Mathematical Modelling and Analysis
Volume 29, Issue 3, 442–459, 2024
https://doi.org/10.3846/mma.2024.18832

Spline Quasi-Interpolation Numerical Methods for Integro-Differential Equations with Weakly Singular Kernels

Abdelmonaim Saou[a], Driss Sbibih[b], Mohamed Tahrichi[a] and Domingo Barrera[c]

[a] *ANAA Team, EST, University Mohammed First*
60000 Oujda, Morocco
[b] *ANO Laboratory, Faculty of Sciences, University Mohammed First*
60000 Oujda, Morocco
[c] *Department of Applied Mathematics, University of Granada*
Campus de Fuentenueva s/n, 18071-Granada, Spain

E-mail(*corresp.*): m.Tahrichi@ump.ac.ma

E-mail: Saou.abdelmonaim@gmail.com

E-mail: Sbibih@yahoo.fr

E-mail: dBarrera@ugr.es

Received March 28, 2023; accepted November 6, 2023

Abstract. In this work, we introduce a numerical approach that utilizes spline quasi-interpolation operators over a bounded interval. This method is designed to provide a numerical solution for a class of Fredholm integro-differential equations with weakly singular kernels. We outline the computational components involved in determining the approximate solution and provide theoretical findings regarding the convergence rate. This convergence rate is analyzed in relation to both the degree of the quasi-interpolant and the grading exponent of the graded grid partition. Finally, we present numerical experiments that validate the theoretical findings.

Keywords: quasi-interpolation operators, numerical methods, Fredholm integro-differential equations, weakly singular kernel, graded grids.

AMS Subject Classification: 41A10; 45G10; 47H30; 65R20.

1 Introduction

In recent years, integro-differential equations have attracted extensive attention in many scientific fields. The applications of these equations are very useful for modeling problems encountered in very different fields [11, 13, 15].

Numerous numerical techniques have emerged to address Fredholm integro-differential equations (FIDE for short) with smooth kernels, [7,9,18,19,20]. Notable among these are the Adomian decomposition [8], the homotopy analysis method [11], and Chebyshev and Taylor collocation methods [22]. Furthermore, the decomposition method was effectively employed to solve the high-order linear Volterra-Fredholm integro-differential equations, as demonstrated in [6]. Recently, in [5], highly accurate pseudo-spectral Galerkin scheme for pantograph type Volterra integro-differential equations having singular kernels are proposed. In [23], the authors generalize a collocation method in the reproducing kernel space in order to solve a weakly singular FIDE. In [14], a spline-based collocation method is proposed to deal with the numerical solution of integro-differential equations with weakly singular kernels. Other authors have worked on this set of equations, among them [2, 3, 7].

Spline quasi-interpolations (abbr. QI) provide approximating splines expressed as a linear combinations of compactly supported (B-splines). These quasi-interpolating splines offer a practical and effective approach to approximating functions due to their straightforward construction and the advantageous property of achieving an optimal convergence rate, all while maintaining a uniformly bounded norm as highlighted in [17].

In this paper, we present a numerical approach grounded in spline quasi-interpolants (QIs) designed to provide a numerical solution for the subsequent FIDE with weakly singular kernel

$$\begin{cases} u'(t) = \displaystyle\int_0^1 k(t,s)u(s)\,ds + a(t)u(t) + g(t)\,, & t \in [0\ ,1], \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where $u_0 \in \mathbb{R}$ and u is the unknown function.

For a non-negative integer m, we assume that $a, g \in C^{m,\nu}[0,1]$ where $C^{m,\nu}[0,1]$ is the space of all continuous functions $u : [0,1] \to \mathbb{R}$ which are m times continuously differentiable in $(0,1)$ and such that the following estimate holds:

$$\text{for all } t \in (0,1), \quad |u^{(m)}(t)| \leqslant c\sigma(t)^{2-m-\nu}, \quad 0 < \nu < 1, \tag{1.2}$$

c being a positive constant and $\sigma(t) := \min\limits_{0<t<1}\{t, 1-t\}$ represents the measure of separation between $t \in (0,1)$ and the extremities of the interval $[0,1]$. Note that $C^{m,\nu}[0,1]$ is a Banach space with respect to the norm (see, e.g., [14])

$$\|u\|_{\infty,v} := \max_{0\leqslant t\leqslant 1} |u(t)| + \sup_{0<t<1} \sigma(t)^{\nu+m-2}|u^{(m)}(t)|, \quad u \in C^{m,\nu}[0,1]\,.$$

For $0 < \nu < 1$, $C^{0,\nu}[0,1]$ reduces to the space of continuous functions defined on $[0,1]$ and endowed with the usual norm $\|u\|_\infty = \max\limits_{0\leqslant x\leqslant 1} |u(x)|$.

We define the domain $\mathit{\Delta}$ by

$$\mathit{\Delta} := \{(t,s) \,:\, 0 \leqslant t \leqslant 1 \,,\, 0 \leqslant s \leqslant 1 \,,\, t \neq s\}.$$

The kernel k belongs to the set $W^{m,\nu}(\mathit{\Delta})$ of all m times continuously differentiable functions $k : \mathit{\Delta} \to \mathbb{R}$ satisfying, for all $(t,s) \in \mathit{\Delta}$ and all non-negative integers i and j such that $i + j \leqslant m$, the condition

$$\Big| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j k(t,s) \Big| \leqslant c_1 |t-s|^{-\nu-i}, \quad 0 < \nu < 1, \tag{1.3}$$

for a positive constant c_1. For $i = j = 0$, condition (1.3) leads to

$$|k(t,s)| \leqslant c_1 |t-s|^{-\nu}, \quad (t,s) \in \mathit{\Delta}.$$

Thus, the kernel $k \in W^{m,\nu}(\mathit{\Delta})$ is at most weakly singular for $0 < \nu < 1$. A particular important kernel is given by

$$k(t,s) = \varphi(t,s)|t-s|^{-\nu}, \quad 0 < \nu < 1, \quad \varphi \in C^m([0,1] \times [0,1]).$$

The paper is organized as follows. In Section 2, we recall the definition and main properties of the spline quasi-interpolation operators, with their convergence properties. In Section 3, we introduce the numerical method based on spline QI operators to solve Equation (1.1). A general framework for the error analysis of the approximate solution is given in Section 4. Lastly, in Section 5, we offer numerical outcomes that exemplify the theoretical approximation characteristics of the proposed method.

2 Spline quasi-interpolation

Let $\mathcal{X}_n := \{0 = x_0 < x_1 < \cdots < x_{2n} = 1\}$ be a non-uniform partition (a graded grid) of the interval $I := [0,1]$ with grid points

$$x_j := \frac{1}{2}\left(\frac{j}{n}\right)^r, \quad j = 0,1,\ldots,n, \quad x_{n+j} := 1 - x_{n-j}, \;\; j = 1,\ldots,n, \tag{2.1}$$

where $r \geqslant 1$ is a real number that ensures the non-uniformity of the grid $\mathcal{X}_n$. The partition (2.1) is uniform for $r = 1$, while it is more densely at the extremities of the interval I for $r > 1$, and the grid points are located symmetrically with respect to the midpoint of the interval I. Moreover, it is easy to see that

$$0 < h_j := x_{j+1} - x_j \leqslant \frac{r}{2n}\left(\frac{j+1}{n}\right)^{r-1} \leqslant \frac{r}{2n}\,, \quad j = 0,\ldots,n-1\,,$$

and similar inequality holds for the grid points on the other half of I.

For a positive integer d, let $\mathcal{S}_d\left(I, \mathcal{X}_n\right)$ be the space of C^{d-1} regular polynomial splines of degree d defined on the partition (2.1). With $\mathcal{J} := \{0,1,\ldots,2n+d-1\}$, its usual basis is composed of $2n+d$ classical normalized B-splines B_k, $k \in \mathcal{J}$. By adding multiple knots $x_{-d} = \cdots = x_0$ and $x_{2n} = \cdots = x_{2n+d}$, the support of each resulting B-spline B_k coincides with the interval $[x_{k-d}, x_{k+1}]$ (see [17]). A discrete quasi-interpolant in this space, abbreviated as dQI can be expressed as

$$\mathcal{Q}_d f := \sum_{k \in \mathcal{J}} \mu_k(f) B_k, \tag{2.2}$$

where the coefficients $\mu_k(f)$ are formed by linear combinations of f values at the specific points within $\mathcal{E}_n := \{\xi_i\, ,\, i = 0\, ,\, \ldots\, ,\, \mathcal{N}\}$, with

$$\begin{cases} \xi_i := t_i\ ,\ \mathcal{N} := 2n+1, & \text{if } \ d \text{ is even}, \\ \xi_i := x_i\ ,\ \mathcal{N} := 2n, & \text{if } \ d \text{ is odd}, \end{cases}$$

where $t_0 = x_0$, $t_{2n+1} = x_{2n}$ and $t_i = \dfrac{1}{2}\,(x_{i-1} + x_i)$ for $i = 1\, ,\, \ldots\, ,\, 2n$. More precisely, for $d+1 \leq k \leq 2n$, the coefficients $\mu_k(f)$ have the following form

$$\mu_k(f) := \begin{cases} \sum_{i=0}^{d} \alpha_{i,k} f\left(\xi_{k-d+i}\right), & \text{if } d \text{ is even}, \\ \sum_{i=1}^{d} \alpha_{i,k} f\left(\xi_{k-d+i-1}\right), & \text{if } d \text{ is odd}. \end{cases}$$

The values of $\alpha_{i,k}$ are determined to ensure that $\mathcal{Q}_d$ accurately reproduces the polynomial space $\mathbb{P}_d$, including all polynomials with total degree of d, i.e.,

$$\mathcal{Q}_d p = p, \quad \text{for all } p \in \mathbb{P}_d.$$

The extremal coefficients $\mu_k(f)$ have particular expressions (more details on the construction of dQIs are given in [10, 17]). The dQI $\mathcal{Q}_d$ can be expressed as

$$\mathcal{Q}_d f = \sum_{j=0}^{\mathcal{N}} f\left(\xi_j\right) L_j,$$

where each L_j is a specific linear combination of B-splines. Due to the continuity of μ_k as linear functionals, the quasi-interpolation operator $\mathcal{Q}_d$ defined from the quas-interpolant $\mathcal{Q}_d f$is uniformly bounded in the space $\mathcal{C}(I)$. By a well-known result in approximation theory (see [4]), it follows that for any $f \in \mathcal{C}^{d+1}(I)$, the following relation holds:

$$\|f - \mathcal{Q}_d f\| \leq C_1 h^{d+1} \|f^{(d+1)}\|,$$

where $h := \max\limits_{0 \leqslant j \leqslant 2n} h_j$ and C_1 is a positive constant independent of h and f.

Subsequently, we present an illustration of a spline dQI, represented as Equation (2.1), when considering the case where $d = 2$. This operator is defined on the space $\mathcal{S}_2(I, \mathcal{X}_n)$ of $\mathcal{C}^1$ quadratic splines (see, e.g., [16]) as

$$\mathcal{Q}_2 f := \sum_{k=0}^{2n+1} \mu_k(f) B_k, \tag{2.3}$$

where the coefficient functionals $\mu_k(f)$ are given by

$$\begin{aligned} &\mu_0(f) = f_0, \quad \mu_{2n+1}(f) = f_{2n+1}, \\ &\mu_j(f) = a_j f_{j-1} + b_j f_j + c_j f_{j+1}, \quad 1 \leq j \leq 2n. \end{aligned} \tag{2.4}$$

with $f_j := f(\xi_j)$ and

$$a_j = -\frac{h_j^2}{(h_{j-1}+h_j)(h_{j-1}+2h_j+h_{j+1})},$$
$$c_j = -\frac{h_j^2}{(h_j+h_{j+1})(h_{j-1}+2h_j+h_{j+1})}, \quad b_j = 1-(a_j+c_j).$$

The following result demonstrates both the boundedness of the operator $\mathcal{Q}_d$ (see [12] ch. 9) and its property as a local approximation of f. Additionally, it highlights that the spline $\mathcal{Q}_d f$ represents the best polynomial approximation of f with a degree d.

Lemma 1. *Let $\mathcal{Q}_d$ be the dQI defined by (2.2) and $f \in \mathcal{C}(I)$. Then for $0 \leq j \leq 2n-1$, we have*

$$\|\mathcal{Q}_d f\|_{\infty,[x_j,x_{j+1}]} \leq k_d \|f\|_{\infty,[x_{j-d+1},x_{j+d}]}, \tag{2.5}$$
$$\|f-\mathcal{Q}_d f\|_{\infty,[x_j,x_{j+1}]} \leqslant (1+k_d)\, d_{\infty,[x_{j-d+1},x_{j+d}]}(f,\mathbb{P}_d), \tag{2.6}$$

where

$$d_{\infty,[u,v]}(f,\mathbb{P}_d) := \inf\left\{\|f-p\|_{\infty,[u,v]}, p \in \mathbb{P}_d\right\},$$
$$\|f-p\|_{\infty,[u,v]} := \max_{x\in[u,v]} |f(x)-p(x)|,$$

and $k_d := (2^d/d!)\,(d(d-1))^d$ is a constant depending only on d.

The main outcome of this section is given next, for which in the rest of this paper c, c_0, c_0', c_1 stand for positive constants which are independent of n and can take different values.

Theorem 1. *Let $\mathcal{Q}_d$ be the dQI defined by (2.2). Then for any function $f \in \mathcal{C}^{d+1,\nu}(I)$*

$$\|f-\mathcal{Q}_d f\|_\infty \leqslant c_1 \begin{cases} n^{-(d+1)}, & r \geqslant (d+1)/(2-\nu), \\ n^{-r(2-\nu)}, & 1 \leqslant r < (d+1)/(2-\nu), \end{cases} \tag{2.7}$$
$$\int_0^1 |f(t)-\mathcal{Q}_d f(t)|dt \leqslant c \begin{cases} n^{-(d+1)}, & r \geqslant (d+1)/(3-\nu), \\ n^{-r(3-\nu)}, & 1 \leqslant r < (d+1)/(3-\nu). \end{cases} \tag{2.8}$$

Proof. Let us first show the estimate on the interval $[x_0, x_d]$. Let $n \geqslant d$ be a non-negative integer and consider Taylor's formula of f at $t = x_{2d-1}$, then we have

$$f(t) = P_d(t) + \frac{1}{d!}\int_{x_{2d-1}}^t (t-s)^d f^{(d+1)}(s)ds,$$

where

$$P_d(t) = \sum_{k=0}^d \frac{(t-x_{2d-1})^k}{k!} f^{(k)}(x_{2d-1}).$$

Since $P_d \in \mathbb{P}_d$, we obtain

$$d_{\infty,[x_0,x_{2d-1}]}(f,\mathbb{P}_d) \leqslant \|f-P_d\|_{\infty,[x_0,x_{2d-1}]}.$$

For $j = 1, \ldots, d-1$ and using inequality (2.6) of Lemma 1, we get

$$\begin{aligned}\|f - \mathcal{Q}_d f\|_{\infty,[x_j,x_{j+1}]} &\leqslant (1+k_d)\,\|f - P_d\|_{\infty,[x_0,x_{d+j}]}\\ &\leqslant (1+k_d)\,\|f - P_d\|_{\infty,[x_0,x_{2d-1}]}.\end{aligned}$$

Then, it holds

$$\|f - \mathcal{Q}_d f\|_{\infty,[x_0,x_d]} \leqslant (1+k_d)\,\|f - P_d\|_{\infty,[x_0,x_{2d-1}]}.$$

Now, we estimate the norm $\|f - P_d\|_{\infty,[x_0,x_{2d-1}]}$. Using (1.2), we have

$$\begin{aligned}&|f(t) - P_d(t)| = \frac{1}{d!}\Big|\int_{x_{2d-1}}^{t} (t-s)^d f^{(d+1)}(s)ds\Big|\\ &\leqslant \frac{1}{d!}\int_{x_{2d-1}}^{t} (t-s)^d |f^{(d+1)}(s)|ds \leqslant \frac{c}{d!}\int_{x_{2d-1}}^{t} (t-s)^d s^{1-d-\nu} ds.\end{aligned}$$

Putting $t = x_{2d-1}\varepsilon$, $s = x_{2d-1}\delta$, and using the fact that $0 < \nu < 1$, we get

$$\begin{aligned}&|f(t) - P_d(t)| \leqslant \frac{c}{d!}\int_{\varepsilon}^{1} x_{2d-1}^{2-\nu}(\delta-\varepsilon)^d \delta^{1-d-\nu} d\delta\\ &\leqslant \frac{c}{d!} x_{2d-1}^{2-\nu}\int_{\varepsilon}^{1} (\delta-\varepsilon)^d \delta^{1-d-\nu} d\delta \leqslant c_0 x_{2d-1}^{2-\nu}.\end{aligned}$$

Since $x_{2d-1} = 0.5\,(2d-1)^r n^{-r}$, we deduce that

$$|f(t) - P_d(t)| \leqslant c_1 n^{-r(2-\nu)},$$

which gives

$$\|f - P_d\|_{\infty,[x_0,x_{2d-1}]} \leqslant c_1 n^{-r(2-\nu)}.$$

Hence,

$$\|f - \mathcal{Q}_d f\|_{\infty,[x_0,x_d]} \leqslant c_1 n^{-r(2-\nu)}\,. \tag{2.9}$$

Let us now show the estimate on the interval $[x_j, x_{j+1}]$ for $j = d, \ldots, n-1$. Consider Taylor's formula of f at $t = x_{j-d+1}$ given by

$$f(t) = P_{d,j}(t) + \frac{1}{d!}\int_{x_{j-d+1}}^{t} (t-s)^d f^{(d+1)}(s)ds,$$

where

$$P_{d,j}(t) = \sum_{k=0}^{d} \frac{(t - x_{j-d+1})^k}{k!} f^{(k)}(x_{j-d+1}).$$

From Lemma 1, we have

$$\begin{aligned}\|f - \mathcal{Q}_d f\|_{\infty,[x_j,x_{j+1}]} &\leqslant (1+k_d) d_{\infty,[x_{j-d+1},x_{j+d}]}\,(f, \mathbb{P}_d)\\ &\leqslant (1+k_d)\|f - P_{d,j}\|_{\infty,[x_{j-d+1},x_{j+d}]},\end{aligned}$$

and using (1.2), we get

$$|f(t)-P_{d,j}(t)| = \frac{1}{d!}\Big|\int_{x_{j-d+1}}^{t}(t-s)^d f^{(d+1)}(s)ds\Big|$$
$$\leqslant \frac{1}{d!}\int_{x_{j-d+1}}^{t}(t-s)^d|f^{(d+1)}(s)|ds \leqslant \frac{c}{d!}\int_{x_{j-d+1}}^{t}(t-s)^d\left(\sigma(s)\right)^{1-d-\nu}ds.$$

First, let us take $j=d,\dots,n-d$. Then, $\sigma(s)=s$, $s\geqslant x_{j-d+1}\geqslant (2d)^{-r}x_{j+d}$ and $1-d-\nu<0$. Using this result we get

$$|f(t)-P_{d,j}(t)| \leqslant \frac{c}{d!}\int_{x_{j-d+1}}^{t}(t-s)^d s^{1-d-\nu}ds \leqslant \frac{c}{d!}\int_{x_{j-d+1}}^{x_{j+d}}|t-s|^d s^{1-d-\nu}ds$$
$$\leqslant \frac{c}{d!}\left(x_{j+d}-x_{j-d+1}\right)^d\int_{x_{j-d+1}}^{x_{j+d}}s^{1-d-\nu}ds \leqslant \frac{c_0'}{d!}\left(x_{j+d}-x_{j-d+1}\right)^{d+1}\left(x_{j+d}\right)^{1-d-\nu}.$$

Since $x_{j-d+1}=\frac{1}{2}\left(\frac{j-d+1}{n}\right)^r$ and $x_{j+d}-x_{j-d+1}\leqslant\frac{(2d-1)r}{2}(j+d)^{r-1}n^{-r}$, it can be shown (by similar reasoning as in [21], chapter 7), that

$$\|f-P_{d,j}\|_{\infty,[x_{j-d+1},x_{j+d}]}\leqslant c_1\begin{cases} n^{-(d+1)}, & r\geqslant (d+1)/(2-\nu),\\ n^{-r(2-\nu)}, & 1\leqslant r\leqslant (d+1)/(2-\nu).\end{cases}$$

Therefore,

$$\|f-\mathcal{Q}_d f\|_{\infty,[x_{j-d+1},x_{j+d}]}\leqslant c_1\begin{cases} n^{-(d+1)}, & r\geqslant (d+1)(2-\nu),\\ n^{-r(2-\nu)}, & 1\leqslant r\leqslant (d+1)/(2-\nu).\end{cases}\tag{2.10}$$

Next, we consider $j=n-d+1,\dots,n-1$. Using $x_{n+d-1}-x_n\leqslant\frac{(d-1)r}{2n}$, it holds

$$|f(t)-P_{d,j}(t)|\leqslant\frac{c}{d!}\int_{x_{j-d+1}}^{t}|t-s|^d s^{1-d-\nu}ds$$
$$\leqslant\frac{c}{d!}\int_{x_{j-d+1}}^{x_n}|t-s|^d s^{1-d-\nu}ds+\frac{c}{d!}\int_{x_n}^{t}|t-s|^d(1-s)^{1-d-\nu}ds$$
$$\leqslant\frac{c}{d!}\int_{x_{n-2d+2}}^{x_n}|t-s|^d s^{1-d-\nu}ds+\frac{c}{d!}\left(x_{n+d}-x_n\right)^d\int_{x_n}^{x_{n+d-1}}(1-s)^{1-d-\nu}ds$$
$$\leqslant\frac{c_0}{d!}\left(x_n-x_{n-2d+2}\right)^{d+1}\left(x_n\right)^{1-d-\nu}+\frac{c}{d!}\left(x_{n+d-1}-x_n\right)^{d+1}\left(1-x_n\right)^{1-d-\nu}$$
$$\leqslant c_1\left(\left(\frac{r}{n}\right)^{d+1}+\left(\frac{r}{2n}\right)\right)^{d+1}\leqslant c_1 n^{-(d+1)}.$$

Then,

$$\|f-\mathcal{Q}_d f\|_{\infty,[x_{n-2d+2},x_{n+d-1}]}\leqslant c_1 n^{-(d+1)}.\tag{2.11}$$

By (2.9), (2.10) and (2.11) we deduce the estimate (2.7), for $j=0,\dots,n-1$.

Due to the symmetry argument the proof is similar, on the other half of the interval $[0,1]$, for $j=n,\dots,2n-1$.

For the estimate (2.8), it is easy to see that the local error associated with $\mathcal{Q}_d$ satisfies the inequality

$$\sup_{x_{j-1}<t<x_j} |f(t)-\mathcal{Q}_d f(t)| \leqslant c\,(x_j-x_{j-1})^{d+1}\sigma(x_j)^{1-d-\nu}, \quad j=1,\dots,2n. \tag{2.12}$$

Using (2.12) and the symmetry of the graded grid partition, we get

$$\begin{aligned}
\int_0^1 |f(t)-\mathcal{Q}_d f(t)|dt &= \sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}|f(t)-\mathcal{Q}_d f(t)|dt + \sum_{j=n+1}^{2n}\int_{x_{j-1}}^{x_j} f|(t)-\mathcal{Q}_d f(t)|dt \\
&\leqslant c\sum_{j=1}^{n}(x_j-x_{j-1})^{d+2}(x_j)^{1-d-\nu} + c\sum_{j=n+1}^{2n}(x_j-x_{j-1})^{d+2}(1-x_j)^{1-d-\nu} \\
&\leqslant 2c\sum_{j=1}^{n}\left(\frac{1}{2}rj^{r-1}n^{-r}\right)^{d+2}\left(\frac{1}{2}\left(\frac{j}{n}\right)^r\right)^{1-d-\nu} \leqslant c_0 n^{-r(3-\nu)}\sum_{j=1}^{n} j^{r(3-\nu)-d-2}.
\end{aligned}$$

From the fact that for a real ℓ the inequality

$$\sum_{\ell=1}^{n}\ell^{\alpha} \leqslant c_0 \begin{cases} n^{\alpha+1}, & \text{if } \alpha > -1, \\ 1, & \text{if } \alpha < -1, \end{cases}$$

holds, we deduce that

$$\int_0^1 |f(t)-\mathcal{Q}_d f(t)|dt \leqslant c \begin{cases} n^{-(d+1)}, & r \geqslant (d+1)/(3-\nu), \\ n^{-r(3-\nu)}, & 1 \leqslant r < (d+1)/(3-\nu), \end{cases}$$

which completes the proof. □

3 Numerical method based on $\mathcal{Q}_d$

Equation (1.1) can be written in operator form as

$$\begin{cases} u'(t) = \mathcal{A}u(t) + \mathcal{K}u(t) + g(t), \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where $\mathcal{A}u(t) := a(t)u(t)$ and $\mathcal{K}u(t) := \int_0^1 k(t,s)u(s)\,ds$ for all $t \in I$.

Let $v \in L^{\infty}(0,1)$ be a function given by $v(t) = u'(t)$. Then, u is given by

$$u(t) = (Jv)(t) + u_0\,, \quad 0 \le t \le 1,$$

where $(Jv)(t) = \int_0^t v(s)ds$. Using the above notations, Equation (3.1) takes the form

$$v = \mathcal{T}v + f, \tag{3.2}$$

where $\mathcal{T} := (\mathcal{K}+\mathcal{A})J$ and

$$f(t) := g(t) + u_0 a(t) + u_0\int_0^1 k(t,s)\,ds\,, \quad t \in I.$$

It is to be noted that $\mathcal{T}$ is a compact operator from $L^\infty(I)$ into $L^\infty(I)$ (see, e.g., [14]).

We propose to solve (3.2) by a collocation method based on $\mathcal{Q}_d$. More precisely, we consider the approximate equation

$$v_n - \mathcal{Q}_d \mathcal{T} v_n = \mathcal{Q}_d f. \tag{3.3}$$

Once v_n is obtained, we calculate the approximate solution u_n of u by

$$u_n(t) = \int_0^t v_n(s)ds + u_0. \tag{3.4}$$

In order to solve (3.3), we transform it into a system of linear equations. Indeed, it is straightforward to prove that v_n is a spline function of the form

$$v_n = \sum_{i=0}^{\mathcal{N}} c_i L_i. \tag{3.5}$$

By replacing v_n with the expression given in (3.3), we obtain

$$\sum_{i=0}^{\mathcal{N}} c_i L_i = \sum_{i=0}^{\mathcal{N}} \mathcal{T}\Bigg(\sum_{j=0}^{\mathcal{N}} c_j L_j \Bigg) (\xi_i)\, L_i + \sum_{i=0}^{\mathcal{N}} f(\xi_i) L_i.$$

By using the linear property of $\mathcal{T}$ and discerning the coefficients corresponding to L_j, we derive the following system

$$c_i - \sum_{j=0}^{\mathcal{N}} \mathcal{T}(L_j)\,(\xi_i)\, c_j = f\,(\xi_i)\ , \quad i = 0, \ldots, \mathcal{N}. \tag{3.6}$$

Let us define the vectors

$$\boldsymbol{\mathcal{C}_{\mathcal{N}}} := (c_0, \ldots, c_{\mathcal{N}})^T \quad \text{and} \quad \boldsymbol{\mathcal{F}_{\mathcal{N}}} := (f(\xi_0), \ldots, f(\xi_{\mathcal{N}}))^T\,,$$

and the matrices

$$\boldsymbol{\mathcal{A}} := (\beta_j\,(\xi_i))_{0 \leqslant i,j \leqslant \mathcal{N}}\,, \quad \boldsymbol{\mathcal{A}_{\mathcal{N}}} := \operatorname{diag}\,(a\,(\xi_i))_{0 \leqslant i \leqslant \mathcal{N}}\, \mathcal{A},$$
$$\boldsymbol{\mathcal{M}_{\mathcal{N}}} := \left(\int_0^1 k\,(\xi_i, s)\, \beta_j(s) ds \right)_{0 \leqslant i,j \leqslant \mathcal{N}},$$

with $\beta_j(s) := \int_0^s L_j(v)dv,\ j = 0, \ldots, \mathcal{N}$. Then, system (3.6) becomes

$$\Big(\boldsymbol{\mathcal{I}_{\mathcal{N}}} - \big(\boldsymbol{\mathcal{A}_{\mathcal{N}}} + \boldsymbol{\mathcal{M}_{\mathcal{N}}} \big) \Big) \boldsymbol{\mathcal{C}_{\mathcal{N}}} = \boldsymbol{\mathcal{F}_{\mathcal{N}}}. \tag{3.7}$$

After the solution $\boldsymbol{\mathcal{C}_{\mathcal{N}}}$ of (3.7) is computed, the respective approximations for v_n and u_n can be expressed as follows:

$$v_n = L(t)\mathcal{C}_{\mathcal{N}}, \quad u_n(t) = \beta(t)\mathcal{C}_{\mathcal{N}} + u_0\,,$$

where

$$L(t) := \big(L_0(t), L_1(t), \ldots, L_{\mathcal{N}}(t) \big), \quad \beta(t) := \big(\beta_0(t), \beta_1(t), \ldots, \beta_{\mathcal{N}}(t) \big).$$

Remark 1. It is crucial to highlight that system (3.7) involves integrals. During the implementation of the method, the integrals $\int_0^1 k(\xi_i,s)\,\beta_j(s)ds$ and $\int_0^s L_j(v)dv$, $0 \leqslant i,j \leqslant \mathcal{N}$ can be computed exactly, thanks to the explicit expression of the piecewise polynomial functions defining L_j. For additional insights, please refer to the appendix section.

4 Error analysis

To ensure thoroughness, we present the theorem concerning the existence and uniqueness of the solution for the problem (1.1).

Theorem 2. *Let $a, g \in C^{m,\nu}[0,1]$, $k \in W^{m,\nu}(\Delta)$, $0 < \nu < 1$. Furthermore, assume that the homogeneous problem associated with* (1.1) *has only the trivial solution $u = 0$. Under these conditions, it can be established that problem* (1.1) *possesses a unique solution $u \in C^{m+1,\nu-1}[0,1]$, and its derivative $v = u'$ belongs to the space $C^{m\,\nu}[0,1]$.*

Proof. See, Pedas [14]. □

The following result establishes both existence and uniqueness of the solution for the linear algebraic system (3.7). Its proof closely resembles that of Theorem 3 in reference [1].

Theorem 3. *Assume that $a, g \in C^{0,\nu}[0,1]$, $k \in W^{0,\nu}(\Delta)$, $0 < \nu < 1$. Then, for h enough small, the linear algebraic system (3.7) has a unique solution in $\mathbb{R}^{\mathcal{N}+1}$, consequently, the Equation (3.3) has a unique solution v_n.*

The following theorem outlines the convergence rate of the presented method. In addition, we represent c as the positive constants that remain unaffected by n and can vary based on distinct inequalities.

Theorem 4. *Assume that $a, g \in C^{d+1,\nu}[0,1]$, $k \in W^{d+1,\nu}(\Delta)$, $0 < \nu < 1$. Let u_n and v_n be the solutions satisfying (3.4) and (3.5), respectively. Then, for n sufficiently large, the following error estimates hold:*

$$\|v - v_n\|_\infty \leqslant c \begin{cases} n^{-r(2-\nu)} & \textit{for } \ 1 \leqslant r < (d+1)/(2-\nu), \\ n^{-(d+1)} & \textit{for } \ r \geqslant (d+1)/(2-\nu), \end{cases} \tag{4.1}$$

$$\|u - u_n\|_\infty \leqslant c \begin{cases} n^{-r(3-\nu)} & \textit{for } \ 1 \leqslant r < (d+1)/(3-\nu), \\ n^{-(d+1)} & \textit{for } \ r \geqslant (d+1)/(3-\nu). \end{cases} \tag{4.2}$$

Proof. It follows from Theorem 3 that the Equations (3.5) and (3.4) determine in a unique way approximate solutions v_n and u_n for v and u respectively. Using Theorem 2 and taking $m = d+1$, we find that $u \in C^{d+2,v-1}[0,1]$ and $v \in C^{d+1\,,v}[0,1]$. Therefore, the estimates

$$\|v - v_n\|_\infty \leqslant c_0 \|v - \mathcal{Q}_d v\|_\infty, \quad \|u - u_n\|_\infty \leqslant c \int_0^1 |v(s) - \mathcal{Q}_d v(s)| \mathrm{d}s,$$

together with (2.7) and (2.8) of Theorem 1 lead to (4.1) and (4.2), which completes the proof. □

5 Numerical results

In order to illustrate the performance of findings outlined in the preceding sections, we examine three examples of Fredholm integro-differential equations that we solve by the numerical method introduced in Section 3 and based on the operator $\mathcal{Q}_2$ defined by (2.3) on the interval $[0\,,1]$ endowed with graded grid partition (2.1).
For different values of n and grading exponent r, we compute the maximum absolute errors

$$E_\infty := \|u - u_n\|_\infty \quad \text{and} \quad E'_\infty := \|v - v_n\|_\infty \,,$$

where u_n and v_n are respectively the approximate solution and its derivative. The obtained results are shown in Tables 1–3, where the notation *NCO* stands for the numerical convergence orders obtained by the logarithm to base 2 of the ratio between two consecutive errors. Furthermore, to enable a straightforward comparison between numerical experiments and theoretical outcomes, we have included the anticipated theoretical convergence orders from Theorem 4 in the final row of each table.

Example 1. Let us consider the equation

$$\begin{cases} u'(t) = \sqrt{t}\, u(t) + g(t) + \displaystyle\int_0^1 |t-s|^{-1/2} u(s)\, ds \,, \quad t \in [0\,,1], \\ u(0) = 0, \end{cases}$$

where the function g is selected so that $u(t) = t^{3/2} + (1-t)^{3/2} - 1$ is the exact solution. It is easy to check that $u \in C^{m,\nu}[0,1]$ and $k \in W^{m,\nu}(\Delta)$ where $\nu = 1/2$ and $m \in \mathbb{N}$. From Theorem 4, the theoretical convergence orders associated to the approximate solutions u_n and v_n are given in the following way

$$E_\infty = \mathcal{O}\left(h^{\delta_r^{(0)}}\right) \quad \text{and} \quad E'_\infty = \mathcal{O}\left(h^{\delta_r^{(1)}}\right),$$

where

$$\delta_r^{(0)} = \begin{cases} 3, & \text{if} \quad r \geqslant 6/5, \\ 2.5\,r, & \text{if} \quad 1 \leqslant r < 6/5, \end{cases} \quad \text{and} \quad \delta_r^{(1)} = \begin{cases} 3, & \text{if} \quad r \geqslant 2, \\ 1.5\,r, & \text{if} \quad 1 \leqslant r < 2. \end{cases}$$

Example 2. Now, we consider the equation

$$\begin{cases} u'(t) = u(t) + g(t) + \displaystyle\int_0^1 |t-s|^{-5/6} u(s)\, ds \,, \quad t \in [0\,,1], \\ u(0) = 1. \end{cases}$$

The function g is selected so that $u(t) = t^{11/6} + (1-t)^{11/6}$ is the exact solution. In this case $u \in C^{m,\nu}[0,1]$ and $k \in W^{m,\nu}(\Delta)$ where $\nu = 5/6$ and $m \in \mathbb{N}$.
By applying Theorem 4, the theoretical convergence orders are

$$E_\infty = \mathcal{O}\left(h^{\delta_r^{(0)}}\right) \quad \text{and} \quad E'_\infty = \mathcal{O}\left(h^{\delta_r^{(1)}}\right),$$

Table 1. Absolute errors E_∞ , E'_∞ and corresponding *NCO*.

n	$r=1$		$r=1.1$		$r=6/5$	
	E_∞	NCO	E_∞	NCO	E_∞	NCO
4	1.02(−01)	−−	8.48(−02)	−−	7.86(−02)	−−
8	2.49(−02)	2.03	1.47(−02)	2.52	1.16(−02)	2.76
16	6.71(−03)	1.89	2.28(−03)	2.37	1.85(−03)	2.64
32	2.01(−03)	1.73	6.44(−04)	2.13	3.56(−04)	2.38
64	6.54(−04)	1.62	1.64(−04)	1.96	7.96(−05)	2.16
128	2.21(−04)	1.56	4.49(−05)	1.87	1.93(−05)	2.04
Theo. Value	−−	2.50	−−	2.75	−−	3.00
n	$r=1$		$r=1.4$		$r=2$	
	E'_∞	NCO	E'_∞	NCO	E'_∞	NCO
4	3.06(−01)	−−	2.32(−01)	−−	2.01(−01)	−−
8	9.36(−02)	1.70	2.22(−02)	3.38	1.31(−02)	3.94
16	1.31(−02)	2.83	5.23(−03)	2.08	7.96(−04)	4.04
32	7.52(−03)	0.79	3.82(−04)	3.77	3.98(−05)	4.31
64	1.30(−03)	2.53	7.58(−05)	2.33	7.10(−06)	2.48
128	4.54(−04)	1.51	1.61(−05)	2.23	8.01(−07)	3.14
Theo. Value	−−	1.50	−−	2.10	−−	3.00

Table 2. Absolute errors E_∞ , E'_∞ and corresponding *NCO*.

n	$r=1$		$r=1.1$		$r=18/13$	
	E_∞	NCO	E_∞	NCO	E_∞	NCO
4	6.95(−02)	−−	6.33(−02)	−−	4.94(−02)	−−
8	1.81(−02)	1.94	1.43(−02)	2.14	8.36(−03)	2.56
16	3.69(−03)	2.29	2.39(−03)	2.58	8.76(−04)	3.25
32	6.57(−04)	2.48	3.33(−04)	2.84	6.40(−05)	3.77
64	1.12(−04)	2.55	4.20(−05)	2.99	1.84(−06)	5.12
128	1.97(−05)	2.50	4.99(−06)	3.07	5.18(−08)	5.15
Theo. Value	−−	2.16	−−	2.38	−−	3.00
n	$r=1$		$r=1.4$		$r=18/7$	
	E'_∞	NCO	E'_∞	NCO	E'_∞	NCO
4	3.45(−01)	−−	3.17(−01)	−−	2.47(−01)	−−
8	1.06(−01)	1.70	8.64(−02)	1.87	5.59(−02)	2.14
16	2.30(−02)	2.20	1.42(−02)	2.60	4.28(−03)	3.70
32	3.58(−03)	2.68	1.82(−03)	2.96	2.69(−04)	3.99
64	6.67(−04)	2.42	2.55(−04)	2.83	1.62(−05)	4.04
128	1.21(−04)	2.45	3.10(−06)	1.78	1.01(−07)	4.01
Theo. Value	−−	1.16	−−	1.63	−−	3.00

where

$$\delta_r^{(0)} = \begin{cases} 3, & \text{if } r \geqslant 18/13, \\ (13/8)\, r, & \text{if } 1 \leqslant r < 18/13, \end{cases} \qquad \delta_r^{(1)} = \begin{cases} 3, & \text{if } r \geqslant 18/7, \\ (7/6)\, r, & \text{if } 1 \leqslant r < 18/7. \end{cases}$$

Table 3. Absolute errors E_∞ , E'_∞ and corresponding *NCO*.

n	$r=1$		$r=1.05$		$r=12/11$	
	E_∞	NCO	E_∞	NCO	E_∞	NCO
4	8.14(−01)	−−	8.14(−01)	−−	8.14(−01)	−−
8	2.62(−03)	8.27	2.31(−03)	8.46	2.09(−03)	8.60
16	5.91(−04)	2.14	4.61(−04)	2.32	3.74(−04)	2.48
32	1.00(−04)	2.55	6.73(−05)	2.77	4.83(−05)	2.95
64	2.05(−05)	2.29	1.24(−05)	2.44	8.01(−06)	2.59
128	5.24(−06)	1.96	2.90(−06)	2.09	1.71(−06)	2.22
Theo. Value	−−	2.75	−−	2.88	−−	3.00
n	$r=1$		$r=1.1$		$r=12/7$	
	E'_∞	NCO	E'_∞	NCO	E'_∞	NCO
4	3.76(−02)	−−	3.55(−02)	−−	2.66(−02)	−−
8	1.29(−02)	1.54	9.67(−03)	1.87	3.23(−03)	3.04
16	2.83(−03)	2.18	3.83(−03)	1.33	5.48(−04)	2.56
32	1.42(−03)	0.99	5.93(−04)	2.69	4.20(−05)	3.70
64	8.76(−05)	4.02	2.41(−05)	4.62	2.04(−06)	4.36
128	1.01(−05)	3.11	3.17(−06)	2.92	2.47(−07)	3.04
Theo. Value	−−	1.75	−−	1.92	−−	3.00

Example 3. As third example, consider the equation

$$\begin{cases} u'(t) = \sqrt{t^5}u(t) + g(t) + \displaystyle\int_0^1 |t-s|^{-1/4} u(s)\,ds\ , \quad t \in [0\ , 1], \\ u(0) = 0. \end{cases}$$

The function g is selected so that $u(t) = t^{7/4} + (1-t)^{7/4} - 1$ is the exact solution. In this case $u \in C^{m,\nu}[0,1]$ and $k \in W^{m,\nu}(\Delta)$, where $\nu = 1/4$ and $m \in \mathbb{N}$.
From Theorem 4, the theoretical convergence orders are

$$E_\infty = \mathcal{O}\left(h^{\delta_r^{(0)}}\right) \quad \text{and} \quad E'_\infty = \mathcal{O}\left(h^{\delta_r^{(1)}}\right),$$

where

$$\delta_r^{(0)} = \begin{cases} 3, & \text{if} \quad r \geqslant \dfrac{12}{11}, \\ \dfrac{11}{4}r, & \text{if} \quad 1 \leqslant r < \dfrac{12}{11}, \end{cases} \quad \text{and} \quad \delta_r^{(1)} = \begin{cases} 3, & \text{if} \quad r \geqslant \dfrac{12}{7}, \\ \dfrac{7}{4}r, & \text{if} \quad 1 \leqslant r < \dfrac{12}{7}. \end{cases}$$

From Tables 1–3 we can see that the used method provides small and interesting errors. In most case, they are bigger than the expected ones according to the theoretical results. This phenomenon is worth examining further in a future paper.

6 Conclusions

In this paper, we have introduced a numerical method based on spline quasi-interpolation operators defined on a bounded interval, designed to numerically solve a specific class of FIDEs with weakly singular kernels. We have thoroughly discussed the computational aspects involved in obtaining the approximate solution. Additionally, we presented and proved theoretical results concerning the convergence order, considering both the quasi-interpolant degree and the grading exponent of the graded grid partition. Throughout our experiments, we observed that the proposed method yields interesting small errors. Furthermore, the numerical convergence orders obtained were generally larger than what was expected based on the theoretical results. This interesting phenomenon warrants further investigation, and we intend to explore it in a future paper. While the majority of the numerical tests have confirmed the theoretical results, we acknowledge that in some cases, when the number of subintervals is sufficiently large, the convergence orders decrease significantly. This could be related to round-off errors caused by the concentration of subdivision nodes at the extremities.

References

[1] C. Allouch, D. Barrera, A. Saou, D. Sbibih and M. Tahrichi. Numerical methods based on spline quasi-interpolation operators for integro-differential equations. *Journal of Mathematical Modeling*, **10**(4):387–401, 2022. https://doi.org/10.22124/jmm.2022.20181.1756.

[2] A.T. Assanova and Sh.N. Nurmukanbet. A solution to a boundary-value problem for integro-differential equations with weakly singular kernels. *Russian Mathematics*, **65**(11):1–13, 2021. https://doi.org/10.3103/s1066369x21110013.

[3] A.T. Assanova and S.N. Nurmukanbet. A solvability of a problem for a Fredholm integro-differential equation with weakly singular kernel. *Lobachevskii Journal of Mathematics*, **43**(1):182–191, 2022. https://doi.org/10.1134/s1995080222040047.

[4] C. de Boor. *A Practical Guide to Splines.* Applied Mathematical Sciences. Springer New York, 2001.

[5] G. Deng, Y. Yang and E. Tohidi. High accurate pseudo-spectral Galerkin scheme for pantograph type Volterra integro-differential equations with singular kernels. *Applied Mathematics and Computation*, **396**:125866, 2021. https://doi.org/10.1016/j.amc.2020.125866.

[6] S.M. El-Sayed, D. Kaya and S. Zarea. The decomposition method applied to solve high-order linear Volterra-Fredholm integro-differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, **5**(2):105–112, 2004. https://doi.org/10.1515/ijnsns.2004.5.2.105.

[7] M. Ganesh and I.H. Sloan. Optimal order spline methods for nonlinear differential and integro-differential equations. *Applied Numerical Mathematics*, **29**(4):445–478, 1999. https://doi.org/10.1016/s0168-9274(98)00067-1.

[8] I. Hashim. Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. *Journal of Computational and Applied Mathematics*, **193**(2):658–664, 2006. https://doi.org/10.1016/j.cam.2005.05.034.

[9] H. Hawsar, M.S. Hari Mohan, H. Mudhafar, O.M. Pshtiwan, Y.A. Musawa and B. Dumitru. Novel algorithms to approximate the solution of nonlinear integro-differential equations of volterra-fredholm integro type. *AIMS Mathematics*, **8**(6):14572–14591, 2023. https://doi.org/10.3934/math.2023745.

[10] M.J. Ibáñez, A. Lamnii, H. Mraoui and D. Sbibih. Construction of spherical spline quasi-interpolants based on blossoming. *Journal of Computational and Applied Mathematics*, **234**(1):131–145, 2010. https://doi.org/10.1016/j.cam.2009.12.010.

[11] H. Jaradat, O. Alsayyed and S. Al-Shara. Numerical solution of linear integro-differential equations. *J. Math. Statist.*, **4**:250–254, 2008.

[12] T. Lyche and K. Morken. *Spline methods draft.* Department of Informatics, Center of Mathematics for Applications, University of Oslo, Oslo, 2008.

[13] S. Nas, S. Yalcinbas and M. Sezer. A Taylor polynomial approach for solving high-order linear Fredholm integro-differential equations. *International Journal of Mathematical Education in Science and Technology*, **31**(2):213–225, 2000. https://doi.org/10.1080/0020739x.2022.12131593.

[14] A. Pedas and E. Tamme. Spline collocation method for integro-differential equations with weakly singular kernels. *Journal of Computational and Applied Mathematics*, **197**(1):253–269, 2006. https://doi.org/10.1016/j.cam.2005.07.035.

[15] A. Saadatmandi and M. Dehghan. Numerical solution of the higher-order linear Fredholm integro-differential-difference equation with variable coefficients. *Computers & Mathematics with Applications*, **59**(8):2996–3004, 2010. https://doi.org/10.1016/j.camwa.2010.02.018.

[16] P. Sablonnière. Quadratic spline quasi-interpolants on bounded domains of $\mathbb{R}^d$, $d = 1, 2, 3$. *Spline and radial functions*, **61**:229–246, 2003.

[17] P. Sablonnière. Univariate spline quasi-interpolants and applications to numerical analysis. *arXiv:math/0504022*, 2005.

[18] X. Shao, L. Yang and A. Guo. A feedforward neural network based on Legendre polynomial for solving linear Fredholm integro-differential equations. *International Journal of Computer Mathematics*, **100**(7):1480–1499, 2023. https://doi.org/10.1080/00207160.2023.2191746.

[19] B. Tair, H. Guebbai, S. Segni and M. Ghiat. An approximation solution of linear Fredholm integro-differential equation using collocation and Kantorovich methods. *Journal of Applied Mathematics and Computing*, **68**(5):3505–3525, 2021. https://doi.org/10.1007/s12190-021-01654-2.

[20] B. Tair, H. Guebbai, S. Segni and M. Ghiat. Solving linear Fredholm integro-differential equation by Nyström method. *Journal of Applied Mathematics and Computational Mechanics*, **20**(3):53–64, 2021. https://doi.org/10.17512/jamcm.2021.3.05.

[21] G. Vainikko. *Multidimensional Weakly Singular Integral Equations.* Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1993. https://doi.org/10.1007/BFb0088979.

[22] S. Yalçinbaş and M. Sezer. A Taylor collocation method for the approximate solution of general linear Fredholm-Volterra integro-difference equations with mixed argument. *Applied Mathematics and Computation*, **175**(1):675–690, 2006. https://doi.org/10.1016/j.amc.2005.07.038.

[23] X. Zhang and H. Du. A generalized collocation method in reproducing kernel space for solving a weakly singular Fredholm integro-differential equations. *Applied Numerical Mathematics*, **156**:158–173, 2020. https://doi.org/10.1016/j.apnum.2020.04.019.

Appendix: The exact computation of the integrals involved in the linear system (3.7)

In this section, we report the exact calculation of the integrals appearing in the linear system (3.7), which must be solved to implement the method presented in this paper. We restrict ourselves to the case where the kernel is defined as

$$k(t,s) := |t-s|^{-\nu}, \quad 0<\nu<1,$$

and the basis functions are given by quadratic B-splines explicitly defined on the partition $\mathcal{X}_n$ (see [4]).

$$B_0(x) = \begin{cases} \dfrac{(x-x_0)^2}{(x_0-x_1)^2}, & \text{if } x_0 \leqslant x < x_1, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_1(x) = \begin{cases} \dfrac{(-x+x_0)(-2x_1x_2+x_0(x_1+x_2)+x(-2x_0+x_1+x_2))}{(x_1-x_0)^2(x_2-x_0)}, & \\ \qquad\qquad \text{if } x_0 \leqslant x < x_1, & \\ \dfrac{(x-x_2)^2}{(x_2-x_1)(x_2-x_0)}, & \text{if } x_1 \leqslant x < x_2, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_j(x) = \begin{cases} \dfrac{(x-x_{j-2})^2}{(x_{j-1}-x_{j-2})(x_j-x_{j-2})}, & \text{if } x_{j-2} \leqslant x < x_{j-1}, \\ \dfrac{x-x_{j-1}}{x_j-x_{j-1}}\left(\dfrac{x-x_{j+1}}{x_{j-1}-x_{j+1}} - \dfrac{x-x_{j-2}}{x_j-x_{j-2}}\right), & \\ \qquad\qquad \text{if } x_{j-1} \leqslant x < x_j,\ 2 \leqslant j \leqslant 2n-1, & \\ \dfrac{(x-x_{j+1})^2}{(x_{j+1}-x_j)(x_{j+1}-x_{j-1})}, & \text{if } x_j \leqslant x < x_{j+1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{2n}(x) = \begin{cases} \dfrac{(x-x_{2n-2})^2}{(x_{2n-1}-x_{2n-2})(x_{2n}-x_{2n-2})}, & \text{if } x_{2n-2} \leqslant x < x_{2n-1}, \\ \dfrac{(x-x_{2n})}{(x_{2n}-x_{2n-1})^2(x_{2n}-x_{2n-2})}(-2x_{2n-2}x_{2n-1}+x_{2n}(x_{2n-1}+x_{2n-2}) & \\ +x(-2x_{2n}+x_{2n-1}+x_{2n-2})), & \text{if } x_{2n-1} \leqslant x < x_{2n}, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{2n+1}(x) = \begin{cases} \dfrac{(x-x_{2n-1})^2}{(x_{2n}-x_{2n-1})^2}, & \text{if } x_{2n-1} \leqslant x < x_{2n}, \\ 0, & \text{otherwise}. \end{cases}$$

First, we calculate for $j=0,\ldots,2n+1$ the integrals β_j defined by $\beta_j(s) = \int_0^s L_j(t)dt$. We know that the quasi-Lagrange functions L_j are linear combinations of a finite number of B-splines. More precisely, it holds

$$L_j(t) = \begin{cases} B_0(t) + a_1 B_1(t), & j = 0, \\ b_1 B_1(t) + a_2 B_2(t), & j = 1, \\ c_{j-1} B_{j-1}(t) + b_j B_j(t) + a_{j+1} B_{j+1}(t), & j = 2, \ldots, 2n-1, \\ c_{2n-1} B_{2n-1}(t) + b_{2n} B_{2n}(t), & j = 2n, \\ c_{2n} B_{2n}(t) + B_{2n+1}(t), & j = 2n+1, \end{cases}$$

where the coefficients a_j, b_j and c_j, $j = 1, \ldots, 2n$, are given by (2.4). Hence, L_j are piecewise polynomial functions with compact support

$$\mathrm{Supp}(L_j) := [x_{j-3}, x_{j+2}], \;\; j = 0, \ldots, 2n+1,$$

where multiple knots at the extremities are properly defined. By integrating each L_j on its support, it is easy to see that

$$\beta_0(s) = \begin{cases} P_{0,\ell}(s), & x_{\ell-1} \leqslant s \leqslant x_\ell \,,\; \ell = 1, 2, \\ \lambda_0, & s \geqslant x_2, \end{cases}$$

$$\beta_1(s) = \begin{cases} P_{1,\ell}(s), & x_{\ell-1} \leqslant s \leqslant x_\ell, \; \ell = 1, 2, 3, \\ \lambda_1, & s \geqslant x_3, \end{cases}$$

$$\beta_2(s) = \begin{cases} P_{2,\ell}(s), & x_{\ell-1} \leqslant s \leqslant x_\ell \,,\; \ell = 1, 2, 3, 4, \\ \lambda_2, & s \geqslant x_4, \end{cases}$$

$$\beta_j(s) = \begin{cases} 0, & s \leqslant x_{j-3}, \\ P_{j,\ell}(s), & x_{\ell-1} \leqslant s \leqslant x_\ell \,,\; \ell = j-2, \ldots, j+2 \,,\; j = 3, \ldots, 2n-2, \\ \lambda_j, & s \geqslant x_{j+2}, \end{cases}$$

$$\beta_{2n-1}(s) = \begin{cases} 0, & s \leqslant x_{2n-4}, \\ P_{2n-1,\ell}(s), & x_{\ell-1} \leqslant s \leqslant x_\ell \,,\; \ell = 2n-3, \ldots, 2n, \\ \lambda_{2n-1}, & s \geqslant x_{2n}, \end{cases}$$

$$\beta_{2n}(s) = \begin{cases} 0, & s \leqslant x_{2n-3}, \\ P_{2n,\ell}(s), & x_{\ell-1} \leqslant s \leqslant x_\ell \,,\; \ell = 2n-2, \ldots, 2n, \\ \lambda_{2n}, & s \geqslant x_{2n}, \end{cases}$$

$$\beta_{2n+1}(s) = \begin{cases} 0, & s \leqslant x_{2n-2}, \\ P_{2n+1,\ell}(s), & x_{\ell-1} \leqslant s \leqslant x_\ell \,,\; \ell = 2n-1, \ldots, 2n, \\ \lambda_{2n+1}, & s \geqslant x_{2n}, \end{cases}$$

where $P_{j,\ell}$, $j = 0, \ldots, 2n+1$, $\ell = 0, \ldots, 2n+1$, are cubic polynomials of the form $P_{j,\ell}(s) = \alpha_3^{j,\ell} s^3 + \alpha_2^{j,\ell} s^2 + \alpha_1^{j,\ell} s + \alpha_0^{j,\ell}$ and λ_j, $j = 0, \ldots, 2n+1$, are real constants. We can now give the expressions of the integrals

$$\mathcal{I}_j(x) := \int_0^1 k\,(x, s)\, \beta_j(s) ds, \quad x \in [0, 1].$$

Indeed, let $\mathcal{B}_\nu^k[a,b]$ be the functions defined for $k \in \{0,1,2,3\}$, $0 \leqslant a\,, b \leqslant 1$ and $0 < \nu < 1$, by

$$\mathcal{B}_\nu^k[a,b](x) := \int_a^b s^k |x-s|^{-\nu} ds, \quad x \in [0,1].$$

These functions are exactly computed with the help of a computer algebra system. Then, from the formulas of β_j given above, it is easy to show that the required integrals are given by

$$\mathcal{I}_0\left(\xi_i\right) = \sum_{\ell=1}^{2} \sum_{k=0}^{3} \alpha_k^{0,\ell} \mathcal{B}_\nu^k[x_{\ell-1}, x_\ell]\left(\xi_i\right) + \lambda_0 \mathcal{B}_\nu^0[x_2, 1]\left(\xi_i\right),$$

$$\mathcal{I}_1\left(\xi_i\right) = \sum_{\ell=1}^{3} \sum_{k=0}^{3} \alpha_k^{1,\ell} \mathcal{B}_\nu^k[x_{\ell-1}, x_\ell]\left(\xi_i\right) + \lambda_1 \mathcal{B}_\nu^0[x_3, 1]\left(\xi_i\right),$$

$$\mathcal{I}_2\left(\xi_i\right) = \sum_{\ell=1}^{4} \sum_{k=0}^{3} \alpha_k^{2,\ell} \mathcal{B}_\nu^k[x_{\ell-1}, x_\ell]\left(\xi_i\right) + \lambda_2 \mathcal{B}_\nu^0[x_4, 1]\left(\xi_i\right),$$

$$\mathcal{I}_j\left(\xi_i\right) = \sum_{\ell=j-2}^{j+2} \sum_{k=0}^{3} \alpha_k^{j,\ell} \mathcal{B}_\nu^k[x_{\ell-1}, x_\ell]\left(\xi_i\right) + \lambda_j \mathcal{B}_\nu^0[x_{j+2}, 1]\left(\xi_i\right), \quad j = 3\,, \ldots\,, 2n-2,$$

$$\mathcal{I}_{2n-1}\left(\xi_i\right) = \sum_{\ell=2n-3}^{2n} \sum_{k=0}^{3} \alpha_k^{2n-1,\ell} \mathcal{B}_\nu^k[x_{\ell-1}, x_\ell]\left(\xi_i\right),$$

$$\mathcal{I}_{2n}\left(\xi_i\right) = \sum_{\ell=2n-2}^{2n} \sum_{k=0}^{3} \alpha_k^{2n,\ell} \mathcal{B}_\nu^k[x_{\ell-1}, x_\ell]\left(\xi_i\right),$$

$$\mathcal{I}_{2n+1}\left(\xi_i\right) = \sum_{\ell=2n-1}^{2n} \sum_{k=0}^{3} \alpha_k^{2n+1,\ell} \mathcal{B}_\nu^k[x_{\ell-1}, x_\ell]\left(\xi_i\right).$$