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Boundary feedback stabilization of quasilinear hyperbolic systems with zero characteristic speed

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Abstract. In this paper, we investigate the boundary feedback stabilization of a quasilinear hyperbolic system with zero characteristic speed and a partially dissipative structure. This structure enables us to construct a Lyapunov function that guarantees exponential stability for the H^2 solution. We also introduce another set of stability conditions by restricting terms corresponding to zero eigenvalues to the dissipative part, which still ensures exponential stability. As an application, we achieve feedback stabilization for the modified model of neurofilament transport in axons.

Keywords: quasilinear hyperbolic system; zero characteristic speed; feedback stabilization; Lyapunov function.

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1 Introduction and main results

The study of stability and stabilization for hyperbolic equations has been a significant area of research for many years, primarily due to its wide applications in physics and mechanics (see [2, 4] and the references therein). Over the decades, various methods have been developed to establish the asymptotic stability of hyperbolic systems, including: characteristic method, backstepping method, Lyapunov function method, etc. The characteristic method is a traditional approach that utilizes the properties of characteristics in hyperbolic equations to analyze stability, notable works include [3, 7, 13, 22]. As a recent technique, the backstepping method has been gaining more attention for its effectiveness in stabilizing hyperbolic systems, we refer to [8, 9, 10, 18, 21] for related investigations. The Lyapunov function method is a fundamental and powerful tool in studying stability of homogeneous hyperbolic systems (see, for example, [5, 25]).

When dealing with inhomogeneous hyperbolic systems, the stability properties change notably due to the presence of a nonzero source term. The Lyapunov function approach, which is still applicable to this kind of systems, requires adjustments to account for this change (see for instance [12,14]). In [1], a necessary and sufficient condition for simple quadratic Lyapunov function was introduced to study a linear 2×2 hyperbolic system. Later in Chapter 6 of [2], the authors gave a sufficient (but a priori non-necessary) condition such that the exponential stability of the system for the H^p ($p \geq 2$) norm is achieved. We also refer to [16] for a relevant result in C^1-norm or C^p-norm.

Actually, these conditions for stability often include an interior condition, which requires *a good coupling structure* of the hyperbolic system, compared to the homogeneous case.

In [26], Yong introduced an important concept named *structural stability condition*, which is satisfied in many physical models. This structure provides a different viewpoint in the study of hyperbolic systems. Subsequently, building upon this structure, they successfully established the boundary feedback stabilization for one-dimensional *linear* hyperbolic systems without vanishing characteristic speed (see [17]). Recently, in [23], the authors of this work studied one-dimensional *nonlinear* quasilinear hyperbolic systems with the same relaxation structure. Through the construction of a strict Lyapunov function together with a perturbation argument, we established the local exponential stability in H^2-norm of these systems without vanishing characteristic speed, moreover, we are also able to provide an explicit and sufficient condition for the gains required in stabilizing boundary feedback control.

The presence of zero characteristic speed in various physical models brings additional challenges in boundary control problems, as boundary conditions may not effectively influence all variables in the system. Several studies have been conducted on this topic, for example, the exact boundary controllability for linear and quasilinear hyperbolic systems with a vanishing characteristic speed was considered in [19] and [6], respectively. In [15], the backstepping method was applied to a 3×3 linear hyperbolic system with zero characteristic speed. In [27], Yong showed that under the *structural stability condition* outlined in [17], the boundary feedback stabilization result is also available for a class of one-dimensional linear hyperbolic system with vanishing characteristic speed.

Inspired by [27], in this paper, we consider a one-dimensional quasilinear hyperbolic system with vanishing characteristic speed and the same relaxation structure as in [23]. Thanks to the partial dissipation in the *structural stability condition*, by introducing an additional assumption regarding internal coupling to address the boundary estimate, we establish the local exponential stability of this nonlinear system for the H^2-norm. Furthermore, by restricting the terms corresponding to the zero eigenvalues to the dissipative part, we propose another type of stability condition which still ensures the achievement of local exponential stability. The main strategy is to construct a strict Lyapunov function together with a perturbation argument based on linear approximation.

Compared to the result in [27], we provide an explicit sufficient condition for the gains of stabilizing boundary feedback control.

Precisely, in this work, we are concerned with the boundary feedback stabilization of the following one-dimensional quasilinear hyperbolic system

$$U_t + \mathbf{A}(U)U_x = \mathbf{Q}(U), \quad t \in (0,\infty), x \in (0,1), \tag{1.1}$$

where $U = (u_1, \dots, u_n)^T$ is the unknown vector function of (t,x). $\mathbf{Q} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth vector function. Let $U^* \in \mathbb{R}^n$ be an equilibrium of (1.1), i.e.,

$$\mathbf{Q}(U^*) = 0.$$

Without loss of generality, we may assume $U^* = 0$, otherwise one can consider $U - U^*$ as the unknown variable. $\mathbf{A} : \mathbb{R}^n \mapsto \mathcal{M}_{n,n}(\mathbb{R})$ is a smooth matrix function. In a neighborhood of $U = 0$, the matrix $\mathbf{A}(U)$ has n real eigenvalues $\mathbf{\Lambda}_i(U)$ $(i = 1, \dots, n)$ satisfying

$$\begin{aligned} &\mathbf{\Lambda}_l(U) < 0 < \mathbf{\Lambda}_s(U) \quad (l = 1, \dots, m; s = n-p+1, \dots, n; p+m \le n), \\ &\mathbf{\Lambda}_k(U) = 0 \quad (k = m+1, \dots, n-p), \end{aligned}$$

and a complete set of left eigenvectors $\mathbf{L}_i(U) = (\mathbf{L}_{i1}(U), \dots, \mathbf{L}_{in}(U))$ $(i = 1, \dots, n)$, i.e.,

$$\mathbf{L}_i(U)\mathbf{A}(U) = \mathbf{\Lambda}_i(U)\mathbf{L}_i(U) \quad (i = 1, \dots, n).$$

Let

$$\mathbf{L}(U) = \begin{pmatrix} \mathbf{L}_-(U) \\ \mathbf{L}_0(U) \\ \mathbf{L}_+(U) \end{pmatrix} = \begin{pmatrix} \mathbf{L}_1(U) \\ \vdots \\ \mathbf{L}_n(U) \end{pmatrix}, \quad \mathbf{\Lambda}(U) = \begin{pmatrix} \mathbf{\Lambda}_-(U) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{\Lambda}_+(U) \end{pmatrix},$$

where

$$\mathbf{\Lambda}_-(U) = \mathrm{diag}\{\mathbf{\Lambda}_1(U), \dots, \mathbf{\Lambda}_m(U)\}, \ \mathbf{\Lambda}_+(U) = \mathrm{diag}\{\mathbf{\Lambda}_{n-p+1}(U), \dots, \mathbf{\Lambda}_n(U)\},$$

and $\mathbf{L}_-(U) \in \mathcal{M}_{m,n}(\mathbb{R})$, $\mathbf{L}_0(U) \in \mathcal{M}_{n-m-p,n}(\mathbb{R})$ and $\mathbf{L}_+(U) \in \mathcal{M}_{p,n}(\mathbb{R})$. Then,

$$\mathbf{L}(U)\mathbf{A}(U) = \mathbf{\Lambda}(U)\mathbf{L}(U).$$

Also denote that

$$\mathbf{R}(U) = \mathbf{L}^{-1}(U) = (\mathbf{R}_-(U), \mathbf{R}_0(U), \mathbf{R}_+(U)),$$

where $\mathbf{R}_-(U) \in \mathcal{M}_{n,m}(\mathbb{R})$, $\mathbf{R}_0(U) \in \mathcal{M}_{n,n-m-p}(\mathbb{R})$ and $\mathbf{R}_+(U) \in \mathcal{M}_{n,p}(\mathbb{R})$.

It is easy to see that system (1.1) is hyperbolic if and only if there is a symmetric positive definite matrix $\mathbf{A}_0(U)$, such that

$$\mathbf{A}_0(U)\mathbf{A}(U) = \mathbf{A}^T(U)\mathbf{A}_0(U). \tag{1.2}$$

Moreover, we assume the system possesses the following partially dissipative structure in a neighborhood of $U = 0$:

There exist invertible matrices $\mathbf{P}(U) \in \mathcal{M}_{n,n}(\mathbb{R})$ and $\mathbf{S}(U) \in \mathcal{M}_{r,r}(\mathbb{R})$ with $0 < r \leq n$, such that

$$\mathbf{P}(U)\mathbf{Q}_U(U) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{S}(U) \end{pmatrix} \mathbf{P}(U), \tag{1.3}$$

$$\mathbf{A}_0(U)\mathbf{Q}_U(U) + \mathbf{Q}_U^T(U)\mathbf{A}_0(U) \leq -\,\mathbf{P}^T(U) \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_r \end{pmatrix} \mathbf{P}(U). \tag{1.4}$$

Here, $\mathbf{Q}_U(U)$ stands for the Jacobian matrix of $\mathbf{Q}$ with respect to U, I_r denotes the $r \times r$ identity matrix.

Let us point out that the above assumptions (1.3) and (1.4) are called *structural stability conditions* in [26]. In our previous work [23], these conditions are sufficient to ensure boundary feedback stabilization for systems without vanishing characteristic speed. However, in this paper, to overcome the difficulties brought by the zero characteristic speed, we introduce two additional coupling conditions as

$$\mathbf{P}(0)\mathbf{A}(U)\mathbf{P}^{-1}(0) = \begin{pmatrix} a(U) & b(U) \\ c(U) & d(U) \end{pmatrix},$$

$$a(0) \text{ has only positive (or only negative) eigenvalues} \tag{1.5}$$

with $a(U) \in \mathcal{M}_{n-r,n-r}(\mathbb{R})$, $b(U) \in \mathcal{M}_{n-r,r}(\mathbb{R})$, $c(U) \in \mathcal{M}_{r,n-r}(\mathbb{R})$, $d(U) \in \mathcal{M}_{r,r}(\mathbb{R})$ and

$$\mathbf{L}_-(U)\mathbf{R}_0(0) = 0, \quad \mathbf{L}_+(U)\mathbf{R}_0(0) = 0. \tag{1.6}$$

Set

$$\xi_-(t,x)=\mathbf{L}_-(0)U(t,x), \quad \xi_0(t,x)=\mathbf{L}_0(0)U(t,x), \quad \xi_+(t,x)=\mathbf{L}_+(0)U(t,x) \tag{1.7}$$

and denote

$$\xi(t,x) = \begin{pmatrix} \xi_-(t,x) \\ \xi_0(t,x) \\ \xi_+(t,x) \end{pmatrix} = \mathbf{L}(0)U(t,x), \tag{1.8}$$

where $\xi_-(t,x) \in \mathbb{R}^m$, $\xi_0(t,x) \in \mathbb{R}^{n-m-p}$ and $\xi_+(t,x) \in \mathbb{R}^p$. According to the theory on the well-posedness of the quasilinear hyperbolic system, the typical boundary conditions are given as follows

$$\begin{pmatrix} \xi_+(t,0) \\ \xi_-(t,1) \end{pmatrix} = \mathbf{K} \begin{pmatrix} \xi_+(t,1) \\ \xi_-(t,0) \end{pmatrix}, \quad t \in (0,\infty), \tag{1.9}$$

where $\mathbf{K}$ is the feedback matrix with constant elements. In Section 5.6 of [2] and [20], the authors proved that it is not always useful if one use local feedback laws to exponentially stabilize the closed-loop system. However, while it is not always the case for local feedback to stabilize the general inhomogeneous hyperbolic system, it is indeed enough to deal with quasilinear hyperbolic system with partially dissipative structure under this kind of classical feedback law.

Finally, the initial condition is prescribed as

$$U(0,x) = U_0(x), \quad x \in (0,1), \tag{1.10}$$

with $U_0 \in H^2((0,1);\mathbb{R}^n)$ in a neighborhood of $U = 0$.

In relation to the well-posedness of solutions to the problem (1.1), (1.9) and (1.10), we present the following proposition, with its proof provided in [5] and [24],

Proposition 1. *There exists* $\delta_0 > 0$ *such that, for every* $U_0 \in H^2((0,1);\mathbb{R}^n)$ *satisfying*

$$\|U_0\|_{H^2((0,1);\mathbb{R}^n)} \le \delta_0,$$

and the C^1 *compatibility conditions at the points* $(t,x) = (0,0)$, $(0,1)$, *the problem* (1.1), (1.9) *and* (1.10) *has a unique maximal classical solution*

$$U \in C^0([0,T); H^2((0,1);\mathbb{R}^n))$$

with some $T \in (0,+\infty]$. *Moreover, if*

$$\|U(t,\cdot)\|_{H^2((0,1);\mathbb{R}^n)} \le \delta_0, \quad \forall t \in [0,T),$$

then $T = +\infty$.

One of our main results is the following theorem.

Theorem 1. *Assume that the hyperbolic system* (1.1) *possesses the partially dissipative structure, i.e.,* (1.2), (1.3) *and* (1.4) *hold, and the coupling conditions* (1.5) *and* (1.6). *Then, there exists a boundary matrix* $\mathbf{K}$ *such that the closed-loop system* (1.1), (1.9) *and* (1.10) *is locally exponentially stable for the* H^2*-norm, i.e., there exist positive constants* δ, C *and* ν, *such that the solution to system* (1.1), (1.9) *and* (1.10) *satisfies*

$$\|U(t,\cdot)\|_{H^2((0,1);\mathbb{R}^n)} \le Ce^{-\nu t}\|U_0\|_{H^2((0,1);\mathbb{R}^n)}, \quad t \in [0,+\infty),$$

provided that $\|U_0\|_{H^2((0,1);\mathbb{R}^n)} \le \delta$ *and the* C^1 *compatibility conditions are satisfied at* $(t,x) = (0,0)$ *and* $(0,1)$.

Remark 1. Theorem 1 still holds if the assumption (1.3) is extended to a more general case that

$$\mathbf{P}(U)\mathbf{Q}_U(U)\mathbf{P}^{-1}(U) = \begin{pmatrix} \mathbf{S}_{11}(U) & \mathbf{S}_{12}(U) \\ \mathbf{S}_{21}(U) & \mathbf{S}_{22}(U) \end{pmatrix}, \tag{1.11}$$

where $|\mathbf{S}_{11}(0)|_\infty$ and $|\mathbf{S}_{21}(0)|_\infty$ are sufficiently small with their upper bounds depending on $A(0)$ and $A_0(0)$. In this case, the non-dissipative part of the boundary term (i.e., the terms corresponding to v_1 in (2.13)) can be absorbed by negative definitive terms. Furthermore, Theorem 1 also holds if the assumption (1.5) is extended to a more general case that

$$\mathbf{P}^{-T}(0)\mathbf{A}_0(U)\mathbf{A}(U)\mathbf{P}^{-1}(0) = \begin{pmatrix} \mathbf{Z}_1(U) & \mathbf{Z}_2(U) \\ \mathbf{Z}_2(U) & \mathbf{Z}_3(U) \end{pmatrix}, \tag{1.12}$$

where $\mathbf{Z}_1(0)$ is a positive (or negative) definite matrix. Essentially, Lyapunov function approach in this paper still works in these two general cases.

Additionally, we propose an alternative type of stability conditions. The conclusion of Theorem 1 remains valid if we modify the assumption (1.3) of the structural stability conditions to a new assumption that $(0\ I_r)\mathbf{P}(0)\mathbf{R}_0(0)$ is full rank, namely,

$$rank\Big((0\ I_r)\mathbf{P}(0)\mathbf{R}_0(0)\Big) = n - m - p, \tag{1.13}$$

and the technical assumption (1.5) is changed into (1.12). Corresponding conclusion is presented in the following theorem.

Theorem 2. *Assume that the hyperbolic system* (1.1) *satisfies the conditions* (1.2), (1.4) *and* (1.13) *and additionally, assumptions* (1.6) *and* (1.12) *hold. Then, there exists a boundary matrix* $\mathbf{K}$ *such that the closed-loop system* (1.1), (1.9) *and* (1.10) *is locally exponentially stable for the* H^2*-norm.*

Remark 2. In constrast to Theorem 1, we find that when condition (1.3) is substituted with (1.13), there is no requirement on the structure of the source term $\mathbf{Q}(U)$ in Theorem 2, while the constraint on $\mathbf{A}(U)$ is stronger. In fact, as we will see in Section 5, (1.13) implies that the dissipative part v_2 in (2.13) contains all the information related to the term ξ_0 corresponding to zero eigenvalues. In the case where $n - m - p = n - r$, v_2 is equivalent to ξ_0.

Remark 3. Condition (1.6) ensures that terms involving ξ_0 corresponding to zero eigenvalues will not appear in the higher-order terms during the estimation of the boundary term. Consequently, this implies that the boundary condition (1.9) is enough to deal with the boundary terms. It is worthy to mention that when $\mathbf{A}(U)$ is a constant matrix or $\mathbf{A}(U)$ is a diagonal matrix, all matrices $\mathbf{L}_+$, $\mathbf{L}_-$ and $\mathbf{R}_0$ can be taken to be constant, condition (1.6) is naturally satisfied. Moreover, for all $\mathbf{A}(U)$ with $\mathbf{L}_\pm(U) \equiv \mathbf{L}_\pm(0)$ or $\mathbf{R}_0(U) \equiv \mathbf{R}_0(0)$, condition (1.6) is also satisfied. However, $\mathbf{A}(U)$ such that $\mathbf{L}_\pm(U)\mathbf{R}_0(0) \neq 0$ exists and in this case, the boundary condition (1.9) would be not enough to control the whole boundary terms, no matter which Lyapunov function it takes.

Remark 4. In Section 3, we may see that the only requirement to $\mathbf{K}$ is to ensure that matrices $\mathbf{G}_0$, $\mathbf{G}_1$ and $\mathbf{G}_2$ defined in (3.7)–(3.9) are positive definite, which can be easily obtained while $|\mathbf{K}|_\infty$ is sufficiently small. Particularly, for certain smooth positive function $\lambda = \lambda(x)$ defined in (3.1)–(3.5), the requirement holds if $\mathbf{K}$ is chosen as

$$\mathbf{K} = \begin{pmatrix} \kappa_+\mathbf{I}_{n-m} & 0 \\ 0 & \kappa_-\mathbf{I}_m \end{pmatrix}$$

with two constants κ_+ and κ_- satisfying

$$\kappa_+^2 < \lambda(1)/\lambda(0), \quad \kappa_-^2 < \lambda(0)/\lambda(1).$$

The paper is organized as follows: in Section 2, we employ a transformation of the unknown variable, which leads to a new quasilinear hyperbolic system with a simpler structure. Then, in Section 3, we construct a weighted H^2-Lyapunov function to prove the exponential stability of the new system which implies immediately the validity of Theorem 1. The proofs of related lemmas are provided in Section 4. In Section 5, we give a sketch of the proof of Theorem 2. Finally, in Section 6, Theorem 1 is applied to a modified model for transport of neurofilaments in axons.

2 Transformation of the system

In this section, by adopting a transformation of the unknown variable, we derive a new hyperbolic system with a *partially dissipative but simpler structure.* Through this approach, we effectively reduce the task of proving the exponential stability of the original system to demonstrating the stability of this new system. Let

$$V = \mathbf{P}(0)U. \tag{2.1}$$

Then, system (1.1) can be reduced to

$$V_t + A(V)V_x = B(V), \tag{2.2}$$

where

$$A(V) = \mathbf{P}(0)\mathbf{A}(\mathbf{P}^{-1}(0)V)\mathbf{P}^{-1}(0) \quad \text{and} \quad B(V) = \mathbf{P}(0)\mathbf{Q}(\mathbf{P}^{-1}(0)V).$$

Clearly, $V = 0$ is an equilibrium of (2.2) and the Jacobian matrix of B with respect to V at this equilibrium can be calculated as

$$B_V(0) = \mathbf{P}(0)\mathbf{Q}_U(0)\mathbf{P}^{-1}(0). \tag{2.3}$$

Let

$$L(V) = \mathbf{L}(\mathbf{P}^{-1}(0)V)\mathbf{P}^{-1}(0), \quad \Lambda(V) = \mathbf{\Lambda}(\mathbf{P}^{-1}(0)V). \tag{2.4}$$

Obviously, we have $\Lambda(0) = \mathbf{\Lambda}(0) = \mathrm{diag}\{\mathbf{\Lambda}_1(0), \dots, \mathbf{\Lambda}_n(0)\}$. It is easy to check that $L(V)$ is the matrix composed of the left eigenvectors of $A(V)$, i.e.,

$$L(V)A(V) = \Lambda(V)L(V), \tag{2.5}$$

which implies that system (2.2) is a hyperbolic system with vanishing characteristic speed. Let

$$R(V) = L^{-1}(V) = \mathbf{P}(0)\mathbf{L}^{-1}(\mathbf{P}^{-1}(0)V) = \mathbf{P}(0)\mathbf{R}(\mathbf{P}^{-1}(0)V).$$

$L(V)$ and $R(V)$ could still be devided into three parts

$$L(V) = \begin{pmatrix} L_-(V) \\ L_0(V) \\ L_+(V) \end{pmatrix}, \quad R(V) = L^{-1}(V) = (R_-(V), R_0(V), R_+(V)), \tag{2.6}$$

where $L_-(V) \in \mathcal{M}_{m,n}(\mathbb{R})$, $L_0(V) \in \mathcal{M}_{n-m-p,n}(\mathbb{R})$ and $L_+(V) \in \mathcal{M}_{p,n}(\mathbb{R})$, $R_-(V) \in \mathcal{M}_{n,m}(\mathbb{R})$, $R_0(V) \in \mathcal{M}_{n,n-m-p}(\mathbb{R})$ and $R_+(V) \in \mathcal{M}_{n,p}(\mathbb{R})$.

Let

$$A_0(V) = (\mathbf{P}^{-1}(0))^T\mathbf{A}_0(\mathbf{P}^{-1}(0)V)\mathbf{P}^{-1}(0). \tag{2.7}$$

Obviously, $A_0(V)$ is a symmetric positive definite matrix satisfying

$$A_0(V)A(V) = A^T(V)A_0(V). \tag{2.8}$$

Thanks to (2.3) and (2.7), the *partially dissipative structure* (1.3)–(1.4) for the original system (1.1) implies the following *partially dissipative but simpler structure* for system (2.2) at the equilibrium $V = 0$

$$B_V(0) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{S}(0) \end{pmatrix}, \tag{2.9}$$

$$A_0(0)B_V(0) + B_V^T(0)A_0(0) \leq -\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_r \end{pmatrix}. \tag{2.10}$$

Additionally, from (1.5) we have

$$A(V) = \begin{pmatrix} a(\mathbf{P}^{-1}(0)V) & b(\mathbf{P}^{-1}(0)V) \\ c(\mathbf{P}^{-1}(0)V) & d(\mathbf{P}^{-1}(0)V) \end{pmatrix}, \tag{2.11}$$

in which $a(0)$ has only positive (or only negative) eigenvalues.

Noting (2.4) and (2.6), we can reduce from (1.6) that

$$L_-(V)R_0(0) = 0, \quad L_+(V)R_0(0) = 0. \tag{2.12}$$

According to the structure in (2.9) and (2.10), we write $V(t,x)$ as

$$V(t,x) = \begin{pmatrix} v_1(t,x) \\ v_2(t,x) \end{pmatrix} \quad \text{with} \quad v_1 \in \mathbb{R}^{n-r}, \ v_2 \in \mathbb{R}^r \tag{2.13}$$

for further use. From (1.7) and (2.4), we can see that the linear diagonal variables $\xi(t,x)$ now becomes

$$\xi(t,x) = L(0)V(t,x),$$

in other words,

$$\xi_-(t,x) = L_-(0)V(t,x), \ \xi_0(t,x) = L_0(0)V(t,x), \ \xi_+(t,x) = L_+(0)V(t,x) \tag{2.14}$$

with (1.8), which implies that the boundary conditions are still given by (1.9).

The initial condition for the variable V is given by

$$V(0,x) = V_0(x) \triangleq \mathbf{P}(0)U_0(x), \quad x \in (0,1). \tag{2.15}$$

In order to prove Thereom 1, it suffices to establish the H^2-stabilization for the system (2.2), (2.15) and (1.9).

3 Proof of Theorem 1

In this section, we can identify appropriate conditions on the feedback matrix $\mathbf{K}$ that ensure the exponential stability of the closed-loop system (2.2), (2.15) and (1.9) for the $H^2(0,1)$-norm. Then, Theorem 1 follows as an immediate consequence.

Let V_0 with small $H^2((0,1);\mathbb{R}^n)$ norm be such that the C^1 compatibility conditions at $(t,x)=(0,0)$ and $(0,1)$ are satisfied. Let also $V\in C^0([0,T),H^2((0,1);\mathbb{R}^n))$ be the maximal classical solution of the problem (2.2), (2.15) and (1.9). We remark that we only prove the stabilization result for smooth solutions while the conclusion follows easily from an density and continuity arguments for distributed solutions. Motivated by [2] and [17], we construct a weighted Lyapunov function as follows:

$$\mathbb{L}(t)\triangleq c_0\mathbb{L}_0(t)+c_1\mathbb{L}_1(t)+c_2\mathbb{L}_2(t)+c_3\mathbb{L}_3(t)+c_4\mathbb{L}_4(t)$$

with

$$\mathbb{L}_0(t)\triangleq\int_0^1\lambda(x)V^TA_0(V)V\,\mathrm{d}x, \tag{3.1}$$

$$\mathbb{L}_1(t)\triangleq\int_0^1\lambda(x)V_x^TA_0(V)V_x\,\mathrm{d}x, \tag{3.2}$$

$$\mathbb{L}_2(t)\triangleq\int_0^1\lambda(x)V_{xx}^TA_0(V)V_{xx}\,\mathrm{d}x, \tag{3.3}$$

$$\mathbb{L}_3(t)\triangleq V^T(t,0)A_0(V(t,0))V(t,0)+V^T(t,1)A_0(V(t,1))V(t,1), \tag{3.4}$$

$$\mathbb{L}_4(t)\triangleq V_x^T(t,0)A_0(V(t,0))V_x(t,0)+V_x^T(t,1)A_0(V(t,1))V_x(t,1), \tag{3.5}$$

where $\lambda(x)>0$ is a continuously differentiable function and c_i $(i=0,1,\dots,4)$ are positive constants to be chosen.

For the simplicity of statements, we denote the $\|\cdot\|_{L^2(0,1)}$ norm as $\|\cdot\|$, $\|\cdot\|_{C^0([0,1])}$ norm as $\|\cdot\|_{C^0}$, $\|\cdot\|_{C^1([0,1])}$ norm as $\|\cdot\|_{C^1}$. Also, we set $|U|_0=|U(t,0)|+|U(t,1)|$ and $|U|_1=|U(t,0)|+|U(t,1)|+|U_x(t,0)|+|U_x(t,1)|$ for any $U=U(t,x)$.

The Sobolev inequality implies that $\|V\|_{C^1}\le C\|V(t,\cdot)\|_{H^2((0,1);\mathbb{R}^n)}$ for a constant $C>0$. Then, by definition of the Lyapunov function $\mathbb{L}(t)$, $\mathbb{L}(t)$ is equivalent to the energy $\|V\|^2+\|V_x\|^2+\|V_{xx}\|^2$ if $\|V\|_{C^0}$ is small. In other words, $\mathbb{L}(t)$ is equivalent to the energy $\|V(t,\cdot)\|^2_{H^2((0,1);\mathbb{R}^n)}$ if $\|V\|_{C^0}$ is small.

With the help of (1.2), it follows that

$$(\mathbf{L}^{-1}(U))^T\mathbf{A}_0(U)\mathbf{L}^{-1}(U)\mathbf{\Lambda}(U)=\mathbf{\Lambda}(U)(\mathbf{L}^{-1}(U))^T\mathbf{A}_0(U)\mathbf{L}^{-1}(U).$$

Consequently, there exist three symmetric positive definite matrices $\mathbf{X}_1(U)\in\mathcal{M}_{m,m}(\mathbb{R})$, $\mathbf{X}_2(U)\in\mathcal{M}_{n-m-p,n-m-p}(\mathbb{R})$ and $\mathbf{X}_2(U)\in\mathcal{M}_{p,p}(\mathbb{R})$ such that

$$(\mathbf{L}^{-1}(U))^T\mathbf{A}_0(U)\mathbf{L}^{-1}(U)=\begin{pmatrix}\mathbf{X}_1(U)&0&0\\0&\mathbf{X}_2(U)&0\\0&0&\mathbf{X}_3(U)\end{pmatrix}. \tag{3.6}$$

It will be shown in Section 4 that the calculations of the time derivative of

$\mathbb{L}(t)$ involve the following three matrices $\mathbf{G}_0$, $\mathbf{G}_1$, $\mathbf{G}_2$ as

$$\mathbf{G}_0 = \mathbf{H}_1 - \mathbf{K}^T\mathbf{H}_2\mathbf{K}, \tag{3.7}$$

$$\mathbf{G}_1 = \mathbf{H}_1 - \mathbf{\Lambda}_\pm\mathbf{K}^T\mathbf{\Lambda}_\pm{}^{-1}\mathbf{H}_2\mathbf{\Lambda}_\pm{}^{-1}\mathbf{K}\mathbf{\Lambda}_\pm, \tag{3.8}$$

$$\mathbf{G}_2 = \mathbf{H}_1 - \mathbf{\Lambda}_\pm{}^2\mathbf{K}^T(\mathbf{\Lambda}_\pm{}^{-1})^2\mathbf{H}_2(\mathbf{\Lambda}_\pm{}^{-1})^2\mathbf{K}\mathbf{\Lambda}_\pm{}^2, \tag{3.9}$$

with $\mathbf{\Lambda}_\pm = \begin{pmatrix} \mathbf{\Lambda}_+(0) & 0 \\ 0 & \mathbf{\Lambda}_-(0) \end{pmatrix}$ and

$$\mathbf{H}_1 \triangleq \begin{pmatrix} \lambda(1)\mathbf{X}_3(0) & 0 \\ 0 & -\lambda(0)\mathbf{X}_1(0) \end{pmatrix}\mathbf{\Lambda}_\pm, \mathbf{H}_2 \triangleq \begin{pmatrix} \lambda(0)\mathbf{X}_3(0) & 0 \\ 0 & -\lambda(1)\mathbf{X}_1(0) \end{pmatrix}\mathbf{\Lambda}_\pm.$$

Let the feedback matrix $\mathbf{K}$ be chosen such that matrices $\mathbf{G}_0$, $\mathbf{G}_1$ and $\mathbf{G}_2$ are all positive definite. Given that both $\mathbf{H}_1$ and $\mathbf{H}_2$ are positive definite, achieving this condition is relatively straightforward if $|\mathbf{K}|_\infty$ is sufficiently small. With this assumption regarding the feedback matrix $\mathbf{K}$, we are able to establish the following lemmas for the estimates of each $\mathbb{L}_i(t)$ $(i = 0, \ldots, 4)$. The proof of these lemmas will be given in Section 4.

Lemma 1. *There exist positive constants α_0, β_0, γ_0 and δ_0 independent of V such that, if $\|V\|_{C^0} \le \delta_0$,*

$$\mathbb{L}_0'(t) \le -\alpha_0\|V\|^2 - \beta_0\left|\begin{pmatrix} \xi_+(t,1) \\ \xi_-(t,0) \end{pmatrix}\right|^2 + \gamma_0\Big(\|V\|_{C^1}\|V\|^2 + |V|_0^3\Big). \tag{3.10}$$

Lemma 2. *There exist positive constants α_1, β_1, η_1, γ_1 and δ_1 independent of V such that, if $\|V\|_{C^0} \le \delta_1$,*

$$\mathbb{L}_1'(t) \le -\alpha_1\|V_x\|^2 - \beta_1\left|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\right|^2 + \eta_1|v_2|_0^2 + \gamma_1\Big(\|V\|_{C^1}\|V_x\|^2 + |V|_0^3\Big), \tag{3.11}$$

where v_2 is defined in (2.13).

Lemma 3. *There exist positive constants α_2, β_2, η_2, γ_2, and δ_2 independent of V such that, if $\|V\|_{C^1} \le \delta_2$,*

$$\mathbb{L}_2'(t) \le -\alpha_2\|V_{xx}\|^2 - \beta_2\left|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}\right|^2 + \eta_2\left(|v_2|_1^2 + \left|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\right|^2\right)$$
$$+ \gamma_2\Big(\|V\|_{C^1}(\|V_{xx}\|^2 + \|V_x\|^2) + |V|_0|V|_1^2\Big). \tag{3.12}$$

Lemma 4. *There exist positive constants γ_3 and δ_3 independent of V such that for any $\varepsilon_3 > 0$, if $\|V\|_{C^0} \le \delta_3$,*

$$\mathbb{L}_3'(t) \le -|v_2|_0^2 + \eta_3\left|\begin{pmatrix} \xi_+(t,1) \\ \xi_-(t,0) \end{pmatrix}\right|^2 + \varepsilon_3\left|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\right|^2 + \gamma_3|V|_0^2|V|_1, \tag{3.13}$$

where η_3 is a constant depends on ε_3.

Lemma 5. *There exist positive constants γ_4 and δ_4 independent of V such that for any $\varepsilon_4 > 0$, if $\|V\|_{C^0} \le \delta_4$,*

$$\mathbb{L}_4'(t) \le -|v_{2x}|_0^2 + \eta_4 \left| \begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix} \right|^2 + \varepsilon_4 \left| \begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix} \right|^2 + \gamma_4 \Big(|V|_1^2 (|V|_0 + |V|_1) + |V|_0 \left| \begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix} \right|^2 \Big), \tag{3.14}$$

where η_4 is a constant depends on ε_4.

With the help of Lemmas 1–5, we are ready to prove Theorem 1.

Let the constant $\delta_5 \le \min\{\delta_0, \delta_1, \delta_2, \delta_3, \delta_4\}$. The order of determining the constants would be $c_2, c_4, \varepsilon_4, c_1, c_3, \varepsilon_3, c_0$, with $\varepsilon_4, \varepsilon_3$ small enough and c_4, c_1, c_3, c_0 large enough. The combination of (3.10)–(3.14) yields that there exist positive constants β and γ such that

$$\mathbb{L}'(t) \le -\beta \mathbb{L}(t) + \gamma \|V\|_{C^1} \mathbb{L}(t). \tag{3.15}$$

Let $\delta_6 \triangleq \min\{\delta_5, \frac{\beta}{2\gamma}\}$. If we assume *in a priori* that $\|V\|_{C^1} \le \delta_6$ for $t \in (0,T)$, we get

$$\mathbb{L}'(t) \le -\frac{\beta}{2} \mathbb{L}(t), \quad t \in (0,T),$$

which implies that $\mathbb{L}(t)$ decays exponentially

$$\mathbb{L}(t) \le e^{-\frac{\beta t}{2}} \mathbb{L}(0), \quad t \in (0,T).$$

Using the equivalence of the energy $\|V(t,\cdot)\|^2_{H^2((0,1);\mathbb{R}^n)}$ and $\mathbb{L}(t)$, we obtain

$$\|V(t,\cdot)\|_{H^2((0,1);\mathbb{R}^n)} \le C_2 e^{-\frac{\beta t}{2}} \|V_0\|_{H^2((0,1);\mathbb{R}^n)}, \quad \forall t \in [0,T) \tag{3.16}$$

for some constant $C_2 > 0$. We also note that Sobolev inequality implies

$$\|V\|_{C^1} \le C_1 \|V(t,\cdot)\|_{H^2((0,1);\mathbb{R}^n)} \le C_1 C_2 \|V_0\|_{H^2((0,1);\mathbb{R}^n)}, \quad \forall t \in [0,T).$$

Let now $\delta = \frac{\delta_6}{C_1 C_2}$. Then, the a priori estimate on $\|V\|_{C^1} \le \delta_6$ indeed holds in $[0,T)$ if $\|V_0\|_{H^2((0,1);\mathbb{R}^n)} \le \delta$. Therefore, (3.16) follows immediately. According to Proposition 1, we finally conclude that inequality (3.16) holds for $T = +\infty$.

Consequently, it follows from (2.1) that the solution U to problem (1.1), (1.9) and (1.10) is locally exponentially stable for the H^2-norm. This concludes the proof of Theorem 1.

4 Proof of the lemmas

4.1 Proof of Lemma 1

We calculate the time-derivative of $\mathbb{L}_0(t)$ defined by (3.1),

$$\mathbb{L}_0'(t) = \int_0^1 \lambda(x)\Big(V^T A_0(V)V_t + V_t^T A_0(V)V\Big)\,\mathrm{d}x + \mathcal{O}\Big(\int_0^1 |V|^2(|V|+|V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big). \tag{4.1}$$

Here and hereafter $\mathcal{O}(X;Y)$ denotes the terms that for $X \geq 0$, $Y \geq 0$, there exist $C > 0$ and $\varepsilon > 0$ independent of X and Y, satisfying

$$Y \leq \varepsilon \Rightarrow |\mathcal{O}(X;Y)| \leq CX.$$

Substituting the system (2.2) into (4.1), we have

$$\begin{aligned}\mathbb{L}_0'(t) =& \int_0^1 2\lambda(x)\Big(V^T A_0(V)B(V) - V^T A_0(V)A(V)V_x\Big)\,\mathrm{d}x \\ &+ \mathcal{O}\Big(\int_0^1 |V|^2(|V|+|V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big), \\ =& \int_0^1 \lambda(x)V^T\Big(A_0(0)B_V(0) + B_V^T(0)A_0(0)\Big)V + \lambda'(x)V^T A_0(0)A(0)V \\ &- \Big(\lambda(x)V^T A_0(V)A(V)V\Big)_x\,\mathrm{d}x + \mathcal{O}\Big(\int_0^1 |V|^2(|V|+|V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big).\end{aligned}$$

With the help of (2.9) and (2.10), positive definite matrix $A_0(0)$ could be proved to have the structure

$$A_0(0) = \begin{pmatrix} X_1(0) & 0 \\ 0 & X_2(0) \end{pmatrix}, \tag{4.2}$$

where $X_1 \in \mathbb{R}^{(n-r)\times(n-r)}$ and $X_2 \in \mathbb{R}^{r\times r}$. Without loss of generality, we assume that the $(n-r)\times(n-r)$ matrix $a = a(0)$ in (2.11) has only positive eigenvalues, then it is easy to obtain that $X_1(0)a(0)$ is positive definite with the help of (2.8). Thus, there exists constant $\mathcal{X} \geq 0$ such that

$$\begin{aligned}V^T A_0(0)A(0)V =& v_1^T X_1(0)a(0)v_1 + v_1^T X_1(0)b(0)v_2 + v_2^T X_2(0)c(0)v_1 \\ &+ v_2^T X_2(0)d(0)v_2 \geq \frac{1}{2}v_1^T X_1(0)a(0)v_1 + \mathcal{X}v_2^T\Big(X_2(0)\mathbf{S}(0) + \mathbf{S}^T(0)X_2(0)\Big)v_2,\end{aligned} \tag{4.3}$$

as $X_1(0)a(0)$ and $-\Big(X_2(0)\mathbf{S}(0) + \mathbf{S}^T(0)X_2(0)\Big)$ are both positive definite matrices. We choose $\lambda = \lambda(x)$ such that

$$\lambda'(x) < 0, \quad -2\mathcal{X}\lambda'(x) \leq \lambda(x).$$

Therefore,

$$
\begin{aligned}
&\lambda(x)V^T\Big(A_0(0)B_V(0)+B_V^T(0)A_0(0)\Big)V+\lambda'(x)V^TA_0(0)A(0)V\\
\le&\frac{1}{2}\lambda'(x)v_1^TX_1(0)a(0)v_1+\Big(\lambda(x)+\mathcal{X}\lambda'(x)\Big)v_2^T\Big(X_2(0)\mathbf{S}(0)+\mathbf{S}^T(0)X_2(0)\Big)v_2.\\
\le&\frac{1}{2}\lambda'(x)v_1^TX_1(0)a(0)v_1+\frac{1}{2}\lambda(x)v_2^T\Big(X_2(0)\mathbf{S}(0)+\mathbf{S}^T(0)X_2(0)\Big)v_2.
\end{aligned}
$$

Using (2.8), (2.10) and integration by parts, we have

$$
\mathbb{L}_0'(t)\le-\alpha_0\|V\|^2+\mathbb{B}_0+\mathcal{O}\Big(\int_0^1|V|^2(|V|+|V_x|)\,\mathrm{d}x;\|V\|_{C^0}\Big),
$$

where α_0 is the smaller value between the smallest eigenvalues of $-\frac{1}{2}\lambda'(x)X_1(0)a(0)$ and the smallest eigenvalues of $-\frac{1}{2}\lambda(x)\Big(X_2(0)\mathbf{S}(0)+\mathbf{S}^T(0)X_2(0)\Big)$, and the boundary term $\mathbb{B}_0$ is

$$
\mathbb{B}_0=\Big[-\lambda(x)V^TA_0(V)A(V)V\Big]\Big|_0^1. \tag{4.4}
$$

It remains to estimate $\mathbb{B}_0$. Noting (2.4), (2.7) and (3.6), we can easily obtain that

$$
(L^{-1}(V))^TA_0(V)L^{-1}(V)=\begin{pmatrix}\mathbf{X}_1(V)&0&0\\0&\mathbf{X}_2(V)&0\\0&0&\mathbf{X}_3(V)\end{pmatrix}. \tag{4.5}
$$

Using (2.5) and (2.14), we have

$$
\begin{aligned}
&V^T(t,x)A_0(V)A(V)V(t,x)\\
&=V^T(t,x)L^T(V)\Big((L^{-1}(V))^TA_0(V)L^{-1}(V)\mathbf{\Lambda}(V)\Big)L(V)V(t,x)\\
&=\xi^T(t,x)R^T(0)L_-^T(V)\mathbf{X}_1(V)\mathbf{\Lambda}_-(V)L_-(V)R(0)\xi(t,x)\\
&\quad+\xi^T(t,x)R^T(0)L_+(V)\mathbf{X}_3(V)\mathbf{\Lambda}_+(V)L_+(V)R(0)\xi(t,x),
\end{aligned} \tag{4.6}
$$

where $L_-(V)$, $L_+(V)$ are defined in (2.6). Condition (2.12) leads to

$$
\begin{aligned}
&V^T(t,x)A_0(V)A(V)V(t,x)=\xi_-^T(t,x)\mathbf{X}_1(0)\mathbf{\Lambda}_-(0)\xi_-(t,x)\\
&+\xi_+^T(t,x)\mathbf{X}_3(0)\mathbf{\Lambda}_+(0)\xi_+(t,x)+\mathcal{O}\Big(|V(t,x)|\Big|\begin{pmatrix}\xi_+(t,x)\\ \xi_-(t,x)\end{pmatrix}\Big|^2;|V(t,x)|\Big).
\end{aligned} \tag{4.7}
$$

Substituting the boundary conditions (1.9) and (4.7) into (4.4), we thus get

$$
\begin{aligned}
\mathbb{B}_0&=\Big[-\lambda(x)V(t,x)A_0(V)A(V)V(t,x)\Big]\Big|_0^1\\
&=-\begin{pmatrix}\xi_+(t,1)\\ \xi_-(t,0)\end{pmatrix}^T\mathbf{G}_0\begin{pmatrix}\xi_+(t,1)\\ \xi_-(t,0)\end{pmatrix}+\mathcal{O}\Big(|V|_0\Big|\begin{pmatrix}\xi_+(t,1)\\ \xi_-(t,0)\end{pmatrix}\Big|^2;|V|_0\Big),
\end{aligned}
$$

with symmetric matrix $\mathbf{G}_0$ defined as (3.7).

As we choose the boundary feedback matrix $\mathbf{K}$ such that the symmetric matrix $\mathbf{G}_0$ is positive definite, it is easy to see that with $\beta_0>0$ being the smallest eigenvalue of $\mathbf{G}_0$, there exist $\delta_0>0$ and $\gamma_0>0$ such that the estimate (3.10) holds if $\|V\|_{C^0}<\delta_0$. This concludes the proof of Lemma 1.

4.2 Proof of Lemma 2

By (3.2), the time-derivative of $\mathbb{L}_1(t)$ can be expressed as

$$\mathbb{L}_1'(t) = \int_0^1 2\lambda(x) V_x^T A_0(V) V_{xt}\,\mathrm{d}x + \mathcal{O}\Big(\int_0^1 |V_x|^2(|V|+|V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big). \quad (4.8)$$

Differentiation of system (2.2) with respect to x gives that

$$V_{tx} + A(V)V_{xx} = B_V(V)V_x - (A'(V)V_x)V_x, \quad (4.9)$$

in which $A'(V)V_x$ is matrix with entries $\frac{\partial a_{ij}(V)}{\partial V}V_x$. Substituting the term of V_{xt} derived from (4.9) into (4.8), and using integrations by parts and some straightforward calculations, we have

$$\begin{aligned}\mathbb{L}_1'(t) =& \int_0^1 \lambda(x)V_x^T\Big(A_0(0)B_V(0)+B_V(0)^T A_0(0)\Big)V_x + \lambda'(x)V_x^T A_0(0)A(0)V_x\,\mathrm{d}x \\ &- \Big[\lambda(x)V_x^T A_0(V)A(V)V_x\Big]\Big|_0^1 + \mathcal{O}\Big(\int_0^1 |V_x|^2(|V|+|V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big).\end{aligned}$$

Similarly as the analysis of $\mathbb{L}_0'(t)$ in the proof of Lemma 1, we obtain that there exists positive constant α_1, such that

$$\mathbb{L}_1'(t) \le -\alpha_1\|V_x\|^2 + \mathbb{B}_1 + \mathcal{O}\Big(\int_0^1 |V_x|^2(|V|+|V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big),$$

where the boundary term $\mathbb{B}_1$ is

$$\begin{aligned}\mathbb{B}_1 =& -\Big[\lambda(x)V_x^T A_0(V)A(V)V_x\Big]\Big|_0^1 = -\lambda(x)\Big[\xi_{-x}^T(t,x)\mathbf{X}_1(0)\boldsymbol{\Lambda}_-(0)\xi_{-x}(t,x) \\ &+ \xi_{+x}^T(t,x)\mathbf{X}_3(0)\boldsymbol{\Lambda}_+(0)\xi_{+x}(t,x)\Big]\Big|_0^1 + \mathcal{O}\Big(|V|_0(|V|_0^2 + \Big|\begin{pmatrix}\xi_{+x}(t,x)\\ \xi_{-x}(t,x)\end{pmatrix}\Big|_0^2); |V|_0\Big),\end{aligned}$$

by using (2.12) similarly as (4.6). From (2.2), (2.14) and with the help of (2.12), we have

$$\begin{aligned}&\begin{pmatrix}\xi_{+t}(t,x)\\ \xi_{-t}(t,x)\end{pmatrix} + \begin{pmatrix}\Lambda_+(0) & 0\\ 0 & \Lambda_-(0)\end{pmatrix}\begin{pmatrix}\xi_{+x}(t,x)\\ \xi_{-x}(t,x)\end{pmatrix} = \begin{pmatrix}L_+(0)B_V(0)V(t,x)\\ L_-(0)B_V(0)V(t,x)\end{pmatrix} \\ &\qquad + \mathcal{O}\Big(|\xi(t,x)|\Big(|\xi(t,x)| + \Big|\begin{pmatrix}\xi_{+x}(t,x)\\ \xi_{-x}(t,x)\end{pmatrix}\Big|\Big); |\xi(t,x)|\Big), \quad (4.10)\end{aligned}$$

in which $L_+(V)$ and $L_-(V)$ are defined in (2.6). Thanks for (2.9), we have

$$B_V(0)V(t,x) = (0, S(0)v_2(t,x))^T. \quad (4.11)$$

Take time derivative on both sides of boundary conditions (1.9) and substitute it into (4.10). Then, with the help of (4.11), he boundary term $\mathbb{B}_1$ can be transformed into

$$\begin{aligned}\mathbb{B}_1 \le& -\begin{pmatrix}\xi_{+x}(t,1)\\ \xi_{-x}(t,0)\end{pmatrix}^T \mathbf{G}_1\begin{pmatrix}\xi_{+x}(t,1)\\ \xi_{-x}(t,0)\end{pmatrix} + C_{11}|v_2|_0^2 + C_{12}\Big|\begin{pmatrix}\xi_{+x}(t,1)\\ \xi_{-x}(t,0)\end{pmatrix}\Big||v_2|_0 \\ &+ \mathcal{O}\Big[|V|_0\Big(|V|_0^2 + \Big|\begin{pmatrix}\xi_{+x}(t,1)\\ \xi_{-x}(t,0)\end{pmatrix}\Big|^2\Big); |V|_0\Big]\end{aligned}$$

with $\mathbf{G}_1$ defined as (3.8) and constants $C_{11}, C_{12} \geq 0$ independent of V.

As $\mathbf{G}_1$ is positive definite with the help of a suitable $\mathbf{K}$, it is easy to see that there exist $\beta_1, \eta_1, \delta_1$ and $\gamma_1 > 0$ such that the estimate (3.11) holds if $\|V\|_{C^0} < \delta_1$. This finishes the proof of Lemma 2.

4.3 Proof of Lemma 3

The time derivative of $\mathbb{L}_2(t)$ gives

$$\mathbb{L}_2'(t) = \int_0^1 2\lambda(x) V_{xx}^T A_0(V) V_{xxt}\,\mathrm{d}x + \mathcal{O}\Big(\int_0^1 |V_{xx}|^2(|V| + |V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big).$$

Differentiating system (4.9) with respect to x and combining (2.2) and (4.9), we have

$$V_{xxt}+A(V)V_{xxx}=B_V(V)V_{xx}+(B_V(V))_xV_x-(A'(V)V_x)V_{xx}-(A'(V)V_x)_xV_x. \tag{4.12}$$

Substituting the term of V_{xxt} derived from (4.12) into $\mathbb{L}_2'(t)$, we do integration by parts and linear approximation, as in the proof of Lemmas 1 and 2, to deduce that

$$\begin{aligned}\mathbb{L}_2'(t) = &\int_0^1 \lambda(x) V_{xx}^T\Big(A_0(0)B_V(0) + B_V(0)^T A_0(0)\Big)V_{xx}\\ &- \lambda'(x)V_{xx}^T A_0(0)A(0)V_{xx}\,\mathrm{d}x - \Big[\lambda(x)V_{xx}^T A_0(V)A(V)V_{xx}\Big]\Big|_0^1\\ &+ \mathcal{O}\Big(\int_0^1 |V_{xx}|^2(|V| + |V_x|) + |V_x|^2|V_{xx}|\,\mathrm{d}x; \|V\|_{C^1}\Big).\end{aligned}$$

Using similar analysis of $\mathbb{L}_0'(t)$ in proof of Lemma 1, we obtain that there exists positive constant α_2 independent of V such that

$$\mathbb{L}_2'(t) \leq -\alpha_2\|V_{xx}\|^2 + \mathbb{B}_2 + \mathcal{O}\Big(\int_0^1 |V_{xx}|^2(|V| + |V_x|) + |V_x|^2|V_{xx}|\,\mathrm{d}x; \|V\|_{C^1}\Big),$$

where $\mathbb{B}_2$ denotes the boundary term derived from integration by parts

$$\begin{aligned}\mathbb{B}_2 = &- \Big[\lambda(x)V_{xx}^T A_0(V)A(V)V_{xx}\Big]\Big|_0^1 = -\lambda(x)\Big[\xi_{-xx}^T(t,x)\mathbf{X}_1(0)\mathbf{\Lambda}_-(0)\xi_{-xx}(t,x)\\ &+ \xi_{+xx}^T(t,x)\mathbf{X}_3(0)\mathbf{\Lambda}_+(0)\xi_{+xx}(t,x)\Big]\Big|_0^1 + \mathcal{O}\Big(|V|_0(|V|_0^2 + \Big|\begin{pmatrix}\xi_{+xx}(t,x)\\ \xi_{-xx}(t,x)\end{pmatrix}\Big|_0^2); |V|_0\Big)\end{aligned}$$

by using (2.12). Differentiating (4.10) with respect to t and x gives that

$$\begin{aligned}&\begin{pmatrix}\xi_{+tt}(t,x)\\ \xi_{-tt}(t,x)\end{pmatrix} - \begin{pmatrix}\Lambda_+(0) & 0\\ 0 & \Lambda_-(0)\end{pmatrix}^2\begin{pmatrix}\xi_{+xx}(t,x)\\ \xi_{-xx}(t,x)\end{pmatrix}\\ &= -\begin{pmatrix}\Lambda_+(0) & 0\\ 0 & \Lambda_-(0)\end{pmatrix}\begin{pmatrix}L_+(0)B_V(0)V_x(t,x)\\ L_-(0)B_V(0)V_x(t,x)\end{pmatrix} + \begin{pmatrix}L_+(0)B_V(0)V_t(t,x)\\ L_-(0)B_V(0)V_t(t,x)\end{pmatrix}\\ &\quad + \mathcal{O}\Big((|\xi(t,x)| + |\xi_x(t,x)|)^2 + |\xi(t,x)|\Big|\begin{pmatrix}\xi_{+x}(t,x)\\ \xi_{-x}(t,x)\end{pmatrix}\Big|; |\xi(t,x)|\Big)\end{aligned} \tag{4.13}$$

with the help of (2.12). Through detailed calculations, and by taking into account (1.9), (2.9) and (4.13), the boundary term $\mathbb{B}_2$ can be rewritten as

$$\begin{aligned}\mathbb{B}_2 \leq &-\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}^T \mathbf{G}_2 \begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix} + C_{21}\Big(|v_2|_1^2 + \Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\Big|^2\Big) \\ &+ C_{22}\Big|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}\Big|\Big(|v_2|_1 + \Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\Big|\Big) \\ &+ \mathcal{O}\Big(|V|_0\big(|V|_1^2 + \Big|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}\Big|^2\big); |V|_0\Big),\end{aligned}$$

where $\mathbf{G}_2$ is defined as (3.9), C_{21} and C_{22} are positive constants. Note that with an appropriately chosen feedback matrix $\mathbf{K}$, there exist $\beta_2, \eta_2, \delta_2$ and $\gamma_2 > 0$ independent of V such that the estimate (3.12) holds if $\|V\|_{C^1} < \delta_2$. This ends the proof of Lemma 3.

4.4 Proof of Lemma 4

According to (3.4), we can decompose the function $\mathbb{L}_3(t)$ as

$$\mathbb{L}_3(t) = \mathbb{I}_3(t) + \mathbb{J}_3(t)$$

with

$$\mathbb{I}_3(t) \triangleq V^T(t,0)A_0(V(t,0))V(t,0), \quad \mathbb{J}_3(t) \triangleq V^T(t,1)A_0(V(t,1))V(t,1).$$

The time derivative of $\mathbb{I}_3(t)$ gives

$$\begin{aligned}\mathbb{I}_3'(t) &= V_t^T(t,0)A_0(V(t,0))V(t,0) \\ &\quad + V^T(t,0)A_0(V(t,0))V_t(t,0) + \mathcal{O}(|V|_0^2|V|_1; |V|_0), \\ &\leq V^T(t,0)\Big(B_V^T(0)A_0(0) + A_0(0)B_V(0)\Big)V(t,0) + \mathcal{O}(|V|_0^2|V|_1; |V|_0) \\ &\quad - \xi_{+x}(t,0)^T \Lambda_+(0)\mathbf{X}_3(0)\xi_+(t,0) - \xi_{-x}(t,0)^T \Lambda_-(0)\mathbf{X}_1(0)\xi_-(t,0) \\ &\quad - \xi_+^T(t,0)\mathbf{X}_3(0)\Lambda_+(0)\xi_{+x}(t,0) - \xi_-^T(t,0)\mathbf{X}_1(0)\Lambda_-(0)\xi_{-x}(t,0)\end{aligned}$$

by taking into account (4.5). With the help of (2.10), we have

$$\begin{aligned}\mathbb{I}_3'(t) \leq &-v_2^T(t,0)v_2(t,0) + O(1)\Big(|\xi_-(t,0)||\xi_{-x}(t,0)| + |\xi_+(t,0)||\xi_{+x}(t,0)|\Big) \\ &+ \mathcal{O}(|V|_0^2|V|_1; |V|_0).\end{aligned} \tag{4.14}$$

Similarly, we have

$$\begin{aligned}\mathbb{J}_3'(t) \leq &-v_2^T(t,1)v_2(t,1) + O(1)\Big(|\xi_-(t,1)||\xi_{-x}(t,1)| + |\xi_+(t,1)||\xi_{+x}(t,1)|\Big) \\ &+ \mathcal{O}(|V|_0^2|V|_1; |V|_0).\end{aligned} \tag{4.15}$$

The combination of (4.14)–(4.15) yields that

$$\begin{aligned}\mathbb{L}_3'(t) \leq &-\Big(v_2^T(t,0)v_2(t,0) + v_2^T(t,1)v_2(t,1)\Big) \\ &+ C_{31}\Big|\begin{pmatrix} \xi_+(t,1) \\ \xi_-(t,0) \end{pmatrix}\Big|\Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\Big| + \mathcal{O}(|V|_0^2|V|_1; |V|_0)\end{aligned}$$

for some constant $C_{31} > 0$. Therefore, for any $\varepsilon_3 > 0$ with $\eta_3 = C_{31}^2/\varepsilon_3$, there exist $\delta_3 > 0$ and $\gamma_3 > 0$ such that the estimate (3.13) holds if $\|V\|_{C^0} < \delta_3$. This ends the proof of Lemma 4.

4.5 Proof of Lemma 5

According to (3.5), we can decompose the function $\mathbb{L}_4(t)$ as

$$\mathbb{L}_4(t) = \mathbb{I}_4(t) + \mathbb{J}_4(t)$$

with

$$\mathbb{I}_4(t) = V_x^T(t,0)A_0(V(t,0))V_x(t,0), \quad \mathbb{J}_4(t) = V_x^T(t,1)A_0(V(t,1))V_x(t,1).$$

Similarly as the analysis of $\mathbb{L}_3'(t)$ in the proof of Lemma 4, we can deduce that

$$\begin{aligned}\mathbb{L}_4'(t) \leq & -v_{2x}^T(t,0)v_{2x}(t,0) - v_{2x}^T(t,1)v_{2x}(t,1) + C_{41}\left|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\right| \\ & \times \left|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}\right| + \mathcal{O}\Big(|V|_1^3 + |V|_0\left|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}\right|^2; |V|_0\Big)\end{aligned}$$

for some constant $C_{41} > 0$. Therefore, for any $\varepsilon_4 > 0$ with $\eta_4 = \frac{C_{41}^2}{\varepsilon_4}$, there exist $\delta_4 > 0$ and $\gamma_4 > 0$ such that the estimate (3.14) holds if $\|V\|_{C^0} < \delta_4$. This ends the proof of Lemma 5.

5 Sketch of the proof of Theorem 2

In this section, we give the sketch of the proof of Theorem 2. In this case, assumption (1.3) for the source terms $\mathbf{Q}(U)$ is no longer required. Instead, we introduce the new structure (1.13). Using the same transformation as detailed in Section 2, this new structure is turned into

$$rank\Big((0\ I_r)R_0(0)\Big) = n - m - p, \tag{5.1}$$

where $R_0(V)$ is defined in (2.6). Here we give a brief explanation to structure (5.1). Actually, since

$$V(t,x) = \begin{pmatrix} v_1(t,x) \\ v_2(t,x) \end{pmatrix} = L^{-1}(0)\xi(t,x) = R(0)\begin{pmatrix} \xi_-(t,x) \\ \xi_0(t,x) \\ \xi_+(t,x) \end{pmatrix},$$

we have

$$v_2 = (0\ I_r)\Big(R_-(0)\xi_- + R_0(0)\xi_0 + R_+(0)\xi_+\Big)$$

with (2.6). It is easy to see that (5.1) implies that the dissipative part v_2 contains all the information related to the term ξ_0 corresponding to zero eigenvalues. In the case that $n - m - p = n - r$, v_2 is equivalent to ξ_0.

Without assumption (1.3) for the source term $\mathbf{Q}(U)$, the structure (4.2) is no longer satisfied. Consequently, the condition (1.5) would not be sufficient to uphold the theorem. In fact, if $\mathbf{S}_{11}(U) = 0$ in (1.11), then we could prove that structure of $A_0(0)$ as described in (4.2) still exists. However, in this theorem, without structural requirement to source term $\mathbf{Q}(U)$, we replaced condition (1.5) with condition (1.12).

With these two new structures, we are ready to give the sketch of the proof of Theorem 2. We still use the same Lyapunov function as in Theorem 1

$$\mathbb{L}(t) = c_0\mathbb{L}_0(t) + c_1\mathbb{L}_1(t) + c_2\mathbb{L}_2(t) + c_3\mathbb{L}_3(t) + c_4\mathbb{L}_4(t)$$

with $\mathbb{L}_i(t)$ $(i = 0, 1, \ldots, 4)$ defined the same as (3.1)–(3.5). For $\mathbb{L}_0(t)$, with the help of (1.12), inequality (4.3) could be turned into

$$V^T A_0(0)A(0)V \geq \frac{1}{2}v_1^T \mathbf{Z_1}(0)v_1 + \mathcal{X}V^T\Big(A_0(0)B_V(0) + B_V^T(0)A_0(0)\Big)V,$$

from which we could easily prove that Lemma 1 still holds. By calculating the time derivative of $\mathbb{L}_1(t)$ in a similar way, we may still get that

$$\mathbb{L}_1'(t) \leq -\alpha_1\|V_x\|^2 + \mathbb{B}_1 + \mathcal{O}\Big(\int_0^1 |V_x|^2(|V| + |V_x|)\,\mathrm{d}x; \|V\|_{C^0}\Big),$$

for some constant $\alpha_1 > 0$. However in this case, without requirement (1.3), we view the boundary term $\mathbb{B}_1$ as

$$\begin{aligned}\mathbb{B}_1 \leq &-\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0)\end{pmatrix}^T \mathbf{G}_1\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0)\end{pmatrix} + C_{11}\Big(\Big|\begin{pmatrix} \xi_{+}(t,1) \\ \xi_{-}(t,0)\end{pmatrix}\Big|^2 + |\xi_0|_0^2\Big)\\ &+ C_{12}\Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0)\end{pmatrix}\Big|\Big(\Big|\begin{pmatrix} \xi_{+}(t,1) \\ \xi_{-}(t,0)\end{pmatrix}\Big| + |\xi_0|_0\Big)\\ &+ \mathcal{O}\Big(|V|_0(|V|_0^2 + \Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0)\end{pmatrix}\Big|^2); |V|_0\Big),\end{aligned}$$

where $\mathbf{G}_1$ is defined the same as (3.8), C_{11} and C_{12} are positive constants. Then, we have

$$\begin{aligned}\mathbb{L}_1'(t) \leq &-\alpha_1\|V_x\|^2 - \beta_1\Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0)\end{pmatrix}\Big|^2 + \eta_1\Big(\Big|\begin{pmatrix} \xi_{+}(t,1) \\ \xi_{-}(t,0)\end{pmatrix}\Big|^2 + |\xi_0|_0^2\Big)\\ &+ \gamma_1\Big(\|V\|_{C^1}\|V_x\|^2 + |V|_0^3\Big)\end{aligned}$$

for some positive constants β_1, η_1, γ_1 and δ_1 provided that $\|V\|_{C^0} \leq \delta_1$. Similarly, for $\mathbb{L}_2(t)$ we may obtain

$$\begin{aligned}\mathbb{L}_2'(t) \leq &-\alpha_2\|V_{xx}\|^2 - \beta_2\Big|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0)\end{pmatrix}\Big|^2 + \eta_2\Big(\Big|\begin{pmatrix} \xi_{+}(t,1) \\ \xi_{-}(t,0)\end{pmatrix}\Big|^2\\ &+ \Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0)\end{pmatrix}\Big|^2 + |\xi_0|_1^2\Big) + \gamma_2(\|V\|_{C^1}(\|V_{xx}\|^2 + \|V_x\|^2 + |V|_0|V|_1^2)\end{aligned}$$

with positive constants α_2, β_2, η_2, γ_2 and δ_2 provided that $\|V\|_{C^1} \leq \delta_2$.

Also, by calculating the time derivative of $\mathbb{L}_3(t)$ and $\mathbb{L}_4(t)$, Lemma 4 and Lemma 5 still hold. However, thanks to the structure (5.1) and (1.9), it is easy to get that

$$|v_2|_0^2 = O(1)\Big(|\xi_0|_0^2 + \Big|\begin{pmatrix} \xi_+(t,1) \\ \xi_-(t,0) \end{pmatrix}\Big|^2\Big),\ |v_{2x}|_0^2 = O(1)\Big(|\xi_{0x}|_0^2 + \Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\Big|^2\Big),$$

which means

$$\mathbb{L}_3'(t) \le -\beta_3|\xi_0|_0^2 + \eta_3\Big|\begin{pmatrix} \xi_+(t,1) \\ \xi_-(t,0) \end{pmatrix}\Big|^2 + \varepsilon_3\Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\Big|^2 + \gamma_3|V|_0^2|V|_1,$$

$$\mathbb{L}_4'(t) \le -\beta_4|\xi_{0x}|_0^2 + \eta_4\Big|\begin{pmatrix} \xi_{+x}(t,1) \\ \xi_{-x}(t,0) \end{pmatrix}\Big|^2 + \varepsilon_4\Big|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}\Big|^2$$

$$+ \gamma_4\Big(|V|_1^3 + |V|_0\Big|\begin{pmatrix} \xi_{+xx}(t,1) \\ \xi_{-xx}(t,0) \end{pmatrix}\Big|^2\Big),$$

with positive constants β_3, β_4, γ_3, γ_4 and δ_3 provided that $\|V\|_{C^0} \le \delta_3$. Therefore, by choosing proper constants c_0, c_1, c_2, c_3 , c_4, ε_3 and ε_4, we may arrive at the inequality (3.15) as well. The rest of the proof is the same as that of Theorem 1. This ends the sketch of the proof of Theorem 2.

6 Application to a modified model for the transport of neurofilaments (NFs)

As a simple application of Theorem 1, we consider a modified dynamical system model for the transport of neurofilaments (NFs) in axons. NFs are neuron-specific cytoskeletal polymers that function as space-filling structures in axons. A model for the transport proceeds of NFs was proposed in [11] as follows: Proteins are stored in NFs as cargos and NFs can move along the axon when they are on track while they can switch on and off track. When off track, NFs pause for long periods until they get back on track. When on track, NFs alternate between short bouts of movement and short pauses. Thus, the NFs are divided into five subpopulations. Denote by $u_k = u_k(t,x)(k = 1, \dots, 5)$ the concentrations at time-space (t,x) of the five subpopulations of NFs along the axon. They evolve according to the following system of first-order partial differential equations (see [11])

$$U_t + AU_x = BU \tag{6.1}$$

with $U \triangleq (u_1, u_2, u_3, u_4, u_5)^T$, $A = diag(v_1, 0, 0, 0, v_5)$ and

$$B = \begin{pmatrix} -k_{21} & k_{12} & 0 & 0 & 0 \\ k_{21} & -k_{12}-k_{32} & k_{23} & 0 & 0 \\ 0 & k_{32} & -k_{23}-k_{43} & k_{34} & 0 \\ 0 & 0 & k_{43} & -k_{34}-k_{54} & k_{45} \\ 0 & 0 & 0 & k_{54} & -k_{45} \end{pmatrix}$$

for $x \in [0,l]$. Here l is the length of the axon, $k_{ij} > 0$ is the rate of change from the i^{th} to the j^{th} subpopulations, and the average retrograde and anterograde velocities v_1 and v_5 are $v_1 = -0.62\mu m/s$, $v_5 = 0.56\mu m/s$. Now, we allow a slight perturbation around (6.1). In other words, we consider

$$U_t + \mathbf{A}(U)U_x = \mathbf{B}(U) \tag{6.2}$$

with, for simplicity, $U = 0$ being a constant equilibrium, $\mathbf{A}(U) = diag(\mathbf{v}_1(U), 0, 0, 0, \mathbf{v}_5(U))$ while $\mathbf{A}(0) = A$ and $\mathbf{B}(U)$ has the form $\mathbf{B}_U(0) = B$. It means that the rate of change k_{ij} and velocities v_1 and v_5 will slightly change while the concentrations of NFs along the axon $u_k (k = 1, \ldots, 5)$ changed on a small scale. It is easy to see that (6.2) is a hyperbolic system with vanishing characteristic speed.

In order to apply Theorem 1, we will verify that the hyperbolic system (6.2) satisfies (1.2)–(1.6). First, since $\mathbf{A}(U)$ is a diagonal matrix, $\mathbf{L}(U)$ and $\mathbf{R}(U) = \mathbf{L}^{-1}(U)$ are both identity matrices, which means (1.6) is naturally satisfied.

Next, for simplicity, we only show that the conditions (1.2)–(1.5) are satisfied at the equilibrium $U = 0$. We choose an invertible matrix $\mathbf{P}(0)$ and a symmetric positive definite matrix $\mathbf{A}_0(0)$, such that the *partially dissipative structure* (1.2)–(1.4) are satisfied. Inspired by [27], we take

$$\mathbf{P}(0) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ k_{21} & -k_{12} & 0 & 0 & 0 \\ 0 & k_{32} & -k_{23} & 0 & 0 \\ 0 & 0 & k_{43} & -k_{34} & 0 \\ 0 & 0 & 0 & k_{54} & -k_{45} \end{pmatrix}$$

and

$$\mathbf{A}_0(0) = diag\left(1, \frac{k_{12}}{k_{21}}, \frac{k_{12}k_{23}}{k_{21}k_{32}}, \frac{k_{12}k_{23}k_{34}}{k_{21}k_{32}k_{43}}, \frac{k_{12}k_{23}k_{34}k_{45}}{k_{21}k_{32}k_{43}k_{54}}\right),$$

which is symmetric and positive-definite. It follows also that $\mathbf{A}_0(0)\mathbf{A}(0) = \mathbf{A}^T(0)\mathbf{A}_0(0)$. Direct computations give that

$$\mathbf{P}(0)\mathbf{B}_U(0)\mathbf{P}^{-1}(0) = diag(0, -e)$$

with

$$e = \begin{pmatrix} k_{21} + k_{12} & -k_{12} & 0 & 0 \\ -k_{32} & k_{32} + k_{23} & -k_{23} & 0 \\ 0 & -k_{43} & k_{43} + k_{34} & -k_{34} \\ 0 & 0 & -k_{54} & k_{54} + k_{45} \end{pmatrix}.$$

Since the eigenvalues of B are non-positive, it is easy to verify (1.4), i.e., the *partially dissipative structure* indeed holds. As in [11], we set

$$\alpha = \frac{k_{21}}{k_{12}}, \quad \beta = \frac{k_{32}}{k_{23}}, \quad \gamma^{-1} = \frac{k_{43}}{k_{34}}, \quad \delta^{-1} = \frac{k_{54}}{k_{45}}.$$

Then, the first column of $\mathbf{P}^{-1}(0)$ can be expressed as

$$\frac{1}{\gamma\delta + \alpha\gamma\delta + \alpha\beta\gamma\delta + \alpha\beta\delta + \alpha\beta}(\gamma\delta, \alpha\gamma\delta, \alpha\beta\gamma\delta, \alpha\beta\delta, \alpha\beta)^T.$$

Thus we derive the following expression for $a(0)$, as defined in (1.5)

$$a(0) = \frac{\gamma\delta v_1 + \alpha\beta v_5}{\gamma\delta + \alpha\gamma\delta + \alpha\beta\gamma\delta + \alpha\beta\delta + \alpha\beta} > 0$$

with $\alpha = \delta = \frac{33}{67}$ and $\beta = \frac{69}{31}\gamma$ from the experimental measurements. Therefore, the additional coupling condition (1.5) is fulfilled.

For the hyperbolic system (6.1), we propose the feedback control laws in the following form

$$\begin{pmatrix} u_5(t,0) \\ u_1(t,1) \end{pmatrix} = \mathbf{K} \begin{pmatrix} u_5(t,1) \\ u_1(t,0) \end{pmatrix} \tag{6.3}$$

with $\mathbf{K}$ being a 2×2 constant matrix. Take a suitable smooth positive function $\lambda = \lambda(x)$. Noting that $\mathbf{\Lambda}_-(0) = v_1$ and $\mathbf{\Lambda}_+(0) = v_5$, we have $\mathbf{X}_1(0) = 1$ and $\mathbf{X}_3(0) = \frac{k_{12}k_{23}k_{34}k_{45}}{k_{21}k_{32}k_{43}k_{54}} = \frac{\delta\gamma}{\alpha\beta} = \frac{31}{69}$. Given these values, the requirements for ensuring the positive definiteness of matrices $\mathbf{G}_0$, $\mathbf{G}_1$ and $\mathbf{G}_2$ are simplified to

$$\begin{pmatrix} 31\lambda(1)v_5^i & 0 \\ 0 & -69\lambda(0)v_1^i \end{pmatrix} \geq \mathbf{K}^T \begin{pmatrix} 31\lambda(0)v_5^i & 0 \\ 0 & -69\lambda(1)v_1^i \end{pmatrix} \mathbf{K},$$

with $i \in \{1, -1, -3\}$. A simple example of a feedback law that satisfies the aforementioned inequalities is:

$$u_1(t,1) = \sqrt{\frac{\lambda(0)}{\lambda(1)}}u_1(t,0), \quad u_5(t,0) = \sqrt{\frac{\lambda(1)}{\lambda(0)}}u_5(t,1).$$

Finally we conclude by Theorem 1 that

Theorem 3. *There exists* $\mathbf{K}$ *such that the modified system for the transport of NFs in axon* (6.1), (6.3) *is locally exponentially stable for the* H^2*-norm.*

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