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Solutions of the attraction-repulsion-chemotaxis system with nonlinear diffusion

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Abstract. In this study, we consider the well-posedness of the attraction-repulsion chemotaxis system. This paper explores the dynamics of species movement in reaction to two chemically opposing substances, incorporating nonlinear diffusion. Our primary objective is to establish the existence of a global-in-time weak solution for the proposed model in an unbounded three-dimensional spatial domain. Our study has confirmed the existence of a global-in-time weak solution for the proposed system in three dimensions. Furthermore, we demonstrate that global-in-time weak solutions are also attainable for the proposed system in a bounded domain with a smooth boundary.

Keywords: attraction-repulsion; chemotaxis; weak solutions; nonlinear diffusion.

AMS Subject Classification: 35D30; 92C17; 35K51; 35K55.

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1 Introduction

Chemotaxis is a biological phenomenon in which microorganisms move in response to chemical signals in their environment. All motile organisms exhibit some form of chemotaxis. This phenomenon is observed across a diverse spectrum, from single-celled bacteria to complex multicellular organisms. Chemotaxis plays crucial roles in many biological processes, such as immune responses, embryonic development, wound healing, and the movement of microorganisms within their environments. Keller and Segel developed a model to describe chemotaxis [15] as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(v) u \nabla v), \\ \epsilon v_t = \Delta v - \beta v + \gamma u, \end{cases} \tag{1.1}$$

where u refers to species density, v refers to concentration of signal, χ refers to sensitivity function and ϵ, β, γ are positive constants. Ishida and Yakota considered the quasilinear degenerate Keller-Segel system of parabolic-parabolic type in [13] and proved the global existence of weak solutions. Sugiyama and Kunii [26] considered the degenerate Keller–Segel model with a power factor in drift term and proved the global-in-time existence of weak solutions and decay properties. Various results on the singularity, local and global existence of solutions are obtained for a simplified form of (1.1) in [9, 11, 22] and the references therein. Mimura and Tsujikawa [21] considered the model as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(v) u \nabla v) + g(u), \\ \epsilon v_t = \Delta v + \beta u - \gamma v, \end{cases} \tag{1.2}$$

where $g(u)$ denotes the growth rate of species. Winkler [30] considered (1.2) with specific growth functions and investigated global solvability and its steady state. Osaki et al. [25] established a non-negative global solution and exponential attractor for (1.2) in a bounded domain of $\mathbb{R}^2$. Tello and Winkler [28] considered (1.2) with $\epsilon = 0$, $\beta = \gamma = 1$ in a bounded domain of $\mathbb{R}^n$ with smooth boundary and established global classical and weak solutions under some assumptions. Kang and Stevens [14] considered (1.2) with the death of species taken into account, which moves chemotactically and establishes the existence of global-in-time regular solutions. Results about the blow-up of the solution, and local and global existence of solutions for various forms of (1.2) are also studied in [19, 24, 29, 31].

In biological processes, the movement of organisms is affected by chemical stimuli such as attraction and repulsion. Chemoattraction involves organisms moving towards increasing signal concentrations, known as chemoattractants, while chemorepulsion involves movement away from increasing signal concentrations, known as chemorepellents. Luca et al. developed a model that emphasizes the chemotactic response of microglial cells in [20], as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(v) u \nabla v) + \nabla \cdot (\xi(w) u \nabla w), \\ \tau v_t = \Delta v + \beta u - \gamma v, \\ \tau w_t = \Delta w + \delta u - \eta w, \end{cases} \tag{1.3}$$

where u refers species density, v refers concentration of chemoattractant, and w refers concentration of chemorepellant. Tao and Wang established global solvability, the existence of steady states, and blow up of (1.3) in [27] for $\tau = 0, 1$. Espejo and Suzuki generalized the above for $\gamma, \eta > 0$ in [7]. Liu and Wang [17] proved the global existence of classical solutions and steady states of (1.3) in one dimension with $\tau = \beta = 1$. Chiyo et al. [4] established the global existence of classical solutions and its boundedness for (1.3) with logistic growth for constant χ and ξ. Tian et al. [10] discussed the large-time behavior of solutions to (1.3) with logistic source. Various results on boundedness and the blow-up of solution for the modified form of (1.3) studied in [1, 3, 12, 16, 23, 32].

In the literature, currently, no study is available on the attraction-repulsion-chemotaxis system with a nonlinear diffusion exponent of $1+\alpha$ in an unbounded domain. Therefore, in this paper, we aim to establish the existence of weak solutions for the proposed system, assuming α is greater than zero and that

$\chi'(\cdot)$ is greater than or equal to a constant χ_0. Specifically, we focus on (1.3) with nonlinear diffusion in $\mathbb{R}^3 \times [0,T)$, where $T > 0$ and it is given below:

$$\begin{cases} u_t = \Delta u^{1+\alpha} - \nabla \cdot (\chi(v)u\nabla v) + \nabla \cdot (\xi(w)u\nabla w), \\ v_t = \Delta v + \beta u - \gamma v, \\ w_t = \Delta w + \delta u - \eta w, \\ (u,v,w)(x,0) = (u_0,v_0,w_0)(x), \end{cases} \tag{1.4}$$

where $u(x,t)$ refers species density, $v(x,t)$ refers concentration of attraction signal, $w(x,t)$ refers concentration of repulsive signal, χ and ξ are non-negative sensitivity functions and $\alpha, \beta, \gamma, \delta, \eta$ are positive constants. In this paper, we establish the existence of weak solutions of (1.4) with the assumptions on the nonlinear exponent and boundedness of singular sensitivity functions.

This paper is organized as follows: In Section 2, we define the weak solution of the proposed system and introduce a suitable approximation system for the proposed system in $\mathbb{R}^3$. Then, we derive local estimates for the approximation problem and extend it to any given time interval $(0,T)$. In Section 3, we establish a weak solution of the proposed system (1.4) in $\mathbb{R}^3$.

2 Weak solutions and uniform energy estimates

In this section, we define the weak solution for (1.4). Then, we introduce the suitable approximation problem for the original model (1.4) and obtain uniform estimates for the approximation problem. Here, we recall some important and basic notations which are used throughout the paper:

1. $L^q(\mathbb{R}^3)$ denotes the set of all Lebesgue integrable functions in $\mathbb{R}^3$ and its norm is denoted by $\|f\|_{L^q(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |f|^q dt\right)^{\frac{1}{q}}$.
2. $W^{k,q}(\mathbb{R}^3)$ denotes the Sobolev space of order k for $1 \le q \le \infty$.
3. The space $L^q(0,T;L^p(\mathbb{R}^3))$, $1 \le q < \infty$, denotes the set of q integrable functions defined on the interval $[0,T]$ with functions in the Lebesgue space $L^p(\mathbb{R}^3)$ and its norm is denoted by
$$\|f\|_{L^q(0,T;L^p(\mathbb{R}^3))} = \left(\int_0^T \|f\|^q_{L^p(\mathbb{R}^3)} dt\right)^{\frac{1}{q}}.$$
4. The space $L^\infty(0,T;L^p(\mathbb{R}^3))$ denotes the set of essentially bounded measurable functions defined on the interval $[0,T]$ with functions in the Lebesgue space $L^p(\mathbb{R}^3)$ and its norm is denoted by
$$\|f\|_{L^\infty(0,T;L^p(\mathbb{R}^3))} = \operatorname*{ess\,sup}_{t\in[0,T]} \|f\|_{L^p(\mathbb{R}^3)}.$$

Definition 1. Let $\alpha > 0$ and $T > 0$ be finite. A triplet (u,v,w) is said to be a weak solution of the system (1.4) if

i) $u, v, w \geq 0$,

ii) $u(1+|x|+|\log u|) \in L^{\infty}(0,T;L^1(\mathbb{R}^3))$, $\quad \nabla u^{\frac{1+\alpha}{2}} \in L^2(0,T;L^2(\mathbb{R}^3))$,

iii) $u \in L^{\infty}(0,T;L^p(\mathbb{R}^3))$, $\quad \nabla u^{\frac{p+\alpha}{2}} \in L^2(0,T;L^2(\mathbb{R}^3))$ for $1 \leq p \leq 1+\alpha$,

iv) $v, w \in L^{\infty}(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3))$, $\quad v, w \in L^{\infty}(\mathbb{R}^3, \times[0,T))$,

v) $\nabla u \in L^2(0,T;L^2(\mathbb{R}^3))$,

vi) For any $\varphi \in L^{\infty}(0,T;L^2(\mathbb{R}^3))$, the following holds

$$\int_{\mathbb{R}^3} u_0\varphi(\cdot,0)dx = \int_0^T \int_{\mathbb{R}^3} (-u\varphi_t + \nabla u^{1+\alpha}\cdot\nabla\varphi - u\chi(v)\nabla v\cdot\nabla\varphi \\ +u\xi(w)\nabla w\cdot\nabla\varphi)\; dx\; dt,$$

$$\int_{\mathbb{R}^3} v_0\varphi(\cdot,0)dx = \int_0^T \int_{\mathbb{R}^3} (-v\varphi_t + \nabla v\cdot\nabla\varphi - \beta u\varphi + \gamma v\varphi)\; dx\; dt,$$

$$\int_{\mathbb{R}^3} w_0\varphi(\cdot,0)dx = \int_0^T \int_{\mathbb{R}^3} (-w\varphi_t + \nabla w\cdot\nabla\varphi - \delta u\varphi + \eta w\varphi)\; dx\; dt.$$

Now, we introduce an approximation problem for the proposed system (1.4) as follows to overcome the strong degeneracy of the diffusion terms:

$$\begin{cases} u_{\epsilon_t} = \Delta(u_\epsilon+\epsilon)^{1+\alpha} - \nabla\cdot(\chi(v_\epsilon)u_\epsilon\nabla v_\epsilon) + \nabla\cdot(\xi(w_\epsilon)u_\epsilon\nabla w_\epsilon), \\ v_{\epsilon_t} = \Delta v_\epsilon + \beta u_\epsilon - \gamma v_\epsilon, \\ w_{\epsilon_t} = \Delta w_\epsilon + \delta u_\epsilon - \eta w_\epsilon, \end{cases} \tag{2.1}$$

with initial conditions

$$u_{0_\epsilon} = \phi_\epsilon * u_0,\; v_{0_\epsilon} = \phi_\epsilon * v_0,\; \text{and } w_{0_\epsilon} = \phi_\epsilon * w_0,$$

where ϕ_ϵ is a usual mollifier with $\epsilon \in (0,1)$. The local regularity of the approximation model (2.1) can be derived as shown in [2]. Therefore, the proof is omitted here.

To prove the existence of weak solutions of (1.4), first, we derive certain bounds on the solutions of regularized problem (2.1). We deduce some uniform estimates, independent of ϵ, of the weak solution locally to the system (2.1). In the subsequent discussion, we use the variables (u, v, w) instead of $(u_\epsilon, v_\epsilon, w_\epsilon)$ for simplicity in notation. We first define the functional as below:

$$E(t) := \int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\; dx + \|u(t)\|_{1+\alpha}^{1+\alpha} + \|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2, \tag{2.2}$$

and

$$D(t) := \|\nabla u^{\frac{1+\alpha}{2}}(t)\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}(t)\|_2^2 + \|\Delta v(t)\|_2^2 + \|\Delta w(t)\|_2^2, \tag{2.3}$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$.

Lemma 1. *Let $\beta, \gamma, \delta, \eta > 0$. Suppose that (u, v, w) is a classical solution of system* (2.1) *for all $\epsilon \in (0,1)$ and initial data $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon})$ satisfies*

i) $u_{0_\epsilon}(1 + |x| + |\log u_{0_\epsilon}|) \in L^1(\mathbb{R}^3)$,

ii) $u_{0_\epsilon} \in L^{1+\alpha}(\mathbb{R}^3)$,

iii) v_{0_ϵ} *and* $w_{0_\epsilon} \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$,

independent of ϵ. Furthermore, we assume that

$$\alpha > 1/6, \quad \chi^{'}, \xi^{'} \in L^\infty_{loc}. \tag{2.4}$$

Then, $0 < t \leq T$, there exists $C > 0$ independent of ϵ such that

$$\sup_{0 \leq \tau \leq t} E(\tau) + \int_0^t D(\tau) d\tau < C, \tag{2.5}$$

where $E(t)$ and $D(t)$ are defined as in (2.2) *and* (2.3) *respectively.*

Proof. Upon integrating $(2.1)_1$, we derive $\|u(t)\|_1 \equiv \|u_0\|_1$, thereby indicating the conservation of u is preserved. Furthermore, $\|v\|_{L^\infty(\mathbb{R}^3 \times [0,T))} \leq \|v_0\|_\infty$ and $\|w\|_{L^\infty(\mathbb{R}^3 \times [0,T))} \leq \|w_0\|_\infty$ can be obtained by applying the maximal principle to $(2.1)_2$ and $(2.1)_3$. The remaining proof of the lemma is splitted into three cases as follows:

(i) $\frac{1}{6} < \alpha \leq \frac{1}{3}$, (ii) $\frac{1}{3} < \alpha \leq 1$, and (iii) $\alpha > 1$.

Case (i): $\frac{1}{6} < \alpha \leq \frac{1}{3}$.

Multiplying $(2.1)_1$ by $\log u$ and integrating the resulting terms, we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} u \log u dx + \int_{\mathbb{R}^3} \nabla \log u \cdot \nabla (u + \epsilon)^{1+\alpha} dx$$
$$= \int_{\mathbb{R}^3} \nabla u \cdot (\chi(v) \nabla v) dx - \int_{\mathbb{R}^3} \nabla u \cdot (\xi(w) \nabla w) dx. \tag{2.6}$$

The second term in LHS of above estimated as follows:

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla \log u \cdot \nabla (u + \epsilon)^{1+\alpha} \, dx &\geq \int_{\mathbb{R}^3} \nabla \log u \cdot (1+\alpha) u^\alpha \nabla u \, dx \\ &= \frac{4}{1+\alpha} \big\| \nabla u^{\frac{1+\alpha}{2}} \big\|_2^2. \end{aligned} \tag{2.7}$$

Using above in (2.6), we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} u \log u \, dx + \frac{4}{1+\alpha} \big\| \nabla u^{\frac{1+\alpha}{2}} \big\|_2^2$$
$$\leq \int_{\mathbb{R}^3} \nabla u \cdot (\chi(v) \nabla v) dx - \int_{\mathbb{R}^3} \nabla u \cdot (\xi(w) \nabla w) dx. \tag{2.8}$$

From $\big|\nabla u^{(1+\alpha)/2}\big|^2 = \big((1+\alpha)/2\big)^2 u^{\alpha-1} |\nabla u|^2$, we have $|\nabla u| = \frac{2}{1+\alpha} u^{\frac{1-\alpha}{2}} \big|\nabla u^{\frac{1+\alpha}{2}}\big|$.

Now, we estimate the first term in RHS of (2.8) using the above equality along with Young's inequality to get

$$\int_{\mathbb{R}^3} \nabla u\cdot(\chi(v)\nabla v)dx \le \frac{2\overline{\chi}}{1+\alpha}\left(\epsilon_1\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C(\epsilon_1)\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 dx\right), \quad (2.9)$$

where $\overline{\chi} := \sup_{\mathbb{R}^3}|\chi(v)|$. From $\left|\nabla u^{1-\alpha}\right|^2 = (1-\alpha)^2 u^{-2\alpha}|\nabla u|^2$ and $|\nabla u|^2 = \frac{4}{(1+\alpha)^2}u^{1-\alpha}\left|\nabla u^{\frac{1+\alpha}{2}}\right|^2$, we have $\left|\nabla u^{1-\alpha}\right| = C_2 u^{\frac{1-3\alpha}{2}}\left|\nabla u^{\frac{1+\alpha}{2}}\right|$, where $C_2 = \frac{2(1-\alpha)}{(1+\alpha)}$. We evaluate the last term in RHS of (2.9) using the above equality and Young's inequality as

$$\begin{aligned}
\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx &= \int_{\mathbb{R}^3} u^{1-\alpha}\nabla v\cdot\nabla v\, dx \\
&\le C_1\left(\int_{\mathbb{R}^3}|\nabla u^{1-\alpha}||\nabla v|\, dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\, dx\right) \\
&\le C_1\left(\int_{\mathbb{R}^3} C_2 u^{\frac{1-3\alpha}{2}}|\nabla u^{\frac{1+\alpha}{2}}||\nabla v|\, dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\, dx\right) \\
&\le C_1C_2\int_{\mathbb{R}^3}\epsilon_2|\nabla u^{\frac{1+\alpha}{2}}|^2 + C(\epsilon_2)u^{1-3\alpha}|\nabla v|^2\, dx + C_1\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\, dx. \quad (2.10)
\end{aligned}$$

Using (2.10) in (2.9), we get

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla u\cdot(\chi(v)\nabla v)\, dx \\
&\qquad\le C_1^{''}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_2^{''}\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\, dx + C_3^{''}\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\, dx, \quad (2.11)
\end{aligned}$$

where $C_i^{''} = \frac{2\overline{\chi}C_i^{'}}{1+\alpha}$ for $i\in\{1,2,3\}$, $C_1^{'} = \epsilon_1 + C_2C_3^{'}\epsilon_2$, $C_2^{'} = C_2C_3^{'}C(\epsilon_2)$ and $C_3^{'} = C_1C(\epsilon_1)$. Following similar procedure as above, we get

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\nabla u\cdot(\xi(w)\nabla w)dx \\
&\qquad\le C_4^{'}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_5^{'}\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2\, dx + C_6^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w|dx, \quad (2.12)
\end{aligned}$$

where $C_i^{'}, i\in\{4,5,6\}$ are positive constants. Using (2.7), (2.11) and (2.12) in (2.6), we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} u\log u dx + C^{'}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 \le C_2^{''}\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2 dx \\
&\quad + C_3^{''}\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|dx + C_5^{'}\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2 dx + C_6^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w|dx, \quad (2.13)
\end{aligned}$$

where $C^{'} = 4/(1+\alpha) - C_1^{''} - C_4^{'}$. Multiplying $(2.1)_1$ by $\langle x\rangle$ and using Young's inequality along with simple algebraic calculations (see (3.6) of [18] and (2.13) of [5]), leads to

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle u\,dx &= \int_{\mathbb{R}^3}(u+\epsilon)^{1+\alpha}\Delta\langle x\rangle\,dx+\int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot u\chi(v)\nabla v\,dx\\ &\quad-\int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot u\xi(w)\nabla w\,dx\\ &\le C_7\big(1+\|\nabla v\|_2^2+\|\nabla w\|_2^2\big)+\big(C_8(\epsilon)+\epsilon\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2\big).\end{aligned}\tag{2.14}$$

Multiplying $(2.1)_1$ by u^α and then integrating, we get

$$\begin{aligned}&\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha}+\int_{\mathbb{R}^3}\nabla u^\alpha\cdot\nabla(u+\epsilon)^{1+\alpha}\,dx\\ &\quad=\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi(v)\nabla v)dx-\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi(w)\nabla w)dx.\end{aligned}\tag{2.15}$$

Using

$$\big|\nabla u^\alpha\big|^2=\alpha^2u^{\alpha-1}\big|\nabla u\big|^2,\quad \big|\nabla u^{\frac{1+2\alpha}{2}}\big|^2=\frac{(1+2\alpha)^2}{4}u^{\frac{2\alpha-1}{2}}\big|\nabla u\big|^2,$$

we get

$$\int_{\mathbb{R}^3}\nabla u^\alpha\cdot\nabla(u+\epsilon)^{1+\alpha}\,dx \ \ge\ \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2.\tag{2.16}$$

Now, we estimate the first term in RHS of (2.16) using Young's inequality along with $|\nabla u|=\frac{2}{1+2\alpha}u^{\frac{1-2\alpha}{2}}\big|\nabla u^{\frac{1+2\alpha}{2}}\big|$ as

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi\nabla v)\,dx\\ &\le C(\chi)\int_{\mathbb{R}^3}\big|\nabla u^{\frac{1+2\alpha}{2}}\big|(u^{\frac12}|\nabla v|)\,dx\\ &\le C(\chi)\Big(\epsilon_3\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+C(\epsilon_3)\int_{\mathbb{R}^3}u|\nabla v|^2\,dx\Big)\\ &\le C(\chi)\Big(C(\epsilon_3)C_9\Big(\int_{\mathbb{R}^3}|\nabla u||\nabla v|\,dx+\int_{\mathbb{R}^3}u|\Delta v|\,dx\Big)+\epsilon_3\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2\Big)\\ &\le C(\chi)\Big(\epsilon_3\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+C(\epsilon_3)C_9\int_{\mathbb{R}^3}u|\Delta v|\,dx\\ &\qquad+C(\epsilon_3)\frac{2C_9}{1+2\alpha}\int_{\mathbb{R}^3}u^{\frac{1-2\alpha}{2}}\big|\nabla u^{\frac{1+2\alpha}{2}}\big||\nabla v|\,dx\Big)\\ &\le C(\chi)\Big(\epsilon_3\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+C(\epsilon_3)\frac{2C_9}{1+2\alpha}\epsilon_4\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2\\ &\qquad+C(\epsilon_3)\frac{2C_9}{1+2\alpha}C(\epsilon_4)\int_{\mathbb{R}^3}u^{1-2\alpha}|\nabla v|^2\,dx+C(\epsilon_3)C_9\int_{\mathbb{R}^3}u|\Delta v|\,dx\Big)\\ &\le C_{10}'\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+C_{11}'\int_{\mathbb{R}^3}u^{1-2\alpha}|\nabla v|^2\,dx+C_{12}'\int_{\mathbb{R}^3}u|\Delta v|\,dx,\end{aligned}\tag{2.17}$$

where $\overline{\chi} := \sup_{\mathbb{R}^3} |\chi(v)|$, $C(\chi) = \frac{2\alpha\overline{\chi}}{1+\alpha}$, $C_{10}^{'} = C(\chi)\big(\epsilon_3 + C(\epsilon_3)\frac{2C_9}{1+2\alpha}\epsilon_4\big)$, $C_{11}^{'} = C(\chi)\frac{2C_9}{1+2\alpha}C(\epsilon_4)C(\epsilon_3)$ and $C_{12}^{'} = C(\chi)C_9C(\epsilon_3)$. Following the same procedure as similar as above, we get

$$-\int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\xi(w)\nabla w)\, dx \leq C_{13}^{'}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_{14}^{'}\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla w|^2 dx + C_{15}^{'}\int_{\mathbb{R}^3} u|\Delta w|dx, \tag{2.18}$$

where $C_i^{'}, i \in \{13, 14, 15\}$ are positive constants. Using (2.16), (2.17) and (2.18) in (2.15), we have

$$\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{15}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 \leq C_{11}^{'}\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\, dx + C_{12}^{'}\int_{\mathbb{R}^3} u|\Delta v|\, dx + C_{14}^{'}\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla w|^2\, dx + C_{15}^{'}\int_{\mathbb{R}^3} u|\Delta w|\, dx, \tag{2.19}$$

where $C_{15} = \frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2} - C_{10}^{'} - C_{12}^{'}$. Multiplying $(2.1)_2$, $(2.1)_3$ by $-\Delta v$, $-\Delta w$ respectively and then integrating and adding the resulting equations, we get

$$\frac{d}{dt}\Big(\|\nabla v\|_2^2 + \|\nabla w\|_2^2\Big) + 2\Big(\|\Delta v\|_2^2 + \|\Delta w\|_2^2\Big) \leq \beta\int_{\mathbb{R}^3} u|\Delta v|dx - \gamma\int_{\mathbb{R}^3} v|\Delta v|dx + \delta\int_{\mathbb{R}^3} u|\Delta w|dx - \eta\int_{\mathbb{R}^3} w|\Delta w|dx, \tag{2.20}$$

Adding (2.13), (2.14), (2.19) and (2.20), we get

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\, dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\Big) \\
&\quad + C_0\Big(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2\Big) \\
&\leq C_0^{'}\Big(\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\, dx + \int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2\, dx + \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\, dx \\
&\qquad + \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla w|^2\, dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\, dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w|\, dx \\
&\qquad + \int_{\mathbb{R}^3} u|\Delta v|dx + \int_{\mathbb{R}^3} u|\Delta w|dx - \int_{\mathbb{R}^3} v|\Delta v|dx - \int_{\mathbb{R}^3} w|\Delta w|dx\Big).
\end{aligned} \tag{2.21}$$

In order to deduce (2.5) from above, we estimate the integrals in RHS as follows: As $0 \leq 1 - 3\alpha < \frac{2}{3}$, using the Hölder inequality and Sobolev embedding, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2 dx &\leq \begin{cases} \int_{\mathbb{R}^3} \big(C(\epsilon_4) + \epsilon_4 u^{\frac{2}{3}}\big)|\nabla v|^2\, dx, & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla v\|_2^2, & \text{if } \alpha = \frac{1}{3}, \end{cases} \\
&\leq \begin{cases} C(\epsilon_4)\|\nabla v\|_2^2 + \epsilon_4\|u_0\|^{\frac{2}{3}}\|\Delta v\|_2^2, & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla v\|_2^2, & \text{if } \alpha = \frac{1}{3}. \end{cases}
\end{aligned} \tag{2.22}$$

Following the same procedure as similar as above, we get

$$\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2 dx \leq \begin{cases} C(\epsilon_5)\|\nabla w\|_2^2 + \epsilon_5 \|u_0\|^{\frac{2}{3}} \|\Delta w\|_2^2, & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla w\|_2^2, & \text{if } \alpha = \frac{1}{3}. \end{cases}$$

$$\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2 \, dx \leq C(\epsilon_6)\|\nabla v\|_2^2 + \epsilon_6 \|u_0\|^{\frac{2}{3}} \|\Delta v\|_2^2.$$

$$\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla w|^2 \, dx \leq C(\epsilon_7)\|\nabla w\|_2^2 + \epsilon_7 \|u_0\|^{\frac{2}{3}} \|\Delta w\|_2^2.$$

Next, we estimate the integral $\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v| \, dx$. Using the Hölder, Young and Gagliardo-Nierenberg inequality in the integral, we get

$$\begin{aligned} \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v| \, dx &\leq C(\epsilon_8)\|u\|_{2-\alpha}^{2-\alpha} + \epsilon_8\|\Delta v\|_2^2 \\ &\leq C(\epsilon_8)C_{16}\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}} \left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^{\frac{6-6\alpha}{2+3\alpha}} + \epsilon_8\|\Delta v\|_2^2 \\ &\leq C(\epsilon_8)C_{16}\left(C(\epsilon_9)\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}} + \epsilon_9\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2\right) + \epsilon_8\|\Delta v\|_2^2 \\ &= C_{17} + C_{18}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + \epsilon_8\|\Delta v\|_2^2, \end{aligned}$$

where $C_{17} = C_{16}C(\epsilon_8)C(\epsilon_9)\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}}$ and $C_{18} = C_{16}C(\epsilon_8)\epsilon_9$. Here, we are using the fact that $\frac{4}{3} \leq \frac{6-6\alpha}{2+3\alpha} < 2$. Following the same procedure as similar as above, we get

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w| \, dx \leq C_{19} + C_{20}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + \epsilon_{10}\|\Delta w\|_2^2.$$

Next, we estimate the integral $\int_{\mathbb{R}^3} u|\Delta v| \, dx$. Using the Young and Gagliardo-Nierenberg inequality in the integral, we get

$$\begin{aligned} \int_{\mathbb{R}^3} u|\Delta v| \, dx &\leq C(\epsilon_{11})\|u\|_2^2 + \epsilon_{11}\|\Delta v\|_2^2 \\ &\leq C(\epsilon_{11})C_{21}\|u_0\|_1^{\frac{1+6\alpha}{2+6\alpha}} \left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^{\frac{6}{2+6\alpha}} + \epsilon_{11}\|\Delta v\|_2^2 \\ &\leq C(\epsilon_{11})C_{21}\left(C(\epsilon_{12})\|u_0\|_1^{\frac{1+6\alpha}{2+3\alpha}} + \epsilon_{12}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2\right) + \epsilon_{11}\|\Delta v\|_2^2 \\ &= C_{21}' + C_{22}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + \epsilon_{11}\|\Delta v\|_2^2, \end{aligned} \tag{2.23}$$

where $C_{21}' = C_{21}C(\epsilon_{11})C(\epsilon_{12})\|u_0\|_1^{\frac{1+6\alpha}{2+3\alpha}}$ and $C_{22} = C_{21}\epsilon_{12}C(\epsilon_{11})$. Here, we are using the fact that $\frac{3}{2} \leq \frac{6}{2+6\alpha} < 2$. Following the same procedure as similar as above, we get

$$\int_{\mathbb{R}^3} u|\Delta w| \, dx \leq C_{23} + C_{24}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + \epsilon_{13}\|\Delta w\|_2^2.$$

Next we estimate the integrals $\int_{\mathbb{R}^3} v|\Delta v|\ dx$, $\int_{\mathbb{R}^3} w|\Delta w|\ dx$. Using the Young and Gagliardo-Nierenberg inequality in the integrals, we get

$$\int_{\mathbb{R}^3} v|\Delta v|\ dx \leq \quad C(\epsilon_{14})\|v\|_2^2 + \epsilon_{14}\|\Delta v\|_2^2 \leq C_{25}\|v_0\|_1^{\frac{8}{7}}\|\Delta v\|_2^{\frac{6}{7}} + \epsilon_{14}\|\Delta v\|_2^2.$$

Similarly, we get

$$\int_{\mathbb{R}^3} w|\Delta w|\ dx \leq C_{26}\|w_0\|_1^{\frac{8}{7}}\|\Delta w\|_2^{\frac{6}{7}} + \epsilon_{15}\|\Delta w\|_2^2. \tag{2.24}$$

Substituting (2.22)–(2.24) in (2.21), we have

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\ dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\right)$$
$$+\overline{C}_0\left(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2+\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2+\|\Delta v\|_2^2+\|\Delta w\|_2^2\right)\leq\overline{C}_0'\left(1+\|\nabla v\|_2^2+\|\nabla w\|_2^2\right).$$

Integrating above with respect to t, we get (2.5).
Case (ii): $1/3 < \alpha \leq 1$.
Multiplying $(2.1)_1$ by $\log u$ and then integrating, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u dx + \int_{\mathbb{R}^3} \nabla \log u \cdot \nabla(u+\epsilon)^{1+\alpha} dx$$
$$= \int_{\mathbb{R}^3} \nabla u\cdot(\chi(v)\nabla v)dx - \int_{\mathbb{R}^3} \nabla u\cdot(\xi(w)\nabla w)dx. \tag{2.25}$$

Now, we evaluate the first term in RHS of (2.25) as follows:

$$\int_{\mathbb{R}^3} \nabla u\cdot(\chi(v)\nabla v)\ dx \leq C_{28}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_{29}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx, \tag{2.26}$$

where $C_{28} = 2\overline{\chi}\epsilon_1/(1+\alpha)$ and $C_{29} = 2\overline{\chi}C(\epsilon_1)/(1+\alpha)$. Following the same procedure as similar as above, we get

$$-\int_{\mathbb{R}^3} \nabla u\cdot(\xi(w)\nabla w)\ dx \leq C_{30}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_{31}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\ dx. \tag{2.27}$$

Using (2.7), (2.26) and (2.27) in (2.25), we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + C_{32}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2$$
$$\leq C_{29}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx + C_{31}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\ dx, \tag{2.28}$$

where $C_{32} = \frac{4}{1+\alpha} - C_{28} - C_{30}$. Multiplying $(2.1)_1$ by u^α and then integrating, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2$$
$$\leq \int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\chi(v)\nabla v)\ dx - \int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\xi(w)\nabla w)\ dx. \tag{2.29}$$

Now, we estimate the first term in RHS of above using Young's inequality along with $|\nabla u| = \frac{2}{1+\alpha} u^{\frac{1-\alpha}{2}} \left|\nabla u^{\frac{1+\alpha}{2}}\right|$ as

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\chi(v)\nabla v)\, dx \le C(\chi) \int_{\mathbb{R}^3} \left|\nabla u^{\frac{1+2\alpha}{2}}\right| (u^{\frac{1}{2}}|\nabla v|)\, dx \\
\le& C(\chi)\left(\epsilon_{18}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C(\epsilon_{18}) \int_{\mathbb{R}^3} u|\nabla v|^2\, dx\right) \\
=& C(\chi)\epsilon_{18}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C(\chi)C(\epsilon_{18}) \int_{\mathbb{R}^3} u|\nabla v|^2\, dx \\
\le& C(\chi)\epsilon_{18}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C(\chi)C(\epsilon_{18})\epsilon_{18}^{'}\left(\int_{\mathbb{R}^3} |\nabla u||\nabla v|\, dx + \int_{\mathbb{R}^3} u|\Delta v|\, dx\right) \\
=& C(\chi)\epsilon_{18}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C(\chi)C(\epsilon_{18})\epsilon_{18}^{'} \int_{\mathbb{R}^3} u|\Delta v|\, dx \\
&+ C(\chi)C(\epsilon_{18})\epsilon_{18}^{'} \int_{\mathbb{R}^3} u^{\frac{1-\alpha}{2}}\left|\nabla u^{\frac{1+\alpha}{2}}\right||\nabla v|\, dx \\
\le& C(\chi)\epsilon_{18}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C(\chi)C(\epsilon_{18})\epsilon_{18}^{'} \int_{\mathbb{R}^3} u|\Delta v|\, dx \\
&+ C(\chi)C(\epsilon_{18})\epsilon_{18}^{'}\left(\epsilon_{19}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + C(\epsilon_{19}) \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx\right) \qquad (2.30) \\
=& C_{33}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C_{34}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + C_{35}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx + C_{36}\int_{\mathbb{R}^3} u|\Delta v|\, dx,
\end{aligned}$$

where $C(\chi) = \frac{2\alpha\overline{\chi}}{1+\alpha}$, $C_{33} = C(\chi)\epsilon_{18}$, $C_{34} = C_{33}\epsilon_{18}^{'}\epsilon_{19}$, $C_{35} = C_{33}\epsilon_{18}^{'}C(\epsilon_{19})$ and $C_{36} = C_{33}\epsilon_{18}^{'}$. Following the same procedure as similar as above, we get

$$\begin{aligned}
-\int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\xi(w)\nabla w)\, dx \le& \epsilon_{20}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C_{37}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 \\
&+ C_{38}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\, dx + C_{39}\int_{\mathbb{R}^3} u|\Delta w|\, dx. \qquad (2.31)
\end{aligned}$$

Using (2.30) and (2.31) in (2.29), we have

$$\begin{aligned}
\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{40}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C_{41}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2& \\
\le\ C_{35}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx + C_{36}\int_{\mathbb{R}^3} u|\Delta v|\, dx& \\
+C_{38}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2 dx + C_{39}\int_{\mathbb{R}^3} u|\Delta w| dx,& \qquad (2.32)
\end{aligned}$$

where $C_{40} = \frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2} - C_{33} - \epsilon_{20}$ and $C_{41} = -C_{34} - C_{37}$. It is clear that $(2.1)_2$, and $(2.1)_3$ are independent of α and therefore, adding (2.20) with (2.28), (2.14), (2.32), we get

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\ dx + \big\|u\big\|_{1+\alpha}^{1+\alpha} + \big\|\nabla v\big\|_2^2 + \big\|\nabla w\big\|_2^2\Big)$$
$$+C_{42}\bigg(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v\big\|_2^2 + \big\|\Delta w\big\|_2^2\bigg)$$
$$\leq C_{43}\Big(\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\ dx + \int_{\mathbb{R}^3} u|\Delta v|\ dx$$
$$+ \int_{\mathbb{R}^3} u|\Delta w|\ dx - \int_{\mathbb{R}^3} v|\Delta v|\ dx - \int_{\mathbb{R}^3} w|\Delta w|\ dx\Big). \tag{2.33}$$

As $0 \leq 1-\alpha < 2/3$, using the Hölder and Sobolev inequality, we get

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 dx \leq \begin{cases} \int_{\mathbb{R}^3} \big(C(\epsilon_{21}) + \epsilon_{21} u^{\frac{2}{3}}\big)|\nabla v|^2\ dx, & \text{if } \frac{1}{3}<\alpha<1, \\ \|\nabla v\|_2^2, \text{ if } \alpha = 1, \end{cases}$$
$$\leq \begin{cases} C(\epsilon_{21})\|\nabla v\|_2^2 + \epsilon_{21}\|u_0\|^{\frac{2}{3}}\|\Delta v\|_2^2, & \text{if } \frac{1}{3} < \alpha < 1, \\ \|\nabla v\|_2^2, \text{if } \alpha = 1. \end{cases} \tag{2.34}$$

Following the same procedure as similar as above, we get

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2 dx \leq \begin{cases} C(\epsilon_{22})\|\nabla w\|_2^2 + \epsilon_{22}\|u_0\|^{\frac{2}{3}}\|\Delta w\|_2^2, & \text{if } \frac{1}{3} < \alpha < 1, \\ \|\nabla w\|_2^2, & \text{if } \alpha = 1. \end{cases} \tag{2.35}$$

As $0 < \frac{6}{2+6\alpha} < 2$ and $1 < \frac{6}{5} < 3+6\alpha$, following the same procedure as we did in (2.23) and from the previous case, we have

$$\begin{aligned} \int_{\mathbb{R}^3} u|\Delta v|\ dx &\leq C_{44} + C_{45}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \epsilon_{23}\|\Delta v\|_2^2, \\ \int_{\mathbb{R}^3} u|\Delta w|\ dx &\leq C_{46} + C_{47}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \epsilon_{24}\|\Delta w\|_2^2, \\ \int_{\mathbb{R}^3} v|\Delta v|\ dx &\leq C_{25}\|v_0\|_1^{\frac{8}{7}}\|\Delta v\|_2^{\frac{6}{7}} + \epsilon_{14}\|\Delta v\|_2^2, \\ \int_{\mathbb{R}^3} w|\Delta w|\ dx &\leq C_{26}\|w_0\|_1^{\frac{8}{7}}\|\Delta w\|_2^{\frac{6}{7}} + \epsilon_{15}\|\Delta w\|_2^2. \end{aligned}$$

Substituting (2.34), (2.35) and above estimates in (2.33), we get

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\ dx + \big\|u\big\|_{1+\alpha}^{1+\alpha} + \big\|\nabla v\big\|_2^2 + \big\|\nabla w\big\|_2^2\Big)$$
$$+\overline{C}_{42}\bigg(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v\big\|_2^2 + \big\|\Delta w\big\|_2^2\bigg)$$
$$\leq \overline{C}_{43}\big(1 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\big).$$

Integrating above with respect to t, we get (2.5).

Case (iii): $\alpha > 1$.

Multiplying $(2.1)_1$ by $\log u$ and then integrating, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + \frac{4}{1+\alpha}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 \le \int_{\mathbb{R}^3} \nabla u\cdot(\chi(v)\nabla v)\ dx - \int_{\mathbb{R}^3} \nabla u\cdot(\xi(w)\nabla w)\ dx. \qquad (2.36)$$

We estimate the RHS terms of above using the assumption $\chi^{'}, \xi^{'} \in L^\infty_{loc}$ as

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla u\cdot(\chi(v)\nabla v)\ dx &\le C_{44}\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_{44}\int_{\mathbb{R}^3} u|\Delta v|\ dx,\\ -\int_{\mathbb{R}^3} \nabla u\cdot(\xi(w)\nabla w)\ dx &\le C_{45}\int_{\mathbb{R}^3} u|\nabla w|^2\ dx + C_{45}\int_{\mathbb{R}^3} u|\Delta w|\ dx.\end{aligned}$$

Using above inequalities in (2.36), we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + \frac{4}{1+\alpha}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 \le C_{44}\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_{44}\int_{\mathbb{R}^3} u|\Delta v|\ dx + C_{45}\int_{\mathbb{R}^3} u|\nabla w|^2\ dx + C_{45}\int_{\mathbb{R}^3} u|\Delta w|\ dx. \qquad (2.37)$$

Multiplying $(2.1)_1$ by u^α and then integrating, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 \le \int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\chi(v)\nabla v)dx - \int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\xi(w)\nabla w)dx. \qquad (2.38)$$

From (2.17), we get

$$\int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\chi(v)\nabla v)\ dx \le C_{50}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C_{51}\int_{\mathbb{R}^3} u|\nabla v|^2\ dx, \qquad (2.39)$$

where $C_{50} = C(\chi)\epsilon_3$ and $C_{51} = C(\chi)C(\epsilon_3)$. Following the same procedure as in (2.39), we get

$$-\int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\xi(w)\nabla w)\ dx \le C_{52}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C_{53}\int_{\mathbb{R}^3} u|\nabla w|^2\ dx. \qquad (2.40)$$

Substituting (2.39) and (2.40) in (2.38), we get

$$\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{49}^{'}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 \le C_{51}\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_{53}\int_{\mathbb{R}^3} u|\nabla w|^2\ dx, \qquad (2.41)$$

where $C_{49}^{'} = \frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2} - C_{50} - C_{52}$. Adding (2.37), (2.14), (2.41) and (2.20), we get

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\ dx + \left\|u\right\|_{1+\alpha}^{1+\alpha} + \left\|\nabla v\right\|_2^2 + \left\|\nabla w\right\|_2^2\right)$$
$$+C_{54}\left(\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + \left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + \left\|\Delta v\right\|_2^2 + \left\|\Delta w\right\|_2^2\right)$$
$$\leq C_{55}\left(\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + \int_{\mathbb{R}^3} u|\nabla w|^2\ dx + \int_{\mathbb{R}^3} u|\Delta v|\ dx\right.$$
$$\left. + \int_{\mathbb{R}^3} u|\Delta w|\ dx - \int_{\mathbb{R}^3} v|\Delta v|\ dx - \int_{\mathbb{R}^3} w|\Delta w|\ dx\right). \tag{2.42}$$

As $\frac{1+\alpha}{2} > 1$, using Young's inequality, we have $u \leq \epsilon_{26} + C(\epsilon_{26})u^{\frac{1+\alpha}{2}}$. We estimate the first term in RHS of (2.42) using previous result with Young's inequality as

$$\int_{\mathbb{R}^3} u|\nabla v|^2 dx \leq \epsilon_{26}\|\nabla v\|_2^2 + C(\epsilon_{26})\int_{\mathbb{R}^3} u^{\frac{1+\alpha}{2}}|\nabla v|^2\ dx$$
$$=\epsilon_{26}\|\nabla v\|_2^2 + C(\epsilon_{26})\int_{\mathbb{R}^3} u^{\frac{1+\alpha}{2}}\nabla v\cdot\nabla v\ dx$$
$$\leq C(\epsilon_{26})C_{56}\left(\int_{\mathbb{R}^3} \nabla u^{\frac{1+\alpha}{2}}\cdot\nabla v + u^{\frac{1+\alpha}{2}}\Delta v\ dx\right) + \epsilon_{26}\|\nabla v\|_2^2$$
$$\leq C(\epsilon_{26})C_{56}\left(\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + \|\nabla v\|_2^2 + \|u\|_{1+\alpha}^{1+\alpha} + \|\Delta v\|_2^2\right) + \epsilon_{26}\|\nabla v\|_2^2. \tag{2.43}$$

Now, we estimate the third term of RHS of above. Using the Gagliardo-Nierenberg and Young inequality, we get

$$\|u\|_{1+\alpha}^{1+\alpha} \leq C_{57}\|u_0\|_1^{\frac{2+2\alpha}{2+3\alpha}}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^{\frac{6\alpha}{2+3\alpha}} \leq C_{57}C(\epsilon_{27}) + \epsilon_{27}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2.$$

Here, we have used the fact $\frac{6\alpha}{2+3\alpha} < 2$. Substituting above in (2.43), we get

$$\int_{\mathbb{R}^3} u|\nabla v|^2\ dx \leq C_{54}' + C_{55}'\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + C_{56}'\|\nabla v\|_2^2 + C_{57}'\|\Delta v\|_2^2, \tag{2.44}$$

where $C_{54}' = C(\epsilon_{26})C(\epsilon_{27})C_{56}C_{57}$, $C_{55}' = C(\epsilon_{26})C_{56} + C(\epsilon_{26})C_{56}\epsilon_{27}$, $C_{56}' = C(\epsilon_{26})C_{56} + \epsilon_{26}$ and $C_{57}' = C(\epsilon_{26})C_{56}$. Following the same procedure as previous estimate and from previous case, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} u|\nabla w|^2\ dx &\leq C_{58}' + C_{59}'\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + C_{60}'\|\nabla w\|_2^2 + C_{61}'\|\Delta w\|_2^2,\\
\int_{\mathbb{R}^3} u|\Delta v|\ dx &\leq C_{44} + C_{45}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + \epsilon_{23}\|\Delta v\|_2^2,\\
\int_{\mathbb{R}^3} u|\Delta w|\ dx &\leq C_{46} + C_{47}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + \epsilon_{24}\|\Delta w\|_2^2,\\
\int_{\mathbb{R}^3} v|\Delta v|\ dx &\leq C_{25}\|v_0\|_1^{\frac{8}{7}}\|\Delta v\|_2^{\frac{6}{7}} + \epsilon_{14}\|\Delta v\|_2^2,\\
\int_{\mathbb{R}^3} w|\Delta w|\ dx &\leq C_{26}\|w_0\|_1^{\frac{8}{7}}\|\Delta w\|_2^{\frac{6}{7}} + \epsilon_{15}\|\Delta w\|_2^2.
\end{aligned}$$

Substituting above estimates and (2.44) in (2.42), we have

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\ dx + \big\|u\big\|_{1+\alpha}^{1+\alpha} + \big\|\nabla v\big\|_2^2 + \big\|\nabla w\big\|_2^2\Big)$$
$$+\ \overline{C}_{54}\Big(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v\big\|_2^2 + \big\|\Delta w\big\|_2^2\Big)$$
$$\le \overline{C}_{55}\big(1 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\big).$$

Integrating above with respect to t, we get (2.5). □

Lemma 2. *Let $\beta, \gamma, \delta, \eta > 0$. Suppose that (u, v, w) is a classical solution of system* (2.1) *for all $\epsilon \in (0, 1)$ and initial data $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon})$ satisfies (i)–(iii) of Lemma 1, independent of ϵ. Furthermore, we assume that*

$$\alpha > 0 \ \text{and}\ \chi^{'}, \xi^{'} \in L^\infty_{loc} \ \text{with}\ \chi^{'}(\cdot) \ge \chi_0 \ \text{and}\ \xi^{'}(\cdot) \ge \xi_0 \tag{2.45}$$

for some constant $\chi_0, \xi_0 > 0$. Then, for any $0 < t \le T$

$$\sup_{0\le\tau\le t} E(\tau) + \int_0^t D(\tau)d\tau < C, \tag{2.46}$$

where $E(t)$ and $D(t)$ are defined as in (2.2) *and* (2.3) *respectively. Here, $C > 0$ a constant independent of ϵ and depends on the given data.*

Proof. It is sufficient to prove only for $0 < \alpha \le \frac{1}{6}$, as for $\alpha > \frac{1}{6}$ result is already shown in the Lemma 1. Multiplying $(2.1)_1$ by $\log u$ and then integrating, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + \frac{4}{1+\alpha}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2$$
$$\le \int_{\mathbb{R}^3} \nabla u\cdot(\chi(v)\nabla v)dx - \int_{\mathbb{R}^3} \nabla u\cdot(\xi(w)\nabla w)dx. \tag{2.47}$$

We estimate the first term in RHS of above using our assumption $\chi^{'}(\cdot) \ge \chi_0$ as

$$\int_{\mathbb{R}^3} \nabla u\cdot(\chi(v)\nabla v)dx = -\int_{\mathbb{R}^3} (\chi^{'}(v)|\nabla v|^2 + \chi(v)\Delta v)u\ dx$$
$$\le -\chi_0\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_{59}\int_{\mathbb{R}^3} |\nabla u||\nabla v|\ dx$$
$$\le -\chi_0\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_{60}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_{61}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx. \tag{2.48}$$

Following procedure as same in (2.48), we get

$$-\int_{\mathbb{R}^3} \nabla u\cdot(\xi(w)\nabla w)dx \ \le\ -\xi_0\int_{\mathbb{R}^3} u|\nabla w|^2\ dx + C_{60}^{'}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2$$
$$+C_{61}^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\ dx. \tag{2.49}$$

Using (2.48) and (2.49) in (2.47), we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + C_{62}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + \chi_0\int_{\mathbb{R}^3} u|\nabla v|^2 dx + \xi_0\int_{\mathbb{R}^3} u|\nabla w|^2 dx$$
$$\leq\ C_{61}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 dx + C_{61}^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2 dx, \tag{2.50}$$

where $C_{62}=\frac{4}{1+\alpha}-C_{60}$. Multiplying $(2.1)_1$ by u^α and then integrating, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2$$
$$\leq \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi(v)\nabla v)dx - \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi(w)\nabla w)dx. \tag{2.51}$$

We estimate the first term in RHS of above using Young's inequality as

$$\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi(v)\nabla v)\ dx \leq C(\chi)\int_{\mathbb{R}^3}\left|\nabla u^{\frac{1+2\alpha}{2}}\right|(u^{\frac{1}{2}}|\nabla v|)\ dx$$
$$\leq C(\chi)\epsilon_{27}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C(\chi)C(\epsilon_{27})\int_{\mathbb{R}^3} u|\nabla v|^2\ dx$$
$$\leq C(\chi)\epsilon_{27}\left(C(\epsilon_{28})\|\nabla v\|_2^2 + \epsilon_{28}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx\right) + C(\chi)\epsilon_{27}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2$$
$$= C_{63}^{'}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C_{64}^{'}\|\nabla v\|_2^2 + C_{65}^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx, \tag{2.52}$$

where $C(\chi)=\frac{2\alpha\overline{\chi}}{1+\alpha}$, $C_{63}^{'}=C(\chi)\epsilon_{27}$, $C_{65}^{'}=C(\chi)C(\epsilon_{27})C(\epsilon_{28})$ and $C_{64}^{'}=C(\chi)C(\epsilon_{27})\epsilon_{28}$. Following the procedure as similar as above, we get

$$-\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi(w)\nabla w)\ dx$$
$$\leq C_{66}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + C_{67}^{'}\|\nabla w\|_2^2 + C_{68}^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\ dx. \tag{2.53}$$

Using (2.52) and (2.53) in (2.51), we get

$$\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{67}\left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 \leq C_{64}^{'}\|\nabla v\|_2^2 + C_{65}^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx$$
$$+ C_{67}^{'}\|\nabla w\|_2^2 + C_{68}^{'}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\ dx, \tag{2.54}$$

where $C_{67}=\frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2}-C_{63}-C_{66}$. As α does not affect $(2.1)_2$, and $(2.1)_3$, from previous case calculation, we have

$$\frac{d}{dt}\left(\left\|\nabla v\right\|_2^2 + \left\|\nabla w\right\|_2^2\right) + 2\left(\left\|\Delta v\right\|_2^2 + \left\|\Delta w\right\|_2^2\right)$$
$$\leq \beta\int_{\mathbb{R}^3} u|\Delta v|\ dx - \gamma\int_{\mathbb{R}^3} v|\Delta v|\ dx + \delta\int_{\mathbb{R}^3} u|\Delta w|\ dx - \eta\int_{\mathbb{R}^3} w|\Delta w|\ dx.$$

Adding (2.50), (2.14), (2.54) and above estimate, we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\ dx + \left\|u\right\|_{1+\alpha}^{1+\alpha} + \left\|\nabla v\right\|_2^2 + \left\|\nabla w\right\|_2^2\right)\\ &\quad + C_{67}\left(\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + \left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + \left\|\Delta v\right\|_2^2 + \left\|\Delta w\right\|_2^2\right)\\ &\leq C_{68}\left(\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2 dx + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\right.\\ &\quad \left. + \int_{\mathbb{R}^3} u|\Delta v| dx + \int_{\mathbb{R}^3} u|\Delta w| dx - \int_{\mathbb{R}^3} v|\Delta v| dx - \int_{\mathbb{R}^3} w|\Delta w| dx\right). \qquad (2.55)\end{aligned}$$

We estimate the first term in RHS of above using $u^{1-\alpha} \leq C(\epsilon_{28}) + \epsilon_{28} u$ and choosing sufficiently small ϵ_{28}, we get

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx \leq C_{69}\|\nabla v\|_2^2. \qquad (2.56)$$

Following the same procedure as above, we get

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\ dx \leq C_{70}\|\nabla w\|_2^2. \qquad (2.57)$$

Substituting (2.56), (2.57) and (2.23)–(2.24) in (2.55), we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\ dx + \left\|u\right\|_{1+\alpha}^{1+\alpha} + \left\|\nabla v\right\|_2^2 + \left\|\nabla w\right\|_2^2\right)\\ &\quad + \overline{C}_{67}\left(\int_{\mathbb{R}^3} u|\nabla v| dx + \left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + \left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_2^2 + \left\|\Delta v\right\|_2^2 + \left\|\Delta w\right\|_2^2\right)\\ &\leq\ \overline{C}_{68}\left(1 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\right).\end{aligned}$$

Integrating above with respect to t, we get (2.46). □

3 Existence of weak solution

In this section, we prove the global weak solution for (1.4) using Lemmas 1–2 proved in the previous section. Furthermore, we extend the same for arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

Theorem 1. *Let $\beta, \gamma, \delta, \eta$ be positive constants. Suppose that* (2.4) *or* (2.45) *hold true and initial data (u_0, v_0, w_0) satisfy*

i) $u_0(1 + |x| + |\log u_0|) \in L^1(\mathbb{R}^3)$,

ii) $u_0 \in L^{1+\alpha}(\mathbb{R}^3)$,

iii) v_0 *and* $w_0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$.

Then, for each $T > 0$, there exists a weak solution (u, v, w) for (1.4) such that

$$\sup_{0\le t\le T}\Big(\int_{\mathbb{R}^3} u(t)(|\log u(t)|+2\langle x\rangle)\ dx+\big\|u(t)\big\|_{L^{1+\alpha}}^{1+\alpha}+\big\|\nabla v(t)\big\|_{L^2}^2+\big\|\nabla w(t)\big\|_{L^2}^2\Big)$$
$$+\int_0^T\Big(\big\|\nabla u^{\frac{1+\alpha}{2}}(t)\big\|_{L^2}^2+\big\|\nabla u^{\frac{1+2\alpha}{2}}(t)\big\|_{L^2}^2+\big\|\Delta v(t)\big\|_{L^2}^2+\big\|\Delta w(t)\big\|_{L^2}^2\Big)dt < C,$$

$C = C(T, \|u_0\|_{L^1\cap L^{1+\alpha}}, \|u_0 \log u_0\|_{L^1}, \|u_0\langle x\rangle\|_{L^1}, \|\nabla v_0\|_{L^1}, \|\nabla w_0\|_{L^1})$.

Proof. The uniformity of the estimates obtained in Lemmas 1 and 2, is independent of ϵ, is ensured by the convergence of $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon})$, for $\epsilon \in (0, 1)$. That is, the constant C in (2.5) chosen independent of ϵ. In the similar manner, there exists constant $C > 0$ such that for $q < \infty$

$$\begin{aligned}
&\|u_\epsilon\|_{L^\infty((0,T)\times\mathbb{R}^3)} + \big\|\nabla u_\epsilon^{\frac{q+\alpha}{2}}\big\|_{L^2((0,T)\times\mathbb{R}^3)} < C,\\
&\|v_\epsilon\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))} + \|v_\epsilon\|_{L^q(0,T;W^{2,q}(\mathbb{R}^3))} + \|v_{\epsilon_t}\|_{L^q(0,T;L^q(\mathbb{R}^3))} < C,\\
&\|w_\epsilon\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))} + \|w_\epsilon\|_{L^q(0,T;W^{2,q}(\mathbb{R}^3))} + \|w_{\epsilon_t}\|_{L^q(0,T;L^q(\mathbb{R}^3))} < C.
\end{aligned}$$

Through derived estimates, the obtained local solution extended to any given time $T > 0$ (as in [6, 8, 18, 26]). Let $k \geq 2 + \alpha$ be chosen. Then u_{ϵ_t} and $u^k_{\epsilon_t} \in L^1(0, T; W^{-2,2}(\mathbb{R}^3))$(as in [18]), where $W^{-2,2}(\mathbb{R}^3)$ denotes the dual space of $W^{2,2}(\mathbb{R}^3)$. Then by Aubin-Lions compactness lemma, we have a weak limit (u, v, w) as $\epsilon \to 0$ which is a weak solution. □

The above theorem can also be proved to the bounded domain in $\mathbb{R}^n$ with the Neumann boundary conditions for u, v and w. Precisely, let Ω be a bounded domain with smooth boundary and consider the system (2.1) in $\Omega \times [0, T)$ with boundary conditions as

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0. \tag{3.1}$$

Theorem 2. *Let $\beta, \gamma, \delta, \eta$ be positive constants. Suppose that (2.4) or (2.45) hold true and initial data (u_0, v_0, w_0) satisfy*

i) $u_0 \in L^1(\Omega) \cap L^{1+\alpha}(\Omega)$,

ii) v_0 *and* $w_0 \in L^\infty(\Omega) \cap H^1(\Omega)$.

Then, for each $T > 0$, there exists a weak solution (u, v, w) for system (1.4) with boundary conditions (3.1) and it satisfies

$$\sup_{0\le t\le T}\Big(\int_{\Omega} u(t)|\log u(t)|\ dx+\big\|u(t)\big\|_{L^{1+\alpha}}^{1+\alpha}+\big\|\nabla v(t)\big\|_{L^2}^2+\big\|\nabla w(t)\big\|_{L^2}^2\Big)$$
$$+\int_0^T\Big(\big\|\nabla u^{\frac{1+\alpha}{2}}(t)\big\|_{L^2}^2+\big\|\nabla u^{\frac{1+2\alpha}{2}}(t)\big\|_{L^2}^2+\big\|\Delta v(t)\big\|_{L^2}^2+\big\|\Delta w(t)\big\|_{L^2}^2\Big)dt < C,$$

where $C = C(T, \|u_0\|_{L^1\cap L^{1+\alpha}}, \|u_0 \log u_0\|_{L^1}, \|\nabla v_0\|_{L^1}, \|\nabla w_0\|_{L^1})$.

Proof. As proof of the theorem is similar to the above, therefore, we give a sketch of the proof. Since, negative part of $\int_{\Omega} u \log u$ is controlled by $\|u\|_{L^1(\Omega)}$, L^1 estimate of $u\langle x\rangle$ (2.15) is not needed. Also, the Gagliardo-Nirenberg inequality modified slightly for the case of bounded domains, that is, (2.23) can be replaced by

$$\|u\|_2^2 \leq C_{21}\|u_0\|_{L^1(\Omega)}^{\frac{1+6\alpha}{2+6\alpha}} \left\|\nabla u^{\frac{1+2\alpha}{2}}\right\|_{L^2(\Omega)}^{\frac{6}{2+6\alpha}} + C_{21}\|u\|_{L^1(\Omega)}^2.$$

□

4 Conclusions

We have proposed a new system that models an attraction-repulsion-chemotaxis with a nonlinear diffusion exponent of $1+\alpha$ in an unbounded domain. The inclusion of nonlinear diffusion in the system captures the impact of the adhesive nature of the cell in its movement reacting to chemical signals, making it more realistic. The existence of a global-in-time weak solution for the proposed system is established for any $\alpha > 0$, and this result is extended to a bounded domain with a smooth boundary.

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