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Terminal value problem for the system of fractional differential equations with additional restrictions

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Abstract. This paper deals with the study of terminal value problem for the system of fractional differential equations with Caputo derivative. Additional conditions are imposed on the solutions of this problem in the form of a linear vector functional. Using the theory of pseudo-inverse matrices, we obtain the necessary and sufficient conditions for the solvability and the general form of the solution of this boundary-value problem. In the one-dimensional case, the obtained results are generalized to the case of a multi-point boundary-value problem. The issue of obtaining similar results for the terminal value problem for the system of fractional differential equations with tempered and Ψ–tempered fractional derivatives of Caputo type is considered.

Keywords: terminal value problem; fractional differential equation; Caputo derivative; pseudoinverse Moore–Penrose matrix; Fredholm integral equation.

AMS Subject Classification: 26A33; 34A08; 34B05; 45B05.

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1 Introduction

The idea of fractional integro-differential calculus emerged almost at the same time as of the theory of conventional integral and differential equations. However, its particularly rapid development began quite recently — in the second half of the last century. This progress was driven not only by the emergence and solution of new interesting questions of a theoretical nature, but also by an increase in practical applications. There exist different definitions of fractional derivatives, each used according to on the specific needs of a particular study. For example, one of the types of fractional derivatives most often used to model processes with memory [13, p. 87], [30, p. 90] is the Caputo derivative [9]. The peculiarity of this derivative lies in the preservation of some properties of ordinary derivatives: the Caputo derivative of the constant function is zero and the initial value problems depend on integer-order derivatives only.

Therefore, as in the theory of ordinary differential equations (see [5,17]), the most studied among different types of boundary-value problems for differential equations with the Caputo fractional derivative is the initial value problem (see [13, 24, 30]). However, other problems, in which other conditions are imposed on the solutions of differential equations, are also of important theoretical and practical significance. Such conditions include, for example, the constraints imposed on the value of the sought solution not at the initial point a of the studied interval $[a, b]$ (the point of definition of the derivative), but at the end point b of this interval — terminal value problem. Such boundary-value problems are used to describe, in particular, the behavior of models of viscoelastic materials [3] and models of financial market dynamics [18]. Various aspects of the theory of the terminal value problems have been considered by numerous researchers, in particular, [1,4,10,12,13,14,15,16,20,21,22,23,34,35,37,38,39,40].

In this paper, we investigate the terminal value problem for the system of differential equations with the Caputo fractional derivative, the solutions of which satisfy additional conditions having the form of a bounded linear vector functional. In [8] the conditions for the solvability and the structure of solutions to such a problem are established, when the condition on the value of the solution at the end point of the interval on which this solution is sought is a constituent part of the bounded linear vector functional. That is, formally, we can apply the results obtained in [8] to the problem considered in this paper. However, singling out the condition at the end point of the interval makes it possible to use a different approach to its study, which has certain advantages. The research approach applied in [8] is significantly based on the concept of the fundamental matrix of a homogeneous system, the finding of which in the case of variable coefficients is a non-trivial problem. An important property of the approach considered in this paper is the reduction of the terminal value problem for the system of differential equations with the Caputo fractional derivative to an equivalent integral equation of the Fredholm type. This approach makes it possible to avoid the use of the notion of the fundamental matrix and, based on the results obtained in [6], to establish the necessary and sufficient conditions for the solvability and the general form of the solution of the given problem.

In the space $C[a, b]$, $-\infty < a < b < +\infty$, we consider a linear terminal value problem for the system of fractional differential equations

$$ {}^{C}\mathrm{D}^{\alpha}_{a+}\boldsymbol{x}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \boldsymbol{f}(t), \tag{1.1} $$

$$ \boldsymbol{x}(b) = \boldsymbol{x}^{*}, \tag{1.2} $$

whose solutions satisfy the conditions

$$ \boldsymbol{l}\boldsymbol{x}(\cdot) = \boldsymbol{q}, \tag{1.3} $$

where $0 < \alpha < 1$, ${}^{C}\mathrm{D}^{\alpha}_{a+}$ is the left Caputo fractional derivative of order α, $\mathbf{A}(t)$ is a $(n \times n)$-matrix and $\boldsymbol{f}(t)$ is a n-vector, whose components are real functions continuous on $[a, b]$, $\boldsymbol{l} = \operatorname{col}\left(l_1, \quad l_2, \quad \dots, \quad l_p\right) : C[a, b] \to \mathbb{R}^p$ is bounded linear vector functional, $l_\nu : C[a, b] \to \mathbb{R}$, $\nu = \overline{1, p}$, $\boldsymbol{x}^{*} = \operatorname{col}\left(x_1^{*}, \quad x_2^{*}, \quad \dots, \quad x_p^{*}\right) \in \mathbb{R}^p$, $\boldsymbol{q} = \operatorname{col}\left(q_1, \quad q_2, \quad \dots, \quad q_p\right) \in \mathbb{R}^p$. Here and further in this paper, the symbol "col" is the notation of a column vector.

Note that sometimes the problem (1.1)–(1.2) is written in a more general form, replacing the condition (1.2) with the condition $\boldsymbol{x}(t^*) = \boldsymbol{x}^*$, $t^* \in (a, b]$ and is called either the terminal value problem [23] or the intermediate value problem [39]. Since, as a rule, a solution is sought on the interval $[a, t^*]$, that is, the condition is imposed on the unknown solution at the end point of the interval of interest, then, without reducing generality, we can consider $t^* = b$ [13, p. 108]. If you need to find a solution to the problem (1.1)–(1.2) on some finite interval $[a, b + b^*]$, $b^* > 0$, then its search takes place in two stages. In the first stage, the solution to the problem (1.1)–(1.2) on the interval $[a, b]$ is sought. In the second stage, the value of the obtained solution is calculated at the point a, i.e., $\boldsymbol{x}(a) = \boldsymbol{x}_a$, and the solution on the interval $[a, b + b^*]$ is found by solving the problem (1.1)–(1.2) with the condition (1.2) replaced by the initial condition $\boldsymbol{x}(a) = \boldsymbol{x}_a$ (see [16]).

2 Preliminaries

We introduce some definitions and preliminary facts of the fractional calculus theory, which will be used in our study. For more details, we refer to the books [13, 24, 30].

Definition 1. [13, p. 9] The function $\Gamma : (0, \infty) \to \mathbb{R}$, defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t$, is called Euler's Gamma function (or Euler's integral of the second kind).

Definition 2. [13, p. 67] Let $\alpha, \beta > 0$. The function $E_{\alpha,\beta}$ defined by

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha + \beta)}$$

whenever the series converges is called the two-parameter Mittag–Leffler function with parameters α and β.

Definition 3. [13, p. 13], [24, p. 69], [30, p. 65] Let $\alpha \in \mathbb{R}_+$. The operator $\mathrm{I}^\alpha_{a+} x(t)$, defined on $L^1[a, b]$ by

$$\mathrm{I}^\alpha_{a+} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(s)\mathrm{d}s}{(t-s)^{1-\alpha}}$$

for $a \le t \le b$, is called the left Riemann–Liouville fractional integral operator of order α.

By $AC^m[a, b]$ we denote the set of functions with an absolutely continuous $(m-1)$st derivative.

Definition 4. [13, p. 27], [24, p. 70], [30, p. 68] Let $\mathrm{I}^{m-\alpha}_{a+} x(t) \in AC^m[a, b]$, $m = [\alpha] + 1$, $t > a$. The left Riemann–Liouville fractional derivative $\mathrm{D}^\alpha_{a+} x(t)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$\mathrm{D}^\alpha_{a+} x(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \mathrm{I}^{m-\alpha}_{a+} x(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_a^t \frac{x(s)\mathrm{d}s}{(t-s)^{1-m+\alpha}}.$$

By $T_{m-1}[x;a]$ we denote the Taylor polynomial of degree $m-1$ for the function x, centered at a.

Definition 5. [13, p. 50], [33, 36] Let $\alpha \geq 0$, $\mathrm{I}_{a+}^{m-\alpha}x(t) \in AC^m[a,b]$ and $T_{m-1}[x;a]$ exists, where $m = [\alpha]+1$. The left Caputo fractional derivative ${}^C\mathrm{D}_{a+}^{\alpha}x(t)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$ {}^C\mathrm{D}_{a+}^{\alpha}x(t) = \mathrm{D}_{a+}^{\alpha}[x - T_{m-1}[x;a]]. \tag{2.1}$$

Lemma 1. *[13, p. 108], [14, 39] Let $0<\alpha<1$, vector function $\boldsymbol{f}(t,\boldsymbol{x}) : [a,b]\times Y \to \mathbb{R}^n$, $Y \subset \mathbb{R}^n$ is continuous with respect to t on $[a,b]$. The terminal value problem*

$$ {}^C\mathrm{D}_{a+}^{\alpha}\boldsymbol{x}(t) = \boldsymbol{f}(t,\boldsymbol{x}(t)), \quad \boldsymbol{x}(b) = \boldsymbol{x}^* $$

is equivalent to a system of weakly singular integral equations

$$ \boldsymbol{x}(t) = \boldsymbol{x}^* + \frac{1}{\Gamma(\alpha)}\int_a^b G(t,s)\boldsymbol{f}(s,\boldsymbol{x}(s))\mathrm{d}s, $$

where

$$ G(t,s) = \begin{cases} -(b-s)^{\alpha-1}, & s>t, \\ (t-s)^{\alpha-1} - (b-s)^{\alpha-1}, & s \leq t. \end{cases} \tag{2.2}$$

Lemma 2. *[13, p. 128] Let $0<\alpha<1$, $h_i \in \mathbb{R}$, $i=\overline{1,3}$, $h_1+h_2 \neq 0$, the function $f(t,x) : [a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous with respect to t on $[a,b]$. The function $x \in C[a,b]$ is a solution of a two-point boundary-value problem*

$$ {}^C\mathrm{D}_{a+}^{\alpha}x(t) = f(t,x(t)), \quad h_1x(a)+h_2x(b) = h_3 $$

if and only if it is a solution of an integral equation

$$ x(t) = \frac{h_3}{h_1+h_2} + \frac{1}{\Gamma(\alpha)}\int_a^b G(t,s)f(s,x(s))\mathrm{d}s, $$

where

$$ G(t,s) = \begin{cases} -\frac{h_2}{h_1+h_2}(b-s)^{\alpha-1}, & s>t, \\ (t-s)^{\alpha-1} - \frac{h_2}{h_1+h_2}(b-s)^{\alpha-1}, & s \leq t. \end{cases} $$

Remark 1. For absolutely continuous functions with an integrable derivative, the Definition 5 coincides with the usual definition of the Caputo derivative

$$ {}^C\mathbb{D}_{a+}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{x'(s)\mathrm{d}s}{(t-s)^{\alpha}}. \tag{2.3}$$

The disadvantage of this definition of ${}^C\mathbb{D}_{a+}^{\alpha}x(t)$ is that the equivalence between a Caputo fractional derivative equation and an integral equation is only valid for the definition ${}^C\mathrm{D}_{a+}^{\alpha}x(t)$ because of the fact that $\mathrm{I}_{a+}^{\alpha}x(t)$ does not map all of $C[a,b]$ into $AC[a,b]$ [10, 33, 36].

3 Criterion of solvability of the terminal value problem with additional restrictions

We will establish the necessary and sufficient conditions for the solvability and give an algorithm for building a family of solutions to the problem under study. To do this, we will show that the boundary-value problem (1.1)–(1.3) is equivalent to a boundary-value problem for a system of Fredholm integral equations of the second kind, and we will use the approach presented in the paper [25] to study it.

By Lemma 1, the terminal value problem (1.1)–(1.2) is equivalent to a system of linear weakly singular equations

$$\boldsymbol{x}(t) = \boldsymbol{x}^* + \frac{1}{\Gamma(\alpha)} \int_a^b G(t,s)(\mathbf{A}(s)\boldsymbol{x}(s) + \boldsymbol{f}(s))\mathrm{d}s, \tag{3.1}$$

where the kernel of $G(t,s)$ has the form (2.2). We rewrite the system (3.1) in the form as follows:

$$\boldsymbol{x}(t) = \boldsymbol{g}(t) + \int_a^b \mathbf{K}(t,s)\boldsymbol{x}(s)\mathrm{d}s, \tag{3.2}$$

where

$$\boldsymbol{g}(t) = \boldsymbol{x}^* + \frac{1}{\Gamma(\alpha)} \int_a^b G(t,s)\boldsymbol{f}(s)\mathrm{d}s, \quad \mathbf{K}(t,s) = \frac{1}{\Gamma(\alpha)} G(t,s)\mathbf{A}(s). \tag{3.3}$$

Therefore, we can shift our focus the study of the boundary-value problem (1.1)–(1.3) to the study of the boundary-value problem for the system of weakly singular equations (3.2), (1.3). Using the approach described in the works [6, 7], we will reduce it to an equivalent problem for a system of integral equations with a square summable kernel $\mathbf{K}_m(t,s)$, $m \in \mathbb{N}$, which is determined using the recurrence relations

$$\mathbf{K}_{m+1}(t,s) = \int_a^b \mathbf{K}(t,\xi)\mathbf{K}_m(\xi,s)\mathrm{d}\xi, \quad \mathbf{K}_1(t,s) = \mathbf{K}(t,s).$$

Indeed, multiplying both sides of the Equation (3.2) by $\mathbf{K}(t,s)$ from the left and integrating the left and right sides of the equality obtained as a result over the interval $[a,t]$, we get

$$\int_a^b \mathbf{K}(t,s)\boldsymbol{x}(s)\mathrm{d}s = \int_a^b \mathbf{K}(t,s)\boldsymbol{g}(s)\mathrm{d}s + \int_a^b \mathbf{K}_2(t,s)\boldsymbol{x}(s)\mathrm{d}s.$$

Continuing this process, we obtain

$$\int_a^b \mathbf{K}_{m-1}(t,s)\boldsymbol{x}(s)\mathrm{d}s = \int_a^b \mathbf{K}_{m-1}(t,s)\boldsymbol{g}(s)\mathrm{d}s + \int_a^b \mathbf{K}_m(t,s)\boldsymbol{x}(s)\mathrm{d}s.$$

Adding all the equations obtained as a result to the Equation (3.2), we find that $\boldsymbol{x}(t)$ is a solution of the system

$$\boldsymbol{x}(t) = \boldsymbol{g}_m(t) + \int_a^b \mathbf{K}_m(t,s)\boldsymbol{x}(s)\mathrm{d}s, \tag{3.4}$$

$$\boldsymbol{g}_m(t) = \boldsymbol{g}(t) + \sum_{l=1}^{m-1} \int_a^b \mathbf{K}_l(t,s)\boldsymbol{g}(s)\mathrm{d}s.$$

The iterated kernels $\mathbf{K}_m(t,s)$ have the same structure as the weakly singular kernel $\mathbf{K}(t,s)$ (3.3), but the number $1-\alpha$ is replaced by the number $1-m\alpha$, which is negative for sufficiently large m. Therefore, (see [28, p. 61]), for all m by which the condition

$$m > 1/(2\alpha), \tag{3.5}$$

is satisfied, the kernels $\mathbf{K}_m(t,s)$ are square summable.

Therefore, according to condition (3.5), after finitely many steps, we arrive at the system (3.4) with square summable kernel $\mathbf{K}_m(t,s)$. Generally speaking, the systems (3.2) and (3.4) are not equivalent. However, it is possible to choose a number m in such a way that condition (3.5) is satisfied, making the systems (3.2) and (3.4) equivalent (see [28, p. 63]). In the future, we assume that the number m is chosen in this way. Thus, we pass from the investigation of the boundary-value problem for the system of integral equation with unbounded kernel (3.2), (1.3) to the investigation of the boundary-value problem for the system of Fredholm integral equation (3.4), (1.3).

In [25], the criterion of solvability of the boundary-value problem (3.4), (1.3) for the case of a single equation ($n=1$) was established. Using the approach described in [25], we present a similar result for a system of equations ($n>1$). By considering the problem (3.4), (1.3) in the Hilbert space $L_2[a,b]$, we will reduce it to an operator equation in the space ℓ_2. Let $\{\varphi_i(t)\}_{i=1}^{\infty}$ be a complete orthonormal system of functions in $L_2[a,b]$. We introduce a n-vectors $\boldsymbol{x}_i$, $\boldsymbol{g}_i$, $i=\overline{1,\infty}$

$$\boldsymbol{x}_i = \int_a^b \boldsymbol{x}(t)\varphi_i(t)\mathrm{d}t, \quad \boldsymbol{g}_i = \int_a^b \boldsymbol{g}_m(t)\varphi_i(t)\mathrm{d}t$$

and a $(n\times n)$-matrices $\mathbf{A}_{ij}$, $i,j=\overline{1,\infty}$,

$$\mathbf{A}_{ij} = \int_a^b \int_a^b \mathbf{K}_m(t,s)\varphi_i(t)\varphi_j(s)\mathrm{d}s\mathrm{d}t.$$

The problem (3.4), (1.3) can be rewritten in the form of a countable system of linear algebraic equations, which is equivalent to an operator equation in the space ℓ_2:

$$\mathbf{U}\boldsymbol{z} = \begin{bmatrix} \boldsymbol{\Lambda} \\ \mathbf{W} \end{bmatrix} \boldsymbol{z} = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{q} \end{bmatrix} = \boldsymbol{h}. \tag{3.6}$$

Here, vectors $\boldsymbol{z}$, $\boldsymbol{g}$ and block matrices $\boldsymbol{\Lambda}$, $\mathbf{W}$ have the form:

$$\boldsymbol{z} = \mathrm{col}\left(\boldsymbol{x}_1, \quad \boldsymbol{x}_2, \quad \ldots, \quad \boldsymbol{x}_i, \quad \ldots\right), \quad \boldsymbol{g} = \mathrm{col}\left(\boldsymbol{g}_1, \quad \boldsymbol{g}_2, \quad \ldots, \quad \boldsymbol{g}_i, \quad \ldots\right),$$

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_{11} & \mathbf{\Lambda}_{12} & \dots & \mathbf{\Lambda}_{1i} & \dots \\ \mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{22} & \dots & \mathbf{\Lambda}_{2i} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{\Lambda}_{i1} & \mathbf{\Lambda}_{i2} & \dots & \mathbf{\Lambda}_{ii} & \dots \\ \vdots & \vdots & \dots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{\Lambda}_{ij} = \begin{cases} \mathbf{I}_n - \mathbf{A}_{ij}, & i = j, \\ -\mathbf{A}_{ij}, & i \neq j, \end{cases}$$

$$\mathbf{W} = \boldsymbol{l\Phi}(\cdot), \quad \boldsymbol{\Phi}(t) = \big(\varphi_1(t), \quad \varphi_2(t), \quad \dots, \quad \varphi_i(t), \quad \dots\big),$$

where, $\mathbf{I}_n$ is the identity matrix of dimensions n. The operator $\mathbf{\Lambda} : \ell_2 \to \ell_2$ appearing on the left-hand side of the operator equation (3.6) has the form $\mathbf{\Lambda} = \mathbf{I} - \mathbf{A}$, where $\mathbf{I} : \ell_2 \to \ell_2$ is the identity operator and $\mathbf{A} : \ell_2 \to \ell_2$ is a compact operator. Thus, $\mathbf{\Lambda} : \ell_2 \to \ell_2$ is a Fredholm operator of index zero ($\dim\ker \mathbf{\Lambda} = \dim\ker \mathbf{\Lambda}^* < \infty$) and $\mathbf{U} : \ell_2 \to \ell_2 \times \mathbb{R}^p$ is a Fredholm operator of nonzero index ($\dim\ker \mathbf{U} < \infty$, $\dim\ker \mathbf{U}^* < \infty$).

Thus, the following theorem is true for Equation (3.6) (see [5]):

Theorem 1. *The homogeneous equation* (3.6) ($\boldsymbol{h} = \mathbf{0}$) *possesses a* d_2*-parameter family of solutions* $\boldsymbol{z} \in \ell_2$

$$\boldsymbol{z} = \mathbf{P}_{\mathbf{\Lambda}_r}\mathbf{P}_{\mathbf{Q}_{d_2}}\boldsymbol{c}_{d_2} \quad \forall \boldsymbol{c}_{d_2} \in \mathbb{R}^{d_2}, \quad d_2 = r - \operatorname{rank} \mathbf{Q}.$$

The inhomogeneous equation (3.6) *is solvable if and only if the following* $r + d_1$ *linearly independent conditions are satisfied:*

$$\mathbf{P}_{\mathbf{\Lambda}_r^*}\boldsymbol{g} = \mathbf{0}, \quad \mathbf{P}_{\mathbf{Q}_{d_1}^*}(\boldsymbol{q} - \mathbf{W}\mathbf{\Lambda}^+\boldsymbol{g}) = \mathbf{0}, \quad d_1 = p - \operatorname{rank} \mathbf{Q} \tag{3.7}$$

and the equation possesses a d_2*-parameter family of solutions* $\boldsymbol{z} \in \ell_2$ *of the form*

$$\boldsymbol{z} = \mathbf{P}_{\mathbf{\Lambda}_r}\mathbf{P}_{\mathbf{Q}_{d_2}}\boldsymbol{c}_{d_2} + \mathbf{P}_{\mathbf{\Lambda}_r}\mathbf{Q}^+(\boldsymbol{q} - \mathbf{W}\mathbf{\Lambda}^+\boldsymbol{g}) + \mathbf{\Lambda}^+\boldsymbol{g}, \quad \forall \boldsymbol{c}_{d_2} \in \mathbb{R}^{d_2}. \tag{3.8}$$

Here, $\mathbf{Q} = \mathbf{W}\mathbf{P}_{\mathbf{\Lambda}_r}$ is a block $(p \times r)$-matrix, $\mathbf{P}_{\mathbf{\Lambda}_r}$ ($\mathbf{P}_{\mathbf{\Lambda}_r^*}$) is a matrix formed by a complete system of r linearly independent columns (rows) of the matrix projector $\mathbf{P}_{\mathbf{\Lambda}}$ ($\mathbf{P}_{\mathbf{\Lambda}^*}$), where $\mathbf{P}_{\mathbf{\Lambda}}$ ($\mathbf{P}_{\mathbf{\Lambda}^*}$) is the projector onto the kernel (cokernel) of the matrix $\mathbf{\Lambda}$, and $\mathbf{P}_{\mathbf{Q}_{d_2}}$ ($\mathbf{P}_{\mathbf{Q}_{d_1}^*}$) is a matrix formed by a complete system of d_2 (d_1) linearly independent columns (rows) of the matrix projector $\mathbf{P}_{\mathbf{Q}}$ ($\mathbf{P}_{\mathbf{Q}^*}$), where $\mathbf{P}_{\mathbf{Q}}$ ($\mathbf{P}_{\mathbf{Q}^*}$) is the projector onto the kernel (cokernel) of the matrix $\mathbf{Q}$ and $\mathbf{\Lambda}^+$ ($\mathbf{Q}^+$) is the pseudoinverse Moore–Penrose matrix for the matrix $\mathbf{\Lambda}$ ($\mathbf{Q}$).

If the conditions (3.7) are satisfied, then, according to the Riesz–Fischer theorem, one can find an element $\boldsymbol{x} \in L_2[a, b]$ such that the quantities $\boldsymbol{x}_i, i = \overline{1, \infty}$, determined from Equation (3.6) are the Fourier coefficients of this element. Thus, the following representation is true:

$$\boldsymbol{x}(t) = \sum_{i=1}^{\infty} \boldsymbol{x}_i\varphi_i(t) = \boldsymbol{\Phi}(t)\boldsymbol{z}. \tag{3.9}$$

The element $\boldsymbol{x}(t)$ given by relations (3.9) is the desired solution of the boundary-value problem (3.4), (1.3), and therefore of the original problem (1.1)–(1.3).

According to [5], the following result is true:

Theorem 2. *The homogeneous boundary-value problem* (1.1)–(1.3) ($\boldsymbol{f}(t) = \boldsymbol{0}, \boldsymbol{q} = \boldsymbol{0}$) *possesses a solution* $\boldsymbol{x} \in C[a, b]$

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\mathbf{P}_{\boldsymbol{\Lambda}_r}\mathbf{P}_{\mathbf{Q}_{d_2}}\boldsymbol{c}_{d_2} \quad \forall \boldsymbol{c}_{d_2} \in \mathbb{R}^{d_2}.$$

The inhomogeneous boundary-value problem (1.1)–(1.3) *is solvable if and only if the following* $r + d_1$ *linearly independent conditions* (3.7) *are satisfied and it possesses a* d_2*-parameter family of solutions* $\boldsymbol{x} \in C[a, b]$ (3.9), *where the vector* $\boldsymbol{z}$ *has the form* (3.8).

Remark 2. In the scalar case ($n = 1$), the terminal value problem (1.1)–(1.2) possesses a unique solution [12,16]. Therefore, the boundary-value problem (1.1)–(1.3) is overdetermined and can have no more than one solution.

Example 1. We illustrate the theoretical results presented above by analyzing a terminal value problem for a system of two fractional differential equations

$$^C\mathrm{D}^{1/2}_{0+}\boldsymbol{x}(t) = \mathbf{A}\boldsymbol{x}(t) + \boldsymbol{f}(t), \quad t \in [0, 1], \tag{3.10}$$

$$\boldsymbol{x}(1) = \boldsymbol{x}^* \tag{3.11}$$

with additional restriction

$$\boldsymbol{x}(0) = \boldsymbol{q}, \tag{3.12}$$

where

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\boldsymbol{f}(t) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 6\sqrt{t} - 6\sqrt{\pi}t + 4\sqrt{\pi} \\ 12\sqrt{t} - 3\sqrt{\pi}t + 2\sqrt{\pi} \end{pmatrix}, \quad \boldsymbol{x}^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{q} = -\begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

By Lemma 1, the terminal value problem (3.10)–(3.11) is equivalent to a system of Fredholm integral equations of the second kind

$$\boldsymbol{x}(t) = \boldsymbol{g}(t) + \int_0^1 \mathbf{K}(t, s)\boldsymbol{x}(s)\mathrm{d}s, \tag{3.13}$$

where

$$\boldsymbol{g}(t) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 3\sqrt{\pi}t - 8t\sqrt{t} + 8\sqrt{t} - 2\sqrt{\pi} \\ 6\sqrt{\pi}t - 4t\sqrt{t} + 4\sqrt{t} - 4\sqrt{\pi} \end{pmatrix},$$

$$\mathbf{K}(t, s) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{cases} -(1-s)^{-1/2}, & s > t, \\ (t-s)^{-1/2} - (1-s)^{-1/2}, & s \le t. \end{cases} \tag{3.14}$$

We have $\alpha = 1/2$ and all iterated kernels starting from $\mathbf{K}_2(t, s)$ are square summable and we can pass from the a system of integral equations (3.13) with unbounded kernel (3.14) to the equivalent (the number 1 is not an eigenvalue of the operator $\mathbf{K}$ (see [28, p. 63])) a system of integral equations with square summable kernel $\mathbf{K}_2(t, s)$

$$\boldsymbol{x}(t) = \boldsymbol{g}_2(t) + \int_0^1 \mathbf{K}_2(t, s)\boldsymbol{x}(s)\mathrm{d}s, \tag{3.15}$$

where

$$\boldsymbol{g}_2(t) = -\frac{1}{2}\begin{pmatrix} 3t^2 - 10t + 5 \\ 6t^2 - 20t + 10 \end{pmatrix},$$

$$\mathbf{K}_2(t,s) = \frac{1}{\pi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{cases} 2\left(1-\sqrt{t}\right)(1-s)^{-1/2} - \pi, & s > t, \\ 2\left(1-\sqrt{t}\right)(1-s)^{-1/2}, & s \le t. \end{cases}$$

Equation (3.15) can be reduced to a system of linear algebraic equations. We introduce the functions $\varphi_i(t) = \sqrt{2i-1}P_{i-1}(t)$, where $P_i(t)$ are the Legendre polynomials. The system $\{\varphi_i(t)\}_{i=1}^{\infty}$ is a complete orthonormal system of functions in $L_2[0,1]$. Equation (3.15) is an equation with degenerate kernel and a polynomial right-hand side and its solution is a polynomial of degree at most 2. This means that, in the construction of the operator equation (3.6), we can restrict ourselves to the functions $\{\varphi_i(t)\}_{i=1}^{3}$

$$\varphi_1(t) = 1, \quad \varphi_2(t) = \sqrt{3}\,(2t-1), \quad \varphi_3(t) = \sqrt{5}\left(6t^2 - 6t + 1\right). \tag{3.16}$$

By using the functions (3.16), we can reduce the Equation (3.15) to the operator equation (3.6) in the form:

$$\begin{bmatrix} \boldsymbol{\Lambda} \\ \mathbf{W} \end{bmatrix} \boldsymbol{z} = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{q} \end{bmatrix},$$

where the vectors $\boldsymbol{z}$, $\boldsymbol{g}$ and block matrices $\boldsymbol{\Lambda}$, $\mathbf{W}$ have the form:

$$\boldsymbol{\Lambda} = \frac{1}{3150\pi}\begin{pmatrix} 525(9\pi-8)\mathbf{I}_2 & -175\sqrt{3}(8-3\pi)\mathbf{I}_2 & -840\sqrt{5}\mathbf{I}_2 \\ 105\sqrt{3}(16-5\pi)\mathbf{I}_2 & 210(8+15\pi)\mathbf{I}_2 & 21\sqrt{15}(16+5\pi)\mathbf{I}_2 \\ -240\sqrt{5}\mathbf{I}_2 & -5\sqrt{15}(16+21\pi)\mathbf{I}_2 & 30(105\pi-8)\mathbf{I}_2 \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{I}_2 & -\sqrt{3}\mathbf{I}_2 & \sqrt{5}\mathbf{I}_2 \end{pmatrix}, \quad \boldsymbol{z} = \operatorname{col}\begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{pmatrix}, \quad \boldsymbol{x}_i = \int_0^1 \boldsymbol{x}(t)\varphi_i(t)\mathrm{d}t,$$

$$\boldsymbol{g} = \frac{1}{60}\operatorname{col}\left(-30\begin{pmatrix}1\\2\end{pmatrix} \quad 35\sqrt{3}\begin{pmatrix}1\\2\end{pmatrix} \quad -3\sqrt{5}\begin{pmatrix}1\\2\end{pmatrix}\right).$$

In this case, the block matrices $\boldsymbol{\Lambda}^+$, $\mathbf{P}_{\boldsymbol{\Lambda}}$, $\mathbf{P}_{\boldsymbol{\Lambda}^*}$, $\mathbf{Q}^+$, $\mathbf{P}_{\mathbf{Q}}$ and $\mathbf{P}_{\mathbf{Q}^*}$ have the form

$$\boldsymbol{\Lambda}^+ = \boldsymbol{\Lambda}^{-1} = \frac{1}{20265\pi - 13792}$$
$$\times\begin{pmatrix} 2(6405\pi+3296)\mathbf{I}_2 & -20\sqrt{3}(105\pi-316)\mathbf{I}_2 & 14\sqrt{5}(15\pi+248)\mathbf{I}_2 \\ 20\sqrt{3}(105\pi-352)\mathbf{I}_2 & 60(315\pi-304)\mathbf{I}_2 & -14\sqrt{15}(45\pi+64)\mathbf{I}_2 \\ 14\sqrt{5}(15\pi+32)\mathbf{I}_2 & 30\sqrt{15}(21\pi-8)\mathbf{I}_2 & 70(285\pi-184)\mathbf{I}_2 \end{pmatrix},$$

$$\mathbf{P}_{\boldsymbol{\Lambda}} = \mathbf{P}_{\boldsymbol{\Lambda}^*} = \widehat{\mathbf{O}}_3, \quad \mathbf{Q} = \begin{pmatrix} \mathbf{O}_2 & \mathbf{O}_2 & \mathbf{O}_2 \end{pmatrix}, \quad \mathbf{Q}^+ = \operatorname{col}\begin{pmatrix} \mathbf{O}_2 & \mathbf{O}_2 & \mathbf{O}_2 \end{pmatrix},$$
$$\mathbf{P}_{\mathbf{Q}} = \widehat{\mathbf{I}}_3, \quad \mathbf{P}_{\mathbf{Q}^*} = \mathbf{I}_2,$$

where $\widehat{\mathbf{O}}_3$ is the zero block matrix of dimension 3, $\widehat{\mathbf{I}}_3$ is identity block matrix of dimensions 3, $\mathbf{I}_2$ is the identity matrix of dimension 2 and $\mathbf{O}_2$ is the zero matrix of dimension 2. Hence, the solvability conditions (3.7) are satisfied and, by Theorem 2, the boundary-value problem (3.10)–(3.12) possesses a unique solution $\boldsymbol{x}(t)$ of the form

$$\begin{aligned}\boldsymbol{x}(t) &= \boldsymbol{\Phi}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{g} \\ &= \Big(-1\cdot\frac{1}{2} + \sqrt{3}\,(2t-1)\,\frac{\sqrt{3}}{2} + \sqrt{5}\left(6t^2-6t+1\right)\cdot 0\Big)\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3t-2\\6t-4\end{pmatrix}.\end{aligned}$$

Note that boundary-value problem (3.10)–(3.12) may not have a solution if condition (3.12) is replaced by another condition. In particular, if the following additional restriction

$$\int_0^1 \boldsymbol{x}(t)\mathrm{d}t = \begin{pmatrix}2\\1\end{pmatrix} \tag{3.17}$$

is imposed on the solution of the terminal value problem (3.10)–(3.11), then it has no solution. Indeed, in this case, the matrix $\mathbf{W}$ and the vector $\boldsymbol{q}$ have the form

$$\mathbf{W} = \begin{pmatrix}\mathbf{I}_2 & \mathbf{O}_2 & \mathbf{O}_2\end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix}2\\1\end{pmatrix}$$

and the solvability conditions (3.7) are not satisfied and, therefore, the boundary-value problem (3.10), (3.11), (3.17) is unsolvable.

Example 2. In the work [12], the authors showed that in the general case ($n \geq 2$), unlike the scalar case ($n = 1$), the terminal value problem (1.1)–(1.2) can have a family of solutions. The imposition of additional restrictions (1.3) on the solutions of the problem (1.1)–(1.2) allows us to single out among them those with more specific properties. We consider a terminal value problem for the system of two fractional differential equations [12]

$$ {}^{C}\mathrm{D}_{0+}^{1/2}\boldsymbol{x}(t) = \mathbf{A}\boldsymbol{x}(t), \quad t\in[0,b], \tag{3.18}$$

$$\boldsymbol{x}(b) = \mathbf{0}, \tag{3.19}$$

where

$$\boldsymbol{x}(t) = \begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$$

and the finite positive number b and the angle φ are such that

$$E_{1/2}(z^*) = E_{1/2,1}(z^*) = 0, \quad z^* = \lambda\sqrt{b}\in\mathbb{C},$$

$$\varphi := \arg(z^*) \in (-\pi,\pi], \quad \lambda := \cos\varphi + \mathrm{i}\sin\varphi.$$

We will impose an additional restriction on the solutions of the terminal value problem (3.18)–(3.19)

$$x_1(0) = 2, \tag{3.20}$$

that is, in our case,

$$\boldsymbol{l}\boldsymbol{x}(\cdot) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\boldsymbol{x}(0), \quad \boldsymbol{q} = \begin{pmatrix}2\\0\end{pmatrix}.$$

By Lemma 1, the terminal value problem (3.18)–(3.19) is equivalent to a system of Fredholm integral equations of the second kind

$$\boldsymbol{x}(t) = \int_0^b \mathbf{K}(t,s)\boldsymbol{x}(s)\mathrm{d}s, \tag{3.21}$$

where

$$\mathbf{K}(t,s) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{cases} -(b-s)^{-1/2}, & s > t, \\ (t-s)^{-1/2} - (b-s)^{-1/2}, & s \le t. \end{cases} \tag{3.22}$$

We have, as in example 1, $\alpha = 1/2$ and all iterated kernels starting from $\mathbf{K}_2(t,s)$ are square summable and we can pass from the a system of integral equations (3.21) with unbounded kernel $\mathbf{K}(t,s)$ (3.22) to a system of integral equations with square summable kernel $\mathbf{K}_2(t,s)$

$$\boldsymbol{x}(t) = \int_0^b \mathbf{K}_2(t,s)\boldsymbol{x}(s)\mathrm{d}s, \tag{3.23}$$

where

$$\mathbf{K}_2(t,s) = \frac{1}{\pi} \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ -\sin 2\varphi & \cos 2\varphi \end{pmatrix} \begin{cases} 2\left(\sqrt{b}-\sqrt{t}\right)(b-s)^{-1/2} - \pi, & s > t, \\ 2\left(\sqrt{b}-\sqrt{t}\right)(b-s)^{-1/2}, & s \le t. \end{cases}$$

It is known [5], that if the system of linearly independent vectors of the kernel of the Fredholm integral operator is known, then this system can be used instead of system $\{\boldsymbol{\varphi}_i(t)\}_{i=1}^r$ to construct the general solution system of equation (3.23). The system (3.18), and hence system of equation (3.23), has two linearly independent solutions of the form [12]

$$\boldsymbol{\varphi}_1(t) = \frac{1}{\beta}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad \boldsymbol{\varphi}_2(t) = \frac{1}{\beta}\begin{pmatrix} -v(t) \\ u(t) \end{pmatrix}, \quad \beta^2 = \int_0^b \left(u^2(t) + v^2(t)\right)\mathrm{d}t, \tag{3.24}$$

where $u, v : \mathbb{R}_+ \to \mathbb{R}$ are given by

$$u(t) = E_{1/2}\left(\lambda\sqrt{t}\right) + E_{1/2}\left(\bar{\lambda}\sqrt{t}\right), \quad v(t) = \mathrm{i}\left(E_{1/2}\left(\lambda\sqrt{t}\right) - E_{1/2}\left(\bar{\lambda}\sqrt{t}\right)\right).$$

By using the vectors (3.24), we get

$$x_i = \int_0^b \boldsymbol{\varphi}_i^T(t)\boldsymbol{x}(t)\mathrm{d}t, \quad A_{ij} = \int_0^b \int_0^b \boldsymbol{\varphi}_i^T(t)\mathbf{K}_2(t,s)\boldsymbol{\varphi}_j(s)\mathrm{d}s\mathrm{d}t, \quad i,j = \overline{1,2}$$

and the vectors $\boldsymbol{z}$, $\boldsymbol{g}$ and matrices $\boldsymbol{\Lambda}$, $\mathbf{W}$, in the Equation (3.6), have the form:

$$\boldsymbol{z} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\Lambda} = \mathbf{O}_2, \quad \mathbf{W} = \frac{2}{\beta}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $r = 2$ and the matrices $\boldsymbol{\Lambda}^+$, $\mathbf{P}_{\boldsymbol{\Lambda}_2}$, $\mathbf{P}_{\boldsymbol{\Lambda}_2^*}$, $\mathbf{Q}$ have the form

$$\boldsymbol{\Lambda}^+ = \mathbf{O}_2, \quad \mathbf{P}_{\boldsymbol{\Lambda}_2} = \mathbf{P}_{\boldsymbol{\Lambda}_2^*} = \mathbf{I}_2, \quad \mathbf{Q} = \mathbf{W}.$$

Since $p = 1$, $\mathrm{rank}\mathbf{Q} = 1$, $d_1 = d_2 = 1$, then the matrices $\mathbf{Q}^+$, $\mathbf{P}_{\mathbf{Q}_1}$ and $\mathbf{P}_{\mathbf{Q}_1^*}$ have the form

$$\mathbf{Q}^+ = \frac{\beta}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \mathbf{P}_{\mathbf{Q}_1} = \begin{pmatrix}0\\ 1\end{pmatrix}, \quad \mathbf{P}_{\mathbf{Q}_1^*} = \begin{pmatrix}0 & 1\end{pmatrix}.$$

Hence, the solvability conditions (3.7) are satisfied and, by Theorem 1, the vector $\boldsymbol{z}$ has the form

$$\boldsymbol{z} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}c + \frac{\beta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}2\\ 0\end{pmatrix} = \begin{pmatrix}\beta\\ c\end{pmatrix},$$

and, by Theorem 2, the boundary-value problem (3.18)–(3.20) possesses a $1-$parameter family of solutions $\boldsymbol{x}(t)$ of the form

$$\boldsymbol{x}(t) = \frac{1}{\beta}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}\beta + \frac{1}{\beta}\begin{pmatrix}-v(t)\\ u(t)\end{pmatrix}c = \begin{pmatrix}u(t) - \hat{c}v(t)\\ v(t) + \hat{c}u(t)\end{pmatrix}, \quad \hat{c} = \frac{c}{\beta}.$$

Example 3. In this example, we consider a two-point boundary-value problem for a scalar equation and illustrate the application of Lemma 2. Such a problem, like the terminal value problem, is well-posed, and imposing an additional restriction on its solutions turns it into an overdetermined problem. We will investigate the solvability of a two-point boundary-value problem for a scalar fractional differential equation

$$^{C}\mathrm{D}^{1/2}_{0+}x(t) = -x(t) + 3\sqrt{\pi}t + 6\sqrt{t} - 2\sqrt{\pi}, \quad t \in [0,1], \tag{3.25}$$

$$x(0) + x(1) = -\sqrt{\pi} \tag{3.26}$$

with additional restriction

$$\int_0^1 x(t)\mathrm{d}t = -\frac{\sqrt{\pi}}{2}. \tag{3.27}$$

By Lemma 2, the problem (3.25)–(3.26) is equivalent to a Fredholm integral equation of the second kind

$$x(t) = 4t\sqrt{t} + 3\sqrt{\pi}t - 4\sqrt{t} - 2\sqrt{\pi} - \frac{1}{\sqrt{\pi}}\int_0^t \frac{x(s)\mathrm{d}s}{\sqrt{t-s}} + \frac{1}{2\sqrt{\pi}}\int_0^1 \frac{x(s)\mathrm{d}s}{\sqrt{1-s}}. \tag{3.28}$$

We have, as in the previous examples, $\alpha = 1/2$ and we can pass from the investigation of the integral equation (3.28) to the investigation of the equivalent the integral equation with square summable kernel $K_2(t,s)$

$$x(t) = g_2(t) + \int_0^1 K_2(t,s)x(s)\mathrm{d}s, \tag{3.29}$$

where

$$g_2(t) = -\frac{\sqrt{\pi}}{4}\left(6t^2 - 20t + 9\right),$$

$$K_2(t,s) = \frac{1}{2\pi}\begin{cases}\left(1 - 2\sqrt{t}\right)(1-s)^{-1/2} - \pi, & s > t,\\ \pi + \left(1 - 2\sqrt{t}\right)(1-s)^{-1/2}, & s \le t.\end{cases}$$

As in Example 1, by using the functions (3.16), we can reduce the equation (3.29) to the operator equation (3.6) in the form:

$$\mathbf{U}\boldsymbol{z} = \begin{bmatrix} \boldsymbol{\Lambda} \\ \mathbf{W} \end{bmatrix} \boldsymbol{z} = \begin{bmatrix} \boldsymbol{g} \\ q \end{bmatrix} = \boldsymbol{h},$$

where

$$\boldsymbol{\Lambda} = \frac{1}{3150\pi} \begin{pmatrix} 1050(3\pi+1) & 175\sqrt{3}(3\pi+2) & 210\sqrt{5} \\ 105\sqrt{3}(8-5\pi) & 210(15\pi+4) & 21\sqrt{15}(5\pi+8) \\ -120\sqrt{5} & -5\sqrt{15}(21\pi+8) & 30(105\pi-4) \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{z} = \operatorname{col}\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}, \quad x_i = \int_0^1 x(t)\varphi_i(t)\mathrm{d}t,$$

$$\boldsymbol{g} = \frac{\sqrt{\pi}}{60} \operatorname{col}\begin{pmatrix} -15 & 35\sqrt{3} & -3\sqrt{5} \end{pmatrix}, \quad q = -\frac{\sqrt{\pi}}{2}.$$

In this case, the matrices $\boldsymbol{\Lambda}^+$, $\mathbf{P}_{\boldsymbol{\Lambda}}$, $\mathbf{P}^*_{\boldsymbol{\Lambda}}$, $\mathbf{Q}^+$, $\mathbf{P}_{\mathbf{Q}}$ and $\mathbf{P}_{\mathbf{Q}^*}$ have the form

$$\boldsymbol{\Lambda}^+ = \boldsymbol{\Lambda}^{-1} = \frac{1}{6(548+1155\pi)} \times \begin{pmatrix} 6405\pi+1648 & -10\sqrt{3}(105\pi+73) & 7\sqrt{5}(15\pi-26) \\ 10\sqrt{3}(105\pi-176) & 60(105\pi+31) & -14\sqrt{15}(15\pi+34) \\ 7\sqrt{5}(15\pi+16) & 10\sqrt{15}(21\pi+11) & 35(195\pi+94) \end{pmatrix},$$

$$\mathbf{P}_{\boldsymbol{\Lambda}} = \mathbf{P}_{\boldsymbol{\Lambda}^*} = \mathbf{O}_3, \quad \mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{Q}^+ = \operatorname{col}\begin{pmatrix} 0 & 0 & 0 \end{pmatrix},$$
$$\mathbf{P}_{\mathbf{Q}} = \mathbf{I}_3, \quad \mathbf{P}_{\mathbf{Q}^*} = 1,$$

where $\mathbf{O}_3$ is the zero matrix of dimension 3 and $\mathbf{I}_3$ is identity matrix of dimensions 3. Hence, the solvability conditions (3.7) are satisfied and, by Theorem 2, the boundary-value problem (3.25)–(3.27) possesses a unique solution $x(t)$ of the form

$$x(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{g} = \frac{\sqrt{\pi}}{12}(-1 + 3(2t-1)\cdot 1 + \sqrt{5}(6t^2-6t+1)\cdot 0) = \sqrt{\pi}(3t-2).$$

4 Multi-point boundary-value problem

The approach described in this paper to study the problem (1.1)–(1.3), in the case of a single equation ($n = 1$), can also be applied to the study of a multi-point boundary-value problem for a linear equation (1.1). The argumentation for this approach is based on the following statement, which is analogous to Lemmas 1 and 2:

Lemma 3. *Let* $0 < \alpha < 1$, $a = t_1 < t_2 < \cdots < t_{k-1} < t_k = b$, $h_i \in \mathbb{R}$, $i = \overline{1, k+1}$, $\beta_k = \sum_{i=1}^{k} h_i \neq 0$, *the function* $f(t,x) : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous* *with respect to t on $[a,b]$. The function $x \in C[a,b]$ is a solution of a multi-point boundary-value problem*

$$^{C}\mathrm{D}_{a+}^{\alpha}x(t) = f(t,x(t)), \tag{4.1}$$

$$\sum_{i=1}^{k} h_i x(t_i) = h_{k+1} \tag{4.2}$$

if and only if it is a solution of a integral equation

$$\begin{aligned} x(t) &= \frac{h_{k+1}}{\beta_k} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s,x(s))\mathrm{d}s}{(t-s)^{1-\alpha}} - \frac{1}{\beta_k \Gamma(\alpha)} \sum_{i=2}^{k} h_i \int_a^{t_i} \frac{f(s,x(s))\mathrm{d}s}{(t_i-s)^{1-\alpha}} \\ &= \frac{h_{k+1}}{\beta_k} + \frac{1}{\Gamma(\alpha)} \int_a^b G(t,s) f(s,x(s))\mathrm{d}s, \end{aligned} \tag{4.3}$$

where

$$G(t,s) = \begin{cases} -\sum\limits_{j=i}^{k} \frac{h_j}{\beta_k}(t_j-s)^{\alpha-1}, & t_i \geq s > t \geq t_{i-1}, \\ (t-s)^{\alpha-1} - \sum\limits_{j=i}^{k} \frac{h_j}{\beta_k}(t_j-s)^{\alpha-1}, & t_{i-1} \leq s \leq t \leq t_i. \end{cases} \tag{4.4}$$

The problem (4.1)–(4.2), as special cases, includes the initial problem at $h_i = 0$, $i = \overline{2,k}$, the terminal value problem at $h_i = 0$, $i = \overline{1,k-1}$, and two-point problem at $k = 2$ (see [13, p. 128]).

Proof. In order to prove the validity of Lemma 3, it is enough to repeat the proof of Lemma 2 given in [13, p. 128]. Indeed, having determined the value of $x(t_i)$ from the Equation (4.3), $i = \overline{1,k}$

$$\begin{aligned} x(t_i) &= \frac{h_{k+1}}{\beta_k} + \frac{1}{\Gamma(\alpha)} \int_a^b G(t_i,s) f(s,x(s))\mathrm{d}s \\ &= \frac{h_{k+1}}{\beta_k} + \frac{1}{\Gamma(\alpha)} \int_a^{t_i} \frac{f(s,x(s))\mathrm{d}s}{(t_i-s)^{1-\alpha}} - \frac{1}{\beta_k \Gamma(\alpha)} \sum_{j=2}^{k} h_j \int_a^{t_j} \frac{f(s,x(s))\mathrm{d}s}{(t_j-s)^{1-\alpha}}, \end{aligned}$$

and substituting them into the condition (4.2), we make sure that it is satisfied. Moreover, an application of the differential operator $^{C}\mathrm{D}_{a+}^{\alpha}$ to both sides of (4.3) yields Equation (4.1). Therefore, $x(t)$ solves the boundary-value problem (4.1)–(4.2) if it solves the integral equation (4.3).

On the contrary, if the function $x(t)$ is a solution of the differential equation (4.1), then it, according to Lemma 2 ($h_1 = 1$, $h_2 = 0$, $h_3 = x_a$), also satisfies the equation

$$x(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s,x(s))\mathrm{d}s}{(t-s)^{1-\alpha}} \tag{4.5}$$

with some (presently unknown) quantity x_a. Taken at $t = t_i$, $i = \overline{2,k}$, we get:

$$x(t_i) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^{t_i} \frac{f(s,x(s))\mathrm{d}s}{(t_i-s)^{1-\alpha}}.$$

Substituting the value of $x(t_i)$ into the condition (4.2), we have

$$\beta_k x_a + \frac{1}{\Gamma(\alpha)} \sum_{i=2}^{k} h_i \int_a^{t_i} \frac{f(s, x(s))\mathrm{d}s}{(t_i - s)^{1-\alpha}} = h_{k+1}. \tag{4.6}$$

Solving (4.5) for x_a and inserting the found result into (4.6), we obtain the equality

$$\beta_k \left(x(t) - \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s, x(s))\mathrm{d}s}{(t - s)^{1-\alpha}} \right) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} h_i \int_a^{t_i} \frac{f(s, x(s))\mathrm{d}s}{(t_i - s)^{1-\alpha}} = h_{k+1},$$

which is equivalent to the Equation (4.3). Therefore, the function $x(t)$ is a solution of the integral equation (4.3), if it is a solution of the boundary-value problem (4.1)–(4.2). □

Recall that Lemma 3, like Lemmas 1 and 2, will be valid only for the definition of the Caputo fractional derivative (2.1), which will coincide with the usual definition of the Caputo derivative (2.3) on the space of absolutely continuous functions.

In the space $C[a, b]$, $-\infty < a < b < +\infty$, we consider a multi-point boundary-value problem for the linear fractional differential equation

$${}^{C}\mathrm{D}^{\alpha}_{a+} x(t) = A(t)x(t) + f(t), \quad \sum_{i=1}^{k} h_i x(t_i) = h_{k+1}, \tag{4.7}$$

that is, a special case of the problem (4.1)–(4.2), the solutions of which are subject to additional constraints (1.3). By Lemma 3, the problem (4.7) is equivalent to a weakly singular integral equation similar to the Equation (3.2), that is, the equation

$$x(t) = g(t) + \int_a^b K(t, s)x(s)\mathrm{d}s, \tag{4.8}$$

where

$$g(t) = \frac{h_{k+1}}{\beta_k} + \frac{1}{\Gamma(\alpha)} \int_a^b G(t, s)f(s)\mathrm{d}s, \quad K(t, s) = \frac{1}{\Gamma(\alpha)} G(t, s)A(s),$$

and the function $G(t, s)$ has the form (4.4).

Hence, we show that the study of the multi-point boundary-value problem (4.7), (1.3) is reduced to the study of the boundary-value problem for the weakly singular integral equation (4.8), (1.3). For the boundary-value problem (4.7), (1.3) the criterion of solvability similar to the Theorem 2 is true. To make sure of this, it is enough to repeat the calculations performed in the case of the terminal value problem, setting $n = 1$ and taking into account the new form of function $g(t)$ and the kernel $K(t, s)$.

5 Terminal value problem for the system of differential equations with the tempered and Ψ-tempered fractional derivatives of Caputo type

The method of researching the boundary-value problem (1.1)–(1.3) presented in this paper can be applied, with minor changes, also for researching the problem of finding the conditions for the solvability and constructing a solution to the terminal value problem for the system of fractional differential equations with tempered Caputo derivative. The tempered derivative, which was first defined in the article [31], is widely used for describes the transition between normal and anomalous diffusions (or the anomalous diffusion in finite time or bounded physical space), as well as in poroelasticity, finance, ground water hydrology, geophysical flows (see works [26, 31] and the literature cited in them).

We will give the definition of tempered Caputo derivative. To do this, we first introduce the definitions of tempered Riemann-Liouville fractional integral and tempered Riemann-Liouville fractional derivative. The properties of these concepts can be found in more detail, for example, in works [19, 26, 27].

Definition 6. [19,27] Suppose that real function $x(t)$ is piecewise continuous, integrable on (a, b), $\alpha > 0$, $\lambda \geq 0$. The tempered Riemann–Liouville fractional integral of order α is defined to be

$$\mathrm{I}_{a+}^{\alpha,\lambda}x(t) = e^{-\lambda t}\mathrm{I}_{a+}^{\alpha}\left(e^{\lambda t}x(t)\right) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{e^{-\lambda(t-s)}x(s)\mathrm{d}s}{(t-s)^{1-\alpha}}.$$

Definition 7. [19, 27] For $x \in C[a, b]$, $0 < \alpha < 1$, $\lambda \geq 0$ the tempered Riemann–Liouville fractional derivative of order α is defined to be

$$\begin{aligned}\mathrm{D}_{a+}^{\alpha,\lambda}x(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right)\mathrm{I}_{a+}^{1-\alpha,\lambda}x(t)\\ &= e^{-\lambda t}\mathrm{D}_{a+}^{\alpha}\left(e^{\lambda t}x(t)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \frac{e^{-\lambda(t-s)}x(s)\mathrm{d}s}{(t-s)^{\alpha}}.\end{aligned}$$

Definition 8. [26] For $x \in AC[a, b]$, $0 < \alpha < 1$, $\lambda \geq 0$ the tempered Caputo fractional derivative of order α is defined to be

$$\begin{aligned}{}^{C}\mathrm{D}_{a+}^{\alpha,\lambda}x(t) &= e^{-\lambda t}\,{}^{C}\mathrm{D}_{a+}^{\alpha}\left(e^{\lambda t}x(t)\right)\\ &= \mathrm{D}_{a+}^{\alpha,\lambda}(x(t) - x(a)) = \mathrm{D}_{a+}^{\alpha,\lambda}x(t) - \frac{e^{-\lambda(t-a)}(t-a)^{-\alpha}}{\Gamma(1-\alpha)}x(a).\end{aligned}$$

Note that if $\lambda = 0$ then the Caputo tempered fractional derivative reduces to the Caputo fractional derivative ${}^{C}\mathrm{D}_{a+}^{\alpha,0}x(t) = {}^{C}\mathrm{D}_{a+}^{\alpha}x(t)$, and therefore, Caputo derivatives can be regarded as a particular case of Caputo tempered derivatives.

In the space $C[a, b]$, we consider a terminal value problems for the system of fractional differential equations with tempered Caputo derivative of order $0 < \alpha < 1$

$$ {}^{C}\mathrm{D}_{a+}^{\alpha,\lambda}\boldsymbol{x}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \boldsymbol{f}(t), \quad e^{\lambda b}\boldsymbol{x}(b) = \boldsymbol{x}^{*}, \tag{5.1}$$

whose solutions satisfy the conditions

$$\boldsymbol{l}\left(e^{\lambda\cdot}\boldsymbol{x}(\cdot)\right) = \boldsymbol{q}, \tag{5.2}$$

and the assumptions on the coefficients of problem (5.1)–(5.2) are the same as for problem (1.1)–(1.3). Since ${}^{C}\mathrm{D}^{\alpha,\lambda}_{a+}\boldsymbol{x}(t) = e^{-\lambda t}\,{}^{C}\mathrm{D}^{\alpha,0}_{a+}\left(e^{\lambda t}\boldsymbol{x}(t)\right)$ and for any function $\boldsymbol{x} \in C[a,b]$ will also be satisfied $e^{\lambda\cdot}\boldsymbol{x}(\cdot) \in C[a,b]$, then we can apply to the study of problem (5.1)–(5.2) the same methodology as when studying problem (1.1)–(1.3). To substantiate this approach, we note that, using the results of works [26, 29], it is possible to show that the function $\boldsymbol{g}(t)$ and the kernel $\mathbf{K}(t,s)$ in the system of weakly singular integral equations (3.2), in this case, will have the form

$$\begin{aligned}
&\boldsymbol{g}(t) = e^{-\lambda(t-a)}\boldsymbol{x}^* + \frac{1}{\Gamma(\alpha)}\int_a^b G(t,s)e^{\lambda s}\boldsymbol{f}(s)\mathrm{d}s,\\
&\mathbf{K}(t,s) = \frac{1}{\Gamma(\alpha)}G(t,s)\mathbf{A}(s)e^{\lambda s},\\
&G(t,s) = \begin{cases} -(b-s)^{\alpha-1}e^{-\lambda b}, & s > t,\\ (t-s)^{\alpha-1}e^{-\lambda t} - (b-s)^{\alpha-1}e^{-\lambda b}, & s \le t. \end{cases}
\end{aligned} \tag{5.3}$$

According to our assumptions, the kernel $\mathbf{K}(t,s)$ (5.3), like the kernel $\mathbf{K}(t,s)$ (3.3), is also a weakly singular kernel. Therefore, after conducting similar considerations, we will obtain a criterion of solvability of the the boundary-value problem (5.1)–(5.2), similar to Theorem 2, which is obviously a more general result.

We also note that the results obtained above can be generalized to the case of the terminal value problem for the system of fractional differential equations with tempered $\varPsi$–Caputo derivative (with some restrictions on the function $\varPsi(t)$). The tempered $\varPsi$–Caputo derivative was first introduced in the paper [27]. It is a generalization of the tempered Caputo derivative ($\varPsi(t) = t$) and covers the well-known fractional derivatives for $\lambda = 0$, like the Caputo–Hadamard fractional derivative ($\varPsi(t) = \ln t$), the Caputo–Erdélyi–Kober fractional derivative ($\varPsi(t) = t^\sigma$). As in the case of the tempered Caputo derivative, we first introduce the definitions of tempered $\varPsi$–fractional integral and tempered $\varPsi$–Riemann–Liouville fractional derivative and then we will give the definition tempered $\varPsi$–Caputo derivative (see [11, 27, 32]).

Definition 9. [11, 27, 32] Let $\alpha > 0$, $\lambda \ge 0$, the real function $x \in C[a,b]$ and $\varPsi \in C^1[a,b]$ is an increasing differentiable function such that $\varPsi'(t) \neq 0$ for all $t \in [a,b]$. Then, the tempered $\varPsi$–fractional integral of order α is defined by

$$\begin{aligned}
\mathrm{I}^{\alpha,\lambda,\varPsi}_{a+}x(t) &= e^{-\lambda\varPsi(t)}\mathrm{I}^{\alpha,\varPsi}_{a+}\left(e^{\lambda\varPsi(t)}x(t)\right)\\
&= \frac{1}{\Gamma(\alpha)}\int_a^t \varPsi'(s)\left[\varPsi(t)-\varPsi(s)\right]^{\alpha-1}e^{-\lambda[\varPsi(t)-\varPsi(s)]}x(s)\mathrm{d}s,
\end{aligned}$$

where $\mathrm{I}_{a+}^{\alpha,\Psi}x(t)$ is the Ψ–Riemann–Liouville fractional integral of order α, which has the form (see [2,27])

$$\mathrm{I}_{a+}^{\alpha,\Psi}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t \Psi'(s)\left[\Psi(t)-\Psi(s)\right]^{\alpha-1}x(s)\mathrm{d}s.$$

Definition 10. [11,27,32] Let the assumptions of the Definition 9 be satisfied and let $0<\alpha<1$. Then the tempered Ψ–Riemann–Liouville fractional derivative of order α is defined to be

$$\begin{aligned}\mathrm{D}_{a+}^{\alpha,\lambda,\Psi}x(t)&=\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda\right)\mathrm{I}_a^{1-\alpha,\lambda,\Psi}x(t)=e^{-\lambda\Psi(t)}\mathrm{D}_{a+}^{\alpha,\Psi}\left(e^{\lambda\Psi(t)}x(t)\right)\\&=\frac{e^{-\lambda\Psi(t)}}{\Gamma(1-\alpha)}\left(\frac{1}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_a^t\Psi'(s)\left[\Psi(t)-\Psi(s)\right]^{-\alpha}e^{\lambda\Psi(s)}x(s)\mathrm{d}s,\end{aligned}$$

where $\mathrm{D}_{a+}^{\alpha,\Psi}x(t)$ is the Ψ–Riemann–Liouville fractional derivative of $x(t)$ of order $0<\alpha<1$, which has the form (see [2,27])

$$\begin{aligned}\mathrm{D}_{a+}^{\alpha,\Psi}x(t)&=\left(\frac{1}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathrm{I}_{a+}^{1-\alpha,\Psi}x(t)\\&=\frac{1}{\Gamma(1-\alpha)}\left(\frac{1}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_a^t\Psi'(s)\left[\Psi(t)-\Psi(s)\right]^{-\alpha}x(s)\mathrm{d}s.\end{aligned}$$

Definition 11. [11,27,32] Let the assumptions of the Definition 10 be satisfied. Then the tempered Ψ–Caputo fractional derivative of $x(t)$ of order α is defined to be

$$\begin{aligned}{}^C\mathrm{D}_{a+}^{\alpha,\lambda,\Psi}x(t)&=e^{-\lambda\Psi(t)}\,{}^C\mathrm{D}_{a+}^{\alpha,\Psi}\left(e^{\lambda\Psi(t)}x(t)\right)=\mathrm{D}_{a+}^{\alpha,\lambda,\Psi}(x(t)-x(a))\\&=\mathrm{D}_{a+}^{\alpha,\lambda,\Psi}x(t)-\frac{e^{-\lambda(\Psi(t)-\Psi(a))}(\Psi(t)-\Psi(a))^{-\alpha}}{\Gamma(1-\alpha)}x(a).\end{aligned}$$

The Ψ–Caputo fractional derivative of $x(t)$ of order α, introduced in the paper [2], is defined by the rule

$${}^C\mathrm{D}_{a+}^{\alpha,\Psi}x(t)=\mathrm{D}_{a+}^{\alpha,\Psi}(x(t)-x(a))=\mathrm{D}_{a+}^{\alpha,\Psi}x(t)-\frac{(\Psi(t)-\Psi(a))^{-\alpha}}{\Gamma(1-\alpha)}x(a).$$

In this case, $e^{\lambda\Psi(\cdot)}x(\cdot)\in C[a,b]$ because it is a composition of two continuous functions. Considering also that

$${}^C\mathrm{D}_{a+}^{\alpha,\lambda,\Psi}x(t)=e^{-\lambda\Psi(t)}\,{}^C\mathrm{D}_{a+}^{\alpha,\Psi}\left(e^{\lambda\Psi(t)}x(t)\right),$$

we can apply the technique, described in this article, to study a terminal value problems for the system of fractional differential equations with tempered Ψ–Caputo derivative of order $0<\alpha<1$ in the space $C[a,b]$

$${}^C\mathrm{D}_{a+}^{\alpha,\lambda,\Psi}\boldsymbol{x}(t)=\mathbf{A}(t)\boldsymbol{x}(t)+\boldsymbol{f}(t),\quad e^{\lambda\Psi(b)}\boldsymbol{x}(b)=\boldsymbol{x}^*,\tag{5.4}$$

whose solutions satisfy the conditions

$$\boldsymbol{l}\Big(e^{\lambda\Psi(\cdot)}\boldsymbol{x}(\cdot)\Big) = \boldsymbol{q} \tag{5.5}$$

with the same assumptions on the coefficients as before. Similarly, as in work [27], it is possible to show that the function $\boldsymbol{g}(t)$ and the kernel $\mathbf{K}(t,s)$ in the system of integral equations (3.2), in this case, will have the form

$$\begin{aligned}
\boldsymbol{g}(t) &= e^{-\lambda(\Psi(t)-\Psi(a))}\boldsymbol{x}^* + \frac{1}{\Gamma(\alpha)}\int_a^b G(t,s)\Psi'(s)e^{\lambda\Psi(s)}\boldsymbol{f}(s)\mathrm{d}s,\\
\mathbf{K}(t,s) &= \frac{1}{\Gamma(\alpha)}G(t,s)\Psi'(s)\mathbf{A}(s)e^{\lambda\Psi(s)},\\
G(t,s) &= \begin{cases} -(\Psi(b)-\Psi(s))^{\alpha-1}e^{-\lambda\Psi(b)}, & s>t,\\ (\Psi(t)-\Psi(s))^{\alpha-1}e^{-\lambda\Psi(t)} - (\Psi(b)-\Psi(s))^{\alpha-1}e^{-\lambda\Psi(b)}, & s\le t. \end{cases}
\end{aligned} \tag{5.6}$$

In order for us to be able to apply to the study of problems (5.4)–(5.5) the same approach as for problems (1.1)–(1.3), it is necessary for the function $\Psi(t)$ to be such that some iterated kernel $\mathbf{K}_m(t,s)$ of the kernel $\mathbf{K}(t,s)$ (5.6) was square summable. So, for example, it will be when $\Psi(t) = t$ (the case of tempered Caputo derivative), as well as when $\Psi(t)$ is a linear function. If this condition is satisfied, after conducting similar considerations as in the previous cases, we will obtain the criterion of solvability of the boundary-value problem (5.4)–(5.5), which generalizes the corresponding results for the cases of the Caputo derivative and the Caputo tempered derivative.

6 Conclusions

We have investigated the terminal value problem for the system of differential equations with the Caputo fractional derivative of order α $(0 < \alpha < 1)$ in the space $C[a,b]$. Additional conditions are imposed on the solutions of this problem in the form of the bounded linear vector functional. The problem under consideration has been reduced to an equivalent Fredholm problem for a system of integral equations with square summable kernels. Necessary and sufficient conditions for solvability have been established and the general form of the solution of the given problem has been found. The one-dimensional case was considered and similar results were obtained for the multi-point boundary-value problem for the differential equation with the Caputo fractional derivative, the solutions of which are subject to additional conditions. For the terminal value problem for the system of fractional differential equations with tempered and Ψ–tempered fractional derivatives of Caputo type, the generalization of the results, obtained in this paper, has been considered.

In the future, the terminal value problem for a multi-term fractional differential equation with additional conditions will be investigated. We want to consider the cases when such a problem is equivalent to the problem studied in this paper and establish a criterion for its solvability.

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