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The Dirichlet Problem for a Class of Anisotropic Schrödinger-Kirchhoff-Type Equations with Critical Exponent

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Abstract. In this paper, our focus lies in addressing the Dirichlet problem associated with a specific class of critical anisotropic elliptic equations of Schrödinger-Kirchhoff type. These equations incorporate variable exponents and a real positive parameter. Our objective is to establish the existence of at least one solution to this problem.

Keywords: Schrödinger-Kirchhoff-type problems, Dirichlet boundary conditions, $\overrightarrow{p}(x)$-Laplacian, anisotropic variable exponent Sobolev spaces, concentration-compactness principle, parameter.

AMS Subject Classification: 35B33; 35D30; 35J50; 35J60; 46E35.

1 Introduction and main results

In this paper, our focus is on a class of critical anisotropic Schrödinger-Kirchhoff-type equations with variable growth conditions. This type of nonlinear partial differential equation describes the behavior of waves in an anisotropic (direction-dependent) system by combining the Schrödinger equation [22], which describes the quantum mechanical behavior of particles, and the Kirchhoff equation [18], which describes the behavior of waves in a medium. This equation includes variable exponent terms, which allow for a more flexible and accurate description of the wave behavior in the anisotropic system. This is especially important in systems that have complex and dynamic properties, such as materials with varying levels of anisotropy or systems that are subject to changes in temperature, pressure, or other external factors. And also the critical nonlinearities in these equations can have important implications for the behavior of the wave, such as the formation of singularities, the collapse of the wave, and the generation of shock waves. These behaviors are important for understanding the behavior of waves in complex media and for predicting how these media will interact with light, for more details, we refer the reader to [1, 20, 21, 23]. More precisely, we show the existence of nontrivial solutions for the following class of equations,

$$
\begin{aligned}
&-M\Big(\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)} + \frac{w(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\Big)\Big(\Delta_{\overrightarrow{p}(x)}(u) \qquad (1.1)\\
&\qquad - w(x)|u|^{p_M(x)-2}u\Big) = |u|^{p^*_m(x)-2}u + \lambda f(x,u) \text{ in } \Omega,\\
&u = 0 \quad \text{on } \partial\Omega, \qquad (1.2)
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $w \in L^\infty(\Omega)$ satisfies $w_0 := \operatorname{ess\,inf}_{x\in\Omega} w(x) > 0$, and λ is a positive parameter. $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ is a vector function defined as $\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x))$, with each component $p_i \in C_+(\overline{\Omega})$ satisfying

$$
\begin{aligned}
1 < p_m^- &:= \inf_{x\in\Omega}\big\{p_m(x)\big\} \leq p_m(x) := \min_{1\leq i\leq N}\{p_i(x)\} \leq p_i(x) \leq p_M(x)\\
&:= \max_{1\leq i\leq N}\{p_i(x)\} \leq p_M^+ := \sup_{x\in\Omega}\big\{p_M(x)\big\} < p_m^*(x) := \frac{Np_m(x)}{N-p_m(x)}, \text{ for all } x\in\overline{\Omega}.
\end{aligned}
$$

The operator

$$
\Delta_{\overrightarrow{p}(x)}(u) := \sum_{i=1}^N \partial_{x_i}\Big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big)
$$

is referred to as the $\overrightarrow{p}(x)$–Laplacian operator, which is a natural extension of the Laplacian operator when all $p_i(x) = 2$. The functions $M : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the following conditions:

($\boldsymbol{H}_{M_1}$) There exists $\mathfrak{M}_0 > 0$ such that $M(s) \geq \mathfrak{M}_0$ for all $s \geq 0$;

($\boldsymbol{H}_{M_2}$) There exists $\gamma \in (\frac{p_M^+}{p_m^{*-}}, 1]$ such that $\widehat{M}(s) \geq \gamma M(s)s$ for all $s \geq 0$, where $\widehat{M}(s) = \int_0^s M(t)dt$;

$(\boldsymbol{H}_{f_1})$ $f(x,s)=o(|s|^{p_M^+-1})$ as $s\to 0$, uniformly for $x\in\Omega$;

$(\boldsymbol{H}_{f_2})$ There exist a positive continuous function $\ell(x)\in(p_M^+,p_m^{*-})$ for all $x\in\Omega$ such that

$$\lim_{|s|\to+\infty}\frac{f(x,s)}{|s|^{\ell^--2}s}=0,\quad \text{uniformly for } x\in\Omega;$$

$(\boldsymbol{H}_{f_3})$ There exists $\alpha\in(p_M^+/\gamma,p_m^{*-})$ such that $0<\alpha F(x,s)\le sf(x,s)$ for all $x\in\Omega$ and $s\neq 0$, where $F(x,s)=\int_0^s f(x,t)dt$ and γ is given by $(\boldsymbol{H}_{M_2})$ below.

Much interest has been generated in problems involving critical exponents, since the publication of the celebrated paper by Brezis and Nirenberg [5], which considers the case $p_i(.)=2$ for all $i\in\{1,2,\dots,N\}$. For further study of problems with critical exponents, we refer the reader to [2,3,4,6,7,8,10,14,15, 16,17], and the references therein.

Our approach to tackling problem (1.1), inspired by the ideas in [3], is primarily variational in nature, and we employ minimax critical point theorems as our primary tool. The main challenge we face arises from the absence of compactness in the embedding $W_0^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow L^{p_m^*(x)}(\Omega)$. Consequently, it becomes unfeasible to directly verify the Palais-Smale condition for the associated energy functional. To address this hurdle, we turn to the new version of the Lions concentration-compactness principle [19], specifically designed for anisotropic variable exponent Sobolev spaces. This adaptation was introduced by Chems Eddine et al. in [9], and it plays a crucial role in addressing this challenge effectively.

So the main result of the paper reads:

Theorem 1. *Assume that assumptions $(\boldsymbol{H}_{M_1})$–$(\boldsymbol{H}_{M_2})$ and $(\boldsymbol{h}_{f_1})$–$(\boldsymbol{H}_{f_3})$ hold. Then, there exists $\lambda_*>0$ such that for all $\lambda\ge\lambda_*$, problem* (1.1) *has at least one nontrivial solution.*

2 Proof of the main result

We define the energy functional associated with problem (1.1) as
$E_\lambda: W_0^{1,\overrightarrow{p}(x)}(\Omega)\to\mathbb{R}$, given by

$$\begin{aligned}E_\lambda(u)=&\widehat{M}\Big(\int_\Omega\sum_{i=1}^N\frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}+\frac{w(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\Big)\\&-\int_\Omega\frac{1}{p_m^*(x)}|u|^{p_m^*(x)}\,\mathrm{d}x-\lambda\int_\Omega F(x,u)\,\mathrm{d}x,\end{aligned}$$

where $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ represents the anisotropic Sobolev space, and its norm is given by $\|u\|=\sum_{i=1}^N\|\partial_{x_i}u\|_{L^{p_i(x)}(\Omega)}$.

Proposition 1 [see [9]]. *The embedding $W_0^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow L^{h(x)}(\Omega)$ is continuous for any $h(x)\in[1,p_m^*(x)]$ such that $p_m\in C_+^{\log}(\overline{\Omega})$. Moreover, the embedding $W_0^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow L^{h(x)}(\Omega)$ is compact for any $h(x)\in[1,p_m^*(x))$.*

The following Poincaré-type inequality holds:

$$\|u\|_{L^{p_M(x)}(\Omega)} \leq C \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} \text{ for all } \ u \in W_0^{1,\overrightarrow{p}(x)}(\Omega), \tag{2.1}$$

where C is a positive constant independent of $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ (see [12, Theorem 2.6]).

Through standard calculus, it can be observed that E_λ is a function in $C^1(W_0^{1,\overrightarrow{p}(x)}(\Omega), \mathbb{R})$ and its Fréchet derivative is expressed as follows:

$$\begin{aligned}\langle E'_\lambda(u), v\rangle =&M\Big(\int_\Omega \sum_{i=1}^{N} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} + \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \mathrm{d}x\Big) \\ &\times \Big(\int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v + w(x)|u|^{p_M(x)-2} uv \, \mathrm{d}x\Big) \\ &- \int_\Omega |u|^{p_m^*(x)-2} uv \, \mathrm{d}x - \lambda \int_\Omega f(x,u) v \, \mathrm{d}x,\end{aligned}$$

for all $u, v \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$. Therefore, the weak solutions of (1.1) coincide with the critical points of E_λ. Consequently, our focus is on establishing the existence of these critical points.

To apply variational methods, we present certain results related to the Palais-Smale compactness condition. It's important to note that a sequence $\{u_n\}_{n\in\mathbb{N}}$ is considered a Palais-Smale sequence of E_λ at the level c_λ if both $E_\lambda(u_n) \to c_\lambda$ and $E'_\lambda(u_n) \to 0$.

Notations: Strong convergence is denoted by $\to$, while weak convergence is denoted by $\rightharpoonup$. Constants are represented by C, C_i, and C'_i, which can vary from one line to another and depend on specific conditions. The symbol δ_{x_j} represents the Dirac mass at x_j. For any $\rho > 0$ and $x \in \Omega$, $B(x,\rho)$ denotes the ball with radius ρ centered at x.

In the following, we prove that the functional E_λ exhibits the mountain pass geometry. This assertion is established in the forthcoming lemmas.

Lemma 1. *Under the assumptions* $(\boldsymbol{H}_{M_1})$, $(\boldsymbol{H}_{f_1})$ *and* $(\boldsymbol{H}_{f_2})$, *there exist positive constants* r *and* ρ *such that for all* u *with* $\|u\| = r$, *it holds that* $E_\lambda(u) \geq \rho > 0$.

Proof. First, from assumptions $(\boldsymbol{H}_{f_1})$ and $(\boldsymbol{H}_{f_2})$, for any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that the following inequality holds for almost every $x \in \Omega$ and all $s \in \mathbb{R}$:

$$|F(x,s)| \leq \varepsilon |s|^{p_M^+} + C(\varepsilon)|s|^{\ell^-}. \tag{2.2}$$

Next, by using [13, Theorem 1.3] and Jensen's inequality on the convex function $q(t) = t^{\bar{p}_{m,M}}$ for $\tilde{p}_{m,M} > 1$, we obtain that

$$\frac{\|u\|^{\bar{p}_{m,M}}}{N^{\bar{p}_{m,M}-1}} = N\left(\frac{\sum_{i=1}^{N} \|u\|_{L^{p_i(x)}(\Omega)}}{N}\right)^{\bar{p}_{m,M}} \leq \sum_{i=1}^{N} \|u\|^{\bar{p}_{m,M}}_{L^{p_i(x)}(\Omega)} \leq \sum_{i=1}^{N} \int_\Omega |u|^{p_i(x)} d\mu_i, \tag{2.3}$$

where $\bar{p}_{m,M} = p_M^+$ if $\|u\| < 1$ and $\bar{p}_{m,M} = p_m^-$ if $\|u\| \geq 1$.

Now, consider $0 < \|u\| < 1$. By using $(\mathbf{M}_1)$, (2.2), (2.3), and Proposition 1, we have

$$\begin{aligned} E_\lambda(u) &\geq \frac{\gamma \mathfrak{M}_0}{p_M^+ N^{p_M^+ - 1}} \|u\|^{p_M^+} - C_1' \|u\|^{p_m^{*-}} - \lambda \varepsilon C_3'^{p_M^+} \|u\|^{p_M^+} - \lambda C(\varepsilon) C_4' \|u\|^{\ell^-} \quad (2.4) \\ &= \|u\| \left[\left(\frac{\gamma \mathfrak{M}_0}{p_M^+ N^{p_M^+ - 1}} - \lambda \varepsilon C_3'^{p_M^+} \right) \|u\|^{p_M^+ - 1} - C_1' \|u\|^{p_m^{*-} - 1} - \lambda C(\varepsilon) C_4' \|u\|^{\ell^- - 1} \right]. \end{aligned}$$

Let $\varepsilon = \gamma \mathfrak{M}_0 / (2\lambda C_3' p_M^+ N^{p_M^+ - 1})$ and define $\Phi(t)$ as follows:

$$\Phi(t) = \frac{\gamma \mathfrak{M}_0}{2 p_M^+ N^{p_M^+ - 1}} t^{p_M^+ - 1} - C_1' t^{p_m^{*-} - 1} - \lambda C(\varepsilon) C_4' t^{\ell^- - 1}.$$

Since $p_M^+ < \ell^- < p_m^{*-}$, there exists $r > 0$ such that $\max_{t \geq 0} \Phi(t) = \Phi(r)$. Consequently, by (2.4), we deduce the existence of $\rho > 0$ such that $E_\lambda(u) \geq \rho$ for all $\|u\| = r$. This completes the proof of Lemma 1. $\square$

Lemma 2. *Under the assumptions* $(\mathbf{H}_{M_2})$ and $(\mathbf{H}_{f_3})$, *for all* $\lambda > 0$, *there exists a nonnegative function* $z \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$, *which is independent of* λ, *such that* $\|z\| > r$ *and* $E_\lambda(z) < 0$.

Proof. Choose a nonnegative function $\phi_0 \in C_0^\infty(\Omega)$ with $\|\phi_0\| = 1$. By integrating $(\boldsymbol{H}_{M_2})$, we obtain

$$\widehat{M}(s) \leq \widehat{M}(s_0) s^{\frac{1}{\gamma}} / s_0^{\frac{1}{\gamma}} \leq c_0 s^{\frac{1}{\gamma}} \quad \text{for all } s \geq s_0 > 0. \quad (2.5)$$

By assumption $(\boldsymbol{H}_{f_3})$, $\int_\Omega F(x, t\phi_0) dx \geq 0$. So, by using Propisition 1, the Poincaré inequality (2.1), (2.5), and the following inequality

$$\sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}^{p_M^+} \leq c \Big(\sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} \Big)^{p_M^+},$$

with c is a positive constant, we obtain

$$E_\lambda(t\phi_0) \leq \frac{C'}{p} t^{p_M^+/\gamma} - \frac{t^{p_m^{*-}}}{p_m^{*+}} \int_\Omega |\phi_0|^{p_m^*(x)} dx \quad \text{for all } t \geq t_0.$$

Given that $p_M^+/\gamma < p_m^{*-}$, the lemma is proved by choosing $z = t_* \phi_0$ with $t_* > 0$ large enough. $\square$

In view of Lemmas 1 and 2, we can employ an version of the Mountain Pass theorem, even without the Palais-Smale condition, to obtain a sequence $\{u_n\} \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$ with the properties

$$E_\lambda(u_n) \to c_\lambda \quad \text{and} \quad E_\lambda'(u_n) \to 0,$$

where

$$c_\lambda = \inf_{\phi \in \Gamma} \max_{t \in [0,1]} E_\lambda(\phi(t)) > 0,$$

with $\Gamma = \big\{ \phi \in C([0,1], W_0^{1,\overrightarrow{p}(x)}(\Omega)) : \phi(0) = 0,\, E_\lambda(\phi(1)) < 0 \big\}$.

Lemma 3. *Under the assumptions* $(\boldsymbol{H}_{M_1})$–$(\boldsymbol{H}_{M_2})$ *and* $(\boldsymbol{H}_{f_1})$–$(\boldsymbol{H}_{f_3})$, *it holds that* $\lim_{\lambda\to\infty} c_\lambda = 0$.

Proof. For a given z as established in Lemma 2, we observe that $\lim_{t\to+\infty} E_\lambda(tz) = -\infty$, which implies the existence of $t_\lambda > 0$ such that $E_\lambda(t_\lambda e) = \max\limits_{t\geq 0} E_\lambda(tz)$. Hence, we have $\langle E'_\lambda(t_\lambda z), t_\lambda z\rangle = 0$, that is

$$M\Big(\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_{x_i}(t_\lambda z)|^{p_i(x)} + \frac{w(x)}{p_M(x)}|t_\lambda z|^{p_M(x)}\mathrm{d}x\Big)\Big(\int_\Omega \sum_{i=1}^N |\partial_{x_i}(t_\lambda z)|^{p_i(x)} + w(x)|t_\lambda z|^{p_M(x)}\,\mathrm{d}x\Big) = \int_\Omega |t_\lambda z|^{p_m^*(x)}\,\mathrm{d}x + \lambda\int_\Omega f(x, t_\lambda z)t_\lambda z\,\mathrm{d}x.$$

Therefore, by using assumption $(\boldsymbol{H}_{f_3})$, Proposition 1, the Poincaré inequality (2.1), and (2.5), it follows that

$$C'\|z\|^{p_M^+/\gamma} t_\lambda^{p_M^+/\gamma} \geq \|z\|^{p_m^{*-}}_{L^{p_m^{*-}}(\Omega)} t_\lambda^{p_m^{*-}} \text{ with } t_0 < t_\lambda.$$

Since $p_M^+/\gamma < p_m^{*-}$, $\{t_\lambda\}_\lambda$ is bounded. Therefore, there exists a sequence $\lambda_n \to +\infty$ and $\delta_0 \geq 0$ such that $t_{\lambda_n} \to \delta_0$ as $n\to\infty$. Hence, by continuity of M, we have $\Big\{M\Big(\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_{x_i}(t_{\lambda_n} z)|^{p_i(x)} + \frac{w(x)}{p_M(x)}|t_{\lambda_n} z|^{p_M(x)}\mathrm{d}x\Big)\Big\}_n$ is bounded, and so, there exists $C > 0$ such that

$$M\Big(\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_{x_i}(t_{\lambda_n} z)|^{p_i(x)} + \frac{w(x)}{p_M(x)}|t_{\lambda_n} z|^{p_M(x)}\mathrm{d}x\Big) \times\Big(\int_\Omega \sum_{i=1}^N |\partial_{x_i}(t_{\lambda_n} z)|^{p_i(x)} + w(x)|t_{\lambda_n} z|^{p_M(x)}\,\mathrm{d}x\Big) \leq C$$

for all $n\in\mathbb{N}$, which implies that,

$$\lambda_n t_{\lambda_n}\int_\Omega f(x, t_{\lambda_n} z)z\,dx + \int_\Omega t_{\lambda_n}^{p_m^*}|z|^{p_m^*(x)}dx \leq C \text{ for all } n\in\mathbb{N}.$$

If $\delta_0 > 0$, the inequality mentioned above implies that

$$\lambda_n t_{\lambda_n}\int_\Omega f(x, t_{\lambda_n} z)z\,dx + \int_\Omega t_{\lambda_n}^{p_m^*}|z|^{p_m^*(x)}dx \to +\infty \leq C, \quad \text{as } n\to\infty,$$

which is impossible, and consequently $\delta_0 = 0$. Let $\phi_*(t) = tz$ for $t\in[0,1]$. Clearly $\phi_* \in \Gamma$, then, by using assumption $(\boldsymbol{H}_{f_3})$, we have

$$\begin{aligned} 0 < c_{\lambda_n} &\leq \max_{t\in[0,1]} E_{\lambda_n}(\phi_*(t)) = E_{\lambda_n}(t_{\lambda_n} z) \\ &\leq \widehat{M}\Big(\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_{x_i}(t_{\lambda_n} z)|^{p_i(x)} + \frac{w(x)}{p_M(x)}|t_{\lambda_n} z|^{p_M(x)}\mathrm{d}x\Big). \qquad (2.6)\end{aligned}$$

Since the function M is continuous and $\delta_0 = 0$, we get

$$\lim_{n\to\infty} \widehat{M}\Big(\int_{\Omega}\sum_{i=1}^{N}\frac{1}{p_i(x)}|\partial_{x_i}(t_{\lambda_n}z)|^{p_i(x)} + \frac{w(x)}{p_M(x)}|t_{\lambda_n}z|^{p_M(x)}\mathrm{d}x\Big) = 0.$$

So, by relation (2.6), it follows that $\lim_{n\to\infty} c_{\lambda_n} = 0$. Moreover, by virtue of assumption $(\boldsymbol{H}_{f_3})$, we can deduce that $\{c_\lambda\}_\lambda$ forms a monotone sequence. Consequently, we can establish that $\lim_{\lambda\to\infty} c_\lambda = 0$. $\square$

Let S_* denote the optimal positive constant of the Sobolev embedding $W_0^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{p_m^*(x)}(\Omega)$, which can be expressed as

$$S_* := \inf_{u\in W_0^{1,\overrightarrow{p}(x)}(\Omega)\setminus\{0\}} \frac{\|u\|}{\|u\|_{L^{p_m^*(x)}(\Omega)}}.$$

Proof. [Proof of Theorem 1] From Lemmas 1, 2 and 3, we can establish the existence of a sequence $\{u_n\}_n \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$ such that $E_\lambda(u_n) \to c_\lambda$ and $E'_\lambda(u_n) \to 0$, with

$$c_\lambda \in \Bigg(0, \Big(\frac{1}{\alpha} - \frac{1}{p_m^{*-}}\Big)\min\Bigg\{\inf_{j\in J}\Big(\mathfrak{M}_0^{\frac{1}{p_M^+}} N^{1-p_M^+} S_*\Big)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}},$$
$$\inf_{j\in J}\Big(\mathfrak{M}_0^{\frac{1}{p_M^+}} N^{1-p_M^+} S_*\Big)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\Bigg\}\Bigg)$$

for $\lambda \ge \lambda_*$. Consequently, we can find a constant $C > 0$ such that $|E_\lambda(u_n)| \le C$. Moreover, by the assumption $(\boldsymbol{H}_{f_3})$ and for sufficiently large n, it follows from the assumptions $(\boldsymbol{H}_{M_1})$ and $(\boldsymbol{H}_{M_2})$ that

$$C + \|u_n\| \ge E_\lambda(u_n) - \frac{1}{\alpha}\langle E'_\lambda(u_n), u_n\rangle \ge \mathfrak{M}_0\Bigg(\int_\Omega\Bigg[\gamma\Big(\sum_{i=1}^{N}\frac{1}{p_i(x)}|\partial_{x_i}u_n|^{p_i(x)}$$
$$+ \frac{w(x)}{p_M(x)}|u_n|^{p_M(x)}\Big) - \frac{1}{\alpha}\Big(\sum_{i=1}^{N}|\partial_{x_i}u_n|^{p_i(x)} + w(x)|u_n|^{p_M(x)}\Big)\Bigg]\mathrm{d}x\Bigg)$$
$$\ge \mathfrak{M}_0\Big(\frac{\gamma}{p_M^+} - \frac{1}{\alpha}\Big)\sum_{i=1}^{N}\int_\Omega|\partial_{x_i}u_n|^{p_i(x)}\mathrm{d}x.$$

On the other hand, for each n, let us denote by $\mathcal{B}_{n_1}$ and $\mathcal{B}_{n_2}$ the indices sets $\mathcal{B}_{n_1} = \Big\{i \in \{1,2,\dots,N\} : \big\|\partial_{x_i}u_n\big\|_{L^{p_i(x)}(\Omega)} \le 1\Big\}$, $\mathcal{B}_{n_2} = \Big\{i \in \{1,2,\dots,N\} : \big\|\partial_{x_i}u_n\big\|_{L^{p_i(x)}(\Omega)} > 1\Big\}$. Then, we have

$$C+\|u_n\| \ge \Big(\frac{\gamma\mathfrak{M}_0}{p_M^+} - \frac{\mathfrak{M}_0}{\alpha}\Big)\Big(\sum_{i\in\mathcal{B}_{n_1}}\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{p_M^+} + \sum_{i\in\mathcal{B}_{n_2}}\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-}\Big)$$
$$= \Big(\frac{\mathfrak{M}_0\gamma}{p_M^+} - \frac{\mathfrak{M}_0}{\alpha}\Big)\Bigg[\sum_{i=1}^{N}\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-} - \sum_{i\in\mathcal{B}_{n_1}}\Big(\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-}$$

$$- \|\partial_{x_i} u_n\|^{p_M^+}_{L^{p_i(x)}(\Omega)}\Big)\Big] \geq \Big(\frac{\mathfrak{M}_0\gamma}{p_M^+} - \frac{\mathfrak{M}_0}{\alpha}\Big)\Big(\sum_{i=1}^N \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} - N\Big).$$

Hence, by using Jensen's inequality (2.3) (applied to the convex function $h : \mathbb{R}^+ \to \mathbb{R}^+, q(t) = t^{p_m^-}$, $p_m^- > 1$), for n large enough we have

$$C + \|u_n\| \geq \Big(\frac{\mathfrak{M}_0\gamma}{p_M^+} - \frac{\mathfrak{M}_0}{\alpha_\lambda}\Big)\Big(\frac{\|u_n\|^{p_m^-}}{N^{p_m^- - 1}} - N\Big).$$

Since $\alpha > p_M^+/\gamma$, $\{u_n\}$ is bounded. Therefore, up to a subsequence, we may assume that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,\overrightarrow{p}(x)}(\Omega), \quad u_n \to u \quad \text{a.e. in } \Omega,\\
&u_n \to u \quad \text{in } L^{h(x)}(\Omega),\ 1 \leq h(x) < p_m^*(x),\\
&\sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} \rightharpoonup \mu = \sum_{i=1}^N \mu_i \quad \text{(weak*-sense of measures)},\\
&|u_n|^{p_m^*(x)} \rightharpoonup \nu, \quad \text{(weak*-sense of measures)},
\end{aligned} \tag{2.7}$$

where μ and ν are nonnegative bounded measures on $\overline{\Omega}$. Then, according to the new version of Lions's concentration–compactness principle for anisotropic variable exponents [9], there exists an index set J which is at most countable, such that

$$\begin{aligned}
&\nu = |u|^{p_m^*(x)} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \nu_j > 0,\\
&\mu \geq \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} + \sum_{j\in J} \mu_j \delta_{x_j}, \quad \mu_j > 0,\\
&N^{1-p_M^+} S_* \nu_j^{\frac{1}{p_m^*(x_j)}} \leq \max\Big\{\big(\mu_j\big)^{1/p_M^+}, \big(\mu_j\big)^{1/p_m^-}\Big\}. \quad \forall j \in J,
\end{aligned} \tag{2.8}$$

with δ_{x_j} is the Dirac measure mass at $x_j \in \overline{\Omega}$.

We consider $\psi \in C_c^\infty(\mathbb{R}^N, [0,1])$ such that $\|\nabla\psi\|_\infty \leq 2$ and

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < 1,\\ 0 & \text{if } |x| \geq 2. \end{cases}$$

For $j \in J$ and $\varepsilon > 0$, let $\psi_{j,\varepsilon}(x) = \psi(\frac{x-x_j}{\varepsilon})$. Given that $E'_\lambda(u_n) \to 0$ and $(\psi_{j,\varepsilon} u_n)$ is bounded, it follows that $\langle E'_\lambda(u_n), \psi_{j,\varepsilon} u_n\rangle \to 0$ as $n \to \infty$. In other words,

$$\begin{aligned}
&M\Big(\int_\Omega \Big(\sum_{i=1}^N \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)} + \frac{w(x)}{p_M(x)} |u_n|^{p_M(x)}\Big)\, dx\Big)\\
&\quad \times \int_\Omega \psi_{j,\varepsilon}\Big(\sum_{i=1}^N \big|\partial_{x_i} u_n\big|^{p_i(x)-2} \partial_{x_i} u_n + w(x)|u_n|^{p_M(x)}\Big)\, \mathrm{d}x\\
&= -M\Big(\int_\Omega \Big(\sum_{i=1}^N \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)} + \frac{w(x)}{p_M(x)} |u_n|^{p_M(x)}\Big)\, dx\Big)
\end{aligned}$$

$$\times \sum_{i=1}^{N} \int_{\Omega} u_n |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n \partial_{x_i} \psi_{j,\varepsilon} \,\mathrm{d}x + \int_{\Omega} |u_n|^{p_m*(x)} \psi_{j,\varepsilon} \,\mathrm{d}x$$
$$+ \lambda \int_{\Omega} f(x,u_n) \psi_{j,\varepsilon} u_n \,\mathrm{d}x + o_n(1). \tag{2.9}$$

First, we will show that

$$\lim_{\varepsilon\to 0} \Big[\limsup_{n\to\infty} M\Big(\int_{\Omega} \sum_{i=1}^{N} \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)} + \frac{w(x)}{p_M(x)} |u_n|^{p_M(x)} \, dx \Big) \times$$
$$| \sum_{i=1}^{N} \int_{\Omega} u_n |\partial_{x_i} u_n|^{p_i(x)-1} \partial x_i \psi_{j,\varepsilon} \,\mathrm{d}x| \Big] = 0. \tag{2.10}$$

By applying the Hölder inequality and considering the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, we obtain

$$\Big| \int_{\Omega} u_n |\partial_{x_i} u_n|^{p_i(x)-1} \partial x_i \psi_{j,\varepsilon} \,\mathrm{d}x \Big| \le \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)-1} |u_n \partial x_i \psi_{j,\varepsilon}| \,\mathrm{d}x$$
$$\le 2 \| |\partial x_i u_n|^{p_i(x)-1} \|_{L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)} \| \partial x_i \psi_{j,\varepsilon} u_n \|_{L^{p_i(x)}(\Omega)}$$
$$\le C \max \Big\{ \Big(\int_{\Omega} |u_n|^{p_i(x)} |\partial x_i \psi_{j,\varepsilon}|^{p_i(x)} \Big)^{\frac{1}{p_i^-}}, \Big(\int_{\Omega} |u_n|^{p_i(x)} |\partial x_i \psi_{j,\varepsilon}|^{p_i(x)} \Big)^{\frac{1}{p_i^+}} \Big\}.$$

Therefore, by Lebesgue's Dominated Convergence Theorem, we get

$$\Big| \int_{\Omega} u_n |\partial_{x_i} u_n|^{p_i(x)-1} \partial x_i \psi_{j,\varepsilon} \,\mathrm{d}x \Big|$$
$$\le C \max \Big\{ \Big(\int_{\Omega} |u|^{p_i(x)} |\partial x_i \psi_{j,\varepsilon}|^{p_i(x)} \Big)^{\frac{1}{p_i^-}}, \Big(\int_{\Omega} |u|^{p_i(x)} |\partial x_i \psi_{j,\varepsilon}|^{p_i(x)} \Big)^{\frac{1}{p_i^+}} \Big\}.$$

Moreover, by Hölder inequality

$$\int_{\Omega} |u|^{p_i(x)} |\partial x_i \psi_{j,\varepsilon}|^{p_i(x)} \,\mathrm{d}x$$
$$\le C \| |u|^{p_i(x)} \|_{L^{\frac{N}{N-p_i(x)}}(\mathbf{B}(x_j,2\varepsilon))} \| |\partial x_i \psi_{j,\varepsilon}|^{p_i(x)} \|_{L^{\frac{N}{p_i(x)}}(\mathbf{B}(x_j,2\varepsilon))}.$$

Note that

$$\int_{\mathbf{B}(x_j,2\varepsilon)} |\partial x_i \psi_{j,\varepsilon}|^N \,\mathrm{d}x \le \Big(\frac{2}{\varepsilon}\Big)^N meas(B(x_j,2\varepsilon)) = \frac{4^N}{N} \mathcal{S}_N,$$

with $\mathcal{S}_N$ is the surface area of an N-dimensional unit sphere. We have

$$\| |\partial x_i \psi_{j,\varepsilon}|^{p_i(x)} \|_{L^{\frac{N}{p_i(x)}}(\mathbf{B}(x_j,2\varepsilon))} \le \max \Big\{ \Big(\int_{\mathbf{B}(x_j,2\varepsilon)} |\partial x_i \psi_{j,\varepsilon}|^N \mathrm{d}x \Big)^{\frac{1}{\left(\frac{N}{p_i(x)}\right)^+}},$$
$$\Big(\int_{\mathbf{B}(x_j,2\varepsilon)} |\partial x_i \psi_{j,\varepsilon}|^N \mathrm{d}x \Big)^{\frac{1}{\left(\frac{N}{p_i(x)}\right)^-}} \Big\} \le C$$

with C is a positive constant that doesn't depend on ε. Therefore,

$$\begin{aligned}&\limsup_{n\to\infty}\Big\|\int_\Omega u_n|\partial_{x_i}u_n|^{p_i(x)-1}\partial x_i\psi_\varepsilon\,\mathrm{d}x\Big\|\\ &\le C\Big\{\||u|^{p_i(x)}\|^{\frac{1}{p_i^-}}_{L^{\frac{N}{N-p_i(x)}}(\mathbf{B}(x_j,2\varepsilon))},\||u|^{p_i(x)}\|^{\frac{1}{p_i^+}}_{L^{\frac{N}{N-p_i(x)}}(\mathbf{B}(x_j,2\varepsilon))}\Big\}.\end{aligned}$$

But,

$$\begin{aligned}\||u|^{p_i(x)}\|_{L^{\frac{N}{N-p_i(x)}}(\mathbf{B}(x_j,2\varepsilon))}&\le\max\Big\{\Big(\int_{\mathbf{B}(x_j,2\varepsilon)}|u|^{p_i^*(x)}\mathrm{d}x\Big)^{\frac{1}{\left(\frac{N}{N-p_i(x)}\right)^+}},\\ &\Big(\int_{\mathbf{B}(x_j,2\varepsilon)}|u|^{p_i^*(x)}\mathrm{d}x\Big)^{\frac{1}{\left(\frac{N}{N-p_i(x)}\right)^-}}\Big\}.\end{aligned}$$

From this, it follows that

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\Big|\int_\Omega u_n|\partial_{x_i}u_n|^{p_i(x)-1}\partial x_i\psi_{j,\varepsilon}\,\mathrm{d}x\Big|=0\text{ for all }i\in\{1,2,\ldots,N\}. \tag{2.11}$$

Since $\{u_n\}_n$ is bounded and the function M is continuous, we can choose a subsequence, and there exists $t_0\ge0$ such that

$$M\Big(\int_\Omega\Big(\sum_{i=1}^N\frac{1}{p_i(x)}|\partial_{x_i}u_n|^{p_i(x)}+\frac{w(x)}{p_M(x)}|u_n|^{p_M(x)}\Big)\,dx\Big)\to M(t_0)\ge\mathfrak{M}_0,$$

as $n\to\infty$. Then, by (2.11), we obtain that (2.10) is proved.

On the other hand, we obtain, by using assumptions $(\boldsymbol{H}_{f_1})$–$(\boldsymbol{H}_{f_3})$, (2.7), and Lebesgue's Dominated Convergence Theorem, that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega f(x,u_n)u_n\psi_{j,\varepsilon}\,\mathrm{d}x&=\int_\Omega f(x,u)u\psi_{j,\varepsilon}\,\mathrm{d}x,\\ \lim_{n\to\infty}\int_\Omega|u_n|^{p_M(x)}\psi_{j,\varepsilon}\,\mathrm{d}x&=\int_\Omega|u|^{p_M(x)}\psi_{j,\varepsilon}\,\mathrm{d}x,\end{aligned}\tag{2.12}$$

once that, when $\epsilon\to0$, we find

$$\lim_{\varepsilon\to0}\int_\Omega f(x,u)u\psi_\varepsilon\,\mathrm{d}x=0,\quad\lim_{\varepsilon\to0}\int_\Omega|u|^{p_M(x)}\psi_{j,\varepsilon}\,\mathrm{d}x=0. \tag{2.13}$$

Since $\psi_{j,\varepsilon}$ has compact support, going to the limit $n\to\infty$ and letting $\epsilon\to0$ in (2.9) we deduce from (2.10), (2.12) and (2.13) that

$$\mathfrak{M}_0\mu_j\le\nu_j\text{ for any }j\in J.$$

Thus, from relation (2.8), we conclude that

$$\nu_j\ge\min\Big\{\Big(\mathfrak{M}_0^{\frac{1}{p_M^+}}N^{1-p_M^+}S_*\Big)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}},\Big(\mathfrak{M}_0^{\frac{1}{p_m^-}}N^{1-p_M^+}S_*\Big)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\Big\}.$$

Let us demonstrate that this inequality cannot hold. Let us assume that

$$\nu_{j_0} \geq \min\left\{ \left(\mathfrak{M}_0^{\frac{1}{p_M^+}} N^{1-p_M^+} S_* \right)^{\frac{p_m^*(x_{j_0}) p_M^+}{p_m^*(x_{j_0}) - p_M^+}}, \left(\mathfrak{M}_0^{\frac{1}{p_m^-}} N^{1-p_M^+} S_* \right)^{\frac{p_m^*(x_{j_0}) p_m^-}{p_m^*(x_{j_0}) - p_m^-}} \right\}$$

for some $j_0 \in J$. From $(\boldsymbol{H}_{M_1})$–$(\boldsymbol{H}_{M_2})$ and $(\boldsymbol{H}_{f_3})$ we see that

$$\begin{aligned} c_\lambda &= E_\lambda(u_n) - \frac{1}{\alpha}\langle E'_\lambda(u_n), u_n\rangle + o_n(1) \geq \left(\frac{1}{\alpha} - \frac{1}{p_m^{*-}}\right) \\ &\times \int_\Omega |u_n|^{p_m^*(x)} dx + o_n(1) \geq \left(\frac{1}{\alpha} - \frac{1}{p_m^{*-}}\right) \int_\Omega \psi_{j_0,\varepsilon} |u_n|^{p_m^*(x)} dx + o_n(1). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} c_\lambda &\geq \left(\frac{1}{\alpha} - \frac{1}{p_m^{*-}}\right) \sum_{j\in J} \psi_{j_0,\varepsilon}(x_j)\nu_j \geq \left(\frac{1}{\alpha} - \frac{1}{p_m^{*-}}\right) \\ &\times \min\left\{ \inf_{j\in J} \left(\mathfrak{M}_0^{\frac{1}{p_M^+}} N^{1-p_M^+} S_* \right)^{\frac{p_m^*(x_j) p_M^+}{p_m^*(x_j) - p_M^+}}, \inf_{j\in J} \left(\mathfrak{M}_0^{\frac{1}{p_m^-}} N^{1-p_M^+} S_* \right)^{\frac{p_m^*(x_j) p_m^-}{p_m^*(x_j) - p_m^-}} \right\}. \end{aligned}$$

This contradicts Lemma 3. Therefore, $J = \emptyset$, and consequently, $u_n \to u$ in $L^{p_m^*(x)}(\Omega)$. By assumptions $(\boldsymbol{H}_{f_1})$–$(\boldsymbol{H}_{f_2})$ and Hölder inequality, we have

$$\begin{aligned} &\int_\Omega |f(x,u_n)(u_n - u)| dx \leq \int_\Omega \left(\varepsilon |u_n|^{p_M^+ - 1} + C_\varepsilon |u_n|^{\ell^- - 1}\right)|u_n - u| dx \leq 2\varepsilon \\ &\times \| |u_n|^{p_M^+ - 1} \|_{L^{\frac{p_M^+}{p_M^+ - 1}}(\Omega)} \|u_n - u\|_{L^{p_M^+}(\Omega)} + 2C_\varepsilon \| |u_n|^{\ell^- - 1} \|_{L^{\frac{\ell^-}{\ell^- - 1}}(\Omega)} \|u_n - u\|_{L^{\ell^-}(\Omega)}. \end{aligned}$$

Then, using again (2.7), we obtain

$$\lim_{n\to\infty} \int_\Omega w(x)|u_n|^{p_M(x)-2} u_n (u_n - u) dx = 0, \quad \lim_{n\to\infty} \int_\Omega f(x,u_n)(u_n - u) dx = 0. \tag{2.14}$$

As $u_n \to u$ in $L^{p_m^*(x)}(\Omega)$, it follows that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p_m^*(x)-2} u_n (u_n - u) dx = 0. \tag{2.15}$$

Additionally, from $\langle E'_\lambda(u_n), u_n - u\rangle = o_n(1)$, we deduce

$$\begin{aligned} \langle E'_\lambda(u_n), u_n - u\rangle &= M\left(\int_\Omega \Big(\sum_{i=1}^N \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)} + \frac{w(x)}{p_M(x)} |u_n|^{p_M(x)} \Big) dx \right) \\ &\times \Big(\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n \partial_{x_i}(u_n - u) + w(x)|u_n|^{p_M(x)-2} u_n (u_n - u) \Big) dx \\ &- \int_\Omega |u_n|^{p_m^*(x)-2} u_n (u_n - u) dx - \lambda \int_\Omega f(x,u_n)(u_n - u) dx = o_n(1). \end{aligned}$$

This, (2.14) and (2.15) imply

$$\lim_{n\to\infty} \mathfrak{M}_0 \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)-2}\partial_{x_i} u_n\big(\partial_{x_i} u_n - \partial_{x_i} u\big)\, dx \le 0. \tag{2.16}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to u in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, we have

$$\lim_{n\to\infty} \mathfrak{M}_0 \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\big(\partial_{x_i} u_n - \partial_{x_i} u\big)\, dx \le 0. \tag{2.17}$$

So, by combining relation (2.16) and relation (2.17), we have

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i} u_n|^{p_i(x)-2}\partial_{x_i} u_n - |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\Big)\big(\partial_{x_i} u_n - \partial_{x_i} u\big) dx \le 0. \tag{2.18}$$

Hence, by applying some elementary inequalities (see, e.g., [11, Chapter I]), for any $\sigma > 1$ there exists a positive constant C_σ such that

$$\langle |\xi|^{\sigma-2}\xi - |\eta|^{\sigma-2}\eta, \xi-\eta\rangle \ge \begin{cases} C_\sigma |\xi-\eta|^\sigma \quad \text{if } \sigma \ge 2, \\ C_\sigma \frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-\sigma}}, (\xi,\eta) \ne (0,0) \text{ if } 1<\sigma<2, \end{cases} \tag{2.19}$$

for any $\xi, \eta \in \mathbb{R}$. Then, by (2.18) and (2.19), we find that

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\, dx = 0.$$

Hence, we can infer that $u_n \to u$ strongly in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, and consequently, $E'_\lambda(u) = 0$, this implies that u is a nontrivial weak solution to problem (1.1) for every $\lambda \ge \lambda_*$. $\square$

3 A special case

Now, we consider a special case of the problem given by Equation (1.1). The problem is described as follows:

$$\begin{aligned} &-\Big(a + b\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_{x_i} u|^{p_i(x)} + \frac{w(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\Big) \\ &\qquad \times \Big(\Delta_{\overrightarrow{p}(x)}(u) - w(x)|u|^{p_M(x)-2}u\Big) = |u|^{p_m^*(x)-2}u + \lambda f(x,u) \text{ in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, a and b are a positive constants.

Assuming $M(s) = a + bs$, it is evident that $M(s) \ge a$. Additionally, we can compute $\widehat{M}(s)$ as follows

$$\widehat{M}(s) = \int_0^1 M(t)dt = as + \frac{1}{2}bs^2 \ge \frac{1}{2}(a+bs)s = \gamma M(s)s,$$

where $\gamma = 1/2$. Therefore, the conditions $(\boldsymbol{H}_{M_1})$ and $(\boldsymbol{H}_{M_2})$ are satisfied.

In this specific case, a typical example of a function $f(x,s)$ that satisfies the conditions $(\boldsymbol{H}_{f_1})$–$(\boldsymbol{H}_{f_3})$ is given as follows:

$$f(x,s) = \sum_{i=1}^{\kappa} g_i(x)|s|^{\ell_i(x)-2}s,$$

where $\kappa \geq 1$, $p_M^+ < \ell_i(x) < p_m^{*-}$ and the nonnegative functions $g_i(x) \in C(\Omega)$. Based on Theorem 1, we can derive the following corollary:

Corollary 1. Under the assumptions $(\boldsymbol{H}_{f_1})$–$(\boldsymbol{H}_{f_3})$, there exists a positive constant $\lambda_* > 0$ such that for any $\lambda \geq \lambda_*$, the problem described by Equation (3.1) has at least one nontrivial solution.

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