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Averaged Reaction for Nonlinear Boundary Conditions on a Grill-Type Winkler Foundation

Delfina Gómez[a] and María-Eugenia Pérez-Martínez[b]

[a] *Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria*
Av. Los Castros s.n., 39005 Santander, Spain
[b] *Departamento de Matemática Aplicada y Ciencias de la Computación, Universidad de Cantabria*
Av. Los Castros s.n., 39005 Santander, Spain
E-mail(*corresp.*): meperez@unican.es
E-mail: gomezdel@unican.es

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Abstract. We consider a homogenization problem for the elasticity operator posed in a bounded domain of the half-space, a part of its boundary being in contact with the plane. This surface is traction-free out of "small regions", where we impose non-linear Winkler-Robin boundary conditions containing "large reaction parameters". Non-periodical distribution of these regions is allowed provided that they have the same area. We show the convergence of solutions towards those of the homogenized problems depending on the relations between the parameters distance, sizes, and reaction.

Keywords: boundary homogenization, elasticity operator, nonlinear Winkler foundations.

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1 Introduction

Linear Winkler-Robin boundary conditions in homogenization frameworks have been approached recently in the literature, cf. [2, 7, 8, 10, 19]. We also refer to the homogenization of Winkler-Steklov type boundary conditions in [6, 13]. However, the case of the homogenization of nonlinear Winkler-Robin boundary conditions remained as an open problem that we address in this paper.

We study the asymptotic behavior of an elastic body which has very large surface reaction terms concentrated in small regions. We assume that the elastic material fills the domain Ω of the upper half space $\mathbb{R}^{3+}$, and a part Σ of its surface lies on the plane $\{x_3 = 0\}$ and contains small regions T^ε of size $O(r_\varepsilon)$, at a distance $O(\varepsilon)$ between them (cf. Figure 1 and (2.3) and (2.5) for precise definition). The boundary conditions are nonlinear Winkler-Robin on T^ε. These regions can have different shapes but must have the same area $|T^\varepsilon| \equiv r_\varepsilon^2|T|$. Outside, the surface Σ is free of forces while the rest of the surface $\partial\Omega \setminus \Sigma$ is assumed to be fixed. Here ε and r_ε are two small parameters such that $\varepsilon^2 \ll r_\varepsilon \leq \varepsilon \ll 1$.

As is well known, from the mechanical viewpoint, the small regions behave as "springs" with a nonlinear elastic behavior represented by a reaction vector function $\beta(\varepsilon)M(x, u^\varepsilon)$, which depends on the point where *the reaction regions* T^ε are placed and on the displacement vector u^ε, while the parameter $\beta(\varepsilon)$, which is referred to as *the reaction parameter*, can range from very small to very large. These regions T^ε are assumed to be domains of the plane $\mathbb{R}^2$ homothetics of any prescribed domain within a set of *isoperimetric* domains with a Lipschitz boundary (cf. (2.1)). For relatively large sizes of these regions (cf. (2.2)), we analyze the different relations between the three parameters of the problem, ε, r_ε and $\beta(\varepsilon)$, and find three possible homogenized problems, which range from Dirichlet to Neumann (the so-called *extreme cases* in the literature) and the intermediate case where an averaged nonlinear Winkler-Robin boundary condition is imposed on Σ. The *averaged reaction term* depends on the unit area $|T|$ and on a constant β^* which links the reaction parameter $\beta(\varepsilon)$ and the total area of the regions T^ε, namely,

$$\lim_{\varepsilon \to 0} \beta(\varepsilon) r_\varepsilon^2 / \varepsilon^2 = \beta^*. \tag{1.1}$$

In the case where $\beta^* = 0$, either the reaction is small or the total area of the reaction regions is small and the homogenized boundary condition is traction-free. In the case where $\beta^* = +\infty$ the reaction or areas are so big that Σ remains, asymptotically, stuck to the plane. The case where $\beta^* > 0$ is referred to as *critical relation between the parameters*. It occurs when the total area of the reaction regions $O(\varepsilon^{-2} r_\varepsilon^2)$ multiplied by the parameter of reaction $\beta(\varepsilon)$ is of order 1. In this way, for a given reaction $\beta(\varepsilon)$ we find a critical size of the reaction regions $r_\varepsilon = O(\beta(\varepsilon)^{1/2}\varepsilon)$ in such a way that the asymptotic behavior is different from the extreme cases. This critical size obviously differs from the classical one $r_\varepsilon = O(\varepsilon^2)$ obtained in the literature with the so-called *strange terms*: cf. [1, 17] for the case where the small regions are stuck to the plane, [13] for Steklov type conditions on the small regions, [7,8] for linear Winkler-Robin boundary conditions; see also references in [5, 7, 8] and [9] for further scalar problems. Also, to each size r_ε corresponds *a critical reaction parameter* $\beta(\varepsilon) = O(\varepsilon^2 r_\varepsilon^{-2})$. The role of $\beta(\varepsilon)$ is crucial in the problem under consideration, and, as a particular case, for the elasticity operator, we provide the averaged nonlinear Winkler-Robin boundary condition when $\beta(\varepsilon)$ is a constant and $r_\varepsilon = O(\varepsilon)$ (cf. [10] for the linear case).

In the periodic case, when $r_\varepsilon = O(\varepsilon)$, the geometrical configuration here considered for the elasticity system is in [6, 8, 10]. The three papers deal with the vibrations of an elastic block. [6] addresses the homogenization of Winkler-Steklov boundary conditions, [8, 10] consider the linear Winkler-Robin ones, in particular, [8] deals with asymptotic expansions. Let us also mention the elasticity operator and similar geometrical configurations in [20] for perforated domains, and [1,12] for different contact laws. Nonlinear restrictions in a scalar problem are in [23]; in this connection, we mention [3, 4] for further Signorini type variational inequalities. For non-periodic geometrical configurations in a scalar problem with nonlinear Robin boundary conditions see [5] and references therein.

Finally, we emphasize that the results in this paper apply to the linear function $M_i(x, u^\varepsilon) \equiv M_{ij}(x)u_j^\varepsilon$ considered in [10] (see Remarks 1 and 3), and therefore, here we extend the results for the linear case to a non-periodic distribution of the T^ε. Also, they extend and complement those for linear and nonlinear scalar problems addressed in [9, 24].

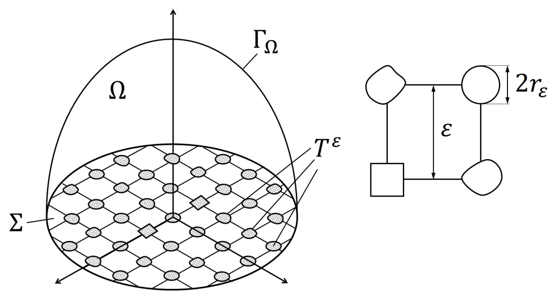

Figure 1. Geometrical configuration of the problem and the grill

Now, let us describe the general structure of the paper. Section 2 contains the setting of the homogenization problem. Section 3 contains the abstract framework of the problem along with the three homogenized problems, which depend on the different relations between the parameters. They are obtained by means of asymptotic expansions in Section 4. Section 5 addresses the convergence of the solutions in $(H^1(\Omega))^3$-weak for the critical relation when $\beta^* > 0$ (cf. Theorem 3), respectively in $(H^1(\Omega))^3$ when dealing with the extreme relations $\beta^* = 0$ or $\beta^* = +\infty$ (cf. Theorems 4 and 5). In particular, it should be noticed that proofs rely on a convergence measure result, cf. Theorem 2, which extends previous ones in the literature (cf. Remark 3). Its proof, along with the preliminary necessary results, is in Section 6. We gather some concluding remarks in Section 7.

2 Setting of the problem

Let Ω be an open bounded domain of $\mathbb{R}^3$ situated in the upper half-space $\mathbb{R}^{3+} = \{x \in \mathbb{R}^3 : x_3 > 0\}$, with a Lipschitz boundary $\partial\Omega$. Let Σ be the part of the boundary in contact with the plane $\{x_3 = 0\}$ which is assumed to be non-empty and let Γ_Ω be the rest of the boundary of Ω: $\partial\Omega = \overline{\Gamma_\Omega} \cup \overline{\Sigma}$. For each $p = 0, 1, 2, \ldots, \mathfrak{M}$, let T^p denote an open bounded domain of the plane

$\{x_3 = 0\}$ with a Lipschitz boundary, and area $|T^p| = |T^0| = |T|$, where $T = T^0$. Let $\mathcal{M}$ denote the set of these $\mathfrak{M} + 1$ isoperimetric domains, namely

$$\mathcal{M} := \{T^p\}_{p=0}^{\mathfrak{M}}, \quad |T^p| = |T|. \tag{2.1}$$

Without any restriction, we can assume that Σ and T^p contain the origin of coordinates, $p = 0, 1, \dots, \mathfrak{M}$ and $T^p \subset B(0, K)$ the ball of radius K for a certain $K < 1/2$.

Let ε be a small parameter $\varepsilon \ll 1$. Let r_ε be an order function such that

$$\varepsilon^2 \ll r_\varepsilon \leq \varepsilon. \tag{2.2}$$

For $\mathrm{k} = (k_1, k_2) \in \mathbb{Z}^2$, we denote by $\widetilde{x}_{\mathrm{k}}^\varepsilon$ the point of the plane $\{x_3 = 0\}$ of coordinates $\widetilde{x}_{\mathrm{k}}^\varepsilon = (k_1\varepsilon, k_2\varepsilon, 0)$, and by $T_{\widetilde{x}_{\mathrm{k}}}^{p,\varepsilon}$ the homothetic domain of a certain T^p of ratio r_ε after translation to the point $\widetilde{x}_{\mathrm{k}}^\varepsilon$:

$$T_{\widetilde{x}_{\mathrm{k}}}^{p,\varepsilon} = \widetilde{x}_{\mathrm{k}}^\varepsilon + r_\varepsilon T^p\,; \tag{2.3}$$

$\widetilde{x}_{\mathrm{k}}^\varepsilon$ is referred to as the center of $T_{\widetilde{x}_{\mathrm{k}}}^{p,\varepsilon}$. If there is no ambiguity, we shall write $\widetilde{x}_{\mathrm{k}}$ instead of $\widetilde{x}_{\mathrm{k}}^\varepsilon$.

In this way, for any fixed ε, we construct the grid of squares in the plane $\{x_3 = 0\}$ whose vertices are the centers of the regions $T_{\widetilde{x}_{\mathrm{k}}}^{p,\varepsilon}$ homothetics after translation of T^p for some $p = 0, 1, 2, \dots, \mathfrak{M}$. We denote by $T_{\widetilde{x}_{\mathrm{k}}}^\varepsilon$ this region which can in fact be any $T_{\widetilde{x}_{\mathrm{k}}}^{p,\varepsilon}$. Let $\mathcal{J}^\varepsilon$ denote $\mathcal{J}^\varepsilon = \{\mathrm{k} \in \mathbb{Z}^2 : T_{\widetilde{x}_{\mathrm{k}}}^\varepsilon \subset \Sigma\}$, while N_ε denotes the number of elements of $\mathcal{J}^\varepsilon$:

$$N_\varepsilon \approxeq |\Sigma|/\varepsilon^2 = O(\varepsilon^{-2}). \tag{2.4}$$

Similarly, for $p = 0, 1, \dots, \mathfrak{M}$, we consider $\mathcal{J}_p^\varepsilon = \{\mathrm{k} \in \mathcal{J}^\varepsilon : T_{\widetilde{x}_{\mathrm{k}}}^\varepsilon = T_{\widetilde{x}_{\mathrm{k}}}^{p,\varepsilon} \subset \Sigma\}$. Finally, if no confusion arises, we denote by $\bigcup T^\varepsilon$ the union of all the T^ε contained in Σ, namely,

$$\bigcup T^\varepsilon \equiv \bigcup_{\mathrm{k} \in \mathcal{J}^\varepsilon} T_{\widetilde{x}_{\mathrm{k}}}^\varepsilon \equiv \bigcup_{p=0}^{\mathfrak{M}} \bigcup_{\mathrm{k} \in \mathcal{J}_p^\varepsilon} T_{\widetilde{x}_{\mathrm{k}}}^{p,\varepsilon}. \tag{2.5}$$

Also, in what follows $x = (x_1, x_2, x_3)$ denotes the usual cartesian coordinates, while by $\hat{x} = (x_1, x_2)$ we refer to the two first components of $x \in \mathbb{R}^3$.

Under the basis that the domain Ω is filled by an elastic material, for $i, j, k, l = 1, 2, 3$, we denote by $a_{ijkl}(x)$ the elastic coefficients of the material, which are assumed to be continuous functions defined in $\overline{\Omega}$ and satisfy the standard symmetry and coercivity properties (cf., e.g., [21]): $\forall x \in \overline{\Omega}$,

$$a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x),\ i, j, k, l = 1, 2, 3, \tag{2.6}$$

$$\exists \alpha_1 > 0 : a_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \alpha_1 \xi_{ij}\xi_{ij},\ \forall \xi = (\xi_{ij})_{i,j=1,2,3}, \xi_{ij} = \xi_{ji}, i, j = 1, 2, 3. \tag{2.7}$$

Also, for a given displacement vector $u(x) = (u_1(x), u_2(x), u_3(x))$ we use the standard notations for stress and strain tensors $\sigma(u)$ and $e(u)$; namely, we denote by $(\sigma_{ij}(u))_{i,j=1,2,3}$ the stress tensor which is related to the strain tensor $(e_{ij}(u))_{i,j=1,2,3}$ by the Hooke's law

$$\sigma_{ij}(u) = a_{ijkl}(x) e_{kl}(u), \quad \text{where} \quad e_{kl}(u) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big). \tag{2.8}$$

Above, and in what follows, we use the convention of summation over repeated indexes.

In connection with the reaction terms in the small regions T^ε, let us introduce the continuous vectorial function $M(x,u) = (M_1(x,u), M_2(x,u), M_3(x,u))$, $M_i(x,u) \equiv M_i(x_1,x_2,x_3,u_1,u_2,u_3)$, $M_i \in C(\overline{\Omega} \times \mathbb{R}^3)$, while $i=1,2,3$, satisfying the following properties (see also Remark 1)

$$M_i(x,0) = 0, \quad \forall x \in \overline{\Omega}, \quad i=1,2,3, \tag{2.9}$$

the monotonicity condition

$$\sum_{i=1}^{3} (M_i(x,u) - M_i(x,v))(u_i - v_i) \geq 0, \quad \forall x \in \overline{\Omega},\ u,v \in \mathbb{R}^3, \tag{2.10}$$

and *the globally Lipschitz condition*

$$\begin{aligned} &|M_i(x,u) - M_i(x',v)| \leq L_i(|x-x'| + |u-v| + |u-v|^{1+\delta}), \\ &\quad \forall x,x' \in \overline{\Omega},\ u,v \in \mathbb{R}^3, \quad i=1,2,3, \end{aligned} \tag{2.11}$$

for certain positive constants L_1, L_2, L_3, $\delta \in [0,2]$.

For $f = (f_1,f_2,f_3) \in (L^2(\Omega))^3$ let us consider the problem

$$\begin{cases} -\dfrac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} = f_i & \text{in } \Omega\,, \\ u^\varepsilon = 0 & \text{on } \Gamma_\Omega\,, \qquad\qquad i=1,2,3. \\ \sigma_{ij}^\varepsilon n_j = 0 & \text{on } \Sigma \setminus \bigcup T^\varepsilon, \\ \sigma_{ij}^\varepsilon n_j + \beta(\varepsilon) M_i(x,u^\varepsilon) = 0 & \text{on } \bigcup T^\varepsilon\,, \end{cases} \tag{2.12}$$

Above, $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ denotes the displacement vector, $\sigma_{ij}^\varepsilon \equiv \sigma_{ij}(u^\varepsilon) = a_{ijkl} e_{kl}(u^\varepsilon)$ (cf. (2.8)), while n stands for the unit outer normal to Ω along Σ, namely, $n = (0,0,-1)$. The boundary conditions on T^ε linking stresses and displacements are referred to as Winkler-Robin boundary conditions. The parameter $\beta(\varepsilon)$ can range from very large to very small or it can be of order 1.

Remark 1. Note that condition (2.11) follows in the case where M_i are smooth functions $M_i \in C^1(\overline{\Omega} \times \mathbb{R}^3)$, satisfying:

$$\left| \frac{\partial M_i}{\partial u_j}(x,u) \right| \leq D_{ij}(1 + |u|^\delta),\ \forall (x,u) \in \overline{\Omega} \times \mathbb{R}^3,\ i,j=1,2,3,$$

for certain positive constants D_{ij}, $i,j = 1,2,3$. Also they follow in the simplified case where $M_i(x,u) \equiv M_i(x,u_i)$, with M_i monotonic and Lipschitz function in the u_i variable (see, e.g., Section 3.3.1 in [22]). In addition, in the linear case, the functions M_i read: $M_i(x,u) \equiv M_{ij}(x) u_j$, where $(M_{ij})_{i,j=1,2,3}$ is a symmetric and positive definite 3×3-matrix, $M_{ij} \in C(\overline{\Omega})$ (see [7,8,10]).

Also, it should be emphasized that hypotheses (2.9)–(2.11) on M_i can be weakened to be defined in $\overline{\Sigma} \times \mathbb{R}^3$ such that $M_i(x,\phi(x)) u_i \in H^1(\Omega)$ for all $\phi \in (C^1(\overline{\Omega}))^3$ and $u \in (H^1(\Omega))^3$ (cf. (3.19) and (6.13)).

3 The abstract framework and the limit problems

Let us denote by $\mathbf{V}$ the space obtained by completion of $\{v \in (C^1(\Omega))^3 \, : \, v = 0 \text{ on } \Gamma_\Omega\}$ with the norm generated by the scalar product:

$$(u, v)_{\mathbf{V}} = \int_\Omega e_{ij}(u) e_{ij}(v) \, dx \, . \tag{3.1}$$

For fixed $\varepsilon > 0$, the weak formulation of problem (2.12) reads: find $u^\varepsilon \in \mathbf{V}$, satisfying

$$\int_\Omega \sigma_{ij}(u^\varepsilon) e_{ij}(v) \, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_i(\hat{x}, u^\varepsilon) v_i \, d\hat{x} = \int_\Omega f_i v_i \, dx, \ \forall v \in \mathbf{V}. \tag{3.2}$$

Above, and if no confusion arises, we have identified the point $\hat{x} = (x_1, x_2)$ in the plane with $(x_1, x_2, 0) \in \Sigma$.

On account of (2.6) and (2.7), the first integral on the left hand side of (3.2) defines a bilinear, symmetric continuous and coercive form on $\mathbf{V} \subset (L^2(\Omega))^3$. As for the second integral, on account of (2.9)–(2.11), we can write

$$0 \le \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_i(\hat{x}, u^\varepsilon) u_i^\varepsilon \, d\hat{x}, \qquad \text{and} \tag{3.3}$$

$$\left| \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_i(\hat{x}, u^\varepsilon) v_i \, d\hat{x} \right| \le C \beta(\varepsilon) \int_{\bigcup T^\varepsilon} (|u^\varepsilon| + |u^\varepsilon|^{1+\delta}) |v| \, d\hat{x}, \tag{3.4}$$

for certain positive constant C independent of ε. Note that the last integral in (3.4) is well defined for $\delta \in [0, 2]$ because of the Hölder's inequality, namely,

$$\begin{aligned} \int_{\bigcup T^\varepsilon} (|u_i^\varepsilon| + |u_i^\varepsilon|^{1+\delta}) |v_j| \, d\hat{x} \le & C_\varepsilon \big(\|u_i^\varepsilon\|_{L^2(\bigcup T^\varepsilon)} \|v_j\|_{L^2(\bigcup T^\varepsilon)} \\ & + \|u_i^\varepsilon\|^{1+\delta}_{L^{(1+\delta)4/3}(\bigcup T^\varepsilon)} \|v_j\|_{L^4(\bigcup T^\varepsilon)} \big), \end{aligned} \tag{3.5}$$

and the continuous embedding $H^1(\Omega) \subset L^4(\Sigma)$.

Also, on account of (2.9)–(2.11), we can define a monotonic, hemicontinuous and coercive operator $A^\varepsilon : \mathbf{V} \longmapsto \mathbf{V}'$ as follows

$$\langle A^\varepsilon u, \, v \rangle = \int_\Omega \sigma_{ij}(u) e_{ij}(v) \, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_i(\hat{x}, u) v_i \, d\hat{x}, \qquad \text{for } u, v \in \mathbf{V}.$$

Solving (3.2) leads us to the variational inequality

$$\langle A^\varepsilon u^\varepsilon, \, v - u^\varepsilon \rangle \ge \big(f, \; v - u^\varepsilon \big)_{(L^2(\Omega))^3}, \quad \forall v \in \mathbf{V}, \tag{3.6}$$

which has a unique solution $u^\varepsilon \in \mathbf{V}$ that also satisfies

$$\int_\Omega \sigma_{ij}(v) e_{ij}(v - u^\varepsilon) dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_i(\hat{x}, v)(v_i - u_i^\varepsilon) \, d\hat{x} \ge \int_\Omega f_i (v_i - u_i^\varepsilon) \, dx, \ \forall v \in \mathbf{V}. \tag{3.7}$$

These results on variational inequalities are a consequence of Theorems 8.2–8.4 in Sections II.8.2 and II.8.3 of [15] (cf. [9] for further references).

To show that the solution $u^\varepsilon \in \mathbf{V}$ of (3.6) is a solution of (3.2) it suffices to take $v = u^\varepsilon \pm w$ in (3.6) for any $w \in \mathbf{V}$. In addition, to show that u^ε is the unique element of $\mathbf{V}$ satisfying (3.2) can be performed by contradiction, on account of (2.10). Indeed, considering that there are two solutions of (3.2), u^ε and $\widetilde{u}^\varepsilon$, we easily get

$$\int_\Omega \sigma_{ij}(u^\varepsilon - \widetilde{u}^\varepsilon) e_{ij}(u^\varepsilon - \widetilde{u}^\varepsilon) dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} \big(M_i(\hat{x}, u^\varepsilon) - M_i(\hat{x}, \widetilde{u}^\varepsilon)\big)(u_i^\varepsilon - \widetilde{u}_i^\varepsilon)\, d\hat{x} = 0,$$

and the positiveness of the second integral provides $u^\varepsilon = \widetilde{u}^\varepsilon$.

In addition, on account of the Poincaré and Korn inequalities, and (2.7), taking $v = u^\varepsilon$ in (3.2), we have

$$\int_\Omega (e_{ij}(u^\varepsilon))^2\, dx \leq C, \quad \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_i(\hat{x}, u^\varepsilon) u_i^\varepsilon\, d\hat{x} \leq C, \tag{3.8}$$

where C is a constant independent of ε. Therefore, we have proved the following result on the weak solution of (2.12).

Theorem 1. *There is a unique solution $u^\varepsilon \in \mathbf{V}$ of the Equation (3.2), which is uniformly bounded*

$$\|u^\varepsilon\|_{\mathbf{V}} \leq C. \tag{3.9}$$

As a consequence of (3.9), for any sequence, still denoted by ε, we can extract a subsequence such that

$$u^\varepsilon \longrightarrow u^0 \text{ in } (H^1(\Omega))^3 - weak, \text{ as } \varepsilon \to 0, \tag{3.10}$$

for some $u^0 \in \mathbf{V}$. The aim of this work is to identify u^0 with the unique solution of certain homogenized problems which depend on the parameter β^* in (1.1).

3.1 The homogenized problems

In order to make the reading of the paper easier, here we state the three homogenized problems (3.11),(3.12); (3.11),(3.13) and (3.11),(3.14), which are obtained depending on the value of β^* in (1.1). We derive all these problems in Section 4, by using the technique of matched asymptotic expansions. For any β^*, u^0 verifies

$$\begin{cases} -\dfrac{\partial \sigma_{ij}(u^0)}{\partial x_j} = f_i & \text{in } \Omega, \qquad i = 1, 2, 3, \\ u^0 = 0 & \text{on } \Gamma_\Omega. \end{cases} \tag{3.11}$$

In addition, if $\beta^* > 0$, the averaged equation on Σ reads

$$\sigma_{ij}(u^0) n_j + \beta^* |T| M_i(\hat{x}, u^0) = 0 \text{ on } \Sigma. \tag{3.12}$$

If $\beta^* = 0$, the boundary condition on Σ reads

$$\sigma_{ij}(u^0) n_j = 0 \text{ on } \Sigma. \tag{3.13}$$

If $\beta^* = +\infty$, the boundary condition on Σ reads

$$u^0 = 0 \text{ on } \Sigma. \tag{3.14}$$

We state the weak formulation of each homogenized problem:

- For problem (3.11),(3.12): find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \sigma_{ij}(u^0)e_{ij}(v)\,dx + \beta^*|T| \int_\Sigma M_i(\hat{x}, u^0)v_i\,d\hat{x} = \int_\Omega f_i v_i\,dx,\ \forall v \in \mathbf{V}. \tag{3.15}$$

The variational inequality satisfied by u^0 is

$$\int_\Omega \sigma_{ij}(v)e_{ij}(v - u^0)\,dx + \beta^*|T| \int_\Sigma M_i(\hat{x}, v)(v_i - u_i^0)\,d\hat{x} \geq \int_\Omega f_i(v_i - u_i^0)\,dx,\ \forall v \in \mathbf{V}. \tag{3.16}$$

- For problem (3.11),(3.13): find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \sigma_{ij}(u^0)e_{ij}(v)\,dx = \int_\Omega f_i v_i\,dx, \quad \forall v \in \mathbf{V}. \tag{3.17}$$

- For problem (3.11),(3.14): find $u^0 \in (H_0^1(\Omega))^3$ satisfying

$$\int_\Omega \sigma_{ij}(u^0)e_{ij}(v)\,dx = \int_\Omega f_i v_i\,dx, \quad \forall v \in (H_0^1(\Omega))^3. \tag{3.18}$$

The existence and uniqueness of solution of problems (3.17) and (3.18) is well known in the literature (cf., for example, Sections I.3.3 and I.3.5 in [21], [1] and [17]), while that of problems (3.15) and (3.16) follow as that of (3.2) and (3.7) with minor modifications (cf. the proof in Theorem 1). Note that, using the same reasoning as in (3.4) and (3.5), we get

$$\int_\Sigma M_i(\hat{x}, u^0)v_i\,d\hat{x} \leq C \sum_{i,j=1}^{3} \big(\|u_i^0\|_{L^2(\Sigma)}\|v_j\|_{L^2(\Sigma)} + \|u_i^0\|^{1+\delta}_{L^{(1+\delta)4/3}(\Sigma)}\|v_j\|_{L^4(\Sigma)}\big). \tag{3.19}$$

4 Asymptotic expansions

In this section, we use the technique on matched asymptotic expansions (cf., e.g., [8] and references therein) to derive the homogenized problems (3.11),(3.12); (3.11),(3.13) and (3.11),(3.14).

For simplicity, throughout the section, we consider the technical restriction that the T^ε are homothetics of T, and that the parameters satisfy $r_\varepsilon \ll \varepsilon$ (cf. (1.1), (2.1), (2.2) and Remark 2 in this connection).

Taking into account (3.9) we consider the asymptotic expansions for the displacement vector u^ε of (2.12) as follows. Assume an outer expansion

$$u^\varepsilon(x) = u^0(x) + r_\varepsilon u^1(x) + \dots, \qquad \text{in } \Omega \cap \{x_3 > d\} \quad \forall d > 0, \tag{4.1}$$

which in fact, is supposed to be valid for x "far" from the regions $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$, namely, at a distance $\rho \gg r_\varepsilon$ from the center $\widetilde{x}_\mathrm{k}$. In addition, we assume a local expansion in a neighborhood of each reaction region $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$

$$u^\varepsilon(x) = V^0(y) + r_\varepsilon V^1(y) + \dots \qquad \text{for } y \in \overline{\mathbb{R}^{3+}}. \tag{4.2}$$

Above, and in what follows, we denote by

$$y = (x - \widetilde{x}_\mathrm{k})/r_\varepsilon \tag{4.3}$$

the local variable in a neighborhood of each center $\widetilde{x}_\mathrm{k}$, $\mathrm{k} \in \mathcal{J}^\varepsilon$, and by u^1, V^1, and dots we denote further terms in the asympotic series containing lower order functions that we are not using in our analysis.

By matching the local and outer expansions for u^ε, at the first order, we can write

$$\lim_{|y|\to\infty} V^0(y) = \lim_{x\to\widetilde{x}_\mathrm{k}} u^0(x). \tag{4.4}$$

By replacing (4.1) in (2.12) we obtain Equations (3.11) for u^0 plus some boundary condition on Σ to be determined. In order to do this, we first determine $V^0(y)$ in the local expansion (4.2). On account of (2.11), let us first obtain formal asymptotics expansions for $M_i(x, u^\varepsilon(x))$, while $i = 1, 2, 3$, in a neighborhood of each region $T^\varepsilon_{\widetilde{x}_k}$:

$$M_i(x, u^\varepsilon(x)) = M_i(r_\varepsilon y + \widetilde{x}_\mathrm{k},\, u^\varepsilon(x)) = M_i(\widetilde{x}_\mathrm{k},\, V^0(y)) + \dots. \tag{4.5}$$

Taking derivatives in (2.12) with respect to y, cf. (4.3), we replace (4.2) and (4.5) in (2.12), and take into account the continuity of the elastic coefficients $a_{ijkl}(x)$ and (4.4). Then, for V^0 we have the equations:

$$\begin{cases} -\dfrac{\partial \sigma^\mathrm{k}_{ij,y}(V^0)}{\partial y_j} = 0 & \text{in } \mathbb{R}^{3+}\,, \\ \sigma^\mathrm{k}_{ij,y}(V^0) n_j = 0 & \text{on } \{y_3 = 0\} \setminus T\,, \\ \sigma^\mathrm{k}_{ij,y}(V^0) n_j + r_\varepsilon \beta(\varepsilon) M_i(\widetilde{x}_\mathrm{k}, V^0) = 0 & \text{on } T\,, \\ V^0(y) \longrightarrow u^0(\widetilde{x}_\mathrm{k}) & \text{as } |y| \to \infty,\, y_3 > 0\,, \end{cases} \qquad i = 1, 2, 3. \tag{4.6}$$

Above, and in what follows, for simplicity, we write the upper index k in the strain tensor to denote:

$$\sigma^\mathrm{k}_{ij,y}(V) = a_{ijkl}(\widetilde{x}_\mathrm{k}) e_{kl,y}(V) \quad \text{where} \quad e_{kl,y}(V) = \frac{1}{2}\Big(\frac{\partial V_k}{\partial y_l} + \frac{\partial V_l}{\partial y_k}\Big). \tag{4.7}$$

We also observe a dependence of V^0 on ε and on the center of the reaction region $\widetilde{x}_\mathrm{k}$.

4.1 The boundary condition on Σ

Considering (3.11), in order to obtain the boundary condition on Σ for u^0, we perform an integration by parts over the equilibrium equations in *coin-like domains*, neglecting the stresses across the lateral surface. We define one of these domains as follows. Let us consider Σ_1 an open domain contained in Σ such that $\partial\Sigma_1$ does not touch any region $T^\varepsilon_{\widetilde{x}_k}$. Let $\tilde{\delta}(\varepsilon)$ be positive, $r_\varepsilon \ll \tilde{\delta}(\varepsilon) \ll 1$. We consider the coin-like domain

$$\Omega^{\tilde{\delta}(\varepsilon)}_{\Sigma_1} = \Omega \cap (\Sigma_1 \times (0, \tilde{\delta}(\varepsilon))). \tag{4.8}$$

Let $\Gamma_{\tilde{\delta}(\varepsilon)}$ denote the lateral boundary of $\Omega^{\tilde{\delta}(\varepsilon)}_{\Sigma_1}$ in such a way that

$$\partial\Omega^{\tilde{\delta}(\varepsilon)}_{\Sigma_1} = \overline{\Gamma_{\tilde{\delta}(\varepsilon)}} \cup \overline{\Sigma_1^{\tilde{\delta}(\varepsilon)}} \cup \overline{\Sigma_1}, \tag{4.9}$$

where $\Sigma_1^{\tilde{\delta}(\varepsilon)}$ denotes the set $\{x : (x_1, x_2, 0) \in \Sigma_1,\ x_3 = \tilde{\delta}(\varepsilon)\}$. On $\Sigma_1^{\tilde{\delta}(\varepsilon)}$, we are "far" from the reaction regions $T^\varepsilon_{\widetilde{x}_k}$ and (4.1) holds. "Near" each region T^ε, we need to use the local expansion (4.2). In particular, on each reaction region $T^\varepsilon_{\widetilde{x}_k}$ we have (cf. (4.3) and (4.7))

$$\sigma_{i3}(u^\varepsilon) = \sigma_{i3}(V^0(y)) \approx a_{i3kh}(\widetilde{x}_k)\frac{1}{r_\varepsilon}e_{kh,y}(V^0(y)) + \ldots = \frac{1}{r_\varepsilon}\sigma^k_{i3,y}(V^0(y)) + \ldots . \tag{4.10}$$

Now, we multiply the divergence vector in (2.12) by e^i with $e^i_j = \delta_{ij}$, and apply the Green formula over $\Omega^{\tilde{\delta}(\varepsilon)}_{\Sigma_1}$ (cf. (4.8) and (4.9)) to obtain

$$\int_{\Sigma_1 \cap \bigcup T^\varepsilon_{\widetilde{x}_k}} \sigma_{i3}(u^\varepsilon)d\hat{x} = \int_{\Omega^{\tilde{\delta}(\varepsilon)}_{\Sigma_1}} f_i dx + \int_{\Gamma_{\tilde{\delta}(\varepsilon)}} \sigma_{ij}(u^\varepsilon)n_j d\Gamma_{\tilde{\delta}} + \int_{\Sigma_1^{\tilde{\delta}(\varepsilon)}} \sigma_{i3}(u^\varepsilon)d\hat{x}. \tag{4.11}$$

We observe that, by construction (cf. (3.9)), the two first integrals on the right-hand side of (4.11) converge towards zero as $\varepsilon \to 0$. For the other integral, we use the approximation (4.1), namely

$$\sigma_{i3}(u^\varepsilon)\Big|_{x_3=\tilde{\delta}(\varepsilon)} = \sigma_{i3}(u^0)\Big|_{x_3=0} + \ldots .$$

Therefore, introducing this and (4.10) in (4.11), and performing the change of variable (4.3), we write

$$\int_{\Sigma_1}\sigma_{i3}(u^0)d\hat{x} = \lim_{\varepsilon\to 0}\sum_{\widetilde{x}_k\in\Sigma_1}\int_{T^\varepsilon_{\widetilde{x}_k}}\sigma_{i3}\big(V^0\big(\frac{x-\widetilde{x}_k}{r_\varepsilon}\big)\big)d\hat{x} = \lim_{\varepsilon\to 0}\sum_{\widetilde{x}_k\in\Sigma_1}\int_T r_\varepsilon\sigma^k_{i3,y}(V^0(y))d\hat{y}. \tag{4.12}$$

Owing to the relation on T in (4.6), we rewrite (4.12) as follows

$$\int_{\Sigma_1} \sigma_{i3}(u^0)d\hat{x} = \lim_{\varepsilon\to 0}\beta(\varepsilon)r_\varepsilon^2 \sum_{\widetilde{x}_k\in\Sigma_1}\int_T M_i(\widetilde{x}_k, V^0(\hat{y}, 0))d\hat{y}. \tag{4.13}$$

Now, using that $\beta(\varepsilon)r_\varepsilon \to 0$, in (4.6), in a first approach we can take $V^0(y) \approx u^0(\widetilde{x}_k)$, $\forall y \in \mathbb{R}^{3+}$, and therefore,

$$\int_{\Sigma_1} \sigma_{i3}(u^0)d\hat{x} = \lim_{\varepsilon\to 0}\beta(\varepsilon)r_\varepsilon^2\varepsilon^{-2} \sum_{\widetilde{x}_k\in\Sigma_1}\varepsilon^2 M_i(\widetilde{x}_k, u^0(\widetilde{x}_k))\int_T d\hat{y}. \tag{4.14}$$

Under assumptions of smoothness for M_i and u^0, when $\beta^* > 0$, (4.14) gives

$$\int_{\Sigma_1} \sigma_{i3}(u^0) d\hat{x} = \beta^* |T| \int_{\Sigma_1} M_i(\hat{x}, 0, u^0(\hat{x}, 0)) d\hat{x}\,,$$

while, when $\beta^* = 0$, (4.14) gives $\int_{\Sigma_1} \sigma_{i3}(u^0)\, d\hat{x} = 0$.

Then, using the somewhat arbitrary choice of $\Sigma_1 \subset \Sigma$, we deduce the following boundary conditions to be added to (3.11) in order to determine the first term of the asymptotic expansion (4.1), namely u^0:

$$\begin{aligned} &\sigma_{ij}(u^0) n_j + \beta^* |T| M_i(\hat{x}, u^0) = 0 \text{ on } \Sigma, \quad \text{ when } \beta^* > 0, \\ &\sigma_{ij}(u^0) n_j = 0 \text{ on } \Sigma, \quad \text{ when } \beta^* = 0. \end{aligned}$$

It should be noted that in both cases $\beta(\varepsilon) r_\varepsilon \to 0$ is satisfied to somehow compensate $r_\varepsilon \varepsilon^{-2} \to +\infty$ and $\beta^* \geq 0$ in (1.1).

Also, when $\beta^* = +\infty$, we multiply by the factor $\varepsilon^2 (\beta(\varepsilon) r_\varepsilon^2)^{-1}$ in both sides of the equality (4.11). Then, taking limits as $\varepsilon \to 0$, since $\varepsilon^2 (\beta(\varepsilon) r_\varepsilon^2)^{-1} \to 0$ as $\varepsilon \to 0$, using the equation on T in (4.6), and the reasoning in (4.12), (4.13) and (4.14), under the basis that $\beta(\varepsilon) r_\varepsilon \to 0$ (cf. Remark 2), we get

$$0 = \lim_{\varepsilon \to 0} \frac{1}{\beta(\varepsilon) r_\varepsilon} \sum_{\widetilde{x}_k \in \Sigma_1} \varepsilon^2 \int_{T^\varepsilon_{\widetilde{x}_k}} \sigma^{\mathrm{k}}_{i3,y}(V^0(y)) d\hat{y} = \lim_{\varepsilon \to 0} \sum_{\widetilde{x}_k \in \Sigma_1} \varepsilon^2 M_i(\widetilde{x}_{\mathrm{k}}, u^0(\widetilde{x}_{\mathrm{k}})) \int_T d\hat{y},$$

and therefore,

$$|T| \int_{\Sigma_1} M_i(\hat{x}, 0, u^0(\hat{x}, 0)) d\hat{x} = 0.$$

Under the new hypothesis

$$L|u| \leq |M(x,u)|, \quad \forall x \in \overline{\Omega}, u \in \mathbb{R}^3, \tag{4.15}$$

for a certain positive constant L, we get $\int_{\Sigma_1} |u^0| d\hat{x} = 0$, and, as above, we have the boundary condition on Σ to be added to (3.11) when $\beta^* = +\infty$: $u^0 = 0$ on Σ.

All of this gives the homogenized problems stated in Section 3.1.

Remark 2. Let us notice that, in the case where $r_\varepsilon = O(\varepsilon)$, the ansatz (4.1)–(4.2) should be replaced by two-scale asymptotic expansions (cf., e.g, [20] for the technique). Similarly, in the case where $\beta^* = +\infty$ and $\lim_{\varepsilon \to 0} \beta(\varepsilon) r_\varepsilon \neq 0$, the reasoning in (4.11)–(4.14) should be suitably modified in order to get $u^0 = 0$ on Σ (cf. [8] for the linear case). We avoid introducing here these formal procedures since the technique used in Sections 5–6 covers these two cases as well as the case of the isoperimetric T^p, $p = 0, 1, \ldots \mathfrak{M}$ (cf. (2.1)).

5 Convergence for solutions

This section is devoted to justifying the asymptotics in Section 4. First, we consider the critical case where $\beta^* > 0$ and show that the limit of u^ε in $(H^1(\Omega))^3$-weak given by (3.10) is the solution of the homogenized problem (3.15); see

Theorem 3. Then, we consider the extreme cases $\beta^* = 0$ and $\beta^* = +\infty$, and show that the limit of u^ε in $(H^1(\Omega))^3$ is the solution of the homogenized problems (3.17) and (3.18) respectively (see Theorem 4 and 5, respect.). The proofs rely on a result of convergence of measures that we state below and prove in Section 6.

Theorem 2. *Assume* (2.2) *and* (2.9)–(2.11). *For* $\phi \in (C^1(\overline{\Omega}))^3$ *and* $v \in (H^1(\Omega))^3$,

$$\left|\frac{\varepsilon^2}{r_\varepsilon^2}\int_{\bigcup T^\varepsilon} M_i(x,\phi)v_i\,d\hat{x} - |T|\int_\Sigma M_i(x,\phi)v_i\,d\hat{x}\right| \le C\Big(\varepsilon^{1/2}+\frac{\varepsilon}{r_\varepsilon^{1/2}}\Big)\|v\|_{(H^1(\Omega))^3}, \quad (5.1)$$

where C is a constant independent of ε and v.

Theorem 3. *For $\beta^* > 0$ in* (1.1), *the solution of* (3.2) *converges in $(H^1(\Omega))^3$-weak towards the solution of* (3.15) *as $\varepsilon \to 0$.*

Proof. In order to prove that the weak limit of u^ε given by (3.10) satisfies (3.15), we take $v = \phi \in (C^1(\overline{\Omega}))^3$ such that $\phi = 0$ on Γ_Ω in (3.7) and pass to the limit as $\varepsilon \to 0$. Because of (3.10), we have

$$\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(\phi)e_{ij}(\phi - u^\varepsilon)\,dx = \int_\Omega \sigma_{ij}(\phi)e_{ij}(\phi - u^0)\,dx \quad \text{and} \quad (5.2)$$

$$\lim_{\varepsilon\to 0}\int_\Omega f_i(\phi_i - u_i^\varepsilon)\,dx = \int_\Omega f_i(\phi_i - u_i^0)\,dx. \quad (5.3)$$

Besides, by Theorem 2, (2.2) and $\beta^* > 0$ in (1.1), (3.9) and (3.10), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\beta(\varepsilon)\int_{\bigcup T^\varepsilon} M_i(\hat{x},\phi)(\phi_i - u_i^\varepsilon)\,d\hat{x} - \beta^*\,|T|\int_\Sigma M_i(\hat{x},\phi)(\phi_i - u_i^0)\,d\hat{x}\\
&= \lim_{\varepsilon\to 0}\Big(\frac{\beta(\varepsilon)r_\varepsilon^2}{\varepsilon^2} - \beta^*\Big)\frac{\varepsilon^2}{r_\varepsilon^2}\int_{\bigcup T^\varepsilon} M_i(\hat{x},\phi)(\phi_i - u_i^\varepsilon)\,d\hat{x}\\
&\quad +\beta^*\lim_{\varepsilon\to 0}\Big(\frac{\varepsilon^2}{r_\varepsilon^2}\int_{\bigcup T^\varepsilon} M_i(\hat{x},\phi)(\phi_i - u_i^\varepsilon)\,d\hat{x} - |T|\int_\Sigma M_i(\hat{x},\phi)(\phi_i - u_i^\varepsilon)\,d\hat{x}\Big)\\
&\quad +\beta^*\,|T|\lim_{\varepsilon\to 0}\int_\Sigma M_i(\hat{x},\phi)(u_i^0 - u_i^\varepsilon)\,d\hat{x} = 0\,, \qquad (5.4)
\end{aligned}$$

where we have applied inequality (5.1) to $v = \phi - u^\varepsilon$. Then, gathering (5.2), (5.3) and (5.4) yields

$$\int_\Omega \sigma_{ij}(\phi)e_{ij}(\phi - u^0)\,dx + \beta^*|T|\int_\Sigma M_i(\hat{x},\phi)(\phi_i - u_i^0)\,d\hat{x} \ge \int_\Omega f_i(\phi_i - u_i^0)\,dx,$$

$\forall\phi \in (C^1(\overline{\Omega}))^3$ such that $\phi = 0$ on Γ_Ω, and, using (3.19), by density it holds for all $\phi \in \mathbf{V}$. Namely, u^0 is the solution of (3.16). By the uniqueness of solution, the whole sequence $u^\varepsilon \to u^0$ as $\varepsilon \to 0$ in the weak topology of $(H^1(\Omega))^3$ and taking $\phi = u^0 \pm v$ for any $v \in \mathbf{V}$ we get that u^0 is also the unique solution of (3.15). Thus, the theorem is proved. □

Theorem 4. *For $\beta^* = 0$ in* (1.1), *the solution of* (3.2) *converges in* $(H^1(\Omega))^3$ *towards the solution of* (3.17) *as* $\varepsilon \to 0$.

Proof. Rewriting the proof of Theorem 3 with minor modifications we get the weak convergence of the solutions in $(H^1(\Omega))^3$, the limit u_0 being the solution of (3.17). To obtain the convergence in the strong topology, we consider $v = u^\varepsilon$ in (3.2) and take limits as $\varepsilon \to 0$, we have:

$$\lim_{\varepsilon\to 0}\Big(\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(u^\varepsilon)\,dx + \beta(\varepsilon)\int_{\bigcup T^\varepsilon} M_i(\hat{x},u^\varepsilon)u_i^\varepsilon\,d\hat{x}\Big)$$
$$= \lim_{\varepsilon\to 0}\int_\Omega f_i u_i^\varepsilon\,dx = \int_\Omega f_i u_i^0\,dx = \int_\Omega \sigma_{ij}(u^0)e_{ij}(u^0)\,dx,$$

where we have used (3.17).

Now taking into account that (3.1) defines a norm in $\mathbf{V}$, the lower semi-continuity of the norm for the weak topology and (3.3), we write

$$\int_\Omega \sigma_{ij}(u^0)e_{ij}(u^0)dx \le \liminf_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(u^\varepsilon)dx$$
$$\le \limsup_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(u^\varepsilon)dx \le \int_\Omega \sigma_{ij}(u^0)e_{ij}(u^0)dx.$$

Consequently,

$$\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(u^\varepsilon)\,dx = \int_\Omega \sigma_{ij}(u^0)e_{ij}(u^0)\,dx,$$

and the convergence of u^ε in $(H^1(\Omega))^3$ holds. The theorem is proved. □

Now, we consider the case where $\beta^* = +\infty$ and we show that the limit of u^ε in $(H^1(\Omega))^3$-weak given by (3.10) is the solution of the Dirichlet homogenized problem (3.18); see Theorem 5.

In addition to the properties of M_i (2.9)–(2.11), in order to obtain the limit we add the hypothesis (4.15). This new assumption gives (see (3.3) and (3.8) to compare)

$$L\,\beta(\varepsilon)\int_{\bigcup T^\varepsilon}\sum_{i=1}^{3}(u_i^\varepsilon)^2\,d\hat{x} \le \beta(\varepsilon)\int_{\bigcup T^\varepsilon} M_i(\hat{x},u^\varepsilon)u_i^\varepsilon\,d\hat{x} \le C. \tag{5.5}$$

Theorem 5. *For $\beta^* = +\infty$ in* (1.1) *and* (4.15), *the solution of* (3.2) *converges in* $(H^1(\Omega))^3$ *towards the solution of* (3.18) *as* $\varepsilon \to 0$.

Proof. Taking $v \in (C_0^\infty(\Omega))^3$ in the variational formulation (3.2), the integral over $\bigcup T^\varepsilon$ vanishes, and passing to the limit in the other terms, it is easy to check that the limit u^0 satisfies

$$-\frac{\partial \sigma_{ij}(u^0)}{\partial x_j} = f_i \quad \text{in } \Omega\,, \quad i = 1,2,3.$$

In order to prove that $u^0 = 0$ on Σ, we first show the following equality

$$\int_{\Sigma} M_i(\hat{x}, \phi) u_i^0 \, d\hat{x} = 0 \qquad \forall \phi \in (C^1(\overline{\Omega}))^3, \quad \phi = 0 \text{ on } \Gamma_{\Omega}. \tag{5.6}$$

Indeed, by Theorem 2, it follows that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{r_\varepsilon^2} \int_{\bigcup T^\varepsilon} M_i(\hat{x}, \phi) u_i^\varepsilon \, d\hat{x} = |T| \int_{\Sigma} M_i(\hat{x}, \phi) u_i^0 \, d\hat{x} \,. \tag{5.7}$$

On the other hand, using (2.9), (2.11), (5.5) and the area of $|\bigcup T^\varepsilon|$, we get

$$\left| \frac{\varepsilon^2}{r_\varepsilon^2} \int\limits_{\bigcup T^\varepsilon} M_i(\hat{x}, \phi) u_i^\varepsilon \, d\hat{x} \right| \leq C \frac{\varepsilon^2}{r_\varepsilon^2 (\beta(\varepsilon))^{\frac{1}{2}}} (\beta(\varepsilon))^{\frac{1}{2}} \| u^\varepsilon \|_{(L^2(\bigcup T^\varepsilon))^3} |\cup T^\varepsilon|^{\frac{1}{2}} \leq C \Big(\frac{\varepsilon^2}{r_\varepsilon^2 \beta(\varepsilon)} \Big)^{\frac{1}{2}}.$$

Hence, from the hypothesis $\beta^* = +\infty$, we obtain

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{r_\varepsilon^2} \int_{\bigcup T^\varepsilon} M_i(\hat{x}, \phi) u_i^\varepsilon \, d\hat{x} = 0. \tag{5.8}$$

Now, gathering (5.7) and (5.8), we obtain (5.6). Then, on account of (3.19), we use a density argument to derive that

$$\int_{\Sigma} M_i(\hat{x}, v) u_i^0 \, d\hat{x} = 0 \qquad \forall v \in \mathbf{V}. \tag{5.9}$$

Finally, (5.9) for $v = u^0$ and (4.15) give $\int_\Sigma (u^0)^2 d\hat{x} = 0$ and consequently, $u^0 = 0$ on Σ. Thus, by the uniqueness of solution, the convergence of the whole sequence u^ε towards u^0 in the weak topology of $(H^1(\Omega))^3$ holds true, as $\varepsilon \to 0$.

To obtain the strong convergence, we rewrite the proof of Theorem 4 with minor modifications: here, we use again the lower semi-continuity of the norm and consider that u^0 is the solution of (3.18). The theorem is proved. $\square$

6 Proof of Theorem 2

In this section, for the sake of completeness, we introduce some auxiliary results that prove to be useful for the proof of Theorem 2. We refer to Lemma 2.4 in Section II.3 of [18] for the proof of Lemma 1. We use the Poincaré inequality (cf. Section I.1 in [21]) and a change of variable to show Lemma 2 and refer to [10] for the proof of Lemma 3.

Lemma 1. *For $w \in H^1(\Omega)$, we have*

$$\|w\|_{L^2(\Omega \cap \{0 < x_3 < \varepsilon\})} \leq C \varepsilon^{1/2} \|w\|_{H^1(\Omega)}, \tag{6.1}$$

$$\Big| \frac{1}{\varepsilon} \int_{\Omega \cap \{0 < x_3 < \varepsilon\}} w \, dx - \int_{\Sigma} w \, d\hat{x} \Big| \leq C \varepsilon^{1/2} \Big\| \frac{\partial w}{\partial x_3} \Big\|_{L^2(\Omega)}. \tag{6.2}$$

Lemma 2. *Let $Y_0^\varepsilon = (-\varepsilon/2, \varepsilon/2)^2 \times (0, \varepsilon)$. If $w \in H^1(Y_0^\varepsilon)$ such that $\int_{Y_0^\varepsilon} w \, dx =$ 0, then,*

$$\|w\|_{L^2(Y_0^\varepsilon)} \leq C \varepsilon \|\nabla w\|_{L^2(Y_0^\varepsilon)}.$$

Lemma 3. *Let Y_0^ε be the domain defined in Lemma 2. Let $K \in \mathbb{R}^+$ such that $T \subset B(0,K)$. If r_ε is a positive constant such that $r_\varepsilon T \subset B(0,Kr_\varepsilon) \Subset (-\varepsilon/2,\varepsilon/2)^2$, then,*

$$\|w\|_{L^2(r_\varepsilon T)} \leq C\big(r_\varepsilon^{1/2}\varepsilon^{-1}\|w\|_{L^2(Y_0^\varepsilon)} + r_\varepsilon^{1/2}\|\nabla w\|_{L^2(Y_0^\varepsilon)}\big) \quad \forall w \in H^1(Y_0^\varepsilon),$$

where C is a constant depending on T but independent of w and ε.

We divide the proof of (5.1) into three steps.

First step: The integral over $\bigcup T^\varepsilon$ on the left-hand side of (5.1) *is transformed into volume integrals (cf.* (6.7)*).*

For each $\mathrm{k} \in \mathcal{J}^\varepsilon$, we denote by $Y_{\widetilde{x}_\mathrm{k}}^\varepsilon$ the homothetic domain of $Y=(-1/2,1/2)^2 \times(0,1)$ of ratio ε after translation to the point $\widetilde{x}_\mathrm{k}^\varepsilon$, that is

$$Y_{\widetilde{x}_\mathrm{k}}^\varepsilon = \widetilde{x}_\mathrm{k}^\varepsilon + \varepsilon Y.$$

For $\phi \in (C(\overline{\Omega}))^3$ fixed, $i=1,2,3$, $p=0,1,\dots,\mathfrak{M}$ and $\mathrm{k} \in \mathcal{J}_p^\varepsilon$, let us introduce the following problem

$$\begin{cases} \Delta\theta_i^{\varepsilon,\mathrm{k}} = \mu_i^{\varepsilon,\mathrm{k}} & \text{in } Y_{\widetilde{x}_\mathrm{k}}^\varepsilon\,, \\ \partial_\nu\theta_i^{\varepsilon,\mathrm{k}} = 0 & \text{on } \partial Y_{\widetilde{x}_\mathrm{k}}^\varepsilon \setminus T_{\widetilde{x}_\mathrm{k}}^{p,\varepsilon}\,, \\ \partial_\nu\theta_i^{\varepsilon,\mathrm{k}} + M_i(\hat{x},\phi) = 0 & \text{on } T_{\widetilde{x}_\mathrm{k}}^{p,\varepsilon}\,, \end{cases} \tag{6.3}$$

where ν denotes the unit outward normal vector to $\partial Y_{\widetilde{x}_\mathrm{k}}^\varepsilon$, in particular, $\nu = n = (0,0,-1)$ on Σ. We find the constants $\mu_i^{\varepsilon,\mathrm{k}}$ from the compatibility conditions for problems in (6.3). Indeed, integrating over $Y_{\widetilde{x}_\mathrm{k}}^\varepsilon$ in (6.3), we get

$$\int_{Y_{\widetilde{x}_\mathrm{k}}^\varepsilon} \Delta\theta_i^{\varepsilon,\mathrm{k}}\,dx = \int_{Y_{\widetilde{x}_\mathrm{k}}^\varepsilon} \mu_i^{\varepsilon,\mathrm{k}}\,dx = \varepsilon^3\mu_i^{\varepsilon,\mathrm{k}}. \tag{6.4}$$

Now, using the Green formula and the boundary conditions in (6.3), we have that the left-hand side of (6.4) satisfies

$$\int_{Y_{\widetilde{x}_\mathrm{k}}^\varepsilon} \Delta\theta_i^{\varepsilon,\mathrm{k}} dx = \int_{\partial Y_{\widetilde{x}_\mathrm{k}}^\varepsilon} \partial_\nu\theta_i^{\varepsilon,\mathrm{k}}\,ds = -\int_{T_{\widetilde{x}_\mathrm{k}}^{p,\varepsilon}} M_i(\hat{x},\phi)\,d\hat{x}, \quad \text{for } \mathrm{k} \in \mathcal{J}_p^\varepsilon,\ p=0,1\dots,\mathfrak{M},$$

and, consequently, we choose

$$\mu_i^{\varepsilon,\mathrm{k}} = -\frac{1}{\varepsilon^3}\int_{T_{\widetilde{x}_\mathrm{k}}^{p,\varepsilon}} M_i(\hat{x},\phi)\,d\hat{x} \quad \text{for } \mathrm{k} \in \mathcal{J}_p^\varepsilon,\ p=0,1\dots,\mathfrak{M},\ i=1,2,3. \tag{6.5}$$

Note that above and in what follows, we avoid writing the dependence on p for $\theta_i^{\varepsilon,\mathrm{k}}$ and $\mu_i^{\varepsilon,\mathrm{k}}$.

Therefore, for each $\mathrm{k} \in \mathcal{J}^\varepsilon$, $i=1,2,3$, and $\mu_i^{\varepsilon,\mathrm{k}}$ defined by (6.5), problem (6.3) has a unique solution $\theta_i^{\varepsilon,\mathrm{k}} \in H^1(Y_{\widetilde{x}_\mathrm{k}}^\varepsilon)$ that is orthogonal to the constants in $L^2(Y_{\widetilde{x}_\mathrm{k}}^\varepsilon)$, namely,

$$\int_{Y_{\widetilde{x}_\mathrm{k}}^\varepsilon} \theta_i^{\varepsilon,\mathrm{k}}\,dx = 0 \qquad \text{for } \mathrm{k} \in \mathcal{J}^\varepsilon,\ i=1,2,3. \tag{6.6}$$

We denote by $Y^\varepsilon = \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} Y^\varepsilon_{\widetilde{x}_\mathrm{k}}$ and by θ^ε the function $\theta^\varepsilon_i(x) = \theta^{\varepsilon,\mathrm{k}}_i(x)$ for $i = 1, 2, 3$, $x \in Y^\varepsilon_{\widetilde{x}_\mathrm{k}}$ with $\mathrm{k} \in \mathcal{J}^\varepsilon$.

Now, we write

$$\int_{\bigcup T^\varepsilon} M_i(\hat{x}, \phi)v_i\, d\hat{x} = -\sum_{p=0}^{\mathfrak{M}} \sum_{\mathrm{k}\in\mathcal{J}^\varepsilon_p} \int_{T^{p,\varepsilon}_{\widetilde{x}_\mathrm{k}}} \partial_\nu \theta^{\varepsilon,\mathrm{k}}_i v_i\, ds = -\sum_{\mathrm{k}\in\mathcal{J}^\varepsilon} \int_{\partial Y^\varepsilon_{\widetilde{x}_\mathrm{k}}} \partial_\nu \theta^{\varepsilon,\mathrm{k}}_i v_i\, ds$$

and use the Green formula and the definition of $\theta^{\varepsilon,\mathrm{k}}_i$ to obtain

$$\int_{\bigcup T^\varepsilon} M_i(\hat{x}, \phi)v_i\, d\hat{x} = -\sum_{\mathrm{k}\in\mathcal{J}^\varepsilon} \int_{Y^\varepsilon_{\widetilde{x}_\mathrm{k}}} \nabla\theta^{\varepsilon,\mathrm{k}}_i \nabla v_i\, dx - \sum_{\mathrm{k}\in\mathcal{J}^\varepsilon} \int_{Y^\varepsilon_{\widetilde{x}_\mathrm{k}}} \mu^{\varepsilon,\mathrm{k}}_i v_i\, dx. \quad (6.7)$$

Second step: Let us prove

$$\Big|\frac{\varepsilon^2}{r_\varepsilon^2} \sum_{\mathrm{k}\in\mathcal{J}^\varepsilon} \int_{Y^\varepsilon_{\widetilde{x}_\mathrm{k}}} \mu^{\varepsilon,\mathrm{k}}_i v_i\, dx + |T| \int_\Sigma M_i(\hat{x}, \phi)v_i\, d\hat{x}\Big| \le C\varepsilon^{1/2}\|v\|_{(H^1(\Omega))^3}. \quad (6.8)$$

To do it, since $|T^p| = |T|$ for $p = 0, 1 \dots, \mathfrak{M}$, we write

$$-\frac{\varepsilon^2}{r_\varepsilon^2} \sum_{\mathrm{k}\in\mathcal{J}^\varepsilon} \int_{Y^\varepsilon_{\widetilde{x}_\mathrm{k}}} \mu^{\varepsilon,\mathrm{k}}_i v_i\, dx - |T| \int_\Sigma M_i(\hat{x}, \phi)v_i\, d\hat{x} = S^\varepsilon_1 + S^\varepsilon_2 + S^\varepsilon_3, \quad (6.9)$$

where

$$\begin{aligned} S^\varepsilon_1 :=& \frac{1}{\varepsilon} \sum_{p=0}^{\mathfrak{M}} \sum_{\mathrm{k}\in\mathcal{J}^\varepsilon_p} \int_{Y^\varepsilon_{\widetilde{x}_\mathrm{k}}} \Big(-\mu^{\varepsilon,\mathrm{k}}_i \frac{\varepsilon^3}{r_\varepsilon^2} - |T^p| M_i(\widetilde{x}^\varepsilon_\mathrm{k}, \phi(\widetilde{x}^\varepsilon_\mathrm{k}))\Big) v_i\, dx, \\ S^\varepsilon_2 :=& \frac{1}{\varepsilon} \sum_{\mathrm{k}\in\mathcal{J}^\varepsilon} \int_{Y^\varepsilon_{\widetilde{x}_\mathrm{k}}} |T| \Big(M_i(\widetilde{x}^\varepsilon_\mathrm{k}, \phi(\widetilde{x}^\varepsilon_\mathrm{k})) - M_i(x, \phi(x))\Big) v_i\, dx, \\ S^\varepsilon_3 :=& |T| \Big(\frac{1}{\varepsilon} \int_{Y^\varepsilon} M_i(x, \phi(x))v_i\, dx - \int_\Sigma M_i(\hat{x}, \phi(\hat{x}))v_i\, d\hat{x}\Big), \end{aligned}$$

and estimate each term S^ε_r for $r = 1, 2, 3$.

For each $\mathrm{k} \in \mathcal{J}^\varepsilon_p$, $p = 0, 1, \dots \mathfrak{M}$, and $i = 1, 2, 3$, the definition of $\mu^{\varepsilon,\mathrm{k}}_i$ (cf. (6.5)) and the change of variable (4.3) yields

$$-\mu^{\varepsilon,\mathrm{k}}_i \frac{\varepsilon^3}{r_\varepsilon^2} = \frac{1}{r_\varepsilon^2} \int_{T^{p,\varepsilon}_{\widetilde{x}_\mathrm{k}}} M_i(\hat{x}, \phi)\, d\hat{x} = \int_{T^p} M_i(\widetilde{x}^\varepsilon_\mathrm{k} + r_\varepsilon \hat{y}, \phi(\widetilde{x}^\varepsilon_\mathrm{k} + r_\varepsilon \hat{y}))\, d\hat{y}.$$

Besides, by condition (2.11), the smoothness of the functions ϕ_j, and the fact that $|T^p| = |T|$ for $p = 0, 1 \dots, \mathfrak{M}$, we get

$$\begin{aligned} &\Big| -\mu^{\varepsilon,\mathrm{k}}_i \frac{\varepsilon^3}{r_\varepsilon^2} - |T^p| M_i(\widetilde{x}^\varepsilon_\mathrm{k}, \phi(\widetilde{x}^\varepsilon_\mathrm{k}))\Big| \\ &\le \int_{T^p} \Big| \Big(M_i(\widetilde{x}^\varepsilon_\mathrm{k} + r_\varepsilon \hat{y}, \phi(\widetilde{x}^\varepsilon_\mathrm{k} + r_\varepsilon \hat{y})) - M_i(\widetilde{x}^\varepsilon_\mathrm{k}, \phi(\widetilde{x}^\varepsilon_\mathrm{k}))\Big)\Big|\, d\hat{y} \le C_1 r_\varepsilon. \quad (6.10) \end{aligned}$$

Thus, (6.10) and (6.1) give

$$|S_1^\varepsilon| \leq \frac{C_1 r_\varepsilon}{\varepsilon} |Y^\varepsilon|^{1/2} \|v\|_{(L^2(\Omega\cap\{0<x_3<\varepsilon\}))^3} \leq \widetilde{C}_1 r_\varepsilon \|v\|_{(H^1(\Omega))^3}. \tag{6.11}$$

Similarly,

$$|S_2^\varepsilon| \leq C_1 |Y^\varepsilon|^{1/2} \|v\|_{(L^2(\Omega\cap\{0<x_3<\varepsilon\}))^3} \leq \widetilde{C}_1 \varepsilon \|v\|_{(H^1(\Omega))^3}. \tag{6.12}$$

In addition, using (6.2), the smoothness of ϕ and the properties of M_i (cf. (2.11)) which imply $\mathfrak{m}_i(x) := M_i(x, \phi(x)) \in W^{1,\infty}(\Omega)$ (cf. [9], [24] and references therein), we deduce

$$|S_3^\varepsilon| \leq C\varepsilon^{1/2} \|v\|_{(H^1(\Omega))^3}. \tag{6.13}$$

Therefore, gathering (6.9), (6.11)–(6.13) and (2.2), we obtain (6.8).

Third step: Let us prove

$$\left| \frac{\varepsilon^2}{r_\varepsilon^2} \sum_{\mathrm{k}\in\mathcal{J}^\varepsilon} \int_{Y^\varepsilon_{\tilde{x}_\mathrm{k}}} \nabla\theta_i^{\varepsilon,\mathrm{k}} \nabla v_i \, dx \right| \leq C \frac{\varepsilon}{r_\varepsilon^{1/2}} \|v\|_{(H^1(\Omega))^3}. \tag{6.14}$$

For each $\mathrm{k} \in \mathcal{J}^\varepsilon$, we first show

$$\int_{Y^\varepsilon_{\tilde{x}_\mathrm{k}}} |\nabla\theta_i^{\varepsilon,\mathrm{k}}|^2 \, dx \leq C r_\varepsilon^3, \tag{6.15}$$

where C is a constant independent of ε and k. From the integral identity for problem (6.3), and (6.6), we have

$$\int_{Y^\varepsilon_{\tilde{x}_\mathrm{k}}} |\nabla\theta_i^{\varepsilon,\mathrm{k}}|^2 dx = -\int_{Y^\varepsilon_{\tilde{x}_\mathrm{k}}} \mu_i^{\varepsilon,\mathrm{k}} \theta_i^{\varepsilon,\mathrm{k}} \, dx - \int_{T^{p,\varepsilon}_{\tilde{x}_\mathrm{k}}} M_i(\hat{x}, \phi) \theta_i^{\varepsilon,\mathrm{k}} d\hat{x} = -\int_{T^{p,\varepsilon}_{\tilde{x}_\mathrm{k}}} M_i(\hat{x}, \phi) \theta_i^{\varepsilon,\mathrm{k}} \, d\hat{x},$$

for $\mathrm{k} \in \mathcal{J}_p^\varepsilon$, $p = 0, 1 \ldots, \mathfrak{M}$. Besides, (2.9), (2.11), the smoothness of the functions ϕ_j, the Cauchy-Schwarz inequality and the fact that $|T^p| = |T|$ for $p = 0, 1 \ldots, \mathfrak{M}$, guarantee for each $i = 1, 2, 3$, and $\mathrm{k} \in \mathcal{J}_p^\varepsilon$,

$$\int_{Y^\varepsilon_{\tilde{x}_\mathrm{k}}} |\nabla\theta_i^{\varepsilon,\mathrm{k}}|^2 \, dx \leq C |T^{p,\varepsilon}_{\tilde{x}_\mathrm{k}}|^{1/2} \|\theta_i^{\varepsilon,\mathrm{k}}\|_{L^2(T^{p,\varepsilon}_{\tilde{x}_\mathrm{k}})} \leq C r_\varepsilon \|\theta_i^{\varepsilon,\mathrm{k}}\|_{L^2(T^{p,\varepsilon}_{\tilde{x}_\mathrm{k}})}. \tag{6.16}$$

Now, we apply Lemma 3 for each T^p, with $p = 0, 1 \ldots, \mathfrak{M}$, to get

$$\|\theta_i^{\varepsilon,\mathrm{k}}\|_{L^2(T^{p,\varepsilon}_{\tilde{x}_\mathrm{k}})} \leq C_p \Big(r_\varepsilon^{1/2} \varepsilon^{-1} \|\theta_i^{\varepsilon,\mathrm{k}}\|_{L^2(Y^\varepsilon_{\tilde{x}_\mathrm{k}})} + r_\varepsilon^{1/2} \|\nabla\theta_i^{\varepsilon,\mathrm{k}}\|_{L^2(Y^\varepsilon_{\tilde{x}_\mathrm{k}})} \Big). \tag{6.17}$$

Moreover, Lemma 2 (cf. (6.6)) and (6.17) allow us to estimate the functions $\theta_i^{\varepsilon,\mathrm{k}}$ in $L^2(T^\varepsilon_{\tilde{x}_\mathrm{k}})$ for $i = 1, 2, 3$ and $\mathrm{k} \in \mathcal{J}_p^\varepsilon$, namely,

$$\|\theta_i^{\varepsilon,\mathrm{k}}\|_{L^2(T^{p,\varepsilon}_{\tilde{x}_\mathrm{k}})} \leq \widetilde{C}_p r_\varepsilon^{1/2} \|\nabla\theta_i^{\varepsilon,\mathrm{k}}\|_{L^2(Y^\varepsilon_{\tilde{x}_\mathrm{k}})}. \tag{6.18}$$

Thus, since $p = 0, 1 \ldots, \mathfrak{M}$, gathering (6.16) and (6.18), we deduce (6.15).

As a result, by definition of θ^ε, (6.15) and (2.4), we get

$$\|\nabla\theta_i^\varepsilon\|^2_{L^2(Y^\varepsilon)} = \sum_{p=0}^{\mathfrak{M}} \sum_{\mathrm{k}\in\mathcal{J}_p^\varepsilon} \int_{Y^\varepsilon_{\tilde{x}_\mathrm{k}}} |\nabla\theta_i^{\varepsilon,\mathrm{k}}|^2\, dx \le C r_\varepsilon^3 \varepsilon^{-2},$$

and, consequently, we obtain (6.14).

Gathering (6.7), (6.8) and (6.14), we conclude the proof of (5.1).

Remark 3. It is worth mentioning that Theorem 2 extends the result in Lemma 1 of [16] to the case of arbitrary plane domains instead of volumetric ones, allowing a diameter $O(r_\varepsilon)$ of smaller order of magnitude than the period of the grid $O(\varepsilon)$, and different shapes of these regions (cf. (2.1)). In particular, it allows to pass to the limit in the integrals

$$\frac{\varepsilon^2}{r_\varepsilon^2} \int_{\bigcup T^\varepsilon} \mathfrak{m}(\hat{x}) v^\varepsilon(\hat{x})\, d\hat{x},$$

for a globally Lipschitz function $\mathfrak{m}$ and a sequence of functions v^ε that converge weakly $H^1(\Omega)$. This is why Theorem 2 applies to the scalar problem in [23] for circular T^ε when $r_\varepsilon \gg \varepsilon^2$ allowing a more general geometry and less restrictive monotonic functions. Also, the results in this paper complement those in [24] and [9] for the scalar problem, but when $r_\varepsilon \ll \varepsilon^2$. Similarly, we complement and extend the results in [7, 10] for linear Winkler-Robin boundary conditions and non-periodic distribution of the regions T^ε. In particular, for a periodic setting and a linear M, Theorem 2 has been announced in [10] without the proof that we provide here.

7 Conclusions

The problem under consideration (2.12) represents a model associated to the displacements in a block of inhomogeneous anisotropic elastic material, which has a part of its boundary Γ_Ω clamped to a rigid support, while the part in contact with the plane $\{x_3 = 0\}$ interacts with the soil via a series of small springs with a nonlinear Winkler law $\sigma_{i3}^\varepsilon = \beta(\varepsilon) M_i(x, u^\varepsilon)$ involving a strong reaction. The properties of function M are somehow optimal (cf. Remarks 1 and 3). Outside these reaction regions, $\bigcup T^\varepsilon$, the surface is free of forces. We provide an averaged reaction of Winkler type, which depends on the relations for the parameters reaction, size and distances between the small regions, and on the unit area $|T|$. As a matter of fact, the results in this paper are new and extend and complement those in the literature for the Laplacian with linear and nonlinear Robin boundary conditions and for the elasticity operator with linear Winkler-Robin boundary conditions. For further relations between parameters with linear Winkler-Robin boundary conditions, we refer to [7,8,10, 11]. For extreme relations between parameters giving rise to (3.17) and (3.18), the technique in [11] can likely be applied. When $r_\varepsilon = O(\varepsilon^2)$ and $\beta^* > 0$, getting the microscopic problem (cf. (4.3)–(4.7)) remains as an open problem under considerations by the authors. Due to the complex geometry of the domain, it is self-evident that obtaining a first approach to the original problem via the homogenized problems can be useful for numerical computations cf., e.g., [14, 22].

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