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Investigation of a Discrete Sturm–Liouville Problem with Two-Point Nonlocal Boundary Condition and Natural Approximation of a Derivative in Boundary Condition

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Abstract. The article investigates a discrete Sturm–Liouville problem with one natural boundary condition and another nonlocal two-point boundary condition. We analyze zeroes, poles and critical points of the characteristic function and how the properties of this function depend on parameters in nonlocal boundary condition. Properties of the Spectrum Curves are formulated and illustrated in figures.

Keywords: discrete Sturm–Liouville problem, natural condition, nonlocal two-point condition, spectrum curves.

AMS Subject Classification: 34B24; 34B10.

1 Introduction

Nonlocal Boundary Value Problems (BVP) are widely used for mathematical modelling of various processes of physics, ecology, chemistry and industry, when it is impossible to determine the boundary values of the unknown function. New applications are found in heat conduction [4], linear thermoelasticity [7].

Consider the following one-dimensional Sturm–Liouville equation

$$-u''(t) = \lambda u(t), \quad t \in (0,1), \tag{1.1}$$

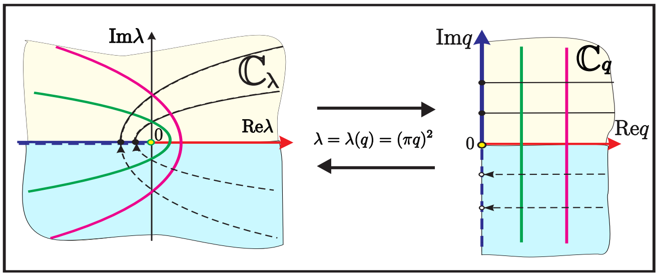


Figure 1. Bijective map: $\lambda = (\pi q)^2$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q$; ○–BP, ○–RP.

$\lambda \in \mathbb{C}_\lambda := \mathbb{C}$ is a complex spectral parameter. The general solution of this equation

$$u(t) = C_1 \cos(\pi q t) + C_2 \sin(\pi q t)/(\pi q), \tag{1.2}$$

and $\lambda = \lambda(q) = (\pi q)^2$. A map $\lambda : \mathbb{C} \to \mathbb{C}$ is not bijection, and the inverse map is multivalued. The point $\lambda = 0$ is the second order Branch Point (BP) and $q = 0$ is Ramification Point (RP) (with index 2). In this article, $q = x + \imath y$, $x, y \in \mathbb{R}$. If $q \in \mathbb{C}_q := \mathbb{R}_q + \mathbb{C}_q^+ + \mathbb{C}_q^-$, where $\mathbb{R}_q := \mathbb{R}_q^- + \mathbb{R}_q^+ + \mathbb{R}_q^0$, $\mathbb{R}_q^- := \{q = x + \imath y \in \mathbb{C} : x = 0, y > 0\}$, $\mathbb{R}_q^+ := \{q = x + \imath y \in \mathbb{C} : x > 0, y = 0\}$, $\mathbb{R}_q^0 := \{q = 0\}$, $\mathbb{C}_q^+ := \{q = x + \imath y \in \mathbb{C} : x > 0, y > 0\}$ and $\mathbb{C}_q^- := \{q = x + \imath y \in \mathbb{C} : \ x > 0,\ y < 0\}$, then the map $\lambda : \mathbb{C}_q \to \mathbb{C}$ is bijection [28] (see, Figure 1).

If we want to find particular solution of Equation (1.1), i.e., to find constants C_1 and C_2 in (1.2), then we must add two additional conditions. For example, the Dirichlet Boundary Condition (BC)

$$u(0) = 0 \tag{1.3_d}$$

gives that $C_1 = 0$ and we have one-parametric family of solutions $u(t) = C_2 \sin(\pi q t)/(\pi q)$. In the case of Neumann BC

$$u'(0) = 0 \tag{1.3_n}$$

gives that $C_2 = 0$ and we have $u(t) = C_1 \cos(\pi q t)$. BC (1.3_d) is essential boundary condition, BC (1.3_n) is natural boundary condition. For finding the second constant we can use two-points Nonlocal Boundary Condition (NBC) [2, 16, 17]:

$$\begin{array}{lll} \text{(Case 1)} & u(1) = \gamma u(\xi), & \xi \in [0,1), \\ \text{(Case 2)} & u'(1) = \gamma u'(\xi), & \xi \in [0,1), \end{array} \tag{1.4}$$

or integral BC [23]

$$u(1) = \gamma \int_{\xi_1}^{\xi_2} u(t)\, dt, \quad 0 \le \xi_1 < \xi_2 \le 1, \tag{1.5}$$

where $\gamma \in \mathbb{R}$. In the (classical) case $\gamma = 0$ we have two local BCs. In this article, we investigate discrete problems which approximating SLP with two-points NBC in Case 1 and Case 2. Green's functions for SLP with NBCs and more general Sturm–Liouville operator $L[u] := -(p(t)u')' + q(t)u$ were investigated in [18, 19, 25].

The Equation (1.1) with two BCs (1.3) and (1.4) defines Sturm–Liouville Problem (SLP). If we have nontrivial solution ($|C_1| + |C_2| \neq 0$) for some value of the parameter λ, then this value is an eigenvalue of SLP, and a solution is a corresponding eigenfunction for this eigenvalue. For each eigenvalue λ there exists eigenpoint $q \in \mathbb{C}_q$. So, our main task is to describe properties of eigenpoints in the domain $\mathbb{C}_q$. For investigation of SLP we can apply method of Characteristic Function (CF) [28]. Some results of such investigation are presented in [2] for SLP with two-point NBC (1.4) and in [1, 23] for SLP with integral BC (1.5). In these two papers the definition of Spectrum Curve was used for describing the dependence of spectrum on parameters γ and ξ.

Asymptotic formulas for eigenvalues and eigenfunctions for Sturm–Liouville operator with local BCs are investigated in the classical book [29]. Asymptotic analysis of eigenvalues and eigenfunctions of SLPs with periodic BCs was obtained in [3]. The SLP with eigenparameter in BCs was investigated in [9, 12]. Discrete Sturm–Liouville problems with eigenparameter in BCs were investigated in [10, 11]. Some results for fractional SLP were published in [8]. We will note paper [27] where the asymptotic properties are studied for some NBCs together with BC (1.3) or with BC (1.4).

In the case of NBCs, properties of discrete SLP (dSLP) were investigated in [1, 2, 5, 6, 13, 15, 22] and results were used for theoretical justification of stability Finite Difference Schemes (FDS) for various elliptic, hyperbolic and parabolic equations [20, 21]. Finite difference method together with asymtotic formulas for eigenvalues [26, 27] gives very good approximation of spectrum for differential SLP.

The article is organized as follows. The statement of the problem and a literature review are given in Section 1. In Section 2, we present results about discrete Sturm–Liouville equation and formulas for solution with one Dirichlet or Neumann classical BC. In Section 3, we investigate discrete SLP with two-points NBC (1.4). We describe properties of Characteristic Function. Then in Section 4 we present Spectrum Curves in complex domain and results about Spectrum of dSLP.

2 Discrete Sturm–Liouville equation and natural approximation of a derivative

2.1 Notation

We use notation: $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$, $\mathbb{N}_{\text{odd}}$ and $\mathbb{N}_{\text{even}}$ are sets for odd and even numbers. Let us denote $\gcd(n_1, n_2)$ the greatest common divisor of $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $n \geq 2$ and $h := 1/n$. We use the notation $\mathbb{N}^h := (0, n) \cap \mathbb{N}$, $\overline{\mathbb{N}}^h := \mathbb{N}^h \cup \{0, n\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Notation $i = \overline{n_1, n_2}$, $n_1, n_2 \in \mathbb{Z}$, means $i \in \mathcal{I}$, where $\mathcal{I} = \varnothing$ for $n_2 < n_1$, $\mathcal{I} = \{n_1, n_1 + 1, \ldots, n_2\}$ for $n_2 \geq n_1$.

We introduce a uniform grids in $[0, 1]$: $\overline{\omega}^h = \{t_j = jh,\ j = \overline{0, n}\}$, $\omega^h = \{t_j = jh,\ j = \overline{1, n-1}\}$ with stepsizes $h_j \equiv h$ and $\omega^h_{1/2} = \{t_{j+1/2} = (t_j + t_{j+1})/2,\ j =$

$\overline{0, n-1}\}$ with stepsizes $h_{j+1/2} = t_{j+1/2} - t_{j-1/2} \equiv h$. Additionally, we use a nonuniform grid $\overline{\omega}^h_{1/2} = \omega^h_{1/2} \cup \{t_{-1/2} = 0, t_{n+1/2} = n\}$, where stepsizes $h_{1/2} = t_{1/2} - t_{-1/2} = h/2$, $h_{n+1/2} = t_{n+1/2} - t_{n-1/2} = h/2$. Also, we make an assumption that ξ is located on the grid $\overline{\omega}^h$, i.e., $\xi = m/n = M/N$, $m \in \overline{\mathbb{N}}^h$, $\gcd(n, m) = K$, $\gcd(N, M) = 1$.

Let us introduce spaces $H(\omega) := \{U \colon \omega \to \mathbb{C}\}$ of grid functions on $\omega = \overline{\omega}^h, \omega^h, \overline{\omega}^h_{1/2}, \omega^h_{1/2}$. We will use the notation $U_j = U(t_j)$, $t_j \in \omega$. We define grid operators:

$$\delta \colon H(\overline{\omega}^h) \to H(\omega^h_{1/2}), \qquad (\delta U)_{j+1/2} := (U_{j+1} - U_j)/h,$$
$$\delta \colon H(\overline{\omega}^h_{1/2}) \to H(\overline{\omega}^h), \qquad (\delta U)_j := (U_{j+1/2} - U_{j-1/2})/h_{j+1/2},$$
$$\delta^2 \colon H(\overline{\omega}^h) \to H(\omega^h),\ (\delta^2 U)_j := \frac{(\delta U)_{j+1/2} - (\delta U)_{j-1/2}}{h_{j+1/2}} = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}.$$

If function $U \in H(\overline{\omega}^h)$ and we know values $(\delta U)_{-1/2}$ or $(\delta U)_{n+1/2}$ (for example, $(\delta U)_{-1/2} := U'(0)$ or $(\delta U)_{n+1/2} := U'(1)$), then the operator δ^2 can be extended to point t_0 and t_n by using formulas:

$$(\delta^2 U)_0 := \frac{(\delta U)_{1/2} - (\delta U)_{-1/2}}{h_{1/2}} = \frac{(U_1 - U_0)/h - (\delta U)_{-1/2}}{h/2}, \tag{2.1}$$
$$(\delta^2 U)_n := \frac{(\delta U)_{n+1/2} - (\delta U)_{n-1/2}}{h_{n+1/2}} = \frac{(\delta U)_{n+1/2} - (U_n - U_{n-1})/h}{h/2}.$$

2.2 Discrete equation

Now, we consider one-dimensional discrete Sturm–Liouville equation

$$-\delta^2 U = \lambda U, \quad t \in \omega^h, \tag{2.2}$$

$\lambda \in \mathbb{C}_\lambda$ is a complex spectral parameter. We rewrite (2.2) in the form

$$U_{j+1} - 2zU_j + U_{j-1} = 0, \quad z = 1 - \lambda h^2/2. \tag{2.3}$$

Equation (2.3) for $j \in \mathbb{Z}$ and its solutions were described in [21] for $z \in \mathbb{R}$. For a general solution of this equation we have expression

$$U_j = C_1 T_j(z) + C_2 \widetilde{T}_{j-1}(z), \quad j \in \mathbb{Z},$$

where

$$T_j(z) = \frac{(z + \sqrt{z^2 - 1})^j + (z - \sqrt{z^2 - 1})^j}{2}, \quad j \in \mathbb{Z},$$

are the Chebyshev polynomials of the first kind of degree j in z,

$$\widetilde{T}_j(z) = \frac{(z + \sqrt{z^2 - 1})^{j+1} - (z - \sqrt{z^2 - 1})^{j+1}}{2\sqrt{z^2 - 1}}, \quad j \in \mathbb{Z},$$

are the Chebyshev polynomial of the second kind of degree j in z. The Chebyshev polynomials can be further extended into (or initially defined as) a polynomials of a complex variable z [14,24]. We can find these Chebyshev polynomials

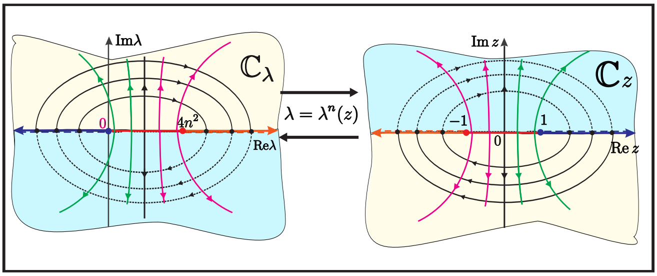


Figure 2. Bijective map: $\lambda = \lambda^n(z) = 2n^2(1-z)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_z$.

as solutions of two Cauchy problems:

$$T_{j+1} - 2zT_j + T_{j-1} = 0, \quad T_0 = 1,\ T_1 = z, \tag{2.4}$$

$$\widetilde{T}_{j+1} - 2z\widetilde{T}_j + \widetilde{T}_{j-1} = 0, \quad \widetilde{T}_{-1} = 0,\ \widetilde{T}_0 = 1. \tag{2.5}$$

Formulas (2.4)–(2.5) written in recursive form allow to find Chebyshev polynomials $T_j(z)$ and $\widetilde{T}_{j-1}(z)$ for all $j \in \mathbb{Z}$. We see that for all $z \in \mathbb{C}$ functions $T_j\colon \mathbb{Z} \to \mathbb{C}$ and $\widetilde{T}_{j-1}\colon \mathbb{Z} \to \mathbb{C}$ are linearly independent.

We have that map $\lambda^n\colon \mathbb{C}_z \to \mathbb{C}_\lambda$:

$$\lambda = \lambda^n(z) := 2n^2(1-z) = 2h^{-2}(1-z) \tag{2.6}$$

is linear. So, we have bijection between two complex planes $\mathbb{C}_\lambda$ and $\mathbb{C}_z$ (see, Figure 2). Note, if $z = -1$, then $\lambda = 4h^{-2} = 4n^2$, if $z = 1$, then $\lambda = 0$.

Let us consider the following map $z\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $z = z(\omega) := (\omega + \omega^{-1})/2$. This function takes the same value z_0 at two different points $\omega_{1,2} = z_0 \pm \sqrt{z_0^2 - 1}$ and $\omega_1\omega_2 = 1$. Let us consider a domain such that ω_1 and ω_2 where $\omega_1\omega_2 = 1$ do not both belong to it. Then function is single valued in such domain. In Figure 3 we see two such domains: $\mathbb{C}_\omega := \{\omega \in \mathbb{C}\colon 0 < |\omega| < 1\} \cup \{\omega = e^{\imath\pi\varphi}\colon 0 \le \varphi \le \pi\}$, $\mathbb{C}_{\omega^*} := \{\omega \in \mathbb{C}\colon \operatorname{Im}\omega > 0\} \cup [-1,0) \cup (0,1]$. If $\omega_1, \omega_2 \notin \mathbb{C}_\omega \cap \mathbb{C}_{\omega^*}$ and $\omega_1\omega_2 = 1$, then these points are symmetrical (according the line $\operatorname{Im}\omega = 0$ and the circle $|\omega| = 1$ to the same point in $\mathbb{C}_\omega \cap \mathbb{C}_{\omega^*}$. The second order BPs $z = \pm 1$ correspond to RPs $\omega = \pm 1$ and $z(-1) = -1$, $z(1) = 1$. Note, that $z(0) = \infty$. We use the notation $\overline{\mathbb{C}}_\omega = \mathbb{C}_\omega \cup \{0\}$. From (2.6) we have (see, Figure 4) $\lambda = \lambda_n(\omega) := (\lambda^n \circ z)(\omega) = n^2\big(2 - (\omega + \omega^{-1})\big)$, and formula for general solution of equation

$$U_{j+1} - (\omega + \omega^{-1})U_j + U_{j-1} = 0, \tag{2.7}$$

is $U_j = C_1 W_j(\omega) + C_2 \widetilde{W}_j(\omega)$, $j \in \mathbb{Z}$, where $W_j(\omega) = (\omega^j + \omega^{-j})/2$, $\widetilde{W}_j(\omega) = (\omega^j - \omega^{-j})(\omega - \omega^{-1})^{-1}$, $j \in \mathbb{Z}$. We will use the domain $\mathbb{C}_\omega$ for investigation of properties of dSLP when λ is near ∞ and corresponding point $\omega = 0$ is isolated point in the domain $\overline{\mathbb{C}}_\omega$.

The main domain in our investigation will be (see, Figure 4) $\mathbb{C}_q^h := \mathbb{R}_y^- + \{0\} + \mathbb{R}_x^h + \{n\} + \mathbb{R}_q^{h+} + \mathbb{C}_y^{h+} + \mathbb{C}_q^{h-}$, where $\mathbb{R}_x^h := \{q = x + \imath y\colon\ 0 < x < n, y = 0\}$,

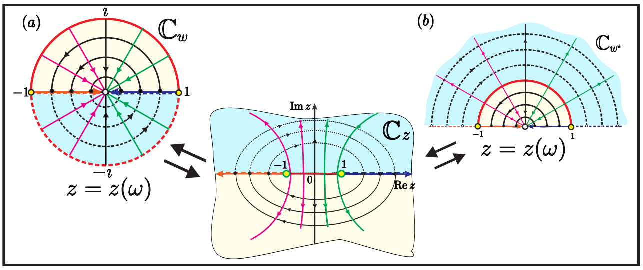


Figure 3. Bijective map: $z = z(\omega) = (\omega + \omega^{-1})/2$ between $\mathbb{C}_z$ and $\mathbb{C}_\omega$.

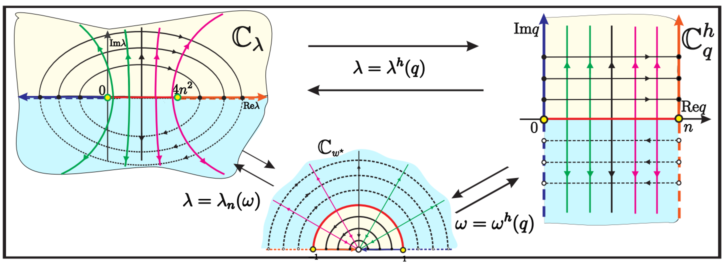


Figure 4. Bijective mappings: $\omega = \omega^h(q) = e^{\imath\pi qh}$ between $\mathbb{C}_{\omega^*}$ and $\mathbb{C}_q^h$, $\lambda = \lambda^h(q) = (4/h^2)\sin^2(\pi qh/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q^h$; ○ – BP, ○ – RP.

$\mathbb{R}_y^- := \{q = \imath y\colon\ y > 0\}$, $\mathbb{R}_y^{h+} := \{q = n + \imath y\colon\ y > 0\}$, $\mathbb{C}_q^{h+} := \{q = x + \imath y\colon\ 0 < x < n,\ y > 0\}$, $\mathbb{C}_q^{h-} := \{q = x + \imath y\colon\ 0 < x < n,\ y < 0\}$. The conformal map $\omega^h\colon \mathbb{C}_q \to \mathbb{C}_{\omega^*}$, $\omega = \omega^h(q) := e^{\imath\pi qh}$, is bijection. Using maps λ_n and ω^h we construct the bijection between complex plane $\mathbb{C}_\lambda$ and domain $\mathbb{C}_q$:

$$\lambda = \lambda^h(q) := (\lambda_n \circ \omega^h)(q) = \frac{2}{h^2}\Big(1 - \frac{e^{\imath\pi qh} + e^{-\imath\pi qh}}{2}\Big) = \frac{4}{h^2}\sin^2\frac{\pi qh}{2}. \tag{2.8}$$

Points $q \in \mathbb{R}_y^-$ correspond to negative λ, $q \in \mathbb{R}_x^h$ – to positive $\lambda < 4n^2$, $q \in \mathbb{R}_y^{h+}$ – to positive $\lambda > 4n^2$, $q \in \mathbb{C}_q^{h\pm}$ – to complex (non-real) λ. The second order BPs $\lambda = 0$ and $\lambda = 4n^2$ correspond to RPs $q = 0$ and $q = n$. We use the notation $\overline{\mathbb{C}}_q^h = \mathbb{C}_q^h \cup \{\infty\}$. Now, Equation (2.7) can be rewritten in the form

$$U_{j+1} - 2\cos(\pi qh)U_j + U_{j-1} = 0, \quad q \in \mathbb{C}_q^h, \tag{2.9}$$

and the general solution of this difference equation is

$$U_j = C_1\cos(\pi q t_j) + C_2\sin(\pi q t_j)/\sin(\pi qh), \quad \text{where } t_j = jh,\ j \in \mathbb{Z}. \tag{2.10}$$

2.3 Solutions of problems with classical BC at the point $t = 0$

Let us consider discrete equation (2.3) or equivalent equation (2.9). If we approximate Dirichlet BC (1.3_d) as $U_0 = 0$, then from (2.10) we get $C_1 = 0$

and discrete equation (2.9) with such condition has solutions [1]:

$$U_j = C_2 \sin(\pi q t_j)/\sin(\pi q h), \quad t_j = jh, \ j \in \mathbb{Z}. \tag{2.11}$$

If we approximate Neumann BC (1.3_n) as

$$(\delta U)_{1/2} = 0, \quad \text{i.e.,} \ U_0 = U_1, \tag{2.12}$$

then we have

$$U_j = \tilde{C} \cos\big(\pi q (t_j - h/2)\big)/\cos(\pi q h/2), \quad t_j = jh, \ j \in \mathbb{Z}. \tag{2.13}$$

Remark 1. The truncation error for condition (2.12) is $\mathcal{O}(h)$. We have a shift $-h/2$ in the argument of function (2.13).

Example 1. Let us consider dSLP with local BCs [21]

$$-\delta^2 U = \lambda U, \quad t \in \omega^h, \quad U_0 = 0, \ U_n = 0.$$

From (2.11) and BC $U_n = 0$ that nontrivial solutions exist for $q_k = k$, $k = 1, \dots, n-1$. So, eigenvalues and eigenfunctions are:

$$\lambda_k = \frac{4}{h^2} \sin^2 \frac{\pi q_k h}{2}, \quad U_j^k = \sin(\pi q_k t_j), \quad q_k = k, \quad k = 1, \dots, n-1.$$

Since $q_k \neq 0$ and $q_k \neq n$, we write more simple expression for eigenfunction than in (2.11).

Example 2. Let us consider dSLP

$$-\delta^2 U = \lambda U, \quad t \in \omega^h, \quad U_0 = U_1, \ U_n = U_{n-1}.$$

Formally, we have two NBCs. From (2.13) and BC $U_n = U_{n-1}$ it follows that nontrivial solutions exists if $\cos\big(\pi q(1 - h/2)\big) = \cos\big(\pi q(1 - 3h/2)\big)$. Roots of this equation are $q_k = (k-1)/(1-2h)$, $k = 1, \dots, n-1$, and roots $q_1 = 0$ and $q_{n-1} = n$ are double. Since these two points are RP, the corresponding eigenvalues are simple. So, eigenvalues and eigenfunctions are:

$$\lambda_k = \frac{4}{h^2} \sin^2 \frac{\pi q_k h}{2}, \ U_j^k = \frac{\cos\big(\pi q_k (t_j - h/2)\big)}{\cos(\pi q h/2)}, \ q_k = \frac{k-1}{1-2h}, \ k = 1, \dots, n-1.$$

In both examples the number of eigenvalues are $n-1$.

2.4 Natural approximation of a derivative

Let us approximate natural condition $u'(0) = 0$ as [21]

$$(\delta U)_{1/2} = -h_{1/2} \lambda U_0. \tag{2.14}$$

Remark 2. If the Sturm–Liouville equation is valid for $t_0 = 0$ and operator δ^2 is defined by formula (2.1) where $h_{1/2} = h/2$, then the condition (2.14) is equivalent to $\delta U_{-1/2} = 0$, and $-\delta^2 U_0 = \lambda U_0$. So, we can say, that condition (2.14) is natural condition for Equation (2.2).

If function $U \in H(\overline{\omega}^h)$ is solution of discrete Sturm—Liouville equation (2.3), then we define grid operators:

$$\delta^+ \colon H(\overline{\omega}^h) \to H(\omega^h \cup \{0\}), \quad (\delta^+ U)_j := \frac{U_{j+1} - zU_j}{h} = \frac{U_{j+1} - \cos(\pi q h)U_j}{h},$$
$$\delta^- \colon H(\overline{\omega}^h) \to H(\omega^h \cup \{n\}), \quad (\delta^- U)_j := \frac{zU_j - U_{j-1}}{h} = \frac{\cos(\pi q h)U_j - U_{j-1}}{h}.$$

From Equation (2.3) we have equality $U_{j+1} - zU_j = zU_j - U_{j-1}$. On the grid ω^h we have

$$(\delta^+ U)_j = (\delta^- U)_j = \big((\delta^+ U)_j + (\delta^- U)_j\big)/2 = \frac{U_{j+1} - U_{j-1}}{2h} =: (\bar{\delta} U)_j.$$

If $(\bar{\delta}U)_0 := (\delta^+ U)_0$, $(\bar{\delta}U)_n := (\delta^- U)_n$, then we have natural approximation $(\bar{\delta}U)_j$ of derivative $u'(t_j)$ on the grid $\overline{\omega}^h$.

Remark 3. If $u \in C^3[0,1]$ satisfies Sturm–Liouville equation (1.1) for $t \in [0,1]$, then truncation error for natural approximation is $\mathcal{O}(h^2)$. For $t \in \omega^h$ this statement is well known. For $j = 0$ we have

$$\varPsi = \frac{u(h) - zu(0)}{h} - u'(0) = \big(u''(0) + \lambda u(0)\big)\frac{h}{2} + \mathcal{O}(h^2) = \mathcal{O}(h^2).$$

For $j = n$ the proof is similar.

If we have natural BC $u'(0) = 0$ for differential equation (1.1), then BC $\bar{\delta}U_0 = 0$ for discrete equation (2.2) is equivalent to (2.14):

$$0 = \bar{\delta}U_0 = \frac{U_1 - zU_0}{h} = \frac{U_1 - (1 - \lambda h^2/2)U_0}{h} = (\delta U)_{1/2} + \lambda h_{1/2} U_0.$$

We can rewrite condition BC $\bar{\delta}U_0 = 0$ (or BC (2.14)) in the form $U_1 = \cos(\pi q h)U_0$. Then the equality $C_1 \cos(\pi q h) + C_2 = \cos(\pi q h) C_1$ is valid. So, $C_2 = 0$ and

$$U_j = C_1 \cos(\pi q t_j), \quad j \in \mathbb{N}_0. \tag{2.15}$$

Example 3. Let us consider dSLP with two local BCs

$$-\delta^2 U = \lambda U, \quad t \in \omega^h, \quad (\bar{\delta}U)_0 = 0,\ U_n = 0.$$

From (2.15) and BC $U_n = 0$ it follows that nontrivial solutions exist for $q_k = k - 1/2$, $k = 1, \ldots, n$. So, eigenvalues and eigenfunctions are:

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\pi q_k h}{2}, \quad U_j^k = \cos(\pi q_k t_j), \quad q_k = k - 1/2, \quad k = 1, \ldots, n.$$

Example 4. Let us consider dSLP with two local natural BCs

$$-\delta^2 U = \lambda U, \quad t \in \omega^h, \quad (\bar{\delta}U)_0 = 0,\ (\bar{\delta}U)_n = 0.$$

BC $(\bar{\delta}U)_n = 0$ is equivalent to $U_{n-1} = \cos(\pi h q)U_n$. From (2.15) it follows that nontrivial solutions exist if $\sin(\pi q)\sin(\pi q h) = 0$. The first multiplier $\sin(\pi q)$ gives $q = 0, 1, \ldots, n-1, n$, the second multiplier $\sin(\pi q h)$ gives $q = 0, n$. Since the points $q = 0, n$ are RPs of the second order, we get that the eigenvalues $\lambda = 0$ and $\lambda = 4n^2$ are simple. So, eigenvalues and eigenfunctions are:

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\pi q_k h}{2}, \quad U_j^k = \cos(\pi q_k t_j), \quad q_k = k, \quad k = 0, \ldots, n.$$

3 Discrete Sturm–Liouville problem

Let us consider discrete Sturm–Liouville equation (2.2)

$$-\delta^2 U = \lambda U, \quad t \in \omega^h, \tag{3.1}$$

with Dirichlet BC or the natural BC:

$$\text{(Case } d) \qquad U_0 = 0, \tag{3.2_d}$$

$$\text{(Case } n) \qquad (\bar{\delta} U)_0 = 0, \tag{3.2_n}$$

and two–point NBC:

$$\text{(Case 1)} \qquad U_n = \ \gamma U_m, \tag{3.3_1}$$

$$\text{(Case 2)} \qquad (\bar{\delta} U)_n = \ \gamma (\bar{\delta} U)_m, \tag{3.3_2}$$

where $0 \le m < n$, $\gamma \in \mathbb{R}$. If $\gamma = \infty$, then we replace BC (3.3_1) with $U_m = 0$ $(0 < m < n)$, BC (3.3_2) with $\bar{\delta} U_m = 0$ $(0 < m < n)$. Our task is to investigate eigenvalues and eigenfunctions of dSLP (3.1)–(3.3). The number $q \in \mathbb{C}_q^h$ is called Eigenvalue Point for eigenvalue λ if $\lambda^h(q) = \lambda$ (see, bijection (2.8)).

Substituting expression (2.11) or (2.15) into BC (3.3) we get the characteristic equation

$$Z^h(q) = \gamma P_\xi^h(q), \quad q \in \mathbb{C}_q^h,$$

where

$$Z^h(q) := \frac{\sin(\pi q)}{\sin(\pi q h)}, \qquad P_\xi^h(q) := \frac{\sin(\pi q \xi)}{\sin(\pi q h)}, \qquad 0 < m < n, \tag{3.4_{d1}}$$

$$Z^h(q) := \cos(\pi q), \qquad P_\xi^h(q) := \cos(\pi q \xi), \qquad 0 \le m < n, \tag{$3.4_{d2,n1}$}$$

$$Z^h(q) := \sin(\pi q h)\sin(\pi q), \quad P_\xi^h(q) := \sin(\pi q h)\sin(\pi q \xi), \ 0 < m < n. \tag{3.4_{n2}}$$

Remark 4. In the differential case we have $Z^0(q) = \gamma P_\xi^0(q), \quad q \in \mathbb{C}_q$, where

$$Z^0(q) = \sin(\pi q)/(\pi q), \qquad P_\xi^0(q) = \sin(\pi q \xi)/(\pi q), \qquad 0 < \xi < 1, \tag{3.5_{d1}}$$

$$Z^0(q) = \cos(\pi q), \qquad P_\xi^0(q) = \cos(\pi q \xi), \qquad 0 \le \xi < 1, \tag{$3.5_{d2,n1}$}$$

$$Z^0(q) = \pi q \sin(\pi q), \qquad P_\xi^0(q) = \pi q \sin(\pi q \xi), \quad 0 < \xi < 1. \tag{3.5_{n2}}$$

3.1 Constant Eigenvalue points

We define a Constant Eigenvalue as the eigenvalue that does not depend on parameter γ for fixed ξ [28]. For any Constant Eigenvalue there exists the Constant Eigenvalue Point (CEP). The notation $\mathcal{C}_\xi$ is used for the set of all CEPs and $n_{ce} = |\mathcal{C}_\xi|$ is the number of CEPs in $\mathcal{C}_\xi$. We can find CEPs as solutions of the following system

$$Z^h(q) = 0, \quad P_\xi^h(q) = 0.$$

Remark 5. If $m = 0$ in Case $d2$ and Case $n1$, then $P^h_0(q) = 1$. So, $\mathcal{C}_0 = \varnothing$. So, in this subsection we assume that $0 < m < n$.

Zeroes of the functions $Z^h(q)$, $q \in \mathbb{C}^h_q$, coincide with Eigenvalue Points in the classical case $\gamma = 0$. On the other hand, function $Z^h = P^h_1$, and zeroes of the function Z^h coincide with zeroes of the function P^h_1. If $\hat{\mathcal{Z}}^h = \check{\mathcal{Z}}^h_1$ and $\check{\mathcal{Z}}^h_\xi$ are sets of zeroes for functions Z^h and P^h_ξ, then

$$\hat{\mathcal{Z}}^h = \{\hat{z}_l = l,\ \ l = \overline{1, n-1}\}, \qquad n_{\hat{z}} = n - 1, \qquad (3.6_{d1})$$
$$\hat{\mathcal{Z}}^h = \{\hat{z}_l = l - 1/2,\ \ l = \overline{1, n}\}, \qquad n_{\hat{z}} = n, \qquad (3.6_{d2,n1})$$
$$\hat{\mathcal{Z}}^h = \{\hat{z}_l = l,\ \ l = \overline{0, n}\}, \qquad n_{\hat{z}} = n + 1; \qquad (3.6_{n2})$$

and

$$\check{\mathcal{Z}}^h_\xi = \{\check{z}_k = k/\xi = k\check{z}_1,\ \ k = \overline{1, m-1}\}, \qquad n_{\check{z}} = m - 1, \qquad (3.7_{d1})$$
$$\check{\mathcal{Z}}^h_\xi = \{\check{z}_k = (k - 1/2)/\xi = (2k-1)\check{z}_1,\ \ k = \overline{1, m}\}, \qquad n_{\check{z}} = m, \qquad (3.7_{d2,n1})$$
$$\check{\mathcal{Z}}^h_\xi = \{\check{z}_k = k/\xi = k\check{z}_1,\ \ k = \overline{0, m}\}, \qquad n_{\check{z}} = m + 1; \qquad (3.7_{n2})$$

where $n_{\hat{z}} = |\hat{\mathcal{Z}}^h|$, $n_{\check{z}} = |\check{\mathcal{Z}}^h_\xi|$. For all cases $n_{\hat{z}} - n_{\check{z}} = n - m$.

Remark 6. We can use formula (3.6) with $n = \infty$ and formula (3.7) with $m = \infty$. Then we get the sets for zeroes in the differential case.

All zeroes are simple, except $\hat{z}_0 = \check{z}_0 = 0$, $\hat{z}_n = \check{z}_m = n$ in Case $n2$ when they are of the second order and real. We note, that $\check{\mathcal{Z}}^h_{1/n} = \varnothing$ in Case $d1$.

We have $\mathcal{C}_\xi = \hat{\mathcal{Z}}^h \cap \check{\mathcal{Z}}^h_\xi$. We use notation $\varkappa$: if $N \cdot M \in \mathbb{N}_{\text{even}}$, then $\varkappa = 0$ in Case $d2, n1$, else $\varkappa = 1$.

Lemma 1. *For dSLP* (3.1)–(3.3) *Constant Eigenvalues are equal to* $\lambda_j = \lambda^h(c_j)$, *where*

$$c_j = \hat{z}_{l_j} = \check{z}_{k_j} = Nj, \qquad j \in \mathcal{J}_\xi := \{j : j = \overline{1, K-1}\}, \qquad (3.8_{d1})$$
$$c_j = \hat{z}_{l_j} = \check{z}_{k_j} = N(j - 1/2), \qquad j \in \mathcal{J}_\xi := \{j : j = \overline{1, \varkappa K}\}, \qquad (3.8_{d2,n1})$$
$$c_j = \hat{z}_{l_j} = \check{z}_{k_j} = Nj, \qquad j \in \mathcal{J}_\xi := \{j : j = \overline{0, K}\}, \qquad (3.8_{n2})$$

and

$$n_{ce} = K - 1, \quad l_j = Nj, \quad k_j = Mj, \qquad (3.9_{d1})$$
$$n_{ce} = \varkappa K, \quad l_j = Nj - (N-1)/2, \quad k_j = Mj - (M-1)/2, \qquad (3.9_{d2,n1})$$
$$n_{ce} = K + 1, \quad l_j = Nj, \quad k_j = Mj. \qquad (3.9_{n2})$$

All Constant Eigenvalues are simple.

Proof. From Remark 6 it follows that CEP for this problem are the CEPs for corresponding differential SLP [28] with $K = \infty$ (if $\varkappa = 0$, then $\mathcal{J}_\xi = \varnothing$). In addition, we must look for $j = 0, K$ in Case $n2$ (points $q = 0, n$). □

Remark 7. In Case $n2$ Constant Eigenvalues $\lambda_0 = 0$, $\lambda_K = 4n^2$ are simple, but corresponding CEPs $c_0 = 0$, $c_K = n$ are of the second order, because points $\lambda = 0, 4n^2$ are BPs ($q = 0, n$ are RPs).

Additionaly, we use notation $\overline{\mathcal{C}}_\xi := \mathcal{C}_\xi \cup \{c_0, c_K\}$, $c_0 = 0$, $c_K = n$, in Case $d1$ and $\overline{\mathcal{C}}_\xi := \mathcal{C}_\xi$ in Cases $d2, n1, n2$. An union of CEPs for all Cases $d1, n1, d2, n2$ we denote $\widetilde{\mathcal{C}}_\xi$. Thus for $x \in \overline{\mathcal{C}}_\xi$ we have $\sin(\pi x) = \sin(\pi x\xi) = 0$ in Case $d1, n2$ and $\cos(\pi q) = \cos(\pi q\xi) = 0$ in Case $d2, n1$. If $x \in \widetilde{\mathcal{C}}_\xi$ we have $\sin(\pi x) = \sin(\pi x\xi) = 0$ or $\cos(\pi x) = \cos(\pi x\xi) = 0$.

3.2 Complex Characteristic Function

Let us consider Complex Characteristic Function (Complex CF) [28]:

$$\gamma_c \colon \overline{\mathbb{C}}_q^h \to \overline{\mathbb{C}}, \quad \gamma_c(q) = \gamma_c^h(q;\xi) := \frac{Z^h(q)}{P_\xi^h(q)}, \quad q \in \overline{\mathbb{C}}_q^h, \quad \xi = m/n.$$

For dSLP (3.1)–(3.3) we have meromorphic functions

$$\gamma_c(q) := \frac{Z^h(q)}{P_\xi^h(q)} = \frac{\cos(\pi q)}{\cos(\pi q\xi)}, \quad 0 \le m < n, \tag{3.10_a}$$

$$\gamma_c(q) := \frac{Z^h(q)}{P_\xi^h(q)} = \frac{\sin(\pi q)}{\sin(\pi q\xi)}, \quad 0 < m < n, \tag{3.10_b}$$

where Case a is for Cases $d2, n1$, Case b is for Case $d1, n2$. We see, that Complex CFs γ_c in the discrete case are the same as Complex CFs in the differential case and γ_c for dSLP is equal to $\gamma_c^0|_{\mathbb{C}_q^h}$ for $\xi \in \mathbb{Q}$, where $\gamma_c^0(q) = Z^0(q)/P_\xi^0(q)$.

3.2.1 Auxilary equations and functions

Let us consider equations

$$\cos z = \gamma \cos(\xi z), \tag{3.11_a}$$

$$\sin z = \gamma \sin(\xi z), \tag{3.11_b}$$

where $z \in \mathbb{C}$, $\xi \in [0,1)$, $\gamma \in \mathbb{R}$.

Lemma 2. *All complex (not real) roots of Equations (3.11_a) and (3.11_b) are simple. If $|\gamma| \notin [1, \xi^{-1}]$, then real roots are simple as well.*

Proof. The lemma is valid for $\xi = 0$ and for $\gamma = 0$. If $|\gamma| < 1$, then all roots of equations (3.11) are real and simple [27,28]. So, we assume that $|\gamma| \ge 1$ and $\xi \in (0,1)$.

Now we claim that not simple complex roots may exist only for $z = x + \imath y$, where $\cos x = 0$ for Equation (3.11_a) and $\sin x = 0$ for Equation (3.11_b).

Let us consider the case of Equation (3.11_a). For not simple root z we have a system of two equations

$$\cos z = \gamma \cos(\xi z), \quad \sin z = \xi\gamma \sin(\xi z). \tag{3.12}$$

If we eliminate $\cos(\xi z)$ and $\sin(\xi z)$, then we get

$$\cos^2 z = \cos^2 x \cosh^2 y - \sin^2 x \sinh^2 y - \imath \sin x \cos x \sinh(2y) = \frac{1-\gamma^2\xi^2}{1-\xi^2}. \quad (3.13)$$

Since the right-hand side of (3.13) is real for real γ, we have $\sin x = 0$ or $\cos x = 0$. In the case $\sin x = 0$ we use (3.13) and for $|\gamma| \geq 1$ to get estimate $\cosh^2 y = (1-\gamma^2\xi^2)(1-\xi^2)^{-1} \leq 1$. From this inequality it follows that $y = 0$ and $\gamma = \pm 1$.

In the case of Equation (3.11_b) we have $\sin z = \gamma \sin(\xi z)$, $\cos z = \xi\gamma\cos(\xi z)$. Then, we derive

$$\sin^2 z = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y + \imath \sin x \cos x \sinh(2y) = \frac{1-\gamma^2\xi^2}{1-\xi^2}. \quad (3.14)$$

We have that $\sin x = 0$ or $\cos x = 0$. In the case $\cos x = 0$ it follows that $y = 0$, $\gamma = \pm 1$. So, we prove that $\sin x = 0$. Thus, we have proven our claim.

If $|\gamma| \leq \xi^{-1}$, then for complex not simple roots we have (see, (3.13) with condition $\cos x = 0$ and (3.14) with condition $\sin x = 0$) that $-\sinh^2 y \geq 0$. So, the complex roots are simple.

Now we investigate the case $|\gamma| > \xi^{-1}$. For real roots $\cos^2 x < 0$ (see, (3.13)) and $\sin^2 x < 0$ (see, (3.14)). So, real roots in this case are simple.

We rewrite Equation (3.11_a) as

$$\cos x \cosh y - \imath \sin x \sinh y = \gamma \cos(\xi x)\cosh(\xi y) - \gamma\imath \sin(\xi x)\sinh(\xi y).$$

If $\cos x = 0$, then $\cos(x\xi) = 0$. From Equation (3.11_b) and $\sin x = 0$ we get $\sin(x\xi) = 0$. It follows (see, Lemma 1, [28]) that system $\cos x = 0$, $\cos(\xi x) = 0$ and system $\sin x = 0$, $\sin(\xi x) = 0$ have solution only for rational $\xi = M/N$:

$$x = x_j = \pi N(j - 1/2), \quad j \in \mathbb{Z}, \quad N, M \in \mathbb{N}_{\mathrm{odd}}, \quad (3.15_a)$$

$$x = x_j = \pi N j, \quad j \in \mathbb{Z}. \quad (3.15_b)$$

We have

$$\cos(\pi N(j-1/2) + \imath y) = \imath(-1)^{(2j-1)N/2+1/2}\sinh y, \quad (3.16_a)$$

$$\sin(\pi N j + \imath y) = \imath(-1)^{Nj}\sinh y. \quad (3.16_b)$$

So, Equations (3.11) have not simple complex roots if and only if an equation

$$\sinh y = \tilde{\gamma}\sinh(\xi y) \quad (3.17)$$

has not simple real root $y > 0$, where $\xi \in \mathbb{Q}$,

$$\tilde{\gamma} = \gamma(-1)^{(N-M)(2j-1)/2}, \quad (3.18_a)$$

$$\tilde{\gamma} = \gamma(-1)^{(N-M)j}, \quad (3.18_b)$$

and $|\tilde{\gamma}| = |\gamma|$. The conditions for existence of such root are

$$\sinh y = \tilde{\gamma}\sinh(\xi y), \quad \cosh y = \xi\tilde{\gamma}\cosh(\xi y).$$

Then, y satisfies equation

$$\varphi(y) := y \cosh y / \sinh y = \xi y \cosh(\xi y)/\sinh(\xi y) = \varphi(\xi y), \tag{3.19}$$

where φ is increasing positive function [17]. So, Equation (3.19) does not have positive roots. Consequently, Equation (3.17) has only simple roots. □

From (3.13) and (3.14) the property follows.

Corollary 1. If $|\gamma| = \xi^{-1}$, then for not simple root z both sides of (3.11) are equal to zero and $z \in \mathbb{R}$.

Complex CFs (3.10) are the partial case of meromorphic function

$$h(z) = f(z)/g(z), \tag{3.20}$$

where $f, g \colon D \to \mathbb{C}$, domain $D \subset \mathbb{C}$, are holomorphic functions and $g \not\equiv 0$. Now, we prove few statements for such function h.

Lemma 3. *If $b \in D$ and $g(b) \neq 0$, then, at the point b the function h has a zero of the same order as function f.*

Proof. For proof we use the general Leibniz rule for the nth derivative of function $f = hg$. □

If $b \in D$ and $g(b) \neq 0$, then, derivatives of the first and the second order of the function (3.20) at the point b are equal to

$$h' = \frac{f'g - fg'}{g^2} = \frac{f'}{g} - h\,\frac{g'}{g}, \quad h'' = \frac{f''}{g} - h'\,\frac{2g'}{g} - h\,\frac{g''}{g}.$$

Lemma 4. *Suppose, that conditions*

$$f'' = \alpha f, \quad g'' = \beta g, \quad z \in D, \tag{3.21}$$

are valid. If $b \in D$, $g(b) \neq 0$ and $h'(b) = 0$, then, $h''(b) = (\alpha - \beta)h(b)$.

Proof. It follows from (3.21) that $h'' = (\alpha - \beta)h - 2h'\,g'/g$. We finish the proof by using condition $h'(b) = 0$. □

If $c \in D$, $f(c) = g(c) = 0$ and $g'(c) \neq 0$, then,

$$h(c) = \lim_{z\to c} h(z) = \lim_{z\to c} \frac{f(z)}{g(z)} = \lim_{z\to c} \frac{f'(z)}{g'(z)} = \frac{f'(c)}{g'(c)}. \tag{3.22}$$

If additionally, functions f and g satisfy conditions (3.21), then,

$$\begin{aligned}\lim_{z\to c}(h'g'/g) =& g'(c) \lim_{z\to c} \frac{f'g - fg'}{g^3} = g'(c) \lim_{z\to c} \frac{f''g - fg''}{3g^2 g'} \\ &= \lim_{z\to c} \frac{\alpha fg - \beta fg}{3g^2} = \frac{\alpha - \beta}{3} \lim_{z\to c} \frac{f}{g} = \frac{\alpha - \beta}{3} \cdot h(c).\end{aligned}$$

Corollary 2. If $c \in D$, $f(c) = g(c) = 0$ and $g'(c) \neq 0$, then, $h'(c) = 0$.

Lemma 5. *Suppose, that conditions* (3.21) *are valid. Let* $c \in D$, $f(c) = g(c) = 0$, $g'(c) \neq 0$, *then,* $h''(c) = \frac{(\alpha-\beta)}{3} h(c)$.

Proof. We find

$$\begin{aligned}\lim_{z\to c} h'' &= \lim_{z\to c}\Big(\frac{f'g - fg'}{g^2}\Big)' = \lim_{z\to c}\frac{f''g - fg''}{g^2} - \lim_{z\to c}(f'g - fg')\frac{2g'}{g^3}\\ &= \lim_{z\to c}\frac{\alpha fg - \beta fg}{g^2} - 2\lim_{z\to c}\frac{h'g'}{g}\\ &=(\alpha-\beta)\,h(c) - \frac{2}{3}(\alpha-\beta)\,h(c) = \frac{(\alpha-\beta)}{3}h(c). \quad \square\end{aligned}$$

3.2.2 Properties of Complex Characteristic Functions

Lemma 6. *Complex CF* $\gamma_c(q)$ *for dSLP* (3.1)–(3.3) *has the properties:* $\overline{\gamma_c(q)} = \gamma_c(\bar{q})$*; if* $\operatorname{Re} q \in [0, N]$*, then,* $\gamma_c(N-q) = (-1)^{N-M}\gamma_c(q)$*;* $\operatorname{Re} q \in [0, n]$*, then,* $\gamma_c(n-q) = (-1)^{n-m}\gamma_c(q)$*; if* $\operatorname{Re} q \in [0, n-N]$*, then,* $\gamma_c(q+N) = (-1)^{N-M}\gamma_c(q)$.

Proof. Elementary proof of this statement is obtained by using the properties of trigonometric functions. $\square$

A set of these Pole Points (PP) of Complex CF at $\mathbb{C}_q^h$ is $\mathcal{P}_\xi := \check{\mathcal{Z}}_\xi^h \setminus \hat{\mathcal{Z}}^h = \check{\mathcal{Z}}_\xi^h \setminus \mathcal{C}_\xi$. In our case all these poles are of the first order, real and $\deg^+(p) = 1$ for $p \in \mathcal{P}_\xi \subset \mathbb{R}_x^h = (0, n)$. So,

$$\mathcal{P}_\xi = \{\check{z}_k \in \check{\mathcal{Z}}_\xi^h \colon k \neq k_j, j \in \mathcal{J}_\xi\} = \{p_i, i = \overline{1, n_p} \colon 0 < p_1 < \ldots < p_{n_p} < n\},$$

where k_j is defined by (3.9), $n_p = |\mathcal{P}_\xi| = n_{\check{z}} - n_{ce}$. The point $p_\infty = \infty \notin \mathbb{C}_q^h$. We denote $\overline{\mathcal{P}}_\xi := \mathcal{P}_\xi \cup \{p_\infty\}$. In the domain $\mathbb{C}_w$, this point corresponds to $w = 0$. We can investigate Complex CF in the neighborhood $w = 0$ ($w = e^{\imath\pi qh}$) in the domain $\mathbb{C}_w$:

$$\gamma_c(w) = \frac{w^n \mp w^{-n}}{w^m \mp w^{-m}} = \frac{1}{w^{n-m}}\big(1 + \mathcal{O}(w^{2m})\big).$$

A lemma follows from this formula.

Lemma 7. *Complex CF* $\gamma_c(w)$ *at the point* $p_\infty = \infty$ *for dSLP* (3.1)–(3.3) *has PP of the* $n-m$*-order, i.e.,* $n_\infty := \deg^+(p_\infty) = n_{\hat{z}} - n_{\check{z}} = n - m$.

A set of zeroes of Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}}^h \setminus \check{\mathcal{Z}}_\xi^h = \hat{\mathcal{Z}}^h \setminus C_\xi$. In our case $\mathcal{Z}_\xi \subset \mathbb{R}_x^h$ and all zeroes are simple, real and $\deg^+(z) = 1$ for $z \in \mathcal{Z}_\xi \subset \mathbb{R}_x^h$. So,

$$\mathcal{Z}_\xi = \{\check{z}_l \in \hat{\mathcal{Z}}_\xi^h \colon l \neq l_j, j \in \mathcal{J}_\xi\} = \{z_i, i = \overline{1, n_z} \colon 0 < z_1 < \ldots < z_{n_z} < n\}, \quad (3.23)$$

where l_j is defined by (3.9), $n_z = |\mathcal{Z}_\xi| = n_{\hat{z}} - n_{ce}$.

If $\gamma_c'(b) = 0$, $b \in \mathbb{C}_q^h$, then, point b is Critical Point (CP) of the function γ_c, and value $\gamma_c(b)$ is a *critical value* of the function γ_c [28]. We denote a set of CPs $\mathcal{B}_\xi := \{b_i, i = \overline{1, n_b}\}$, where $n_b = |\mathcal{B}_\xi|$ is the number of CPs at $\mathbb{C}_q^h$. If $b \in \mathcal{B}_\xi$ then, $\deg^+(b)$ is one unit larger than the order of CP b.

If the point $q \in \overline{\mathcal{C}}_\xi$, then it is Removable Singularity Point of Complex CF. We can calculate values of Complex CF and its derivatives at $c_j \in \overline{\mathcal{C}}_\xi$:

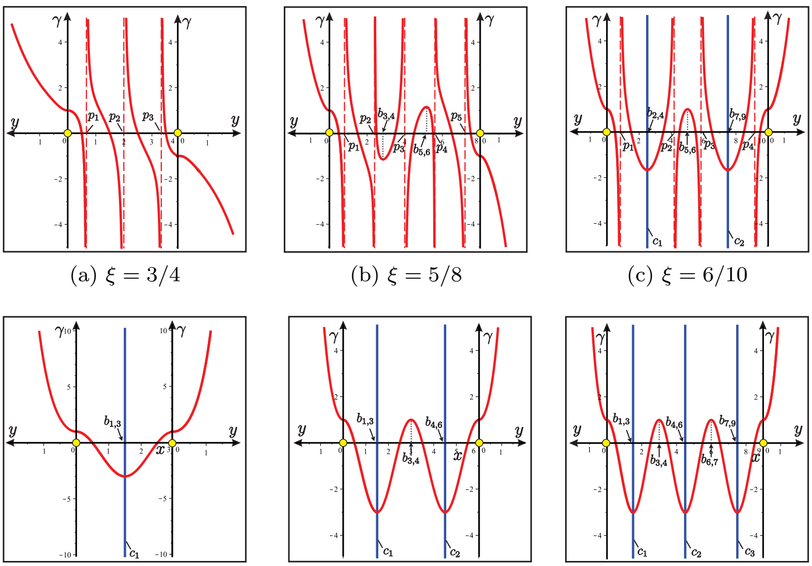
$$\gamma_j := \gamma_c(c_j;\xi) = (-1)^{(N-M)/2}\xi^{-1}; \tag{3.24$_a$}$$

$$\gamma_j := \gamma_c(c_j;\xi) = (-1)^{(1-\xi)c_j}\xi^{-1}; \tag{3.24$_b$}$$

$$\gamma_j' := \gamma_c'(c_j;\xi) = 0; \tag{3.24$'_{a,b}$}$$

$$\gamma_j'' := \gamma_c''(c_j;\xi) = -\frac{1}{3}\pi^2(1-\xi^2)\gamma_j(\xi). \tag{3.24$''_{a,b}$}$$

Formula (3.24) follows from (3.22). We have ($3.24'_{a,b}$) from Corollary 2 and we get ($3.24''_{a,b}$) from Lemma 5. From formula($3.24'_{a,b}$) we get that every CEP is CP. The contrary statement is not valid (see, Figure 5(e)).



(a) $\xi = 3/4$ (b) $\xi = 5/8$ (c) $\xi = 6/10$

(d) $\xi = 1/3$ (e) $\xi = 2/6$ (f) $\xi = 3/9$

Figure 5. Real CF for various ξ in Case a.

Taylor series for $\gamma_c(q)$ at RP $q = 0$ are

$$\gamma_c(q) = 1 - \tfrac{1-\xi^2}{2}\pi^2 q^2 + \mathcal{O}(q^4) = 1 - \tfrac{n^2-m^2}{2n^2}\pi^2 q^2 + \mathcal{O}(q^4), \tag{3.25$_a$}$$

$$\gamma_c(q) = \tfrac{1}{\xi} - \tfrac{1-\xi^2}{6\xi}\pi^2 q^2 + \mathcal{O}(q^4) = \frac{n}{m} - \tfrac{n^2-m^2}{6nm}\pi^2 q^2 + \mathcal{O}\big(q^4\big), \tag{3.25$_b$}$$

and they follow from Taylor series formulas for $\cos q$ and $\sin q$. Formula (3.25) is valid for the differential case, too.

Using Lemma 6 we derive formulas for $\gamma_c(q)$ at RP $q = n$:

$$\gamma_c(q) = (-1)^{n-m} - (-1)^{n-m}\tfrac{n^2-m^2}{2n^2}\pi^2(q-n)^2 + \mathcal{O}\big((q-n)^4\big), \tag{3.26$_a$}$$

$$\gamma_c(q) = (-1)^{n-m}\tfrac{n}{m} - (-1)^{n-m}\tfrac{n^2-m^2}{6nm}\pi^2(q-n)^2 + \mathcal{O}\big((q-n)^4\big). \tag{3.26$_b$}$$

Remark 8. Formulas (3.25) and (3.26) show that at RPs $q = 0$ and $q = n$ we have CPs of the first order ($\gamma_c''(q) \neq 0$). We note, that for such CPs, the derivative of CF $\gamma_c \circ (\lambda^h)^{-1}$ at the corresponding point $\lambda = 0$ or $\lambda = 4n^2$ is not equal to zero, and we do not have a CP.

3.3 Complex-Real Characteristic Function

All nonconstant eigenvalues (which depend on the parameter $\gamma \in \mathbb{R}$) are γ-points of Complex-Real Characteristic Function (CF) [28]. CF $\gamma(q)$ is the restriction of Complex CF $\gamma_c(q)$ on a set $\mathcal{D}_\xi := \{q \in \mathbb{C}_q^h \colon \operatorname{Im}\gamma_c(q) = 0\}$ [1]. If $q \in \mathcal{D}_\xi$, then, $\lambda = 4/h^2 \sin^2(\pi q h/2)$ is nonconstant eigenvalue of dSLP for some real γ. We can extend CF to domain $\mathbb{C}_q^h$: $\gamma(\infty) = \infty$. We call set $\mathcal{D}_\xi$ Spectrum Domain. We use notation $\overline{\mathcal{D}}_\xi := \mathcal{D}_\xi \cup \overline{\mathcal{P}}_\xi$ for extended Spectrum Domain.

The condition $\operatorname{Im}\gamma_c(q) = 0$ ($q \notin \mathcal{P}_\xi$) is equivalent to

$$\begin{aligned}\sin(\pi x)\cos(\pi x\xi)\tfrac{\sinh(\pi y)}{\cosh(\pi y)} &= \cos(\pi x)\sin(\pi x\xi)\tfrac{\sinh(\pi y\xi)}{\cosh(\pi y\xi)},\\ \cos(\pi x)\sin(\pi x\xi)\tfrac{\sinh(\pi y)}{\cosh(\pi y)} &= \sin(\pi x)\cos(\pi x\xi)\tfrac{\sinh(\pi y\xi)}{\cosh(\pi y\xi)}.\end{aligned} \tag{3.27}$$

If $q \in \mathcal{D}_\xi$, then, $\gamma(q) = \operatorname{Re}\gamma_c(q) = \gamma_c(q)$, and

$$\gamma(q) = \tfrac{\sin(\pi x)}{\sin(\pi x\xi)}\tfrac{\sinh(\pi y)}{\sinh(\pi y\xi)}, x \in \overline{\mathcal{C}}_\xi,\ \gamma(q) = \tfrac{\cos(\pi x)}{\cos(\pi x\xi)}\tfrac{\cosh(\pi y)}{\cosh(\pi y\xi)},\ x \notin \overline{\mathcal{C}}_\xi, \tag{3.28$_a$}$$

$$\gamma(q) = \tfrac{\cos(\pi x)}{\cos(\pi x\xi)}\tfrac{\sinh(\pi y)}{\sinh(\pi y\xi)}, x \in \overline{\mathcal{C}}_\xi,\ \gamma(q) = \tfrac{\sin(\pi x)}{\sin(\pi x\xi)}\tfrac{\cosh(\pi y)}{\cosh(\pi y\xi)},\ x \notin \overline{\mathcal{C}}_\xi. \tag{3.28$_b$}$$

Complex-Real CF $\gamma(q)$ for dSLP (3.1)–(3.3) and for SLP in the differential case have the property of symmetry $\gamma(\bar{q}) = \gamma(q)$ (see, Lemma 6). If $q \in \mathcal{D}_\xi$, $\operatorname{Re} q + N \in [0, n]$, then $q + N \in \mathcal{D}_\xi$.

A restriction of a Complex CF or a Complex-Real CF on the $\mathbb{R}_q^h$ is called Real Characteristic Function (Real CF) $\gamma_r(q)$. Real CF describes real nonconstant eigenvalues. We can plot the graph of Real CF for Eigenvalue Points: $0 < q = x < n$ in the middle graph; $q = 0 + \imath y$, $y > 0$ in the left half plane and $q = n + \imath y$, $y > 0$ in the right half plane. Two γ-axes correspond to RPs $q = 0, n$. The graphs of Real CF in Case a are presented in Figure 5. We add vertical lines $x = c_j$, $j \in \mathcal{J}_\xi$, to represent CEPs for each γ. Thus, each horizontal line crossing the CF graph and these vertical lines give all eigenvalue points (including their multiplicity). The graphs of Real CF in Case b can be found in [2, 16, 17, 28].

Lemma 8. *There exist only real CPs for Complex-Real CF (or for corresponding CF in the case of differential problem) $\gamma(q)$, and they can exist only for γ values: $|\gamma| \in [1, \xi^{-1}]$. Every CP b such that $|\gamma(b)| = \xi^{-1}$ is CEP.*

Proof. *Case a.* If $\cos(\pi\xi q) \neq 0$, then, $\cos(\pi q) = \gamma\cos(\pi\xi q)$ and a condition

$$0 = -\gamma'(q)/\pi = \frac{\sin(\pi q)\cos(\pi\xi q) - \xi\cos(\pi q)\sin(\pi\xi q)}{\cos^2(\pi\xi q)} = \frac{\sin(\pi q) - \xi\gamma\sin(\pi\xi q)}{\cos(\pi\xi q)}$$

is equivalent to the system (3.12), where $z = \pi q$, $q \in \mathbb{C}_q^h$ or $q \in \mathbb{C}_q^0$. So, CP q for CF corresponds to not simple root $z = \pi q$ of the Equation (3.11$_a$). Then,

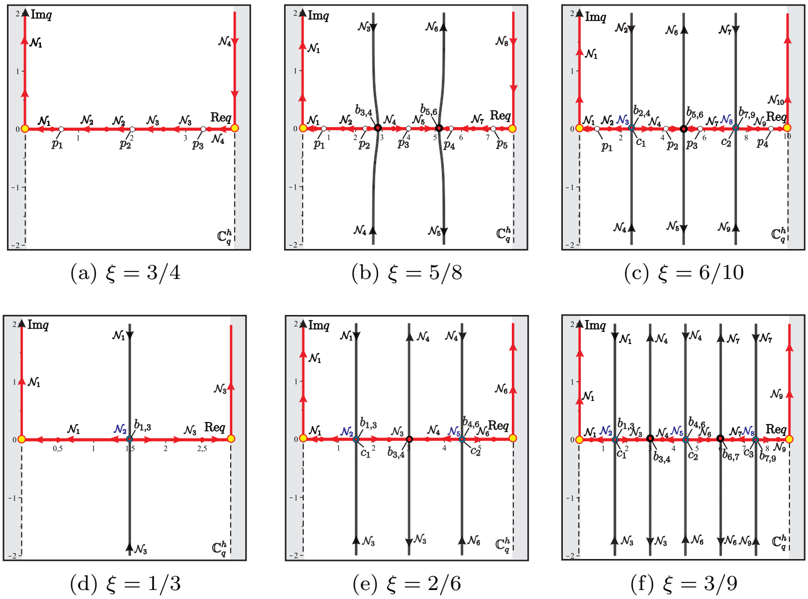


Figure 6. Specrum Curves for various ξ in Case $d2, n1$. ● – CP, ◉ – CEP and CP, ○ – eigenvalue point at RP, ∘ – PP.

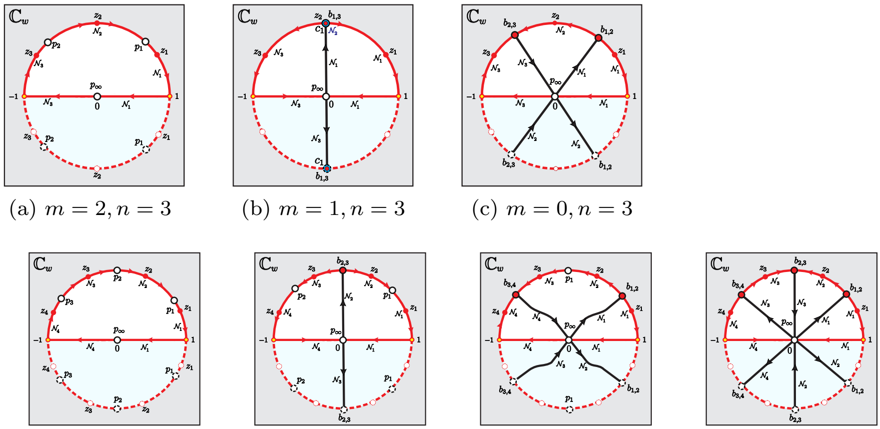
we use Lemma 2 and Corollary 1. If $\cos(\pi\xi q) = 0$, then, $q = x$ is real. In this case $x \in \mathcal{C}_\xi$. In *Case b* the proof is the same. □

So, all CPs of CF are CPs of Real CP. For our problems all CPs are of the first order and $n_b = n - m - 1$.

4 Spectrum curves

The definition of Spectrum Curve was introduced in articles [1,2,23], but as a mathematical object it was already used before [28].

The Spectrum Domain $\mathcal{D}_\xi := \{q \in \mathbb{C}_q^h : \operatorname{Im}\gamma_c(q) = 0\}$ is a contour line (isoline) of function $\operatorname{Im}\gamma_c : \mathbb{C}_q^h \to \mathbb{R}$ for fixed ξ. The contour line is a smooth curve outside the critical points $b \in B_\xi$. For CF $\gamma(q) = \operatorname{Re}\gamma_c(q)|_{\mathcal{D}_\xi}$ at point $a \in \mathcal{D}_\xi \setminus \mathcal{B}_\xi$ we have $\gamma(q) = \gamma(a) + \operatorname{Re}\big(\gamma_c'(a)(q-a)\big) + o(q-a)$, $\gamma_c'(a) \neq 0$. Thus, we can parametrize each such curve using the parameter γ. We add arrow on this curve $\mathcal{N}(\gamma)$ (arrows show the direction in which $\gamma \in \mathbb{R}$ is increasing). At a CP $b \in \mathcal{B}_\xi$ we have $\operatorname{Im}\big(\gamma_c^{(r)}(b)(q-b)^r/r!\big) + o\big((q-b)^r\big) = 0$, $\gamma_c^{(r)}(b) \neq 0$, $r > 1$. So, for CP the structure of contour line is as for function $w = (q-b)^r$. For our problems CPs are of the first order and real. In this case $\operatorname{Im}\big((q-b)^2/2\big) = (x-b)y$ and we have intersection of lines $x = b$ and $y = 0$ at point b (see, [28]). At CP $b \in \mathcal{B}_\xi$, these curves change direction and the angle between the old and the new direction is $\pi/\deg^+(b)$. We use the "right-hand rule". So, the curve turns to the right. For the $\gamma \to \pm\infty$, curve $\mathcal{N}(\gamma)$ approaches PP $p \in \overline{\mathcal{P}}_\xi$. If $q \in \overline{\mathcal{P}}_\xi + \mathcal{B}_\xi$, then $\deg^+(q)$ corresponds to the



(a) $m=2, n=3$ (b) $m=1, n=3$ (c) $m=0, n=3$

(d) $m=3, n=4$ (e) $m=2, n=4$ (f) $m=1, n=4$ (g) $m=0, n=4$

Figure 7. Spectrum Curves in $\mathbb{C}_w$ for $n=3$ and $n=4$ in Case $d2, n1$.

number of outgoing curves at that point. Note that incoming curves alternate with outgoing, so $\deg(q) = \deg^+(q) = \deg^-(q)$.

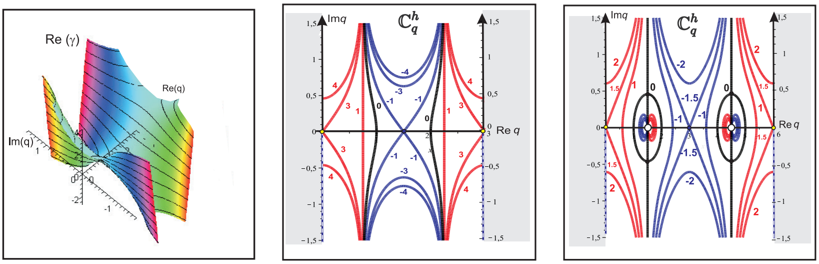
We can enumerate curves $\mathcal{N}(\gamma)\colon \mathbb{R}_\gamma = (-\infty, +\infty) \to D_\xi \subset \mathbb{C}_q^h$ for our problem by classical case ($\gamma = 0$): $\mathcal{N}_l(0) = \hat{z}_l \in \mathcal{Z}_\xi$ (see, (3.23)). These curves are *regular Spectrum Curves*. The regular Spectrum Curves form Spectrum Domain $\mathcal{D}_\xi = \cup_{\hat{z}_l \in \mathcal{Z}_\xi} \mathcal{N}_l$ (see, Figure 6 in Case $d2, n1$ and article [28] in Case $d1, n2$). The index of $b \in \mathcal{B}_\xi$ is formed of the indices of the regular Spectrum Curves which intersect at this CP. If CP of the first order is real, then the left index coincides with the index of Spectral Curve which is defined by the smaller real λ values, and the right index is defined by greater λ values.

Note that the part of the Spectrum Domain $\mathcal{D}_\xi$ in domain $\mathbb{C}_q^{h+}$ is symmetric to the part in domain $\mathbb{C}_q^{h-}$. More properties about symmetricity of $\mathcal{D}_\xi$ we can get using Lemma 6.

For every CEP $c_j \in \mathcal{C}_\xi$, $j \in \mathcal{J}_\xi$, (see, (3.8)), we define *nonregular Spectrum Curve* $\mathcal{N}_{l_j} = \{c_j\}$ and such Spectrum Curve is one point the same for all γ. We note that nonregular Spectrum Curves can overlap with a point of a regular Spectrum Curve. So, we have $n_{\hat{z}} = n_z + n_{ce}$ *Spectrum Curves*: n_z regular and n_{ce} nonregular and we have a collection $\overline{\mathcal{N}}_\xi := \{\mathcal{N}_l \colon \hat{z}_l \in \hat{\mathcal{Z}}^h\}$ of all Spectrum Curves. In our cases $\mathcal{C}_\xi \subset \mathcal{D}_\xi$ and Spectrum Domain $\mathcal{D}_\xi = \overline{\mathcal{D}}_\xi$. Function γ_c has real values on $\mathcal{D}_\xi$ except pole points. If $\mathcal{C}_\xi \neq \varnothing$, then we will always add points c_j, $j \in \mathcal{J}_\xi$, in the domain $\mathbb{C}_q^h$.

The domain $\mathbb{C}_w$ is useful for investigation of Spectrum Curves near the point $q = \infty$ ($w = 0$). Lemma 7 states that this point is PP of the $(n-m)$-order. We see Spectrum Curves in $\mathbb{C}_w$ in Figure 7 (Case $d2, n1$).

Spectrum Curves describe a qualitative view of the spectrum for a fixed ξ. Each of Spectrum Curves $\mathcal{N}_l$ describes eigenvalue points for eigenvalue λ_l, and we know how eigenvalue point is moving in the domain $\mathbb{C}_q^h$ when γ is changing from $-\infty$ to $+\infty$. If $y=0$, then, equalities (3.27) are valid, and $[0, n] \smallsetminus \mathcal{P}_\xi \subset \mathcal{D}_\xi$.



(a) $m = 1, n = 2$ (b) $m = 1, n = 3$ (c) $m = 4, n = 6$

Figure 8. Surfacecontours and contour lines for the function $\operatorname{Re}\gamma_c(q)$ in Case b.

If $x \in \widetilde{\mathcal{C}}_\xi$ (i.e., $\sin(\pi x) = \sin(\pi x\xi) = 0$ or $\cos(\pi x) = \cos(\pi x\xi) = 0$), then, equalities (3.27) are valid, too. So, vertical lines $x + \imath y$, $x \in \widetilde{\mathcal{C}}_\xi$, belong to the Spectrum Domain $\mathcal{D}_\xi$ (see, Figure 6) and all $x \in \widetilde{\mathcal{C}}_\xi \setminus \{0, n\}$ are CPs. Particulary, $\mathbb{R}_y^-, \mathbb{R}_y^{h+} \subset \mathcal{D}_\xi$. Another Spectrum Curves parts in complex domain $\mathbb{C}_q^{h+} + \mathbb{C}_q^{h-}$ are not vertical lines. From Lemma 8 it folows that in this domain Spectrum Curves do not intersect (there is no CPs). PPs are eigenvalue points for $\gamma = \infty$.

For $m = n - 1$ CPs do not exist ($n_b = 0$) and there exist real eigenvalues only (see, Figure 6(a) and Figure 7(a),(d)).

4.1 Spectrum of dSLP

If we want to get information about dSLP spectrum, then we must know the values of CF on Spectrum Curve. First of all, we must calculate values of CF for special points: RP (see, (3.25)), CEP (see, (3.24)) and CPs. If CP is not RP or CEP, then, in general there is no formula for CP (we can find CP only numerically). If CP $b = \tilde{c}_j$ belongs to $\widetilde{\mathcal{C}}_\xi \setminus \overline{\mathcal{C}}_\xi$, then $|\gamma(b)| = 1$:

$$\tilde{c}_j = Nj, \;\; \tilde{\gamma}_j := \gamma(\tilde{c}_j) = (-1)^{(1-\xi)\tilde{c}_j}, \quad j = \overline{0, K}, \tag{4.1$_a$}$$

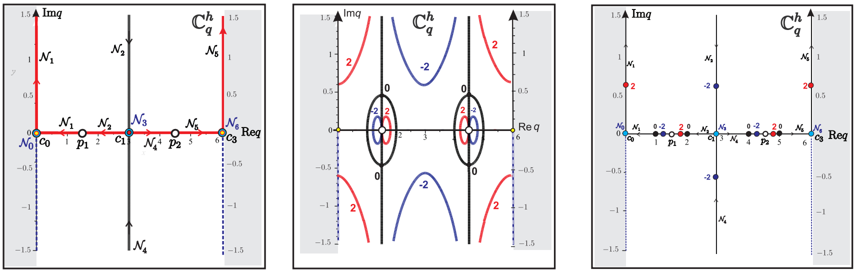
$$\tilde{c}_j = N(j - 1/2), \;\; \tilde{\gamma}_j := \gamma(\tilde{c}_j) = (-1)^{(N-M)/2}, \quad j = \overline{1, \varkappa K}. \tag{4.1$_b$}$$

If $x \in \widetilde{\mathcal{C}}_\xi$, then Spectrum Curves parts belongs to vertical lines $x + \imath y$, $y \in \mathbb{R}$, $x = c_j$ (see, (3.8)) or $x = \tilde{c}_j$ (see, (4.1)) and we have simple formulas between γ and y:

$$\gamma(c_j + \imath y) = \gamma_j \, \frac{\sinh(\pi y)}{\sinh(\pi y\xi)}, \quad \gamma(\tilde{c}_j + \imath y) = \tilde{\gamma}_j \, \frac{\cosh(\pi y)}{\cosh(\pi y\xi)}, \tag{4.2$_a$}$$

$$\gamma(c_j + \imath y) = \gamma_j \, \frac{\sinh(\pi y)}{\sinh(\pi y\xi)}, \quad \gamma(\tilde{c}_j + \imath y) = \tilde{\gamma}_j \, \frac{\cosh(\pi y)}{\cosh(\pi y\xi)}. \tag{4.2$_b$}$$

Because $\{0, n\} \subset \widetilde{\mathcal{C}}_\xi$ we have function $\gamma(\imath y) = \cosh(\pi y)/\cosh(\pi y\xi)$ in Case a and function $\gamma(\imath y) = \sinh(\pi y)/\sinh(\pi y\xi)$ in Case b for description negative eigenvalues. For positive eigenvalues $\lambda \geq 4n^2$ we can use function $\gamma(n + \imath y) = (-1)^{(1-\xi)n}\cosh(\pi y)/\cosh(\pi y\xi)$ in Case a and function $\gamma(n + \imath y) = (-1)^{(n-m)}\sinh(\pi y)/\sinh(\pi y\xi)$ in Case b. For positive eigenvalues λ, $0 \leq$



(a) Spectrum Curves (b) Contours $\gamma = -2, 0, 2$ (c) Eigenvalue Points

Figure 9. Spectrum of dSLP in Case $n2$, $m = 4, n = 6$. ◉ – CEP at RP, ● – CEP.

$\lambda \leq 4n^2$, we can investigate real CF: (a) $\gamma(x) = \cos(\pi x)/\cos(\pi x\xi)$ or (b) $\gamma(x) = \sin(\pi x)/\sin(\pi x\xi)$, $x \in [0, n] \smallsetminus \mathcal{P}_\xi$.

Let us return to general case. We can look for contour lines for the function $\operatorname{Re}\gamma_c(q)$ (see, Figure 8). Every γ-contour line has intersections with Spectrum Domain $\mathcal{D}_\xi$. On every regular Spectrum Curve we have one intersection point. If we add CEPs, then we get full Spectrum of dSLP (see, Figure 9). If γ-contour line intersects CP, then we have not simple eigenvalue (the geometrical multiplicity is 1, algebraic multiplicity is 2, and if additionally CP is CEP, then algebraic multiplicity is 3). In Figure 9(c) we see eigenvalue points for $\gamma = -2, 0, 2$. We have 4 eigenvalues points on regular Spectrum Curves $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_4, \mathcal{N}_5)$, and 3 CEPs $(\mathcal{N}_0 = \{c_0\}, \mathcal{N}_3 = \{c_1\}, \mathcal{N}_6 = \{c_3\})$. CEPs c_0 and c_2 are double and they are in RPs. So, corresponding eigenvalues $\lambda = 0$ and $\lambda = 4n^2 = 146$ are simple. These eigenvalues will be not simple (double) for $\gamma = 1.5$, when coresponding 1.5-contour line intersects both RPs (see, Figure 8(c)) and give additional eigenvalue point on regular Spectrum Curves $\mathcal{N}_1, \mathcal{N}_5$.

5 Conclusions

In this paper, we investigated dSLP with four cases (d1,d2,n1,n2) of BCs where one of the BCs is nonlocal. We introduced a natural approximation of the derivative and obtained a second-order SLP approximation using dSLP. The approximation with a natural discrete derivative in BCs, which we considered in this article, gives the same CFs as in the differential case.

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