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Dissipative measure-valued solutions to the magnetohydrodynamic equations

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Abstract. In this paper, we study the dissipative measure-valued solution to the magnetohydrodynamic equations of 3D compressible isentropic flows with the adiabatic exponent $\gamma > 1$ and prove that a dissipative measure-valued solution is the same as the standard smooth classical solution as long as the latter exists, provided they emanate from the same initial data (weak–strong uniqueness principle).

Keywords: compressible magnetohydrodynamic equations; measure-valued solution; weak-strong uniqueness.

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1 Introduction

In this paper, we consider the following compressible magnetohydrodynamic equations which describe the motion of electrically conducting media in the presence of a magnetic field in the isentropic case:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$
$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div} \mathbb{S}(\nabla \mathbf{u}),$$
$$\partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (v \nabla \times \mathbf{H}), \quad \operatorname{div} \mathbf{H} = 0 \tag{1.2}$$

for $(t,x) \in [0,+\infty) \times \Omega$, where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain. System (1.1)–(1.2) is supplemented by the following initial conditions and boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \mathbf{H}|_{\partial\Omega} = 0, \nabla\rho \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.3}$$
$$(\rho, \mathbf{u}, \mathbf{H})|_{t=0}(t,x) = (\rho_0, \mathbf{u}_0, \mathbf{H}_0)(x) \text{ for } x \in \Omega, \tag{1.4}$$

where $\mathbf{n}$ is the unit outward normal vector. The unknowns ρ, $\mathbf{u} \in \mathbb{R}^3$ and $\mathbf{H} \in \mathbb{R}^3$ denote the particle density, the velocity, and the magnetic field, respectively. The pressure $p(\rho) = A\rho^\gamma$ is assumed to depend on the particle density with constant $A > 0$ and the adiabatic exponent $\gamma > 1$. $\mathbb{S}$ denotes the standard Newtonian viscous stress

$$\mathbb{S}(\nabla\mathbf{u}) = \mu\left(\nabla\mathbf{u} + \nabla^t\mathbf{u} - (2/3)\,\mathrm{div}\,\mathbf{u}\mathbb{I}\right) + \eta\,\mathrm{div}\,\mathbf{u}\mathbb{I}, \quad \mu > 0, \;\; \eta \geq 0.$$

Note that $\mathbb{S}$ depends only on the symmetric part of its argument. $v > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. The symbol $\otimes$ denotes the Kronecker tensor product. $(\nabla\times\mathbf{H})\times\mathbf{H}$ is the magnetic force per volume of fluid. The electric field can be written in terms of the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$:

$$\mathbf{E} = \nu\nabla\times\mathbf{H} - \mathbf{u}\times\mathbf{H}.$$

Although the electric field $\mathbf{E}$ does not appear in system (1.1)–(1.2), it is indeed induced according to the above relation by the moving conductive flow in the magnetic field.

The isentropic compressible magnetohydrodynamic equations (1.1)–(1.2) received a great deal of attention from physicists and mathematicians because of their physical importance, complexity, rich phenomena, and mathematical challenges, such as [1, 4, 5, 8, 9, 10, 17] and the references cited therein. In particular, Hu-Wang [12] established the global weak solutions to the nonlinear compressible magnetohydrodynamic equations (1.1)–(1.2) with general initial data. Hu-Wang [11] and Jiang et al. [14] studied the low Mach limit (or incompressible limit) problem of (1.1)–(1.2) for different cases. Yang-Dou-Ju [16] established the weak-strong uniqueness in the class of finite energy weak solutions to (1.1)–(1.2).

The concept of measure-valued solutions to partial differential equations (more precisely for hyperbolic conservation laws) was first introduced by DiPerna [3]. Later, Neustupa [15] analyzed the measure-valued solutions to compressible Navier-Stokes equations but his theory does not involve the energy. Demoulini et al. [2] defined a notion of a dissipative measure-valued solution for Polyconvex Elastodynamics and showed that such a solution agrees with a classical solution with the same initial data, when such a classical solution exists (weak-strong uniqueness principle). Recently Feireisl et al. [6] introduced a new concept of dissipative measure-valued solutions to compressible Navier-Stokes system and gave the first instance of weak-strong uniqueness for measure-valued solutions of a viscous fluid model. Huang [13] obtain the global existence of dissipative measure-valued solutions to the three-dimensional compressible micropolar fluid system and gave the weak-strong uniqueness principle to this system.

Motivated by the above results, in the present paper, we shall prove the global existence of dissipative measure-valued solutions to the isentropic compressible magnetohydrodynamic equations (1.1)–(1.2) and obtain its weak-strong uniqueness principle. Because the term $\int_0^T\int_\Omega \left\langle Y_{t,x}; |\mathbf{v}-\tilde{\mathbf{u}}|^2\right\rangle \mathrm{d}x\mathrm{d}t$, where $\tilde{\mathbf{u}}$ is the corresponding strong solution, is not included in the relative entropy, some new techniques are required to decouple the velocity. By using generalized Poincaré's inequality, Sobolev's inequality, and a Korn-type inequality to decouple the velocity and magnetic field, we have solved the difficulties caused by the complicated interaction between velocity and magnetic field. To our best knowledge, there is no result on the dissipative measure-valued solutions to the compressible magnetohydrodynamic equations.

The rest of the paper is organized as follows. Section 2 will give the precise definition of dissipative measure-valued solutions to the isentropic compressible magnetohydrodynamic equations (1.1)–(1.2) and state our main results. Section 3 gives a detailed proof of Theorem 1 by constructing an approximate system. In Section 4, we will show the weak-strong uniqueness principle to the compressible magnetohydrodynamic equations (1.1)–(1.4) by fine energy estimation.

2 Main results

Let

$$\mathcal{Q} = \left\{ [\rho, \mathbf{u}, \mathbf{H}] \mid \rho \geq 0, \ \mathbf{u} \in \mathbb{R}^3, \ \mathbf{H} \in \mathbb{R}^3 \right\}$$

be the natural phase space. The symbol $\mathcal{P}(\mathcal{Q})$ denotes the space of (Borel) probability measures, i.e., for $\mu \in \mathcal{P}(\mathcal{Q})$, we have $\mu(\mathcal{Q}) = 1$.

2.1 Measure-valued solutions to system (1.1)–(1.4)

We shall first focus on the existence of dissipative measure-valued solutions to the compressible magnetohydrodynamic equations (1.1)–(1.4). To do that, let's state the definition of dissipative measure-valued solutions to (1.1)–(1.4).

Definition 1. We say that a parameterized measure $\left\{Y_{t,x}\right\}_{(t,x)\in[0,T]\times\Omega}$,

$$Y_{t,x} \in L^\infty_{\text{weak- }(\star)}\left([0,T]\times\Omega; \mathcal{P}(\mathcal{Q})\right), \left\langle Y_{t,x}; s\right\rangle = \rho, \left\langle Y_{t,x}; \mathbf{v}\right\rangle = \mathbf{u}, \left\langle Y_{t,x}; \mathbf{B}\right\rangle = \mathbf{H}$$

is a dissipative measure-valued solution of the compressible magnetohydrodynamic equations (1.1)–(1.4) in $[0,T]\times\Omega$, with the initial conditions $Y_{0,x}$ and dissipation defect $\mathcal{D}(\tau)$, with $\mathcal{D} \in L^\infty[0,T], \mathcal{D} \geq 0$, if the following holds.

1. **Equation of continuity.**

   For any $\phi \in C^1([0,T]\times\bar{\Omega})$ and $\tau \in [0,T]$,

$$\begin{aligned} &\int_\Omega \left\langle Y_{\tau,x}; s\right\rangle \phi(\tau,\cdot)\mathrm{d}x - \int_\Omega \left\langle Y_{0,x}; s\right\rangle \phi(0,\cdot)\mathrm{d}x \\ &\quad = \int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; s\right\rangle \partial_t\phi + \left\langle Y_{t,x}; s\mathbf{v}\right\rangle \cdot \nabla\phi\right] \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.1}$$

2. **Momentum equation.**

$$\mathbf{u} = \left\langle Y_{t,x}; \mathbf{v}\right\rangle \in L^2\left(0,T; H^1_0\left(\Omega;\mathbb{R}^3\right)\right)$$

and for any $\psi \in C^1([0,T]\times\bar{\Omega})$ with $\psi|_{\partial\Omega}=0$,

$$
\begin{aligned}
&\int_\Omega \langle Y_{\tau,x}; s\mathbf{v}\rangle \cdot \psi(\tau,\cdot)\mathrm{d}x - \int_\Omega \langle Y_{0,x}; s\mathbf{v}\rangle \cdot \psi(0,\cdot)\mathrm{d}x \\
&= \int_0^\tau\int_\Omega \Big(\langle Y_{t,x}; s\mathbf{v}\rangle \cdot \partial_t\psi + \langle Y_{t,x}; s(\mathbf{v}\otimes\mathbf{v})\rangle : \nabla\psi + \langle Y_{t,x}; As^\gamma\rangle \operatorname{div}\psi\Big)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau\int_\Omega \Big(\langle Y_{t,x}; \mathbf{B}\otimes\mathbf{B}\rangle : \nabla\psi - \frac{1}{2}\Big\langle Y_{t,x}; |\mathbf{B}|^2\Big\rangle \operatorname{div}\psi\Big)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla\psi\mathrm{d}x\mathrm{d}t + \int_0^\tau \Big\langle r^V; \nabla\psi\Big\rangle\,\mathrm{d}t
\end{aligned} \tag{2.2}
$$

with the measure $r^V(\tau)=\{r^V_{i,j}(\tau)\}\in L^\infty_{\text{weak}}(0,T;\mathcal{M}(\Omega))$ satisfying

$$\Big|\Big\langle r^V(\tau);\nabla\psi\Big\rangle\Big| \le C\mathcal{D}(\tau)\|\psi\|_{C^1(\Omega)}.$$

3. **Magnetic field equation.**
For any $\omega\in C^1([0,T]\times\bar{\Omega})$ with $\omega|_{\partial\Omega}=0$,

$$
\begin{aligned}
&\int_\Omega \langle Y_{\tau,x}; \mathbf{B}\rangle(\tau,\cdot)\cdot\omega\mathrm{d}x - \int_\Omega \langle Y_{0,x}; \mathbf{B}\rangle \cdot \omega(0,\cdot)\mathrm{d}x \\
&= \int_0^\tau\int_\Omega \Big[\langle Y_{t,x}; \mathbf{B}\rangle(\tau,\cdot)\cdot\partial_t\omega + \langle Y_{t,x}; (\mathbf{v}\times\mathbf{B})\rangle\cdot(\nabla\times\omega)\Big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \nu\int_0^\tau\int_\Omega \nabla\mathbf{H} : \nabla\omega\mathrm{d}x\mathrm{d}t - \int_0^\tau \Big\langle r^H; \nabla\omega\Big\rangle\,dt
\end{aligned}
$$

with the measure $r^H(\tau)=\{r^H_{i,j}(\tau)\}\in L^\infty_{weak}(0,T;\mathcal{M}(\Omega))$ satisfying

$$\Big|\Big\langle r^H(\tau);\nabla\psi\Big\rangle\Big| \le C\mathcal{D}(\tau)\|\psi\|_{C^1(\Omega)}.$$

4. **Energy inequality.** For any $\tau\in[0,T]$,

$$
\begin{aligned}
&\int_\Omega \Big\langle Y_{\tau,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{As^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{B}|^2\Big\rangle(\tau,\cdot)\mathrm{d}x + \mathcal{D}(\tau) \\
&\quad + \int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}^\epsilon) : \nabla\mathbf{u}^\epsilon\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \nu|\nabla\mathbf{H}|^2\mathrm{d}x\mathrm{d}t \\
&\le \int_\Omega \Big\langle Y_{0,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{As^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{B}|^2\Big\rangle(0,\cdot)\mathrm{d}x.
\end{aligned} \tag{2.3}
$$

5. **Generalized Poincaré's inequality.** For any $\tau\in[0,T]$,

$$\int_0^\tau\int_\Omega \Big\langle Y_{t,x}; |\mathbf{v}-\mathbf{u}|^2 + |\mathbf{B}-\mathbf{H}|^2\Big\rangle\mathrm{d}x\mathrm{d}t \le C\mathcal{D}(\tau).$$

Now, we state the result on the existence of dissipative measure-valued solutions to the compressible magnetohydrodynamic equations (1.1)–(1.4).

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. If $(\rho_0, \mathbf{u}_0, \mathbf{H}_0)$ is the initial data with finite energy, then there exists a dissipative measure-valued solution to the compressible magnetohydrodynamic equations* (1.1)–(1.4) *with initial data*

$$Y_{0,x} = \delta_{(\rho_0, \mathbf{u}_0, \mathbf{H}_0)}.$$

Remark 1. For simplicity, here the pressure function $p(\rho)$ is given by the isentropic pressure-density state $p = A\rho^\gamma$ with $\gamma > 1$, although more general cases can be treated as well.

2.2 Weak-strong uniqueness to the compressible magnetohydrodynamic equations (1.1)–(1.4)

For the weak-strong uniqueness to the compressible magnetohydrodynamic equations (1.1)–(1.4), i.e., a measure-valued solution and a standard smooth classical solution originating from the same initial data coincide, we have the following result.

Theorem 2. *Let $\Omega \subset \mathrm{R}^3$ be a smooth bounded domain. Let $\{Y_{t,x}; \mathcal{D}(t)\}$ be a dissipative measure-valued solution to the compressible magnetohydrodynamic equations* (1.1)–(1.4) *with the initial state $Y_{0,x}$ in the sense specified in Definition 1. Suppose that $\{\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{H}}\}$ is a strong solution to the compressible magnetohydrodynamic equations* (1.1)–(1.3) *with the initial data*

$$\begin{cases} \tilde{\rho}(0,\cdot) = \tilde{\rho}(0) = \rho_0 \in C^1(\bar{\Omega}), \quad \rho_0 \geq \underline{\rho} > 0, \\ \tilde{\mathbf{u}}(0,\cdot) = \tilde{\mathbf{u}}_0 = \mathbf{u}_0 \in C^2(\bar{\Omega}), \qquad \tilde{\mathbf{H}}(0,\cdot) = \tilde{\mathbf{H}}_0 = \mathbf{H}_0 \in C^2(\bar{\Omega}), \quad \operatorname{div} \mathbf{H}_0 = 0 \end{cases}$$

in $[0,T] \times \Omega$ belonging to the class

$$(\tilde{\rho}, \nabla\tilde{\rho}, \tilde{\mathbf{u}}, \nabla\tilde{\mathbf{u}}, \tilde{\mathbf{H}}, \nabla\tilde{\mathbf{H}}) \in C([0,T]\times\overline{\Omega}), \ \left(\partial_t\tilde{\mathbf{u}}, \partial_t\tilde{\mathbf{H}}\right) \in L^2\left(0,T; C\left(\overline{\Omega}; \mathbb{R}^3\right)\right), \tag{2.4}$$

$$\tilde{\rho} \geq \underline{\rho} > 0, \quad \tilde{\mathbf{u}}|_{\partial\Omega} = 0, \quad \tilde{\mathbf{H}}\Big|_{\partial\Omega} = 0. \tag{2.5}$$

Then, there exists a constant $C(T)$, such that

$$\begin{aligned} &\int_\Omega \left\langle Y_{\tau,x}; \frac{1}{2}s|\mathbf{v} - \tilde{\mathbf{u}}|^2 + \frac{A}{\gamma-1}(s^\gamma - \gamma\tilde{\rho}^{\gamma-1}(s-\tilde{\rho}) - \tilde{\rho}^\gamma) + \frac{1}{2}|\mathbf{B} - \tilde{\mathbf{H}}|^2 \right\rangle \mathrm{d}x \\ &\quad + \mathcal{D}(\tau) + \int_0^\tau \int_\Omega (\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\tilde{\mathbf{u}})) : (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}}) \mathrm{d}x \ \mathrm{d}t \\ &\quad + \nu \int_0^\tau \int_\Omega |\nabla\mathbf{H} - \nabla\tilde{\mathbf{H}}|^2 \ \mathrm{d}x \ \mathrm{d}t \leq C(T) \\ &\times \int_\Omega \left\langle Y_{0,x}; \frac{1}{2}s|\mathbf{v} - \tilde{\mathbf{u}}_0|^2 + \frac{A}{\gamma-1}(s^\gamma - \gamma\tilde{\rho}_0^{\gamma-1}(s-\tilde{\rho}_0) - \tilde{\rho}_0^\gamma) + \frac{1}{2}|\mathbf{B} - \tilde{\mathbf{H}}_0|^2 \right\rangle \mathrm{d}x, \end{aligned}$$

for any $\tau \in [0,T]$. In particular, if the initial states coincide, meaning

$$Y_{0,x} = \delta_{[\tilde{\rho}_0, \tilde{\mathbf{u}}_0, \tilde{\mathbf{H}}_0]} \text{ for a.a. } x \in \Omega,$$

then one gets that $\mathcal{D}(\tau) = 0$ and

$$Y_{\tau,x} = \delta_{[\tilde{\rho}(\tau,x), \tilde{\mathbf{u}}(\tau,x), \tilde{\mathbf{H}}(\tau,x)]} \text{ for a.a. } (\tau,x) \in [0,T] \times \Omega.$$

3 Proof of Theorem 1

To obtain the existence of measure-valued solution of compressible magnetohydrodynamic equations (1.1)–(1.2), we consider the following model of a viscous compressible fluid, where the density ρ, the velocity $\mathbf{u}$ and the magnetic filed $\mathbf{H}$ satisfy

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = \epsilon\Delta\rho, \tag{3.1}$$

$$\begin{aligned}&(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = (\nabla\times\mathbf{H})\times\mathbf{H} + \operatorname{div}\mathbf{S}(\nabla\mathbf{u}) + \epsilon\operatorname{div}(\mathbf{u}\otimes\nabla\rho),\\ &\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(v\nabla\times\mathbf{H}), \quad \operatorname{div}\mathbf{H} = 0,\end{aligned} \tag{3.2}$$

where $\epsilon > 0$ is a parameter. System (3.1)–(3.2) is supplemented by the following boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0,\ \mathbf{H}|_{\partial\Omega} = 0,\ \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{3.3}$$

where $\mathbf{n}$ is the unit outward normal. It is easy to know that the sufficiently smooth solutions of system (3.1)–(3.3) obey the total energy balance

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{A\rho^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{H}|^2\right]\mathrm{d}x\\ &\quad + \int_\Omega \left[\mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u} + \frac{4A\epsilon}{\gamma}\left|\nabla\rho^{\frac{\gamma}{2}}\right|^2 + \nu|\nabla\mathbf{H}|^2\right]\mathrm{d}x = 0,\end{aligned}$$

where we have used the following equality

$$\nabla\times(\nabla\times\mathbf{H}) = \nabla\operatorname{div}\mathbf{H} - \Delta\mathbf{H} = -\Delta\mathbf{H}$$

due to $\operatorname{div}\mathbf{H} = 0$.

We assume that the energy of the initial data is bounded

$$\int_\Omega \left[\frac{1}{2}\rho_0^\epsilon|\mathbf{u}_0^\epsilon|^2 + \frac{A\left(\rho_0^\epsilon\right)^\gamma}{\gamma-1} + \frac{1}{2}\left|\mathbf{H}_0^\epsilon\right|^2\right]\mathrm{d}x \le C$$

uniformly for $\epsilon \to 0$. So, we can obtain the following energy inequality

$$\begin{aligned}&\int_\Omega \left[\frac{1}{2}\rho^\epsilon|\mathbf{u}^\epsilon|^2 + \frac{A(\rho^\epsilon)^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{H}^\epsilon|^2\right]\mathrm{d}x\\ &\quad + \int_0^T\int_\Omega \left[\mathbb{S}(\nabla\mathbf{u}^\epsilon) : \nabla\mathbf{u}^\epsilon + \frac{4\epsilon A}{\gamma}\left|\nabla(\rho^\epsilon)^{\frac{\gamma}{2}}\right|^2 + \nu|\nabla\mathbf{H}^\epsilon|^2\right]\mathrm{d}x\mathrm{d}t\\ &\le \int_\Omega \left[\frac{1}{2}\rho_0^\epsilon|\mathbf{u}_0^\epsilon|^2 + \frac{A\left(\rho_0^\epsilon\right)^\gamma}{\gamma-1} + \frac{1}{2}\left|\mathbf{H}_0^\epsilon\right|^2\right]\mathrm{d}x \le C.\end{aligned} \tag{3.4}$$

It is easy to deduce the following fact:

$$\begin{cases}\rho^\epsilon\ln\rho^\epsilon \in \left(-1/e, 0\right), & 0 < \rho^\epsilon < 1,\\ \rho^\epsilon\ln\rho^\epsilon \leqslant C_0(\rho^\epsilon)^\gamma, & \rho^\epsilon \geqslant 1\end{cases}$$

with $C_0 = \max\left\{\frac{1}{\gamma}, \frac{1}{\gamma(\gamma-1)}\right\}$. Then, due to the boundedness of domain Ω and the energy balance (3.4), by using Korn inequality and Poincaré inequality, we can obtain the following bounds

$$\sup_{\tau\in[0,T]}\int_\Omega (\rho^\epsilon)^\gamma(\tau,\cdot)\mathrm{d}x \le C\Big(\Longrightarrow \sup_{\tau\in[0,T]}\int_\Omega \rho^\epsilon\log\rho^\epsilon(\tau,\cdot)\mathrm{d}x \le C\Big), \tag{3.5}$$

$$\sup_{\tau\in[0,T]}\int_\Omega |\mathbf{H}^\epsilon|^2\mathrm{d}x \le C, \tag{3.6}$$

$$\sup_{\tau\in[0,T]}\int_\Omega \rho^\epsilon|\mathbf{u}^\epsilon|^2\mathrm{d}x \le C, \tag{3.7}$$

$$\int_0^T\!\!\int_\Omega \mathbb{S}(\nabla\mathbf{u}^\epsilon):\nabla\mathbf{u}^\epsilon\mathrm{d}x\mathrm{d}t \le C\bigg(\Longrightarrow \int_0^T\!\!\int_\Omega\Big(|\mathbf{u}^\epsilon|^2+|\nabla\mathbf{u}^\epsilon|^2\Big)\,\mathrm{d}x\mathrm{d}t\le C\bigg), \tag{3.8}$$

$$\epsilon\int_0^T\int_\Omega\left|\nabla(\rho^\epsilon)^{\frac{\gamma}{2}}\right|^2\mathrm{d}x\mathrm{d}t\le C,\quad \int_0^T\int_\Omega|\nabla\mathbf{H}^\epsilon|^2\,\mathrm{d}x\mathrm{d}t\le C, \tag{3.9}$$

uniformly for $\epsilon\to 0$.

Using the following calculus formula

$$(\nabla\times\mathbf{a})\times\mathbf{a} = (\mathbf{a}\cdot\nabla)\mathbf{a}-\frac{1}{2}\nabla|\mathbf{a}|^2,\ \ \mathrm{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$$

for vectors $\mathbf{a}$ and $\mathbf{b}$, we rewrite system (3.1)–(3.2) in the following weak form

$$\left[\int_\Omega \rho^\epsilon\phi\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega\left[\rho^\epsilon\partial_t\phi+\rho^\epsilon\mathbf{u}^\epsilon\cdot\nabla\phi-\epsilon\nabla\rho^\epsilon\cdot\nabla\phi\right]\mathrm{d}x\mathrm{d}t \tag{3.10}$$

for any $\phi\in C^1([0,T]\times\bar\Omega)$,

$$\begin{aligned}\left[\int_\Omega\rho^\epsilon\mathbf{u}^\epsilon\cdot\psi\mathrm{d}x\right]_{t=0}^{t=\tau} =& \int_0^\tau\int_\Omega\left[\rho^\epsilon\mathbf{u}^\epsilon\cdot\partial_t\psi+\rho^\epsilon\left(\mathbf{u}^\epsilon\otimes\mathbf{u}^\epsilon\right):\nabla\psi+A(\rho^\epsilon)^\gamma\,\mathrm{div}\,\psi\right]\mathrm{d}x\mathrm{d}t\\ &-\int_0^\tau\int_\Omega\left[\mathbf{H}^\epsilon\otimes\mathbf{H}^\epsilon:\nabla\psi-\frac{1}{2}|\mathbf{H}^\epsilon|^2\cdot\mathrm{div}\,\psi\right]\mathrm{d}x\mathrm{d}t\\ &-\int_0^\tau\int_\Omega\left[\mathbb{S}\left(\nabla\mathbf{u}^\epsilon\right):\nabla\psi+\epsilon\left(\mathbf{u}^\epsilon\otimes\nabla\rho^\epsilon\right):\nabla\psi\right]\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.11}$$

for any $\psi\in C^1([0,T]\times\bar\Omega)$ with $\psi|_{\partial\Omega}=0$,

$$\left[\int_\Omega\mathbf{H}^\epsilon\cdot\omega\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega\left[\mathbf{H}^\epsilon\cdot\partial_t\omega+\left(\mathbf{u}^\epsilon\times\mathbf{H}^\epsilon\right):\nabla\times\omega-\nu\nabla\mathbf{H}^\epsilon:\nabla\omega\right]\mathrm{d}x\mathrm{d}t \tag{3.12}$$

for any $\omega\in C^1([0,T]\times\bar\Omega)$ with $\omega|_{\partial\Omega}=0$.

We find that the ϵ-dependent quantities

$$\epsilon\int_0^\tau\int_\Omega\nabla\rho^\epsilon\cdot\nabla\phi\mathrm{d}x\mathrm{d}t\quad\text{and}\quad\epsilon\int_0^\tau\int_\Omega\left(\mathbf{u}^\epsilon\otimes\nabla\rho^\epsilon\right):\nabla\psi\mathrm{d}x\mathrm{d}t$$

vanish in the asymptotic limit $\epsilon \to 0$ as long as (3.5)–(3.9) hold. To see this, note that at least on the set $\Omega_1 = \{x \in \Omega | \rho^\epsilon \geq 1\}$, the two integrals above can be controlled by (3.5)–(3.9),

$$
\begin{aligned}
&\epsilon \int_0^\mathcal{T} \int_{\Omega_1} |\nabla \rho^\epsilon \cdot \nabla \phi| \,\mathrm{d}x\mathrm{d}t = \frac{2\sqrt{\epsilon}}{\gamma} \int_0^\mathcal{T} \int_{\Omega_1} \left| \sqrt{\epsilon} \nabla (\rho^\epsilon)^{\frac{\gamma}{2}} \cdot \frac{(\rho^\epsilon)^{\frac{\gamma}{2}} \nabla \phi}{(\rho^\epsilon)^{\gamma-1}} \right| \mathrm{d}x\mathrm{d}t \\
&\leq C\sqrt{\epsilon} \left[\epsilon \int_0^T \int_{\Omega_1} \left| \nabla (\rho^\epsilon)^{\frac{\gamma}{2}} \right|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_1} (\rho^\epsilon)^\gamma \mathrm{d}x\mathrm{d}t \right] \to 0 \text{ (when } \epsilon \to 0), \quad (3.13) \\
&\epsilon \int_0^\mathcal{T} \int_{\Omega_1} \left| (\mathbf{u}^\epsilon \otimes \nabla \rho^\epsilon) : \nabla \psi \right| \mathrm{d}x\mathrm{d}t \\
&\quad = \frac{2\sqrt{\epsilon}}{\gamma} \int_0^\mathcal{T} \int_{\Omega_1} \left| \left(\sqrt{\rho^\epsilon} \mathbf{u}^\epsilon \otimes \frac{\sqrt{\epsilon} \nabla (\rho^\epsilon)^{\frac{\gamma}{2}}}{(\rho^\epsilon)^{\frac{\gamma-1}{2}}} \right) : \nabla \psi \right| \mathrm{d}x\mathrm{d}t \leq C\sqrt{\epsilon} \\
&\times \left[\int_0^\mathcal{T} \int_{\Omega_1} \rho^\epsilon |\mathbf{u}^\epsilon|^2 \mathrm{d}x\mathrm{d}t + \epsilon \int_0^T \int_{\Omega_1} \left| \nabla (\rho^\epsilon)^{\frac{\gamma}{2}} \right|^2 \mathrm{d}x\mathrm{d}t \right] \to 0 \text{ (when } \epsilon \to 0). \qquad (3.14)
\end{aligned}
$$

Now, we begin to estimate $\nabla \rho^\epsilon$ on the set $\Omega_2 = \{x \in \Omega | \rho^\epsilon < 1\}$. Multiplying (3.1) on ρ^ϵ and integrating the result equations, we get

$$
\begin{aligned}
&\epsilon \int_0^\mathcal{T} \int_{\Omega_2} |\nabla \rho^\epsilon|^2 \mathrm{d}x\mathrm{d}t \leq \frac{1}{2} \int_{\Omega_2} |\rho_0^\epsilon|^2 \mathrm{d}x + \frac{1}{2} \int_0^\mathcal{T} \int_{\Omega_2} |\rho^\epsilon|^2 |\operatorname{div} \mathbf{u}^\epsilon| \mathrm{d}x\mathrm{d}t \\
&\leq \frac{1}{2} \int_{\Omega_2} |\rho_0^\epsilon|^2 \mathrm{d}x + \frac{1}{2} \int_0^\mathcal{T} \int_{\Omega_2} |\operatorname{div} \mathbf{u}^\epsilon| \mathrm{d}x\mathrm{d}t \\
&\leq \frac{1}{2} \int_{\Omega_2} |\rho_0^\epsilon|^2 \mathrm{d}x + \int_0^\mathcal{T} \int_{\Omega_2} (1 + |\nabla \mathbf{u}^\epsilon|^2) \mathrm{d}x\mathrm{d}t \leq C.
\end{aligned}
$$

Then the ϵ-dependent quantities

$$
\epsilon \int_0^\mathcal{T} \int_\Omega \nabla \rho^\epsilon \cdot \nabla \phi \mathrm{d}x\mathrm{d}t \quad \text{and} \quad \epsilon \int_0^\mathcal{T} \int_\Omega (\mathbf{u}^\epsilon \otimes \nabla \rho^\epsilon) : \nabla \psi \mathrm{d}x\mathrm{d}t
$$

on the set $\Omega_2 = \{\rho^\epsilon < 1\}$ can be controlled by (3.5)–(3.9),

$$
\begin{aligned}
&\epsilon \int_0^\mathcal{T} \int_{\Omega_2} |\nabla \rho^\epsilon \cdot \nabla \phi| \,\mathrm{d}x\mathrm{d}t = \sqrt{\epsilon} \int_0^\mathcal{T} \int_{\Omega_2} \left| \sqrt{\epsilon} \nabla \rho^\epsilon \cdot \nabla \phi \right| \mathrm{d}x\mathrm{d}t \\
&\leq C\sqrt{\epsilon} \left[\epsilon \int_0^T \int_{\Omega_2} |\nabla \rho^\epsilon|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_2} |\nabla \phi|^2 \mathrm{d}x\mathrm{d}t \right] \to 0 \quad \text{(when} \quad \epsilon \to 0), \\
&\hfill (3.15) \\
&\epsilon \int_0^\mathcal{T} \int_{\Omega_2} \left| (\mathbf{u}^\epsilon \otimes \nabla \rho^\epsilon) : \nabla \psi \right| \mathrm{d}x\mathrm{d}t = \sqrt{\epsilon} \int_0^\mathcal{T} \int_{\Omega_2} \left| (\mathbf{u}^\epsilon \otimes \sqrt{\epsilon} \nabla \rho^\epsilon) : \nabla \psi \right| \mathrm{d}x\mathrm{d}t \\
&\leq C\sqrt{\epsilon} \left[\int_0^\mathcal{T} \int_{\Omega_2} |\mathbf{u}^\epsilon|^2 \mathrm{d}x\mathrm{d}t + \epsilon \int_0^T \int_{\Omega_2} |\nabla \rho^\epsilon|^2 \,\mathrm{d}x\mathrm{d}t \right]
\end{aligned}
$$

$$\leq C\sqrt{\epsilon}\left[\int_0^T\int_{\Omega_2}|\nabla\mathbf{u}^\epsilon|^2\mathrm{d}x\mathrm{d}t+\epsilon\int_0^T\int_{\Omega_2}|\nabla\rho^\epsilon|^2\,\mathrm{d}x\mathrm{d}t\right]\to 0\ (\text{when } \epsilon\to 0). \quad (3.16)$$

Next, we can get that the following quantities are relative compactness in Lebesgue norm L^1.

1. ρ^ϵ is equi-integrable in L^1_x. In fact, we use (3.5) to have

$$\log G\int_{\{x\in\Omega|\rho^\epsilon\geq G\}}\rho^\epsilon\mathrm{d}x\leq\int_{\{x\in\Omega|\rho^\epsilon\geq G\}}\rho^\epsilon\log\rho^\epsilon\mathrm{d}x\leq C,$$

with $G\in(0,+\infty)$. So, when $G\to+\infty$, it gives that $\sup_\epsilon\int_{[x\in\Omega|\rho^\epsilon\geq G]}\rho^\epsilon\mathrm{d}x\to 0.$

2. $\rho^\epsilon\mathbf{u}^\epsilon$ is equi-integrable in L^1_x. In fact, in view of (3.5) and (3.7), we get

$$\int_{\{x\in\Omega|\rho^\epsilon\mathbf{u}^\epsilon|\geq G\}}|\rho^\epsilon\mathbf{u}^\epsilon|\,\mathrm{d}x\leq\frac{1}{\sqrt{\log G}}\int_{\{x\in\Omega|\rho^\epsilon\geq\sqrt{G}\}}\sqrt{\rho^\epsilon\log\rho^\epsilon}\sqrt{\rho^\epsilon}|\mathbf{u}^\epsilon|\mathrm{d}x$$
$$+\frac{1}{\sqrt{G}}\int_{\{x\in\Omega||\mathbf{u}^\epsilon|\geq\sqrt{G}\}}\sqrt{\rho^\epsilon}|\mathbf{u}^\epsilon|^2\mathrm{d}x$$
$$\leq\frac{1}{\sqrt{\log G}}\left(\int_{\{x\in\Omega|\rho^\epsilon\geq G\}}\rho^\epsilon\log\rho^\epsilon\mathrm{d}x+\int_\Omega\rho^\epsilon|\mathbf{u}^\epsilon|^2\mathrm{d}x\right)\leq\frac{C}{\sqrt{\log G}}.$$

Therefore, one gets $\sup_\epsilon\int_{\{x\in\Omega|(\rho^\epsilon\mathbf{u}^\epsilon)(x)|\geq G\}}|\rho^\epsilon\mathbf{u}^\epsilon|\,\mathrm{d}x\to 0$ as $G\to+\infty$.

3. $\mathbf{H}^\epsilon$ is equi-integrable in L^1_x. By (3.6), we have

$$G\int_{\{x\in\Omega||\mathbf{H}^\epsilon|\geq G\}}|\mathbf{H}^\epsilon|^2\mathrm{d}x\leq\int_{\{x\in\Omega||\mathbf{H}^\epsilon|\geq G\}}|\mathbf{H}^\epsilon|^2\mathrm{d}x\leq\int_\Omega|\mathbf{H}^\epsilon|^2\mathrm{d}x\leq C.$$

So, we get $\sup_\epsilon\int_{\{x\in\Omega||\mathbf{H}^\epsilon(x)|\geq G\}}|\mathbf{H}^\epsilon|\,\mathrm{d}x\to 0$ as $G\to+\infty$.

4. $\mathbf{u}^\epsilon$ and $\mathbf{H}^\epsilon$ are also equi-integrable in $L^1_{t,x}$. Using (3.8) and (3.9), we obtain that

$$\int_{\{(t,x)||\mathbf{u}^\epsilon(t,x)|\geq G\}}|\mathbf{u}^\epsilon|\,\mathrm{d}x\mathrm{d}t\leq\frac{1}{G}\int_{\{(t,x)||\mathbf{u}^\epsilon(t,x)|\geq G\}}|\mathbf{u}^\epsilon|^2\,\mathrm{d}x\ \mathrm{d}t$$
$$\leq\frac{1}{G}\int_0^T\int_\Omega|\mathbf{u}^\epsilon|^2\,\mathrm{d}x\mathrm{d}t\leq\frac{C}{G}\int_0^T\int_\Omega|\nabla\mathbf{u}^\epsilon|^2\,\mathrm{d}x\mathrm{d}t\leq\frac{C}{G}$$

and

$$\int_{\{(t,x)||\mathbf{H}^\epsilon(t,x)|\geq G\}}|\mathbf{H}^\epsilon|\,\mathrm{d}x\mathrm{d}t\leq\frac{1}{G}\int_{\{(t,x)||\mathbf{H}^\epsilon(t,x)|\geq G\}}|\mathbf{H}^\epsilon|^2\,\mathrm{d}x\mathrm{d}t$$
$$\leq\frac{1}{G}\int_0^T\int_\Omega|\mathbf{H}^\epsilon|^2\,\mathrm{d}x\mathrm{d}t\leq\frac{C}{G}\int_0^T\int_\Omega|\nabla\mathbf{H}^\epsilon|^2\,\mathrm{d}x\mathrm{d}t\leq\frac{C}{G}.$$

So, as $G \to +\infty$, we have that $\sup_{\epsilon} \int_{\{(t,x)||\mathbf{u}^\epsilon(t,x)|\geq G\}} |\mathbf{u}^\epsilon| \,\mathrm{d}x\mathrm{d}t \to 0$ and $\sup_{\epsilon} \int_{\{(t,x)||\mathbf{H}^\epsilon(t,x)|\geq G\}} |\mathbf{H}^\epsilon| \,\mathrm{d}x\mathrm{d}t \to 0$.

Now, we use the well-developed framework of parametrized measures associated with the family of equi-integrable functions $\{\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon\}_{\epsilon>0}$ generating Young measure

$$Y_{t,x} \in \mathcal{P}(\mathcal{Q}), \qquad \text{a.a.} \qquad (t,x) \in (0,T) \times \Omega.$$

For simplicity, we use the notation $\rho(t,x) = \langle Y_{t,x}; s\rangle, \mathbf{u}(t,x) = \langle Y_{t,x}; \mathbf{v}\rangle$ and $\mathbf{H}(t,x) = \langle Y_{t,x}; \mathbf{B}\rangle$. Now, we perform the limit $\epsilon \to 0$ in the weak form of the approximate system.

For the weak formulation (3.10) of continuity equation, in view of (3.13) and (3.15), the last term in (3.10) vanishes as $\epsilon \to 0$. So, we have

$$\begin{aligned}
&\int_\Omega \langle Y_{\tau,x}; s\rangle \phi(\tau,\cdot)\mathrm{d}x - \int_\Omega \langle Y_{0,x}; s\rangle \phi(0,\cdot)\mathrm{d}x \\
&= \int_0^\tau \int_\Omega \left[\langle Y_{t,x}; s\rangle \partial_t\phi + \langle Y_{t,x}; s\mathbf{v}\rangle \cdot \nabla\phi\right] \mathrm{d}x\mathrm{d}t
\end{aligned}$$

for any $\phi \in C^1([0,T] \times \bar{\Omega})$.

Using the no-slip boundary conditions, the viscous dissipation term can be rewritten in the following form

$$\begin{aligned}
&\int_0^\tau \int_\Omega \mathbb{S}(\nabla\mathbf{u}^\epsilon) : \nabla\mathbf{u}^\epsilon \mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \left[\mu\left(\nabla\mathbf{u}^\epsilon + \nabla^t\mathbf{u}^\epsilon - \frac{2}{3}\operatorname{div}\mathbf{u}^\epsilon\mathbb{I}\right) + \eta \operatorname{div}\mathbf{u}\mathbb{I}\right] : \nabla\mathbf{u}^\epsilon)\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \left[\mu|\nabla\mathbf{u}^\epsilon|^2 + \left(\frac{\mu}{3} + \eta\right)|\operatorname{div}\mathbf{u}^\epsilon|^2\right] \mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \left(\mu|\nabla\mathbf{u}^\epsilon|^2 + \lambda|\operatorname{div}\mathbf{u}^\epsilon|^2\right) \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\lambda = \frac{\mu}{3} + \eta > 0$. For energy inequality (3.4), in view of the L^1- boundedness in (3.5)–(3.9) and, one gets that

$$\frac{1}{2}\rho^\epsilon|\mathbf{u}^\epsilon|^2 + \frac{A(\rho^\epsilon)^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{H}^\epsilon|^2 \to E \quad \text{weakly- } (*) \text{ in } L^\infty_{\text{weak}}\left(0,T; \mathcal{M}(\bar{\Omega})\right)$$

with non-negative measure $E_\infty = E - \left\langle Y_{t,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{as^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{B}|^2\right\rangle \mathrm{d}x$;

$$\mu|\nabla\mathbf{u}^\epsilon|^2 + \lambda|\operatorname{div}\mathbf{u}^\epsilon|^2 + \nu|\nabla\mathbf{H}^\epsilon|^2 \to \sigma \text{ weakly- } (*) \text{ in } \mathcal{M}^+([0,T] \times \bar{\Omega})$$

with non-negative measure

$$\sigma_\infty = \sigma - \left(\mu\left|\nabla\langle Y_{t,x}; \mathbf{v}\rangle\right|^2 + \lambda\left|\operatorname{div}\langle Y_{t,x}; \mathbf{v}\rangle\right|^2 + \nu\left|\nabla\langle Y_{t,x}; \mathbf{B}\rangle\right|^2\right) \mathrm{d}x\mathrm{d}t.$$

Then, we may perform the limit $\epsilon \to 0$ in the energy inequality (3.4) obtaining

$$\begin{aligned}&\int_\Omega \left\langle Y_{\tau,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{As^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{B}|^2 \right\rangle \mathrm{d}x + \mathcal{D}(\tau)\\ &\quad + \int_0^\tau\int_\Omega \Big(\mu \left|\nabla \left\langle Y_{t,x}; \mathbf{v}\right\rangle\right|^2 + \lambda \left|\operatorname{div}\left\langle Y_{t,x}; \mathbf{v}\right\rangle\right|^2 + \nu \left|\nabla\left\langle Y_{t,x}; \mathbf{B}\right\rangle\right|^2 \Big)\mathrm{d}x\mathrm{d}t\\ &\le \int_\Omega \left\langle Y_{0,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{As^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{B}|^2 \right\rangle \mathrm{d}x\end{aligned}$$

with $\mathcal{D}(\tau) = E_\infty(\tau)[\overline{\Omega}] + \sigma_\infty[[0,\tau]\times\overline{\Omega}]$ for any $\tau \in [0,T]$.

For the weak formulation (3.11) of momentum equation, noting that $|\rho^\epsilon(\mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon)| \le \rho^\epsilon |\mathbf{u}^\epsilon|^2$ and $|\mathbf{H}^\epsilon \otimes \mathbf{H}^\epsilon| \le |\mathbf{H}^\epsilon|^2$, using (3.14), (3.16) and the equi-integrability of other terms, we derive from Lemma 2.1 in [6] that

$$\begin{aligned}&\int_\Omega \left\langle Y_{\tau,x}; s\mathbf{v}\right\rangle \cdot \psi(\tau,\cdot)\mathrm{d}x - \int_\Omega \left\langle Y_{0,x}; s\mathbf{v}\right\rangle \cdot \psi(0,\cdot)\mathrm{d}x\\ &= \int_0^\tau\int_\Omega \Big(\left\langle Y_{t,x}; s\mathbf{v}\right\rangle \cdot \partial_t\psi + \left\langle Y_{t,x}; s(\mathbf{v}\otimes\mathbf{v})\right\rangle : \nabla\psi + A\left\langle Y_{t,x}; s^\gamma\right\rangle \operatorname{div}\psi \Big)\mathrm{d}x\mathrm{d}t\\ &\quad - \int_0^\tau\int_\Omega \left(\left\langle Y_{t,x}; \mathbf{B}\otimes\mathbf{B}\right\rangle : \nabla\psi - \frac{1}{2}\left\langle Y_{t,x}; |\mathbf{B}|^2\right\rangle \operatorname{div}\psi \right)\mathrm{d}x\mathrm{d}t\\ &\quad - \int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla\psi \mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^V; \nabla\psi\right\rangle \mathrm{d}t\end{aligned}$$

for any $\psi \in C^1([0,T]\times\bar{\Omega})$ with $\psi|_{\partial\Omega} = 0$, where the measure $\left|r^V(\tau)\right| = \left|\left\{r^V_{i,j}(\tau)\right\}_{i,j=1,2,3}\right| \le CE_\infty(\tau), \left\{r^V_{i,j}(\tau)\right\} \in L^\infty_{\text{weak}}(0,T;\mathcal{M}(\Omega))$ for a.e. $\tau \in (0,T)$.

For the weak formulation (3.12) of magnetic field equation, noting that $|\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon| \le \frac{1}{2}(|\mathbf{u}^\epsilon|^2 + |\mathbf{H}^\epsilon|^2)$, we can obtain that

$$\begin{aligned}&\int_\Omega \left\langle Y_{\tau,x}; \mathbf{B}\right\rangle \cdot \omega(\tau,\cdot)\mathrm{d}x - \int_\Omega \left\langle Y_{0,x}; \mathbf{B}\right\rangle \cdot \omega(0,\cdot)\mathrm{d}x = -\int_0^\tau \left\langle r^H; \nabla\omega\right\rangle dt\\ &\quad + \int_0^\tau\int_\Omega \Big[\left\langle Y_{t,x}; \mathbf{B}\right\rangle \cdot \partial_t\omega + \left\langle Y_{t,x}; \mathbf{v}\times\mathbf{B}\right\rangle : \nabla\times\omega - \nu\nabla\mathbf{H} : \nabla\omega \Big]\mathrm{d}x\mathrm{d}t\end{aligned}$$

for any $\omega \in C^1([0,T]\times\bar{\Omega})$ with $\omega|_{\partial\Omega} = 0$, where the measure

$$\left|r^H(\tau)\right| = \left|\left\{r^H_{i,j}(\tau)\right\}_{i,j=1,2,3}\right| \le CE_\infty(\tau), \left\{r^H_{i,j}(\tau)\right\} \in L^\infty_{weak}(0,T;\mathcal{M}(\Omega))$$

for a.e. $\tau \in (0,T)$.

For generalized Poincaré's inequality, we have

$$\begin{aligned}&\int_0^\tau\int_\Omega \left\langle Y_{t,x}; |\mathbf{v}-\mathbf{u}|^2 + |\mathbf{B}-\mathbf{H}|^2 \right\rangle \mathrm{d}x\mathrm{d}t\\ &= \lim_{\epsilon\to 0}\int_0^\tau\int_\Omega \left(|\mathbf{u}^\epsilon - \mathbf{u}|^2 + |\mathbf{H}^\epsilon - \mathbf{H}|^2\right)\mathrm{d}x\mathrm{d}t\\ &\le C\lim_{\epsilon\to 0}\int_0^\tau\int_\Omega \left(|\nabla\mathbf{u}^\epsilon - \nabla\mathbf{u}|^2 + |\nabla\mathbf{H}^\epsilon - \nabla\mathbf{H}|^2\right)\mathrm{d}x\mathrm{d}t \le C\mathcal{D}(\tau).\end{aligned}$$

4 Proof of Theorem 2

We assume that $(\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{H}})$ is a strong solution to the compressible magnetohydrodynamic equations, satisfying

$$\partial_t \tilde{\rho} + \operatorname{div}(\tilde{\rho}\tilde{\mathbf{u}}) = 0, \tag{4.1}$$

$$\partial_t \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} + A\gamma\tilde{\rho}^{\gamma-2}\nabla\tilde{\rho} = \frac{1}{\tilde{\rho}}(\nabla \times \tilde{\mathbf{H}}) \times \tilde{\mathbf{H}} + \frac{1}{\tilde{\rho}} \operatorname{div} \mathbb{S}(\nabla\tilde{\mathbf{u}}), \tag{4.2}$$

$$\partial_t \tilde{\mathbf{H}} - \nabla \times (\tilde{\mathbf{u}} \times \tilde{\mathbf{H}}) = \nu \Delta \tilde{\mathbf{H}}, \quad \operatorname{div} \tilde{\mathbf{H}} = 0 \tag{4.3}$$

with the property (2.4)–(2.5).

Now, we introduce the following relative entropy inequality for the measure-valued solutions to the compressible magnetohydrodynamic equations,

$$\begin{aligned}
&\mathcal{E}_{mv}(\tau) = \mathcal{E}_{mv}(\rho, \mathbf{u}, \mathbf{H}|\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{H}})(\tau) \\
=& \int_\Omega \left\langle Y_{\tau,x}; \frac{1}{2}s|\mathbf{v} - \tilde{\mathbf{u}}|^2 + \frac{A}{\gamma-1}(s^\gamma - \gamma\tilde{\rho}^{\gamma-1}(s-\tilde{\rho}) - \tilde{\rho}^\gamma) + \frac{1}{2}|\mathbf{B} - \tilde{\mathbf{H}}|^2 \right\rangle \mathrm{d}x \\
=& \int_\Omega \left\langle Y_{\tau,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{As^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{B}|^2 \right\rangle \mathrm{d}x - \int_\Omega \left\langle Y_{\tau,x}; sv \right\rangle \cdot \tilde{\mathbf{u}} \mathrm{d}x \\
&+ \int_\Omega \frac{1}{2} \left\langle Y_{\tau,x}; s \right\rangle |\tilde{\mathbf{u}}|^2 \mathrm{d}x - \int_\Omega \left\langle Y_{\tau,x}; s \right\rangle \frac{A\gamma\tilde{\rho}^{\gamma-1}}{\gamma-1} \mathrm{d}x + \int_\Omega A\tilde{\rho}^\gamma \mathrm{d}x \\
&- \int_\Omega \left\langle Y_{\tau,x}; \mathbf{B} \right\rangle \cdot \tilde{\mathbf{H}} \mathrm{d}x + \int_\Omega \frac{1}{2}|\tilde{\mathbf{H}}|^2 \mathrm{d}x.
\end{aligned} \tag{4.4}$$

Take $\psi = \tilde{\mathbf{u}}$ as a test function in momentum equation (2.2) to obtain that

$$\begin{aligned}
&- \int_\Omega \left\langle Y_{\tau,x}; s\mathbf{v} \right\rangle \cdot \tilde{\mathbf{u}}(\tau, \cdot) \mathrm{d}x + \int_\Omega \left\langle Y_{0,x}; s\mathbf{v} \right\rangle \cdot \tilde{\mathbf{u}}(0, \cdot) \mathrm{d}x \\
=& - \int_0^\tau \int_\Omega \Big(\left\langle Y_{t,x}; s\mathbf{v} \right\rangle \cdot \partial_t \tilde{\mathbf{u}} + \left\langle Y_{t,x}; s(\mathbf{v} \otimes \mathbf{v}) \right\rangle : \nabla\tilde{\mathbf{u}} + \left\langle Y_{t,x}; As^\gamma \right\rangle \operatorname{div} \tilde{\mathbf{u}} \Big) \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \Big(\left\langle Y_{t,x}; \mathbf{B} \otimes \mathbf{B} \right\rangle : \nabla\tilde{\mathbf{u}} - \frac{1}{2} \left\langle Y_{t,x}; |\mathbf{B}|^2 \right\rangle \operatorname{div} \tilde{\mathbf{u}} \Big) \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla\tilde{\mathbf{u}} \mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^V; \nabla\tilde{\mathbf{u}} \right\rangle \mathrm{d}t.
\end{aligned}$$

Next, we use $\phi = \frac{|\tilde{\mathbf{u}}|^2}{2}$ as a test function in continuity equation (2.1) to yield the following equality

$$\begin{aligned}
&\int_\Omega \left\langle Y_{\tau,x}; \frac{1}{2}s|\tilde{\mathbf{u}}|^2 \right\rangle (\tau, \cdot) \mathrm{d}x - \int_\Omega \left\langle Y_{0,x}; \frac{1}{2}s|\tilde{\mathbf{u}}|^2 \right\rangle (0, \cdot) \mathrm{d}x \\
=& \int_0^\tau \int_\Omega \Big[\left\langle Y_{t,x}; s\tilde{\mathbf{u}} \right\rangle \partial_t \tilde{\mathbf{u}} + \left\langle Y_{t,x}; s\tilde{\mathbf{u}} \otimes \mathbf{v} \right\rangle : \nabla\tilde{\mathbf{u}} \Big] \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where we have used $s\mathbf{v} \cdot \nabla \left(\frac{|\tilde{\mathbf{u}}|^2}{2} \right) = s(\tilde{\mathbf{u}} \otimes \mathbf{v}) : \nabla\tilde{\mathbf{u}}$. Similarly, we chose test function $\phi = A\gamma\tilde{\rho}^{\gamma-1}/(\gamma-1)$ in continuity equation (2.1) to yield that

$$-\int_\Omega \left\langle Y_{\tau,x}; s\right\rangle \frac{A\gamma\tilde{\rho}^{\gamma-1}}{\gamma-1}(\tau,\cdot)\mathrm{d}x + \int_\Omega \left\langle Y_{0,x}; s\right\rangle \frac{A\gamma\tilde{\rho}^{\gamma-1}}{\gamma-1}(0,\cdot)\mathrm{d}x$$
$$= -\int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; s\right\rangle A\gamma\tilde{\rho}^{\gamma-2}\partial_t\tilde{\rho} + \left\langle Y_{t,x}; s\mathbf{v}\right\rangle \cdot A\gamma\tilde{\rho}^{\gamma-2}\nabla\tilde{\rho}\right]\mathrm{d}x\mathrm{d}t.$$

From Equation (4.1), one gets that

$$\int_\Omega A\tilde{\rho}^\gamma(\tau,\cdot)\mathrm{d}x - \int_\Omega A\tilde{\rho}^\gamma(0,\cdot)\mathrm{d}x = \int_0^\tau \int_\Omega A\gamma\tilde{\rho}^{\gamma-1}\partial_t\tilde{\rho}\ \mathrm{d}x\mathrm{d}t.$$

Using $\tilde{\mathbf{H}}$ as a test function to the equation of magnetic equation, one gets that

$$-\int_\Omega \left\langle Y_{\tau,x}; \mathbf{B}\right\rangle \cdot \tilde{\mathbf{H}}(\tau,\cdot)\mathrm{d}x + \int_\Omega \left\langle Y_{0,x}; \mathbf{B}\right\rangle \cdot \tilde{\mathbf{H}}(0,\cdot)\mathrm{d}x$$
$$= -\int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; \mathbf{B}\right\rangle \cdot \partial_t\tilde{\mathbf{H}} + \left\langle Y_{t,x}; \nabla\times(\mathbf{v}\times\mathbf{B})\right\rangle \cdot \tilde{\mathbf{H}})\right]\mathrm{d}x\mathrm{d}t$$
$$+\nu\int_0^\tau \int_\Omega \nabla\mathbf{H} : \nabla\tilde{\mathbf{H}}\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^H; \nabla\tilde{\mathbf{H}}\right\rangle dt.$$

Recalling that the smooth functions $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{u}}$ solve (4.3), we use $\tilde{\mathbf{H}}\Big|_{\partial\Omega} = 0$ to deduce that

$$\frac{1}{2}\int_\Omega |\tilde{\mathbf{H}}|^2(\tau,\cdot)\mathrm{d}x - \frac{1}{2}\int_\Omega |\tilde{\mathbf{H}}|^2(0,\cdot)\mathrm{d}x$$
$$= -\nu\int_0^\tau \int_\Omega |\nabla\tilde{\mathbf{H}}|^2\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau \int_\Omega \nabla\times(\tilde{\mathbf{u}}\times\tilde{\mathbf{H}})\cdot\tilde{\mathbf{H}}\mathrm{d}x\mathrm{d}t. \tag{4.5}$$

Combining (2.3) with (4.4)–(4.5), we obtain that

$$\mathcal{E}_{mv}(\tau) + \mathcal{D}(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla\mathbf{u}) : (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \nu|\nabla\mathbf{H}|^2\mathrm{d}x\mathrm{d}t$$
$$\le \int_\Omega \left\langle Y_{0,x}; \frac{1}{2}s|\mathbf{v}-\tilde{\mathbf{u}}_0|^2 + \frac{A}{\gamma-1}(s^\gamma - \gamma\tilde{\rho}_0^{\gamma-1}(s-\tilde{\rho}_0) - \tilde{\rho}_0^\gamma) + \frac{1}{2}|\mathbf{B}-\tilde{\mathbf{H}}_0|^2\right\rangle \mathrm{d}x$$
$$\underbrace{-\int_0^\tau \int_\Omega \left(\left\langle Y_{t,x}; s\mathbf{v}\right\rangle \cdot \partial_t\tilde{\mathbf{u}} + \left\langle Y_{t,x}; s(\mathbf{v}\otimes\mathbf{v})\right\rangle : \nabla\tilde{\mathbf{u}}\right)\mathrm{d}x\mathrm{d}t}_{K_1}$$
$$\underbrace{+\int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; s\tilde{\mathbf{u}}\right\rangle \partial_t\tilde{\mathbf{u}} + \left\langle Y_{t,x}; s\tilde{\mathbf{u}}\otimes\mathbf{v}\right\rangle : \nabla\tilde{\mathbf{u}}\right]\mathrm{d}x\mathrm{d}t}_{K_2}$$
$$\underbrace{-\int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; s\right\rangle A\gamma\tilde{\rho}^{\gamma-2}\partial_t\tilde{\rho} + \left\langle Y_{t,x}; s\mathbf{v}\right\rangle \cdot A\gamma\tilde{\rho}^{\gamma-2}\nabla\tilde{\rho}\right]\mathrm{d}x\mathrm{d}t}_{K_3}$$

$$\underbrace{+\int_0^\tau\int_\Omega A\gamma\tilde\rho^{\gamma-1}\partial_t\tilde\rho\,\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_\Omega\left\langle Y_{t,x};As^\gamma\right\rangle\mathrm{div}\,\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t}_{K_4}$$
$$\underbrace{+\int_0^\tau\int_\Omega\left(\left\langle Y_{t,x};\mathbf{B}\otimes\mathbf{B}\right\rangle:\nabla\tilde{\mathbf{u}}-\frac12\left\langle Y_{t,x};|\mathbf{B}|^2\right\rangle\mathrm{div}\,\tilde{\mathbf{u}}\right)\mathrm{d}x\mathrm{d}t}_{K_5}$$
$$\underbrace{-\int_0^\tau\int_\Omega\left[\left\langle Y_{t,x};\mathbf{B}\right\rangle\cdot\partial_t\tilde{\mathbf{H}}+\left\langle Y_{t,x};\nabla\times(\mathbf{v}\times\mathbf{B})\right\rangle\cdot\tilde{\mathbf{H}})\right]\mathrm{d}x\mathrm{d}t}_{K_6}$$
$$\underbrace{+\int_0^\tau\int_\Omega\nabla\times(\tilde{\mathbf{u}}\times\tilde{\mathbf{H}})\cdot\tilde{\mathbf{H}}\mathrm{d}x\mathrm{d}t}_{K_7}+\nu\int_0^\tau\int_\Omega\nabla\mathbf{H}:\nabla\tilde{\mathbf{H}}\mathrm{d}x\mathrm{d}t$$
$$-\nu\int_0^\tau\int_\Omega|\nabla\tilde{\mathbf{H}}|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\left\langle r^V;\nabla\tilde{\mathbf{u}}\right\rangle\mathrm{d}t+\int_0^\tau\left\langle r^H;\nabla\tilde{\mathbf{H}}\right\rangle dt. \tag{4.6}$$

Using the momentum equation (4.2), we have that

$$\begin{aligned}
K_1+K_2=&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};s(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle\cdot\partial_t\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};s(\mathbf{v}-\tilde{\mathbf{u}})\otimes\mathbf{v}\right\rangle:\nabla\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t\\
=&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};s(\mathbf{v}-\tilde{\mathbf{u}})\otimes(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle:\nabla\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\left\langle Y_{t,x};s(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle\cdot A\gamma\tilde\rho^{\gamma-2}\nabla\tilde\rho\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};(s-\tilde\rho)(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle\cdot\frac{1}{\tilde\rho}\left((\nabla\times\tilde{\mathbf{H}})\times\tilde{\mathbf{H}}+\mathrm{div}\,\mathbb{S}(\nabla\tilde{\mathbf{u}})\right)\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};\mathbf{v}-\tilde{\mathbf{u}}\right\rangle\cdot\left((\nabla\times\tilde{\mathbf{H}})\times\tilde{\mathbf{H}}\right)\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega\mathbb{S}(\nabla\tilde{\mathbf{u}}):(\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t\\
\le& C\int_0^\tau\mathcal{E}_{mv}(\rho,\mathbf{u},\mathbf{H}|\tilde\rho,\tilde{\mathbf{u}},\tilde{\mathbf{H}})(t)\mathrm{d}t+\int_0^\tau\int_\Omega\left\langle Y_{t,x};s(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle\cdot A\gamma\tilde\rho^{\gamma-2}\nabla\tilde\rho\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};(s-\tilde\rho)(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle\cdot\frac{1}{\tilde\rho}\left((\nabla\times\tilde{\mathbf{H}})\times\tilde{\mathbf{H}}+\mathrm{div}\,\mathbb{S}(\nabla\tilde{\mathbf{u}})\right)\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};\tilde{\mathbf{H}}\cdot\nabla\tilde{\mathbf{H}}\cdot(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega\left\langle Y_{t,x};(\mathbf{v}-\tilde{\mathbf{u}})\cdot\nabla\tilde{\mathbf{H}}\cdot\tilde{\mathbf{H}}\right\rangle\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\mathbb{S}(\nabla\tilde{\mathbf{u}}):(\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Using the equation of continuity (4.1), we get that

$$\begin{aligned}
K_3+K_4=&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};As^\gamma\right\rangle\mathrm{div}\,\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_\Omega A\gamma\tilde\rho^{\gamma-1}\,\mathrm{div}(\tilde\rho\tilde{\mathbf{u}})\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\left\langle Y_{t,x};s\right\rangle A\gamma\tilde\rho^{\gamma-2}\,\mathrm{div}(\tilde\rho\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_\Omega\left\langle Y_{t,x};s\mathbf{v}\right\rangle\cdot A\gamma\tilde\rho^{\gamma-2}\nabla\tilde\rho\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= -A\int_0^\tau\int_\Omega \left\langle Y_{t,x}; s^\gamma - \gamma\tilde{\rho}^{\gamma-1}(s-\tilde{\rho}) - \tilde{\rho}^\gamma\right\rangle \operatorname{div}\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau\int_\Omega \left\langle Y_{t,x}; s(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \cdot A\gamma\tilde{\rho}^{\gamma-2}\nabla\tilde{\rho}\mathrm{d}x\mathrm{d}t \\
&\le C\int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t - \int_0^\tau\int_\Omega \left\langle Y_{t,x}; s(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \cdot A\gamma\tilde{\rho}^{\gamma-2}\nabla\tilde{\rho}\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.7}
$$

due to

$$
\begin{aligned}
&-\int_0^\tau\int_\Omega A\gamma\tilde{\rho}^{\gamma-1}\operatorname{div}(\tilde{\rho}\tilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t = -\int_0^\tau\int_\Omega A\gamma\tilde{\rho}^{\gamma}\operatorname{div}\tilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \\
&\quad -\int_0^\tau\int_\Omega A\gamma\tilde{\rho}^{\gamma-1}\ \tilde{\mathbf{u}}\cdot\nabla\tilde{\rho}\ \mathrm{d}x\mathrm{d}t = -\int_0^\tau\int_\Omega A(\gamma-1)\tilde{\rho}^{\gamma}\operatorname{div}\tilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Using the following calculus formulas

$$
\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\operatorname{div}\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{a} + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}
$$

and the magnetic filed equation (4.3), by integration by parts, we deduce that

$$
\begin{aligned}
K_5 &= \int_0^\tau\int_\Omega \left\langle Y_{t,x}; \mathbf{B}\otimes\mathbf{B} : \nabla\tilde{\mathbf{u}} - \frac{1}{2}|\mathbf{B}|^2\operatorname{div}\tilde{\mathbf{u}}\right\rangle \mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau\int_\Omega \left\langle Y_{t,x}; \mathbf{B}\cdot\nabla\tilde{\mathbf{u}}\cdot\mathbf{B} - \frac{1}{2}|\mathbf{B}|^2\operatorname{div}\tilde{\mathbf{u}}\right\rangle \mathrm{d}x\mathrm{d}t, \\
K_6 &= -\int_0^\tau\int_\Omega \left\langle Y_{t,x}; \mathbf{B}\right\rangle \cdot \left[\nabla\times(\tilde{\mathbf{u}}\times\tilde{\mathbf{H}}) + \nu\Delta\tilde{\mathbf{H}}\right]\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau\int_\Omega \left\langle Y_{t,x}; \nabla\times(\mathbf{v}\times\mathbf{B})\right\rangle \cdot \tilde{\mathbf{H}}\mathrm{d}x\mathrm{d}t \\
&= -\int_0^\tau\int_\Omega \left\langle Y_{t,x}; \tilde{\mathbf{H}}\cdot\nabla\tilde{\mathbf{u}}\cdot\mathbf{B} - \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{H}}\cdot\mathbf{B} - \mathbf{B}\cdot\tilde{\mathbf{H}}\operatorname{div}\tilde{\mathbf{u}}\right\rangle \mathrm{d}x\mathrm{d}t \\
&\quad + \nu\int_0^\tau\int_\Omega \nabla\mathbf{H} : \nabla\tilde{\mathbf{H}}\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \left\langle Y_{t,x}; \mathbf{B}\cdot\nabla\tilde{\mathbf{H}}\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\tilde{\mathbf{H}}\cdot\mathbf{B}\right\rangle \mathrm{d}x\mathrm{d}t, \\
K_7 &= \int_0^\tau\int_\Omega (\tilde{\mathbf{H}}\cdot\nabla\tilde{\mathbf{u}}\cdot\tilde{\mathbf{H}} - \frac{1}{2}|\tilde{\mathbf{H}}|^2\operatorname{div}\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.8}
$$

Let $\tilde{\mathbf{m}} = (\nabla\times\tilde{\mathbf{H}})\times\tilde{\mathbf{H}} + \operatorname{div}\mathbb{S}(\nabla\tilde{\mathbf{u}})$. Putting (4.7)–(4.8) into (4.6), we have that

$$
\begin{aligned}
&\mathcal{E}_{mv}(\tau) + \mathcal{D}(\tau) + \int_0^\tau\int_\Omega \nu|\nabla\mathbf{H} - \nabla\tilde{\mathbf{H}}|^2\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau\int_\Omega (\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\tilde{\mathbf{u}})) : (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t \\
&\le \int_\Omega \left\langle Y_{0,x}; \frac{1}{2}s|\mathbf{v}-\tilde{\mathbf{u}}_0|^2 + \frac{A}{\gamma-1}(s^\gamma - \gamma\tilde{\rho}_0^{\gamma-1}(s-\tilde{\rho}_0) - \tilde{\rho}_0^\gamma) + \frac{1}{2}|\mathbf{B}-\tilde{\mathbf{H}}_0|^2\right\rangle \mathrm{d}x \\
&\quad + C\int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t - \int_0^\tau\int_\Omega \left\langle Y_{t,x}; (s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \cdot \frac{\tilde{\mathbf{m}}}{\tilde{\rho}}\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau\int_\Omega \left\langle Y_{t,x}; \tilde{\mathbf{H}}\cdot\nabla\tilde{\mathbf{H}}\cdot(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \left\langle Y_{t,x}; (\mathbf{v}-\tilde{\mathbf{u}})\cdot\nabla\tilde{\mathbf{H}}\cdot\tilde{\mathbf{H}}\right\rangle \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^\tau\int_\Omega\left\langle Y_{t,x};\mathbf{B}\cdot\nabla\tilde{\mathbf{u}}\cdot\mathbf{B}-\frac{1}{2}|B|^2\operatorname{div}\tilde{\mathbf{u}}\right\rangle \mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};\tilde{\mathbf{H}}\cdot\nabla\tilde{\mathbf{u}}\cdot\mathbf{B}-\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{H}}\cdot\mathbf{B}-\mathbf{B}\cdot\tilde{\mathbf{H}}\operatorname{div}\tilde{\mathbf{u}}\right\rangle \mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\left\langle Y_{t,x};\mathbf{B}\cdot\nabla\tilde{\mathbf{H}}\cdot\mathbf{v}-\mathbf{v}\cdot\nabla\tilde{\mathbf{H}}\cdot\mathbf{B}\right\rangle \mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega(\tilde{\mathbf{H}}\cdot\nabla\tilde{\mathbf{u}}\cdot\tilde{\mathbf{H}}-\frac{1}{2}|\tilde{\mathbf{H}}|^2\operatorname{div}\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t+\int_0^\tau\left\langle r^V;\nabla\tilde{\mathbf{u}}\right\rangle \mathrm{d}t+\int_0^\tau\left\langle r^H;\nabla\tilde{\mathbf{H}}\right\rangle dt\\
&=\int_\Omega\left\langle Y_{0,x};\frac{1}{2}s|\mathbf{v}-\tilde{\mathbf{u}}_0|^2+\frac{A}{\gamma-1}(s^\gamma-\gamma\tilde{\rho}_0^{\gamma-1}(s-\tilde{\rho}_0)-\tilde{\rho}_0^\gamma)+\frac{1}{2}|\mathbf{B}-\tilde{\mathbf{H}}_0|^2\right\rangle \mathrm{d}x\\
&+C\int_0^\tau\mathcal{E}_{mv}(\rho,\mathbf{u},\mathbf{H}|\tilde{\rho},\tilde{\mathbf{u}},\tilde{\mathbf{H}})(t)\mathrm{d}t-\int_0^\tau\int_\Omega\langle Y_{t,x};(s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\rangle\cdot\frac{\tilde{\mathbf{m}}}{\tilde{\rho}}\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\left\langle Y_{t,x};(\mathbf{B}-\tilde{\mathbf{H}})\cdot\nabla\tilde{\mathbf{H}}\cdot(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};(\mathbf{v}-\tilde{\mathbf{u}})\cdot\nabla\tilde{\mathbf{H}}\cdot(\mathbf{B}-\tilde{\mathbf{H}})\right\rangle \mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\left\langle Y_{t,x};(\mathbf{B}-\tilde{\mathbf{H}})\cdot\nabla\tilde{\mathbf{u}}\cdot(\mathbf{B}-\tilde{\mathbf{H}})\right\rangle \mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left\langle Y_{t,x};\frac{1}{2}|\mathbf{B}-\tilde{\mathbf{H}}|^2\right\rangle \operatorname{div}\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t+\int_0^\tau\left\langle r^V;\nabla\tilde{\mathbf{u}}\right\rangle \mathrm{d}t+\int_0^\tau\left\langle r^H;\nabla\tilde{\mathbf{H}}\right\rangle dt\\
&=\int_\Omega\left\langle Y_{0,x};\frac{1}{2}s|\mathbf{v}-\tilde{\mathbf{u}}_0|^2+\frac{A}{\gamma-1}(s^\gamma-\gamma\tilde{\rho}_0^{\gamma-1}(s-\tilde{\rho}_0)-\tilde{\rho}_0^\gamma)+\frac{1}{2}|\mathbf{B}-\tilde{\mathbf{H}}_0|^2\right\rangle \mathrm{d}x\\
&+C\int_0^\tau\mathcal{E}_{mv}(t)\mathrm{d}t+\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4, \qquad (4.9)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R}_1&=-\int_0^\tau\int_\Omega\langle Y_{t,x};(s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\rangle\cdot\frac{\tilde{\mathbf{m}}}{\tilde{\rho}}\mathrm{d}x\mathrm{d}t,\\
\mathcal{R}_2&=\int_0^\tau\int_\Omega\left\langle Y_{t,x};(\mathbf{B}-\tilde{\mathbf{H}})\cdot\nabla\tilde{\mathbf{H}}\cdot(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\left\langle Y_{t,x};(\mathbf{v}-\tilde{\mathbf{u}})\cdot\nabla\tilde{\mathbf{H}}\cdot(\mathbf{B}-\tilde{\mathbf{H}})\right\rangle \mathrm{d}x\mathrm{d}t,\\
\mathcal{R}_3&=\int_0^\tau\int_\Omega\left\langle Y_{t,x};(\mathbf{B}-\tilde{\mathbf{H}})\cdot\nabla\tilde{\mathbf{u}}\cdot(\mathbf{B}-\tilde{\mathbf{H}})\right\rangle \mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\left\langle Y_{t,x};\frac{1}{2}|\mathbf{B}-\tilde{\mathbf{H}}|^2\right\rangle \operatorname{div}\tilde{\mathbf{u}}\mathrm{d}x\mathrm{d}t,\\
\mathcal{R}_4&=\int_0^\tau\left\langle r^V;\nabla\tilde{\mathbf{u}}\right\rangle \mathrm{d}t+\int_0^\tau\left\langle r^H;\nabla\tilde{\mathbf{H}}\right\rangle dt.
\end{aligned}
$$

Now, we begin to estimate $\mathcal{R}_i (i=1,2,3,4)$. We rewrite $\mathcal{R}_1$ to

$$
\mathcal{R}_1=-\int_0^\tau\int_\Omega\langle Y_{t,x};(s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\rangle\cdot\frac{\tilde{\mathbf{m}}}{\tilde{\rho}}\mathrm{d}x\mathrm{d}t
$$

$$= -\int_0^\tau \int_{\{0<s\leq\tilde{\rho}/2\}\cup\{\tilde{\rho}/2<s<2\tilde{\rho}\}} \left\langle Y_{t,x}; (s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \cdot \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau \int_{\{s\geq 2\tilde{\rho}\}} \left\langle Y_{t,x}; (s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \cdot \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \mathrm{d}x\mathrm{d}t. \tag{4.10}$$

Similar to [7], we use Hölder's inequality, Sobolev's inequality, and a Korn-type inequality to get that

$$\left| -\int_0^\tau \int_{\{0<s\leq\tilde{\rho}/2\}\cup\{\tilde{\rho}/2<s<2\tilde{\rho}\}} \left\langle Y_{\tau,x}; (s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \cdot \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \mathrm{d}x\mathrm{d}t \right|$$
$$\leq C(\delta) \left\| \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \right\|^2_{C([0,T]\times\Omega)} \int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t + \delta \int_0^\tau \int_\Omega \left\langle Y_{\tau,x}; |\mathbf{v}-\tilde{\mathbf{u}}|^2 \right\rangle \mathrm{d}x\mathrm{d}t$$
$$\leq C(\delta) \int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t + \delta \int_0^\tau \int_\Omega \left\langle Y_{\tau,x}; |\mathbf{v}-\mathbf{u}|^2 + |\mathbf{u}-\tilde{\mathbf{u}}|^2 \right\rangle \mathrm{d}x\mathrm{d}t$$
$$\leq C(\delta) \int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t + \delta \int_0^\tau \int_\Omega (\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\tilde{\mathbf{u}})) : (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t + \delta\mathcal{D}(\tau)$$

for any $\delta > 0$. Here, we have used the following equality:

$$\int_\Omega (\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\tilde{\mathbf{u}})) : (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}})\mathrm{d}x = \int_\Omega \left(\mu|\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}|^2 + \lambda|\operatorname{div}\mathbf{u} - \operatorname{div}\tilde{\mathbf{u}}|^2\right)\mathrm{d}x.$$

On the other hand, it is easy to find that the following inequalities

$$s^\gamma - \gamma\tilde{\rho}^{\gamma-1}(s-\tilde{\rho}) - \tilde{\rho}^\gamma \geq \left[1 - (\gamma+1)\left(\frac{1}{2}\right)^\gamma\right] s^\gamma, \quad \left|\frac{s-\tilde{\rho}}{s\tilde{\rho}}\right| s^{\frac{1}{2}-\frac{\gamma}{2}} \leq C$$

hold for $s > 2\tilde{\rho} > 2\underline{\rho} > 0$ and $\gamma > 1$. Then, using Hölder's inequality, we have that

$$\left| -\int_0^\tau \int_{\{s\geq 2\tilde{\rho}\}} \left\langle Y_{\tau,x}; (s-\tilde{\rho})(\mathbf{v}-\tilde{\mathbf{u}})\right\rangle \cdot \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \mathrm{d}x\mathrm{d}t \right|$$
$$\leq \int_0^\tau \int_{\{s\geq 2\tilde{\rho}\}} \left\langle Y_{\tau,x}; \left|\frac{s-\tilde{\rho}}{s\tilde{\rho}}\right| s^{\frac{1}{2}-\frac{\gamma}{2}} \cdot s^{\frac{\gamma}{2}} \cdot s^{\frac{1}{2}}(\mathbf{v}-\tilde{\mathbf{u}}) \right\rangle \cdot \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \mathrm{d}x\mathrm{d}t$$
$$\leq C \|\tilde{\mathbf{m}}\|_{C([0,T]\times\Omega)} \int_0^\tau \int_{\{s\geq 2\tilde{\rho}\}} \left\langle Y_{\tau,x}; s^\gamma + s|\mathbf{v}-\tilde{\mathbf{u}}|^2 \right\rangle \mathrm{d}x\mathrm{d}t \leq C \int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t.$$

For $\mathcal{R}_2$, we deduce from Hölder's inequality and Soblev's inequality that

$$\mathcal{R}_2 \leq 2 \left\|\nabla\tilde{\mathbf{H}}\right\|_{C([0,T]\times\Omega)} \int_0^\tau \int_\Omega \left\langle Y_{\tau,x}; |\mathbf{B}-\tilde{\mathbf{H}}||\mathbf{v}-\tilde{\mathbf{u}}| \right\rangle \mathrm{d}x\mathrm{d}t$$
$$\leq \delta \int_0^\tau \int_\Omega (\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\tilde{\mathbf{u}})) : (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t + \delta\mathcal{D}(\tau) + C(\delta) \int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t$$

for all $\delta > 0$. For $\mathcal{R}_3$, we get that

$$\mathcal{R}_3 \leq C\|\nabla\tilde{\mathbf{u}}\|_{C([0,T]\times\Omega)}\int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t \leq C\int_0^\tau \mathcal{E}_{mv}(t)\mathrm{d}t$$

for all $\delta > 0$. Moreover, in view of the Definition 1, we have that

$$\mathcal{R}_4 \leq C(\|\nabla\tilde{\mathbf{u}}\|_{C([0,T]\times\overline{\Omega})} + \|\nabla\tilde{\mathbf{H}}\|_{C([0,T]\times\overline{\Omega})})\int_0^\tau \mathcal{D}(t)\mathrm{d}t \leq C\int_0^\tau \mathcal{D}(t)\mathrm{d}t. \tag{4.11}$$

Combining (4.10)–(4.11) with (4.9) and using the smallness of δ, one gets that

$$\begin{aligned}
&\mathcal{E}_{mv}(\tau) + \mathcal{D}(\tau) + \frac{1}{2}\int_0^\tau\int_\Omega \nu|\nabla\mathbf{H} - \nabla\tilde{\mathbf{H}}|^2\mathrm{d}x\mathrm{d}t \\
&+\frac{1}{2}\int_0^\tau\int_\Omega(\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\tilde{\mathbf{u}})) : (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}})\mathrm{d}x\mathrm{d}t \\
\leq &\int_\Omega \left\langle Y_{0,x}; \frac{1}{2}s|\mathbf{v} - \tilde{\mathbf{u}}_0|^2 + \frac{A}{\gamma-1}(s^\gamma - \gamma\tilde{\rho}_0^{\gamma-1}(s-\tilde{\rho}_0) - \tilde{\rho}_0^\gamma) + \frac{1}{2}|\mathbf{B} - \tilde{\mathbf{H}}_0|^2\right\rangle \mathrm{d}x \\
&+ C\int_0^\tau \left[\mathcal{E}_{mv}(t) + \mathcal{D}(t)\right]\mathrm{d}t.
\end{aligned} \tag{4.12}$$

Applying Grönwall inequality to (4.12), we conclude the proof of Theorem 2.

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