D^{0.8}x(t) = \dfrac{|x(t)|}{250(t^2+1)(1+|x(t)|)} + \dfrac{\sin(y(t))}{300(t^4+1)}y(t) + \dfrac{e^{t^2}}{20(1+e^{t^2})}, \\ D^{0.85}D^{0.97}y(t) = \dfrac{\sin(2x(t)+y(t))}{400e^{t^2}} + \dfrac{\cos(t)}{15(e^{t^2}+1)}, \end{cases}$$

under the fractional closed conditions:

$$\begin{cases} x(1) = \frac{1}{8}y(0) + \frac{1}{9}D^{0.85}y(0), \\ D^{0.8}x(1) = \frac{1}{2}y(0) + \frac{1}{5}D^{0.85}y(0), \\ y(1) = \frac{1}{5}x(0) + \frac{1}{9}D^{0.8}x(0), \\ D^{0.85}y(1) = \frac{1}{3}x(0) + \frac{1}{4}D^{0.8}x(0). \end{cases}$$

We see that

$$f_1(t,x(t),y(t)) = \frac{|x(t)|}{250(t^2+1)(1+|x(t)|)} + \frac{\sin(y(t))}{300(t^4+1)}y(t) + \frac{e^{t^2}}{20(1+e^{t^2})},$$
$$f_2(t,x(t),y(t)) = \frac{\sin(2x(t)+y(t))}{400e^{t^2}} + \frac{\cos(t)}{15(e^{t^2}+1)},$$
$$U_1 = 1/250,\ \ U_2 = 1/200,\ \ V_1 = 1/300,\ \ V_2 = 1/400,$$
$$\alpha_1 = 0.9,\ \alpha_2 = 0.8,\ \beta_1 = 0.85,\ \beta_2 = 0.97,\ \alpha^* = 0.8,\ \beta^* = 0.85,$$
$$\delta = 0.6474,\ \delta^* = 1.0398,\ \vartheta = 0.586,\ \vartheta^* = 1.0125,\ A_1 = 1/8,\ A_2 = 1/10$$
$$B_1 = \frac{1}{5},\ B_1 = \frac{1}{9},\ \varepsilon_1 = \frac{1}{2},\ \varepsilon_2 = \frac{1}{3},\ \varsigma_1 = 2.9046,\ \varsigma_2 = -0.3902,\ \nu = 1,$$
$$\Delta = -0.3750,\ \varphi = 0.1285.$$

We have $\eth_1 = 0.0804$, $\eth_2 = 0.1016$, $\eth_1 + \eth_2 = 0.1821 < 1$. So, thanks to Theorem 2 this system has at least one solution $(x(t), y(t))$ defined over $[0,1]$.

5 Travelling waves for the conformable coupled system of Burgers equations

The coupled system of Burgers equations that we study is given by [26, 27]:

$$\begin{cases} K_x^{2\beta} u = K_t^{\alpha} u + 2uK_x^{\beta} u + mK_x^{\beta}(uv), \\ K_x^{2\beta} v = K_t^{\alpha} v + 2vK_x^{\beta} v + nK_x^{\beta}(uv), \end{cases}$$

where, $0 < \alpha, \beta \leq 1$, n, m are real constants with $nm \neq 1$ and K_z^{α} is the conformable fractional derivative of order α with respect to z given by the following expression, (see [11, 14]):

$$\left(K_t^{\alpha} u\right)(x,t) = \lim_{\varepsilon \to 0} \Big(\big(u(x, t + \varepsilon t^{1-\alpha}) - u(x,t) \big) / \varepsilon \Big), \quad t > 0, \quad 0 < \alpha \leq 1.$$

The same definition in the case of $z = x$. To search for traveling wave solutions for this system, we use the transformation [5, 8]:

$$u(x,t) = U(\xi); \;\; \xi = wx^{\beta}/\beta + kt^{\alpha}/\alpha.$$

So, we have the following three quantities

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = kU_{\xi}, \quad \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = wU_{\xi}, \quad \text{and} \quad \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} = w^2 U_{\xi\xi}.$$

Using these quantities, the above time and space fractional system can easily be converted into the following ordinary differential system:

$$\begin{cases} kU_{\xi} - \omega^2 U_{\xi\xi} + 2\omega U U_{\xi} + m\omega V U_{\xi} + m\omega U V_{\xi} = 0, \\ kV_{\xi} - \omega^2 V_{\xi\xi} + 2\omega V V_{\xi} + n\omega V U_{\xi} + n\omega U V_{\xi} = 0. \end{cases} \tag{5.1}$$

Now, we use the transformation $\eta = \tanh(\xi)$ and we suppose that

$$u(x,t) = U(\xi) = F(\eta) = \sum_{i=0}^{m} a_i \eta^i, v(x,t) = V(\xi) = F^{\star}(\eta) = \sum_{i=0}^{n} b_i \eta^i.$$

By substitution in the two Equations of (5.1), we can write

$$\begin{aligned} &k\left(1-\eta^2\right)\frac{dF}{d\eta} - \omega^2\Big[-2\eta\left(1-\eta^2\right)\frac{dF}{d\eta} + \left(1-\eta^2\right)^2\frac{d^2F}{d\eta^2}\Big] \\ &+ 2\omega F\left(1-\eta^2\right)\frac{dF}{d\eta} + m\omega F^{\star}\left(1-\eta^2\right)\frac{dF}{d\eta} + m\omega F\left(1-\eta^2\right)\frac{dF^{\star}}{d\eta} = 0 \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &k\left(1-\eta^2\right)\frac{dF^{\star}}{d\eta} - \omega^2\Big[-2\eta\left(1-\eta^2\right)\frac{dF^{\star}}{d\eta} + \left(1-\eta^2\right)^2\frac{d^2F^{\star}}{d\eta^2}\Big] \\ &+ 2\omega F^{\star}\left(1-\eta^2\right)\frac{dF^{\star}}{d\eta} + n\omega F\left(1-\eta^2\right)\frac{dF^{\star}}{d\eta} + n\omega F^{\star}\left(1-\eta^2\right)\frac{dF}{d\eta} = 0. \end{aligned} \tag{5.3}$$

The balance technique [14, 25] allows us to propose the following form of solution:

$$\begin{cases} F(\eta) = a_0 + a_1\eta, \\ F^\star(\eta) = b_0 + b_1\eta. \end{cases} \tag{5.4}$$

Substituting (5.4) into (5.2), we can get

$$\begin{aligned} &k\left(1-\eta^2\right)a_1 + 2\omega^2\eta\left(1-\eta^2\right)a_1 + 2\omega(a_0+a_1\eta)\left(1-\eta^2\right)a_1 \\ &+ m\omega(b_0+b_1\eta)\left(1-\eta^2\right)a_1 + m\omega(a_0+a_1\psi)\left(1-\eta^2\right)b_1 = 0. \end{aligned}$$

Also, by substitution of (5.4) into (5.3), we can write

$$\begin{aligned} &k\left(1-\eta^2\right)b_1 + 2\omega^2\eta\left(1-\psi^2\right)b_1 + 2\omega(b_0+b_1\psi)\left(1-\eta^2\right)b_1 \\ &+ n\omega(a_0+a_1\eta)\left(1-\eta^2\right)b_1 + n\omega(b_0+b_1\eta)\left(1-\eta^2\right)a_1 = 0, \end{aligned}$$

Then, we have:

$$\begin{cases} \eta^0 : m\omega a_0 b_1 + m\omega a_1 b_0 + 2\omega a_0 a_1 + k a_1 = 0, \\ \eta^1 : 2m\omega a_1 b_1 + 2\omega^2 a_1 + 2\omega a_1^2 = 0, \\ \eta^2 : m\omega a_0 b_1 - m\omega a_1 b_0 - 2\omega a_0 a_1 - k a_1 = 0, \\ \eta^3 : -2m\omega a_1 b_1 - 2\omega^2 a_1 - 2\omega a_1^2 = 0, \end{cases} \tag{5.5}$$

and

$$\begin{cases} \eta^0 : n\omega b_0 a_1 + n\omega b_1 a_0 + 2\omega b_0 b_1 + k b_1 = 0, \\ \eta^1 : 2n\omega a_1 b_1 + 2\omega^2 b_1 + 2\omega b_1^2 = 0, \\ \eta^2 : -n\omega a_0 b_1 - n\omega a_1 b_0 - 2\omega a_0 a_1 - k a_1 = 0, \\ \eta^3 : -2n\omega a_1 b_1 - 2\omega^2 b_1 - 2\omega b_1^2 = 0. \end{cases} \tag{5.6}$$

Solving (5.5) and (5.6) with the aid of Maple, we obtain

$$a_0 = -\frac{(m-1)k}{2\omega(mn-1)}, a_1 = -\frac{(m-1)\omega}{mn-1)}, b_0 = -\frac{(n-1)k}{2\omega(mn-1)}, b_1 = -\frac{(n-1)\omega}{mn-1)}.$$

Hence, the following traveling wave solution is obtained:

$$u(x,t) = \frac{(1-m)k}{2\omega(mn-1)} + \frac{(1-m)\omega}{mn-1)}\tanh(\xi), \tag{5.7}$$

$$v(x,t) = -\frac{(n-1)k}{2\omega(mn-1)} - \frac{(n-1)\omega}{mn-1)}\tanh(\xi). \tag{5.8}$$

The graphical illustration of the Equations (5.7)–(5.8) can be seen in Figures 1 (A) and (B), respectively.

(A) Solution (5.7) (B) Solution (5.8)

Figure 1. Plot of (u, v); $0 \leq x \leq 10$, $0 \leq t \leq 30$, $m = 2, n = -3, \alpha = \frac{7}{10}, \beta = \frac{9}{10}$.

6 Conclusions

In summary, we have studied two parts:

In the first part, we have investigated the above class of nonlinear differential systems with Caputo derivatives, augmented by the introduction of coupled closed fractional boundary conditions. The establishment of the two first main theorems on existence of solutions shows the efficacy of our proposed conditions, providing a good framework for modeling complex phenomena. Moreover, the stability in the sense of Ulam-Hyers with the detailed examples add a layer of practical significance to the main results. Our research, on this first part, contributes a foundational understanding of coupled closed fractional boundary conditions.

In the second part of our work, we have oriented our focus to the application of the tanh numerical method to obtain a new travelling wave solution for the coupled system of Burgers equations, incorporating conformable derivatives in accordance with Khalil framework.

By bridging these two aspects, we have aimed to provide an understanding of the system's behavior in a large sense, linking the theoretical foundation of Caputo derivatives with the practical implications revealed through the tanh method for the coupled Burgers system in the context of Khalil approach.

By integrating high-order fractional operators and new weighted approaches, [1, 19] future research can continue to enhance the applicability and precision of fractional calculus in capturing the dynamics of the above studied system.

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