Mathematical Modelling and Analysis
Volume 29, Issue 3, 480–492, 2024
https://doi.org/10.3846/mma.2024.19588

A Note on Fractional-Type Models of Population Dynamics

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Received July 23, 2023; accepted December 19, 2023

Abstract. The fractional exponential function is considered. General expansions in fractional powers are used to solve fractional population dynamics problems. Laguerre-type exponentials are also considered, and an application to Laguerre-type fractional logistic equation is shown.

Keywords: fractional exponential function, Mittag-Leffler functions and generalizations, fractional population dynamic models, Laguerre-type exponentials, fractional Laguerre-type models.

AMS Subject Classification: 26A33; 33B10; 33E12; 34A08; 92D25.

1 Introduction

Significant efforts have been devoted to the study of fractional derivative operators and their applications to numerous scientific fields. There are different definitions of this form of differentiation. The different types of this operator have been examined and compared in the book by S. Samko, A.A. Kilbas and O. Marichev [29], where applications of fractional calculus to ordinary and partial differential equations are also shown. One can also see the works [19, 20, 21, 22, 23, 25] by V. Kyriakova, the work of Gorenflo-Mainardi [17] as well as Mainardi et al. [24], and the articles [15, 31]. An elementary approach, based on Euler's classical definition, which falls under Caputo's [2] and makes use of power series with fractional exponent has been considered in recent articles [4,18]. According to that method, it is natural to consider, for the exponential function, an expansion in fractional powers of a fixed number α, with $0 < \alpha < 1$.

The introduction of such a function, which verifies the invariant property of the classical exponential in relation to the fractional derivative of the same order, makes it possible to extend to the fractional field many of the families of special polynomials. What's more, the technique of fractional power series expansions allows solutions to some differential problems of population dynamics. Several articles in this direction appeared in recent times (see e.g., [6,12,13,30]). A different approach can be found in [5].

Another possible generalization of the exponential has been considered in the past in connection with the Laguerre-type derivative $D_L = DxD$ and its iterates as $D_{nL} = DxDxDx \cdots DxD$, [7,26,28].

However, the Laguerre-type derivatives are not completely new, since they can be considered as particular cases of the hyper-Bessel differential operators when $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 1$ (the special case considered in operational calculus by Ditkin and Prudnikov [10]). In general, the *Bessel-type differential operators of arbitrary order n* were introduced by Dimovski, in 1966 [9] and later called by Kiryakova *hyper-Bessel operators*, because are closely related to their eigenfunctions, called hyper-Bessel by Delerue [8], in 1953. These operators were studied in 1994 by Kiryakova in her book [25], Ch. 3.

Since the Laguerre-type exponentials, on the positive semi-axis of the abscissas, are convex increasing functions, with a growth lower than the exponential one, a natural application was made in the context of population dynamics [3]. This topic is combined with the new type of fractional exponential, and a fractional Laguerrian model is presented in the final section.

2 The fractional exponentials

In what follows, we use the Euler definition for the fractional derivative of powers, which falls as a special case of the fractional derivative introduced by Caputo,

$$\begin{cases} D_t^\nu t^n = \dfrac{\Gamma(n+1)}{\Gamma(n+1-\nu)}\, t^{n-\nu}, \quad (n \geq [\nu]), \\ D_t^\nu t^n = 0, \quad (n < [\nu]), \end{cases}$$

where n and ν are rational numbers, and $[\nu]$ denotes the integral part of ν. If c is a constant, then $D_t^\nu c = 0$.

We recall that the Caputo derivative is defined as follows [2]

$$D_{a+}^\nu f(t) = \frac{1}{\Gamma(n-\nu)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu-n+1}}\, d\tau, \quad (n \geq [\nu]),$$

and reduces to the preceding equation when $a = 0$ and $f(t) = t^n$.

A natural extension of the exponential function in the framework of fractional Taylor expansions is given, for any α, such that $0 < \alpha < 1$, by

$$\mathrm{Exp}_\alpha(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots .$$

Note that this function satisfies, for any fixed complex number x,

$$D_t^\alpha \mathrm{Exp}_\alpha(xt) = x^\alpha \, \mathrm{Exp}_\alpha(xt) . \tag{2.1}$$

The same happens interchanging t and x in the Equation (2.1).

For any α, such that $0 < \alpha < 1$, this expansion is convergent in the whole complex plane, as the same holds for the classical exponential $\exp(t)$. Actually, this is the classical Mittag-Leffler function $E_\alpha(t^\alpha)$, but as we deal with populatiton dynamics problems in which exponential functions are involved, it is convenient to mantain an exponential-type symbol.

In particular, for $\alpha = 1/2$, we find

$$\mathrm{Exp}_{1/2}(t) = E_{1/2}(t^{1/2}) = 1 + \frac{t^{1/2}}{\Gamma(3/2)} + \frac{t}{\Gamma(2)} + \cdots + \frac{t^{n/2}}{\Gamma(n/2+1)} + \cdots .$$
$$D_t^{1/2} \mathrm{Exp}_{1/2}(xt) = x^{1/2} \, \mathrm{Exp}_{1/2}(xt) .$$

Remark 1. The Mittag-Leffler function writes, in general

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)} , \quad \forall\, x \in \mathbf{C}, \forall\, \alpha, \beta \in \mathbf{R}^+ ,$$

and for $\beta = 1$

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)} , \quad \forall\, x \in \mathbf{C} .$$

It results $E_\alpha(x^\alpha) = \mathrm{Exp}_\alpha(x)$ and $D_x^\alpha E_\alpha(x^\alpha) = E_\alpha(x^\alpha)$, so that the fractional exponentials reduce to particular cases of the Mittag-Leffler function. In particular, we have:

$$E_{1/2}(x^{1/2}) = \sum_{n=0}^{\infty} \frac{x^{n/2}}{\Gamma(n/2+1)} = \mathrm{Exp}_{1/2}(x) .$$

Asymptotics for a variant of the Mittag-Leffler function are contained in [16], and extensions of the same function to the multi-index case can be found in [27].

3 The fractional Malthus model

Consider the fractional Cauchy's initial value problem

$$\begin{cases} D_t^\alpha P(t) = \gamma^\alpha P(t), \\ P(0) = P_0 , \end{cases}$$

where $0 < \alpha < 1$, and γ is a positive constant. The solution is given by $P(t) = \mathrm{Exp}_\alpha(\gamma t)P_0$. In fact,

$$D_t^\alpha \mathrm{Exp}_\alpha(\gamma t)P_0 = \gamma^\alpha \mathrm{Exp}_\alpha(t)P_0 = \gamma^\alpha P(t)\,.$$

As the parameter α is less than 1, the population number $P(t)$ growth is less than exponential growth.

This classical topic was recently considered in the fractional framework [1], where an application to modelling World population growth was examined. See also [30] and the references therein.

4 The fractional-order logistic equation

We consider the fractional-order logistic initial value problem [14]

$$\begin{cases} D_t^\alpha P(t) = r\,P(t)\left[1 - \frac{1}{K}\,P(t)\right], \quad (0<\alpha<1)\,, \\ P(0) = p_0\,. \end{cases} \tag{4.1}$$

The expansion in fractional powers, introduced in the preceding sections, is exploited to derive the following result.

Theorem 1. *Putting*

$$P(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha\, n+1)}\,, \tag{4.2}$$

the solution of the fractional-order logistic initial value problem in Equation (4.1) *is obtained by computing the a_n coefficients through the following recursion*

$$\begin{cases} a_0 = p_0, \\ a_{n+1} = r\left[a_n - \dfrac{1}{K}\displaystyle\sum_{k=0}^{n}, \frac{a_k\, a_{n-k}\,\Gamma(n\,\alpha+1)}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha\, k+1)}\right]. \end{cases} \tag{4.3}$$

Proof. As it results

$$D_t^\alpha P(t) = \sum_{n=1}^{\infty} a_n \frac{t^{(n-1)\,\alpha}}{\Gamma((n-1)\,\alpha+1)} = \sum_{n=0}^{\infty} a_{n+1} \frac{t^{n\,\alpha}}{\Gamma(n\,\alpha+1)}\,,$$

$$P(t)\cdot\frac{1}{K}\,P(t) = \frac{1}{K}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, a_{n-k}\frac{t^{n\,\alpha}}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha\,k+1)}\,,$$

substituting into Equation (4.1) we find

$$\begin{aligned} &\sum_{n=0}^{\infty} a_{n+1}\frac{t^{n\,\alpha}}{\Gamma(n\,\alpha+1)} \\ =&r\left[\sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha\,n+1)} - \frac{1}{K}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, a_{n-k}\frac{t^{n\,\alpha}}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha\,k+1)}\right] \\ =&r\sum_{n=0}^{\infty}\left[\frac{a_n}{\Gamma(n\,\alpha+1)} - \frac{1}{K}\sum_{k=0}^{n}\frac{a_k\,a_{n-k}}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha\,k+1)}\right] t^{n\,\alpha}\,, \end{aligned}$$

so that the recursion (4.3) for the a_n coefficients follows. □

Example 1. Assuming $\alpha = 0.75, r = 1, K = 1.5$, and putting $y_0 = 0.5$, we find the a_n $(0 \le n \le 20)$ coefficients reported in Table 1.

Table 1. The a_n coefficients for $0 \le n \le 20$.

0.5, 0.333333, 0.111111, -0.0795398, -0.129554, 0.019401, 0.237329, 0.141486, -0.548324, -1.00183, 1.21113, 6.08445, 0.512067, -37.0785, -52.3417, 215.583, 763.847, -874.775, -9288.89, -5554.9, 103580.0

The corresponding graph is shown in Figure 1.

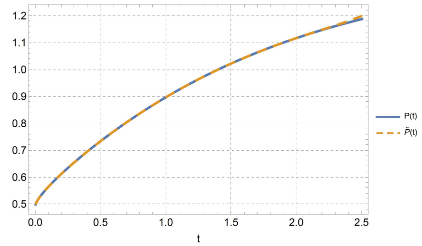

Figure 1. Graph of the solution, using the a_n coefficients, *vs* the approximation using the predictor-corrector method.

5 The minimum threshold fractional logistics model

We consider the fractional-order logistic initial value problem

$$\begin{cases} D_t^\alpha P(t) = r\,P(t)\,[1 - P(t)/K]\,[1 - m/P(t)]\,, \quad (0 < \alpha < 1)\,, \\ P(0) = p_0\,, \end{cases}$$

where m denotes the minimum population threshold that guarantees the population survival. Since

$$D_t^\alpha P(t) = r\,\left[P(t) + mP(t)/K - P^2(t)/K - m\right]\,, \tag{5.1}$$

considering the expansion of $P(t)$ (equation (4.2) in Theorem 4.1), we can state the theorem

Theorem 2. *The solution of the minimum threshold fractional logistics model is obtained by computing the a_n coefficients through the following recursion*

$$\begin{cases} a_0 = p_0, \quad a_1 = r\left[\left(1 + \dfrac{m}{K}\right)p_0 - \dfrac{1}{K}p_0^2 - m\right], \\ a_{n+1} = r\,\left[\left(1 + \dfrac{m}{K}\right)a_n - \dfrac{1}{K}\displaystyle\sum_{k=0}^{n} \dfrac{a_k\,a_{n-k}\,\Gamma(n\,\alpha + 1)}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha\,k + 1)}\right], \ (n \ge 1)\,. \end{cases}$$

Proof. Substituting the fractional expansion (4.2) of $P(t)$ into Equation (5.1), we find

$$\sum_{n=0}^{\infty} a_{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = r \left[\left(1+\frac{m}{K}\right) \sum_{n=0}^{\infty} a_n \frac{t^{n\alpha}}{\Gamma(\alpha n+1)} - \frac{1}{K} \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k\, a_{n-k} \frac{t^{n\alpha}}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha k+1)} \right] - r\,m\,,$$

that is

$$\sum_{n=0}^{\infty} a_{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - r\left(1+\frac{m}{K}\right) \sum_{n=0}^{\infty} a_n \frac{t^{n\alpha}}{\Gamma(\alpha n+1)} + \frac{r}{K} \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k\, a_{n-k} \frac{t^{n\alpha}}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha k+1)} = -r\,m\,,$$

By considering first the monomial when $n = 0$, and then equating the monomials with the same fractional exponent, we find the result. □

Example 2. Assuming $\alpha = 0.5, \mu = 0.1, r = 0.2, K = 1.0$, and putting $y_0 = 0.5$, we find the a_n $(0 \le n \le 40)$ coefficients reported in Table 2.

Table 2. The a_n coefficients for $0 \le n \le 40$.

0.5, 0.04, 0.0008, -0.000391437, -0.0000270287, 9.8358×10^{-6}, 1.32073×10^{-6}, -3.72283×10^{-7}, -8.32023×10^{-8}, 1.8003×10^{-8}, 6.36674×10^{-9}, -1.0045×10^{-9}, -5.69914×10^{-10}, 5.7712×10^{-11}, 5.8111×10^{-11}, -2.53961×10^{-12}, -6.61331×10^{-12}, -1.16546×10^{-13}, 8.26082×10^{-13}, 7.07962×10^{-14}, -1.11595×10^{-13}, -1.83611×10^{-14}, 1.60719×10^{-14}, 4.18353×10^{-15}, -2.42948×10^{-15}, -9.32435×10^{-16}, 3.77902×10^{-16}, 2.10285×10^{-16}, -5.86548×10^{-17}, -4.86023×10^{-17}, 8.53499×10^{-18}, 1.15645×10^{-17}, -9.57398×10^{-19}, -2.83428×10^{-18}, -2.00953×10^{-20}, 7.14197×10^{-19}, 7.13301×10^{-20}, -1.8436×10^{-19}, -3.71495×10^{-20}, 4.84798×10^{-20}, 1.54998×10^{-20}

The corresponding graph is shown in Figure 2.

6 The fractional Volterra-Lotka model

Let us consider the fractional Volterra-Lotka model

$$\begin{cases} D_t^{\alpha} X(t) = X(t)[s-\gamma Y(t)], \\ D_t^{\alpha} Y(t) = Y(t)[\beta X(t) - r]. \end{cases} \tag{6.1}$$

The orbits are given by the parametric equations $X = X(t), Y = Y(t)$ with $X(0) = x_0$, $Y(0) = y_0$, respectively. The equilibrium point is the same as the classical mode, that is:

$$X_0 = r/\beta, \quad Y_0 = s/\gamma.$$

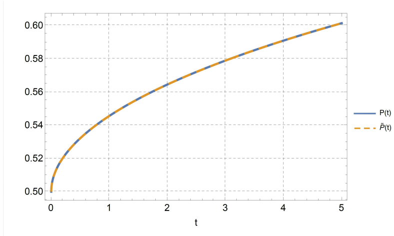

Figure 2. Graph of the solution, using the a_n coefficients, *vs* the approximation using the predictor-corrector method.

The following theorem holds true.

Theorem 3. *Upon putting*

$$X(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)}, \quad Y(t) = \sum_{n=0}^{\infty} b_n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)},$$

the solution of the fractional Volterra-Lotka model (6.1) *under the initial conditions* $X(0) = x_0$ *and* $Y(0) = y_0$ *is obtained by computing the coefficients* a_n *and* b_n *through the following recursive formulas*

$$\begin{cases} a_0 = x_0, \quad b_0 = y_0, \\ a_{n+1} = s\, a_n - \gamma\, \Gamma(n\,\alpha + 1) \displaystyle\sum_{k=0}^{n} \frac{a_k\, b_{n-k}}{\Gamma(\alpha\, k + 1)\, \Gamma(\alpha(n-k) + 1)}, \\ b_{n+1} = -r\, b_n + \beta\, \Gamma(n\,\alpha + 1) \displaystyle\sum_{k=0}^{n} \frac{a_k\, b_{n-k}}{\Gamma(\alpha\, k + 1)\, \Gamma(\alpha(n-k) + 1)}. \end{cases} \tag{6.2}$$

Proof. As it results

$$D_t^{\alpha} X(t) = \sum_{n=0}^{\infty} a_{n+1} \frac{t^{n\,\alpha}}{\Gamma(n\,\alpha + 1)}, \quad D_t^{\alpha} Y(t) = \sum_{n=0}^{\infty} b_{n+1} \frac{t^{n\,\alpha}}{\Gamma(n\,\alpha + 1)},$$

$$X(t) \cdot Y(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k\, b_{n-k} \frac{t^{n\,\alpha}}{\Gamma(\alpha\, k + 1)\, \Gamma(\alpha(n-k) + 1)},$$

upon substituting into Equation (6.1) we find

$$\begin{aligned} &\sum_{n=0}^{\infty} a_{n+1} \frac{t^{n\,\alpha}}{\Gamma(n\,\alpha + 1)} \\ &= s \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha\, n + 1)} - \gamma \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k\, b_{n-k} \frac{t^{n\,\alpha}}{\Gamma(\alpha\, k + 1)\, \Gamma(\alpha(n-k) + 1)} \\ &= \sum_{n=0}^{\infty} \left[s \frac{a_n}{\Gamma(n\,\alpha + 1)} - \gamma \sum_{k=0}^{n} \frac{a_k\, b_{n-k}}{\Gamma(\alpha\, k + 1)\, \Gamma(\alpha(n-k) + 1)} \right] t^{n\,\alpha}, \end{aligned}$$

$$\sum_{n=0}^{\infty} b_{n+1}\,\frac{t^{n\,\alpha}}{\Gamma(n\,\alpha+1)}$$
$$= -r\sum_{n=0}^{\infty} b_n\,\frac{t^{\alpha n}}{\Gamma(\alpha\,n+1)} + \beta\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\,b_{n-k}\,\frac{t^{n\,\alpha}}{\Gamma(\alpha\,k+1)\,\Gamma(\alpha(n-k)+1)}$$
$$= \sum_{n=0}^{\infty}\left[-r\,\frac{b_n}{\Gamma(n\,\alpha+1)} + \beta\sum_{k=0}^{n}\frac{a_k\,b_{n-k}}{\Gamma(\alpha\,k+1)\,\Gamma(\alpha(n-k)+1)}\right] t^{n\,\alpha}\,,$$

so that the recursion (6.2) for the coefficients a_n and b_n follows. □

6.1 Numerical results

Assuming $\alpha = 0.75, r = 1.0, s = 1.0, \beta = 2.0, \gamma = 2.0$, and putting $x_0 = 0.5, y_0 = 0.75$, we find the a_n and b_n coefficients reported in Tables 3 and 4.

Table 3. The a_n coefficients for $0 \le n \le 100$.

0.5, -0.25, 0.125, 0.3125, -0.90625, 0.359375, 6.57031, -22.668, -28.4629, 543.257,
-1380.2, -10538.3, 97130.2, -22377.8, -4.70768×10^{6}, 2.73618×10^{7}, 1.47377×10^{8},
-2.95171×10^{9}, 7.12904×10^{9}, 2.2628×10^{11}, -2.39001×10^{12}, -7.69477×10^{12},
3.63224×10^{14}, -1.96953×10^{15}, -3.76386×10^{16}, 6.63894×10^{17}, 4.34674×10^{17},
-1.31595×10^{20}, 1.24262×10^{21}, 1.61746×10^{22}, -4.70681×10^{23}, 1.15781×10^{24},
1.14212×10^{26}, -1.69378×10^{27}, -1.42465×10^{28}, 7.30506×10^{29}, -4.57342×10^{30},
-2.04896×10^{32}, 4.56169×10^{33}, 1.88959×10^{34}, -2.21651×10^{36}, 2.42486×10^{37},
6.76272×10^{38}, -2.24302×10^{40}, 5.53085×10^{39}, 1.20445×10^{43}, -2.02428×10^{44},
-3.6925×10^{45}, 1.88509×10^{47}, -9.48764×10^{47}, -1.09122×10^{50}

Table 4. The b_n coefficients for $0 \le n \le 100$.

0.75, 0.0, -0.375, 0.1875, 1.03125, -2.29688, -3.91406, 33.1523, -27.3574, -544.362,
2467.82, 6690.27, -114359.0, 233867.0, 4.45144×10^{6}, -3.65209×10^{7}, -8.34939×10^{7},
3.18258×10^{9}, -1.32633×10^{10}, -2.05887×10^{11}, 2.82218×10^{12}, 2.48258×10^{12},
-3.73401×10^{14}, 2.70615×10^{15}, 3.29629×10^{16}, -7.34496×10^{17}, 9.63715×10^{17},
1.31066×10^{20}, -1.50528×10^{21}, -1.34267×10^{22}, 5.00282×10^{23}, -2.12877×10^{24},
-1.10925×10^{26}, 1.91891×10^{27}, 1.06338×10^{28}, -7.55386×10^{29}, 6.05931×10^{30},
1.94263×10^{32}, -4.96085×10^{33}, -9.37331×10^{33}, 2.24478×10^{36}, -2.87099×10^{37},
-6.23313×10^{38}, 2.37298×10^{40}, -5.16908×10^{40}, -1.19872×10^{43}, 2.26459×10^{44},
3.26361×10^{45}, -1.95465×10^{47}, 1.33274×10^{48}, 1.06841×10^{50}

The corresponding graph is shown in Figure 3.

7 The fractional Laguerre-type exponentials

In this section, we consider the fractional Laguerre-type exponential

$$_{L_1}\mathrm{Exp}_{1/2} := E_{(1/2,1/2),(1,1)}(\sqrt{t})\,,$$

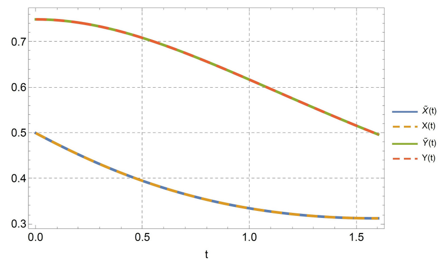

Figure 3. Graph of the solution, using the a_n and b_n coefficients, *vs* the approximation using the predictor-corrector method.

where $E_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)}(t)$ denotes the 2×2 Mittag-Leffler function by M.M. Dzrbashjan [11]. Since

$$\begin{aligned} {}_{L_1}\mathrm{Exp}_{1/2}(t) =& 1 + \frac{1}{[\Gamma(3/2)]^2} t^{1/2} + \frac{1}{[\Gamma(2)]^2} t \\ &+ \frac{1}{[\Gamma(5/2)]^2} t^{3/2} + \cdots + \frac{1}{[\Gamma(n/2+1)]^2} t^{n/2} + \dots , \end{aligned} \tag{7.1}$$

introducing the fractional order hyper-Bessel differential operator $D^{1/2}x^{1/2}D^{1/2}$ [25], it results

$$D^{1/2}\, x^{1/2}\, D^{1/2}\, {}_{L_1}\mathrm{Exp}_{1/2}(t) = {}_{L_1}\mathrm{Exp}_{1/2}(t)\,.$$

More generally, putting

$${}_{L_n}\mathrm{Exp}_{1/2} := E^n_{(\frac{1}{2},\frac{1}{2}),(1,1)}(\sqrt{t})\,,$$

where

$$E^n_{(\alpha_i),(\beta_i)}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\prod_{i=1}^{n} \Gamma(\alpha_i k + \beta_i)}\,, \quad t \in \mathbf{C},\ \alpha_i > 0,\ \beta_i \in \mathbf{R},$$

we find

$$\begin{aligned} {}_{L_n}\mathrm{Exp}_{1/2}(t) =& 1 + \frac{1}{[\Gamma(3/2)]^{n+1}}\, t^{1/2} + \frac{1}{[\Gamma(2)]^{n+1}}\, t \\ &+ \frac{1}{[\Gamma(5/2)]^{n+1}}\, t^{3/2} + \cdots + \frac{1}{[\Gamma(n/2+1)]^{n+1}}\, t^{n/2} + \dots , \end{aligned}$$

and considering the hyper-Bessel differential operator, containing $n+1$ fractional derivatives, it results

$$D^{1/2}\, x^{1/2}\, D^{1/2}\, x^{1/2}\, D^{1/2}\ \cdots\ x^{1/2}\, D^{1/2}\, {}_{L_n}\mathrm{Exp}_{1/2}(t) = {}_{L_n}\mathrm{Exp}_{1/2}(t)\,.$$

In what follows, we use the typographical more simple notation considered in Equation (7.1).

8 The Laguerre-type fractional-order logistic equation

We consider the Laguerre-type fractional-order logistic initial value problem

$$\begin{cases} D_t^{1/2} t^{1/2} D_t^{1/2} P(t) = r\, P(t) \left[1 - \frac{1}{K}\ P(t)\right] , \\ P(0) = p_0 \,. \end{cases}$$

We prove the result

Theorem 4. *Putting*

$$P(t) = \sum_{n=0}^{\infty} a_n \, \frac{t^{n/2}}{\Gamma(n/2+1)} \,,$$

the solution of the considered Laguerre-type fractional-order logistic initial value problem is obtained computing the a_n coefficients using the recursion

$$\begin{cases} a_0 = p_0, \\ a_{n+1} = r \left[a_n \, \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+3}{2})} - \frac{1}{K} \sum_{k=0}^{n} \frac{a_k \, a_{n-k} \, [\Gamma(\frac{n+2}{2})]^2}{\Gamma(\frac{n+3}{2})\, \Gamma(\frac{n-k}{2}+1)\, \Gamma(\frac{k}{2}+1)} \right] . \end{cases}$$

Proof. Using the fractional differentiation, we find

$$D_t^{1/2} t^{1/2} D_t^{1/2} P(t) = \sum_{n=0}^{\infty} a_{n+1} \, \frac{\Gamma(\frac{n+3}{2})\, t^{n/2}}{[\Gamma(\frac{n+2}{2})]^2} \,,$$

$$P(t)\, \frac{1}{K}\ P(t) = \frac{1}{K} \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \, a_{n-k} \, \frac{t^{n/2}}{\Gamma(\frac{n-k}{2}+1)\, \Gamma(\frac{k}{2}+1)} \,.$$

Substituting into the equation, we find

$$\begin{aligned} &\sum_{n=0}^{\infty} a_{n+1} \, \frac{\Gamma(\frac{n+3}{2})\, t^{n/2}}{[\Gamma(\frac{n+2}{2})]^2} \\ &= r \left[\sum_{n=0}^{\infty} a_n \, \frac{t^{n/2}}{\Gamma(\frac{n+2}{2})} - \frac{1}{K} \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \, a_{n-k} \, \frac{t^{n/2}}{\Gamma(\frac{n-k}{2}+1)\, \Gamma(\frac{k}{2}+1)} \right] \\ &= r \sum_{n=0}^{\infty} \left[\frac{a_n}{\Gamma(n/2+1)} - \frac{1}{K} \sum_{k=0}^{n} \frac{a_k \, a_{n-k}}{\Gamma(\frac{n-k}{2}+1)\, \Gamma(\frac{k}{2}+1)} \right] t^{n/2} \,, \end{aligned}$$

so that the recursion for the a_n coefficients follows. □

8.1 Numerical results

Assuming $r = 0.5, K = 2.0$, and putting $y_0 = 1.6$, we find the a_n coefficients reported in Table 5. The corresponding graph is shown in Figure 4.

Table 5. The a_n coefficients for $0 \le n \le 80$.

1.6, 0.324973, -0.04032, -0.0159838, 0.00671283, 0.0011658, -0.0013536, 0.0000165536, 0.000308317, -0.0000683558, -0.0000738482, 0.0000385159, 0.0000163458, -0.0000184599, -1.85623×10^{-6}, 8.464187×10^{-6}, -1.459437×10^{-6}, -3.73171×10^{-6}, 1.83676×10^{-6}, 1.49532×10^{-6}, -1.51334×10^{-6}, -4.31732×10^{-7}, 1.09636×10^{-6}, -7.18573×10^{-8}, -7.33593×10^{-7}, 3.01059×10^{-7}, 4.44805×10^{-7}, -3.91019×10^{-7}, -2.18061×10^{-7}, 4.06177×10^{-7}, 3.73562×10^{-8}, -3.76615×10^{-7}, 1.10092×10^{-7}, 3.14918×10^{-7}, -2.32259×10^{-7}, -2.24182×10^{-7}, 3.31876×10^{-7}, 1.02018×10^{-7}, -4.05676×10^{-7}, 5.70566×10^{-8}, 4.42730×10^{-7}, -2.59304×10^{-7}, -4.21556×10^{-7}, 5.0813×10^{-7}, 3.05864×10^{-7}, -7.97158×10^{-7}, -3.92065×10^{-8}, 1.09682×10^{-6}, -4.59916×10^{-7}, -1.32949×10^{-6}, 1.29790×10^{-6}, 1.32517×10^{-6}, -2.58925×10^{-6}, -7.4736×10^{-7}, 4.39667×10^{-6}, -1.02107×10^{-6}, -6.57178×10^{-6}, 5.02947×10^{-6}, 8.38448×10^{-6}, -0.0000128957, -7.73228×10^{-6}, 0.000026728, -4.1954×10^{-7}, -0.0000482524, 0.0000268543, 0.000075668, -0.0000925551, -0.0000947775, 0.000233987, 0.0000569883, -0.000503748, 0.000170192, 0.000946659, -0.000903899, -0.0014917, 0.00282404, 0.00160145, -0.00720912, 0.000702611, 0.0159745, -0.0112644

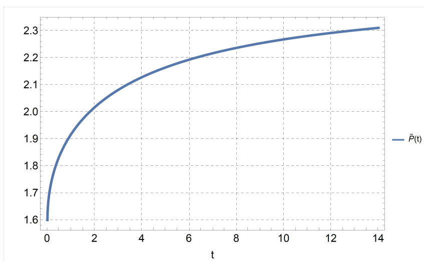

Figure 4. Graph of the solution, using the a_n coefficients.

9 Conclusions

We have shown that the use of expansions in fractional power series and, in particular, the use of the fractional exponential allows solving the fractional case of classical population dynamic problems. This approach has proved to be useful also in the case of the Laguerre-type derivative. Further applications to the solution of fractional differential equations and to the study of special fractional polynomials will be considered in subsequent articles.

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