Mathematical Modelling and Analysis
Volume 29, Issue 3, 460–479, 2024
https://doi.org/10.3846/mma.2024.18696

Existence Results in Weighted Sobolev Space for Quasilinear Degenerate $p(z)-$Elliptic Problems with a Hardy Potential

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Received February 27, 2023; accepted October 9, 2023

Abstract. The objective of this work is to establish the existence of entropy solutions to degenerate nonlinear elliptic problems for L^1-data f with a Hardy potential, in weighted Sobolev spaces with variable exponent, which are represented as follows

$$-\operatorname{div}\big(\Phi(z,v,\nabla v)\big)+g(z,v,\nabla v)=f+\rho\frac{|v|^{p(z)-2}v}{|v|^{p(z)}},$$

where $-\operatorname{div}(\Phi(z,v,\nabla v))$ is a Leray-Lions operator from $W_0^{1,p(z)}(\Omega,\omega)$ into its dual, $g(z,v,\nabla v)$ is a non-linearity term that only meets the growth condition, and $\rho>0$ is a constant.

Keywords: nonlinear elliptic equations, entropy solutions, Hardy potential, weighted variable exponent Sobolev space.

AMS Subject Classification: 35J60; 35J70; 46E35.

1 Introduction

Our focus is to investigate the existence of entropy solutions for a specific set of weighted quasilinear degenerated elliptic equations, which includes a Hardy potential term. The presence of this term often poses significant challenges in finding a solution. To overcome this obstacle, we rely on the use of weighted Sobolev spaces, which have proven to be an effective tool for our study. We will examine a new framework that comprises Sobolev spaces with variable exponents and weights, as elaborated in Section 2. Initially, we will assume that Ω is a bounded open subset of $\mathbb{R}^N$, where $N \geq 2$, and that $\partial\Omega$ denotes the boundary of this subset. Additionally, the variable exponent $p(z) : \Omega \to [1, \infty]$ is a continuous Log-Hölder function that depends solely on the space variable x (definitions are provided below). Furthermore, there exists a weight function ω that is measurable and strictly positive at almost all points in ω, satisfying certain integrability conditions defined in Section 2. We are interested in studying a specific model of the problem expressed as:

$$\begin{cases} \mathcal{L}v + g(z, v, \nabla v) = f + \rho \frac{|v|^{p(z)-2}v}{|z|^{p(z)}} \text{ in } \Omega, \\ v = 0 \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is Leray-Lions operator operating from $W_0^{1,p(z)}(\Omega, \omega)$ to its dual space $W^{-1,p'(z)}(\Omega, \omega^*)$, where $\omega^* = \omega^{1-p'(z)}$ defines as follows

$$\mathcal{L}v = -\mathrm{div}\big(\Phi(z, v, \nabla v)\big),$$

such that Φ is also a Carathéodory function setting of $\Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfying the ellipticity, strict monotonicity and growth assumptions, while $g: \ \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a nonlinear term with natural growth terms with respect to ∇v and $f \in L^1(\Omega)$.

The weighted Sobolev spaces $W^{k,p}(\Omega)$ appear in general as solution spaces for parabolic and elliptic partial differential equations. For degenerate partial differential equations, it quite natural to try to find solutions in weighted Sobolev spaces (see [5, 11, 16, 18, 20, 24] for more details).

In various practical applications, we may encounter boundary value problems for elliptic equations that have perturbations in their ellipticity due to the presence of a degeneracy or singularity. Such unfavorable behavior can stem from both the coefficients of the corresponding differential operator and the solution it self. One example of such an operator is the p-Laplacian, which is characterized by a degeneracy or singularity of the classical Laplace operator (where $p = 2$). These differential equations arise in a variety of practical problems, including glaciology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, oil production, and reaction-diffusion problems. For further examples of the practical applications of degenerate elliptic equations, see [14, 17].

To carry out our analysis, we will review some previous studies that have dealt with a particular case of the problem (1.1). Indeed, we will focus on results related to the elliptic equation (1.1) when f in $L^1(\Omega)$. We will also examine the following problem

$$\begin{cases} \mathcal{L}v + g(z, v, \nabla v) = f + \rho \frac{|v|^{p(z)-2}v}{|z|^{p(z)}} \quad \text{in} \quad \mathcal{D}'(\Omega), \\ v \in W_0^{1,p(z)}(\Omega, \omega), \quad g(z, v, \nabla v) \in L^1(\Omega). \end{cases} \tag{1.2}$$

When p is constant and without weight, there is a considerable amount of literature on the problem represented by (1.2). In [23] the author realised a new existence result for a class of nonlinear elliptic equations containing a p-Laplacian operator, without any restriction on the sign of g, and with bounded measures data. In [12] the authors studied the problem when the second term f belongs to $W^{-1,p'}(\Omega)$ and obtained an existence result. In [15] the authors obtained existence results in the case where $g(z,v)$ is monotone and increasing in v. In [13] the authors also treated the same topic in the case where $f \in L^1(\Omega)$. In [28] the authors have extended these results to the one-sided case. Likewise to the previous work, Del Vecchio [26] was the first to prove an existence result for the problem when g is not necessarily equal to zero and does not depend on the gradient, which was later extended in [22] by using rearrangement techniques. In their study mentioned in [6], the authors treated the problem using Sobolev spaces with weight $W_0^{1,p}(\Omega,\omega)$ but with a fixed value of p.

Moreover, in [8] the authors studied the obstacle problems associated with equation (1.2) in the non-classical case, by considering non-standard weightless anisotropic Sobolev spaces $W_0^{1,p(z)}(\Omega)$ and proving an existence result without sign condition on g. Recent studies on elliptic problems, Hardy potential and entropy theory can be found in several papers by Di Fazio, Hjiej, Ragusa, such as [7, 10, 19, 25]. More information on degenerate nonlinear elliptic equations can be found in [1, 2, 3].

The objective of this paper is to investigate the existence of entropy solutions to nonlinear elliptic problems that are less regular than weak solutions, under certain assumptions outlined in problem (1.2), which were first addressed by Benilan. The key contribution of this study is to address the non-coercivity of $\mathcal{L}v$ by introducing a penalty term of the form $\frac{1}{\eta}|v|^{p(z)-2}v$ in approximate problems. This approach enables the circumvention of the singularity of $\frac{|v|^{p(z)-2}v}{|z|^{p(z)}}$, which would otherwise prevent the existence of solutions. The regularization effect of $g(z,v,\nabla v)$ is also utilized in this work. Additionally, this study extends previous research on entropy solutions in unweighted spaces to encompass weighted spaces $W_0^{1,p(z)}(\Omega,\omega)$.

The rest of the paper is organized as follows. In Section 2, we first recall some useful definitions and properties of weighted Lebesgue and Sobolev spaces with variable exponents. We then describe the functional framework in which our work takes place and make some basic assumptions. Section 3 is devoted to the main results of this paper. Based on the theory of pseudo-monotone operators and the strong convergence of truncations of approximate solutions, we prove the existence of at least one entropic solution to (1.1). Finally, the proof of the Lemma 5 is given in the Appendix.

2 Assumptions and background mathematics

2.1 Basic assumptions

In this paper, we maintain the following assumptions throughout: Consider a weight function denoted as ω, subject to the following condition

$$\omega \in L^1_{loc}(\Omega) \quad \text{and} \quad \omega^{\frac{-1}{p(z)-1}} \in L^1_{loc}(\Omega). \tag{2.1}$$

Additionally, we make the assumption that the weight function satisfies the following condition

$$\omega^{-s(z)} \in L^1_{loc}(\Omega), \text{ where } \quad s(z) \in \Big]\frac{N}{p(z)}, \infty\Big[\cap \Big[\frac{1}{p(z)-1}, \infty\Big[. \tag{2.2}$$

Here, s is a positive function, which will be specified later. We also introduce the Leray-Lions operator $\mathcal{L}$, defined from $W_0^{1,p(z)}(\Omega,\omega)$ to its dual space $W^{-1,p'(z)}(\Omega,\omega^*)$, given by $\mathcal{L}v = -\mathrm{div}\big(\Phi(z,v,\nabla v)\big)$. It's worth noting that the function Φ in the following hypotheses satisfies

$$|\Phi(z,s,\zeta)| \leq \beta\omega^{\frac{1}{p(z)}}\Big[\xi(z) + |s|^{\frac{p(z)}{p'(z)}} + \omega^{\frac{1}{p'(z)}}|\zeta|^{p(z)-1}\Big], \tag{2.3}$$

$$[\Phi(z,s,\zeta) - \Phi(z,s,\overline{\zeta})]\,(\zeta - \overline{\zeta}) > 0, \quad \text{ for all } \zeta \neq \overline{\zeta} \in \mathbb{R}^N, \tag{2.4}$$

$$\Phi(z,s,\zeta)\,\zeta \geq \alpha\omega|\,\zeta|^{p(z)}. \tag{2.5}$$

Such that Ω is a measurable set in $\mathbb{R}^N$, and $\xi(z)$ is a positive function in $L^{p'(z)}(\Omega)$, where $p'(z)$ is the conjugate exponent of $p(z)$, and α and β are positive constants. Furthermore, g is a Carathéodory function with $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which satisfies, almost everywhere for $z \in \Omega$ and for every $s \in \mathbb{R}$ and $\zeta \in \mathbb{R}^N$, the following condition:

$$|g(z,s,\zeta)| \leq h(|s|)\omega(z)|\,\zeta|^{p(z)} + \mathcal{C}(z), \tag{2.6}$$

where $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a non-decreasing, continuous and positive function, while $\mathcal{C}(z)$ is a positive function in $L^1(\Omega)$. We assume that

$$f \in L^1(\Omega) \quad \text{ and } \quad |v|^{p(z)-2}v/|z|^{p(z)} \in L^1(\Omega). \tag{2.7}$$

We will use the truncation function $T_k(s)$, which is defined by

$$T_k(s) = \max\big\{-k, \min\{s,k\}\big\}.$$

2.2 Background mathematics

Let Ω be a bounded open subset of $\mathbb{R}^N$, where $N \geq 2$. We shall denote the following:

$$C_+(\overline{\Omega}) = \big\{p(z) : \Omega \to \mathbb{R} \text{ such that } p^- \leq p(z) \leq p^+ < +\infty\big\},$$

where

$$p^- := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} p(z); \quad p^+ := \operatorname*{ess\,sup}_{z\in\overline{\Omega}} p(z)$$

and $\omega(z)$ is a weight function, i.e., ω is a measurable function that is strictly positive a.e. in Ω. Consider p belongs to $C_+(\overline{\Omega})$ and ω as a weighted function in Ω. We set

$$\mathcal{T}_0^{1,p(z)}(\Omega,\omega) := \{v : \Omega \to \mathbb{R} \text{ measurable, such that } T_k(v) \in W_0^{1,p(z)}(\Omega,\omega) \text{ for each } k > 0\}.$$

First of all, we can give a simpler definition of an entropy solution of (1.2) as follows.

Definition 1. A measurable function v is called an entropy solution of the problem (1.1) if $v \in \mathcal{T}_0^{1,p(z)}(\Omega,\omega)$, $g(z,v,\nabla v) \in L^1(\Omega)$, $\frac{|v|^{p(z)-2}v}{|z|^{p(z)}} \in L^1(\Omega)$, and for each $\varphi \in W_0^{1,p(z)}(\Omega,\omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \Phi(z,v,\nabla v)\nabla T_k(v-\varphi)dz + \int_\Omega g(z,v,\nabla v)T_k(v-\varphi)dz \leq \int_\Omega fT_k(v-\varphi)dz + \rho\int_\Omega \frac{|v|^{p(z)-2}v}{|z|^{p(z)}}T_k(v-\varphi)dz.$$

We define the Lebesgue space with weights and variable exponents $L^{p(z)}(\Omega,\omega)$, as follows

$$L^{p(z)}(\Omega,\omega) = \{v : \Omega \to \mathbb{R}, \text{ measurable} : \int_\Omega \omega(z)\,|v|^{p(z)}dz < \infty\},$$

endowed with the norm $\|v\|_{p(z),\omega} = \inf\big\{\mu > 0 : \int_\Omega \omega(z)|\frac{u}{\mu}|^{p(z)}dz \leq 1\big\}$.

Proposition 1. *[4]The space $\big(L^{p(z)}(\Omega,\omega), \|\cdot\|_{p(z),\omega}\big)$ is of Banach.*

Lemma 1. *Let $v \in L^{p(z)}(\Omega,\omega)$. There are the following assertions*

1. *If* $\rho_\omega(v) > 1 \quad (= 1; < 1) \Leftrightarrow \|v\|_{p(z),\omega} > 1 \quad (= 1; < 1)$, *respectively.*
2. *If* $\|v\|_{p(z),\omega} > 1$, *then* $\|v\|^{p_-}_{p(z),\omega} \leq \rho_\omega(v) \leq \|v\|^{p^+}_{p(z),\omega}$.
3. *If* $\|v\|_{p(z),\omega} < 1$, *then* $\|v\|^{p^+}_{p(z),\omega} \leq \rho_\omega(v) \leq \|v\|^{p_-}_{p(z),\omega}$.

Proposition 2. *[4] Let Ω be a bounded open subset of $\mathbb{R}^N$, and ω be a weight function on Ω, If (2.1) is verified then $L^{p(z)}(\Omega,\omega) \hookrightarrow L^1_{loc}(\Omega)$.*

We define $W^{1,p(z)}(\Omega,\omega)$ as follows

$$W^{1,p(z)}(\Omega,\omega) = \big\{v \in L^{p(z)}(\Omega) : |\nabla v| \in L^{p(z)}(\Omega,\omega)\big\},$$

equipped with the norm

$$\|u\|_{1,p(z),\omega} = \|v\|_{p(z)} + \sum_{i=1}^N \|\frac{\partial v}{\partial z_i}\|_{p(z),\omega}, \tag{2.8}$$

which is equivalent to the Luxembourg norm

$$|||v||| = \inf\Big\{\mu > 0 : \int_\Omega \Big(\Big|\frac{v}{\mu}\Big|^{p(z)} + \omega(z)\sum_{i=1}^N \Big|\frac{1}{\mu}\,\frac{\partial v}{\partial z_i}\Big|^{p(z)}\Big)dz \leq 1\Big\}.$$

Proposition 3. *[4] The space $\big(W^{1,p(z)}(\Omega,\omega), \|\cdot\|_{1,p(z),\omega}\big)$ is Banach, if ω is a weighted function in Ω satisfying the condition (2.1).*

We also define $W_0^{1,p(z)}(\Omega,\omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(z)}(\Omega,\omega)$ with respect to the norm (2.8). It can be observed that the space $\big(W_0^{1,p(z)}(\Omega,\omega), \||\cdot|\|\big)$ is a reflexive Banach space (see [4]). Notice that the assumptions (2.1) and (2.2) imply

$$\|v\|_{W_0^{1,p(z)}(\Omega,\omega)} = \sum_{i=1}^N \Big\|\frac{\partial v}{\partial z_i}\Big\|_{p(z),\omega},$$

which is a norm defined on $W_0^{1,p(z)}(\Omega,\omega)$ and its equivalent to (2.8).

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p(z)}(\Omega,\omega)$ is equivalent to $W^{-1,p'(z)}\big(\Omega,\omega^*\big)$, where $\omega^* = \omega^{1-p'(z)}$, $p'(z)$ is the conjugate of $p(z)$; i.e., $p'(z) = \frac{p(z)}{p(z)-1}$. For more results, we refer the reader to [1,2,3,4].

Proposition 4. *[9, proposition 2.1] Let us consider ω a weight function which satisfies* (2.1) *and $p_s \in C_+(\overline{\Omega})$. Then $W^{1,p(z)}(\Omega,\omega) \hookrightarrow W^{1,p_s(z)}(\Omega)$.*

Corollary 1. Let $p_s \in C_+(\overline{\Omega})$ and ω a fuction weight which satisfies (2.1). Then $W^{1,p(z)}(\Omega,\omega) \hookrightarrow\hookrightarrow L^{r(z)}(\Omega)$, for $1 \le r(z) < p_s^*(z)$, where $p_s^*(z) = \frac{Np_s(z)}{N-p_s(z)}$.

Lemma 2. *[4]Let $r \in C_+(\overline{\Omega})$, ω be a weighted function in Ω, $g \in L^{r(z)}(\Omega,\omega)$ and $(g_\eta)_\eta \subset L^{r(z)}(\Omega,\omega)$ such that $\|g_\eta\|_{r(z),\omega} \le C$. If $g_\eta \to g$ a.e. in Ω, then $g_\eta \rightharpoonup g$ weakly in $L^{r(z)}(\Omega,\omega)$.*

Lemma 3. *Suppose that* (2.3)–(2.7) *hold, let $v \in W_0^{1,p(z)}(\Omega,\omega)$ and $(v_\eta)_\eta$ a sequence in $W_0^{1,p(z)}(\Omega,\omega)$, if $v_\eta \rightharpoonup v$ weakly in $W_0^{1,p(z)}(\Omega,\omega)$, and*

$$\int_\Omega \big(|v_\eta|^{p(z)-2}v_\eta - |v|^{p(z)-2}v\big)\big(v_\eta - v\big)dz + \int_\Omega \big(\Phi(z,v_\eta,\nabla v_\eta) - \Phi(z,v_\eta,\nabla v)\big)(\nabla v_\eta - \nabla v)dz \to 0,$$

then, $v_\eta \longrightarrow v$ strongly in $W_0^{1,p(z)}(\Omega,\omega)$.

Proof. The proof follows the usual techniques developed in [9, Lemma 4.1] for the case of anisotropic weighted Sobolev space. □

Lemma 4. *[4]Let $(v_\eta)_\eta$ a sequence from $W_0^{1,p(z)}(\Omega,\omega)$ such that $v_\eta \rightharpoonup v$ weakly in $W_0^{1,p(z)}(\Omega,\omega)$. Then $T_k(v_\eta) \rightharpoonup T_k(v)$ weakly in $W_0^{1,p(z)}(\Omega,\omega)$.*

3 Main result

This section is intended to derive the next existence Theorem.

Theorem 1. *Assume that* (2.1)–(2.7) *holds, then the problem* (1.1) *admits at least one entropy solution $v \in W_0^{1,p(z)}(\Omega,\omega)$.*

Proof of Theorem 1

Step 1: Approximate problems

Let us consider a sequence of smooth functions defined by $(f_\eta)_{\eta\in\mathbb{N}}$ which converges to f in $L^1(\Omega)$ and which satisfies $|f_\eta|\leq|f|$ and $g_\eta(z,s,\xi)=T_\eta(g(z,s,\xi))$. We take into account the approached problem

$$A_\eta v_\eta + g_\eta(z, v_\eta, \nabla v_\eta) = f_\eta + \rho\frac{|T_\eta(v_\eta)|^{p(z)-2}T_\eta(v_\eta)}{|z|^{p(z)}+1/\eta}, \tag{3.1}$$

such that $A_\eta u = -\mathrm{div}\big(\Phi(z, T_\eta(u), \nabla u)\big) + \frac{1}{\eta}|u|^{p(z)-2}u$. We consider $G_\eta : W_0^{1,p(z)}(\Omega,\omega) \to W^{-1,p'(z)}(\Omega,\omega^*)$ such that

$$\langle G_\eta v, \varphi\rangle = \int_\Omega g_\eta(z,v,\nabla v)\varphi dx - \rho\int_\Omega \frac{|T_\eta(v)|^{p(z)-2}T_\eta(v)}{|z|^{p(z)}+1/\eta}\varphi dz$$

for any $v, \varphi \in W_0^{1,p(z)}(\Omega,\omega)$.

Proposition 5. *The operator G_η is bounded.*

Proof. To begin the proof of this result, we apply (2.6), (2.7) and Hölder's inequality. Then, for v and φ belonging to $W_0^{1,p(z)}(\Omega,\omega)$, we can deduce that

$$\begin{aligned}
|\langle G_\eta v, \varphi\rangle| &\leq \int_\Omega |g_\eta(z,v,\nabla v)||\varphi| dz + \rho\int_\Omega \frac{|T_\eta(v)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}|\varphi| dz\\
&\leq \Big(\int_\Omega |g_\eta(z,v,\nabla v)| dz\Big)^{\frac{1}{(p^+)'}}\|\varphi\|_{p(z)} + \rho\Big(\int_\Omega \frac{|T_\eta(v)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}dz\Big)^{\frac{1}{(p^+)'}}\|\varphi\|_{p(z)}\\
&\leq (\eta^{p^+} + \rho\eta^{(p^+)'})(\mathrm{meas}(\Omega))^{\frac{1}{(p^+)'}}\|\varphi\|_{p(z)} \leq C_0\|\varphi\|_{W_0^{1,p(z)}(\Omega,\omega)}.
\end{aligned}$$

□

Lemma 5. *The operator $B_\eta = A_\eta + G_\eta : W_0^{1,p(z)}(\Omega,\omega) \to W^{-1,p'(z)}(\Omega,\omega^*)$ is a pseudo-monotone. Additionally, B_η is said to be coercive in the following sense*

$$\frac{\langle B_\eta u, u\rangle}{\|u\|_{W_0^{1,p(z)}(\Omega,\omega)}} \to +\infty \text{ as } \|u\|_{W_0^{1,p(z)}(\Omega,\omega)} \to \infty.$$

Proof. See in Appendix. □

According to Lemma 5 (cf. [21, Theorem 8.2]), there exists at least one weak solution, v_η in $W_0^{1,p(z)}(\Omega,\omega)$ for the problem defined in Equation (3.1).

Step 2: A priori estimates

Lemma 6. *Let us suppose that v_η is a weak solution of the problem* (3.1). *In this case, the regularity results stated below hold.*

$$v \in W_0^{1,q(z)}(\Omega,\omega), \tag{3.2}$$

such that $1 \leq q(z) < p(z)$ *and* $\omega^{-p(z)/(1-p(z))} \in L^1(\Omega)$,

$$\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda(z)}} dz \leq C \quad \textit{for each } 1 < \Lambda(z) < \frac{Np(z)-q(z)}{Nq(z)}, \tag{3.3}$$

$$\int_\Omega \omega|\nabla T_k(v_\eta)|^{p(z)} dz \leq C(1+k)^{\Lambda^+} \quad \textit{for all } \eta > 0, \tag{3.4}$$

where C *is a positive constant independent of* η *and* k.

Proof. In this step, we will use some methods of [27]. We choose $\Lambda(z) > 1$ and define the function $\Theta(y)$, which defines from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$\Theta(y) = sign(y)\Big(1 - \frac{1}{(1+|y|)^{\Lambda(z)-1}}\Big).$$

It is evident that $\Theta(v_\eta)$ belongs to $W_0^{1,p(z)}(\Omega,\omega)$. Using $\Theta(v_\eta)$ as the test function in Equation (3.1), we obtain

$$\Big(\Lambda^- - 1\Big)\int_\Omega \frac{\Phi(z,T_\eta(v_\eta),\nabla v_\eta)\nabla v_\eta}{(1+|v_\eta|)^{\Lambda(z)}} dz + \int_\Omega |g_\eta(z,v_\eta,\nabla v_\eta)||\Theta(v_\eta)| dz$$
$$+ \frac{1}{\eta}\int_\Omega |v_\eta|^{p(z)-2} v_\eta \Theta(v_\eta) dz = \int_\Omega f_\eta \Theta(v_\eta) dz + \rho\int_\Omega \frac{|T_\eta(v_\eta)|^{p(z)-2}T_\eta(v_\eta)}{|z|^{p(z)}+1/\eta}\Theta(v_\eta) dz.$$

Additionally, the sign of $\Theta(v_\eta)$ is the same as that of v_η, which makes the third term of the previous inequality positive. Furthermore, based on the Equation (2.5) and $|\Theta(\cdot)| \leq 1$ we conclude that

$$\Big(\Lambda^- - 1\Big)\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda(z)}} dz + \int_\Omega |g_\eta(z,v_\eta,\nabla v_\eta)|\Theta(v_\eta)| dz$$
$$\leq \rho\int_\Omega \frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta} dz + \int_\Omega |f| dz.$$

Using the growth assumption (2.6), we get

$$\Big(\Lambda^- - 1\Big)\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda(z)}} dz \leq C_1 + C_2\int_\Omega \omega|\nabla v_\eta|^{p(z)} dz$$
$$+ \rho\int_\Omega \frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta} dz + \int_\Omega |f| dz.$$

Thanks to Young's inequality, we get

$$\rho\int_\Omega \frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}} dz \leq \frac{1}{p'_-}\int_\Omega |T_\eta(v_\eta)|^{p(z)} dz + C_{p(z)}\int_\Omega \frac{dz}{|z|^{p^2(z)}}.$$

According to Corollary 1, we deduce that

$$(\Lambda^- - 1)\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda^-}} dz + \frac{1}{p_-}\int_\Omega |T_\eta(v_\eta)|^{p(z)} dz$$
$$\leq C_1 + C_{p(z)}\int_\Omega \frac{dz}{|z|^{p^2(z)}} + \int_\Omega |f| dz. \tag{3.5}$$

Since $p(z)>1$, the integral $\int_\Omega \frac{dz}{|z|^{p^2(z)}}$ is finite. As a result, Equation (3.3) can be deduced. Also, we get $\int_\Omega |T_\eta(v_\eta)|^{p(z)}dz \le C$. Using Hölder's inequality, we can conclude that if $q(z)$ is taken such that $1\le q(z)<p(z)$, then

$$\begin{aligned}\int_\Omega \omega|\nabla v_\eta|^{q(z)}dz \le & \left\| \frac{\omega^{\frac{q(z)}{p(z)}}|\nabla v_\eta|^{q(z)}}{\big(1+|v_\eta|\big)^{\frac{\Lambda(z)q(z)}{p(z)}}}\right\|_{\frac{p(z)}{q(z)}} \left\|\omega^{1-\frac{q(z)}{p(z)}}\big(1+|v_\eta|\big)^{\frac{\Lambda(z)q(z)}{p(z)}}\right\|_{\frac{p(z)}{p(z)-q(z)}}\\ \le & \Big(\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda(z)}}dz\Big)^{\frac{q^-}{p^-}}\Big(\int_\Omega \omega(1+|v_\eta|)^{\frac{q(z)\Lambda(z)}{p(z)-q(z)}}dz\Big)^{1-\frac{q^+}{p^+}}.\end{aligned}$$

From (2.1) and Hölder's inequality, we have

$$\begin{aligned}\int_\Omega \omega|\nabla v_\eta|^{q(z)}dz \le & \Big(\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda(z)}}dz\Big)^{\frac{q^-}{p^-}} \|\omega\|_{p'(z)}^{1-\frac{q^+}{p^+}} \left\|(1+|v_\eta|)^{\frac{q(z)\Lambda(z)}{p(z)-q(z)}}\right\|_{p(z)}^{1-\frac{q^+}{p^+}}\\ \le C_6 & \Big(\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda(z)}}dz\Big)^{\frac{q^-}{p^-}}\Big(\int_\Omega (1+|v_\eta|)^{\frac{p(z)\Lambda(z)q(z)}{p(z)-q(z)}}dz\Big)^{\frac{1}{p^-}(1-\frac{q^+}{p^+})}. \end{aligned}\tag{3.6}$$

We select $\Lambda(z)>1$ so that $\frac{q(z)\Lambda(z)p(z)}{p(z)-q(z)}<1$, where $\Lambda(z)$ exists if $q(z)<\frac{p(z)}{1+p(z)}$. In view of (3.5) and (3.6), we get the desired estimates (3.2). On the other hand, to get (3.4), we can write

$$\int_\Omega \omega|\nabla T_k(v_\eta)|^{p(z)}dz = \int_{\{|v_\eta|<k\}} \omega|\nabla v_\eta|^{p(z)}dz \le (1+k)^{\Lambda^+}\int_\Omega \frac{\omega|\nabla v_\eta|^{p(z)}}{(1+|v_\eta|)^{\Lambda(z)}}dz.$$

□

Step 3: Weak convergence of $(T_k(v_\eta))_\eta$ in $W_0^{1,p(z)}(\Omega,\omega)$

We first show that $(v_\eta)_\eta$ is a Cauchy sequence. This is possible because of the Equation (3.4).

$$\int_\Omega |\nabla T_k(v_\eta)|^{p(z)}\omega dz \le C(1+k)^{\Lambda^+}+k^{p^+}|\Omega| \quad for\ k\ge 1.$$

Consequently, if the sequence $(T_k(v_\eta))_\eta$ is bounded in $W_0^{1,p(z)}(\Omega,\omega)$, then it is possible to identify a specific subsequence denoted by $(T_k(v_\eta))_\eta$ such that

$$T_k(v_\eta)\rightharpoonup \delta_k \text{ in } W_0^{1,p(z)}(\Omega,\omega) \quad\text{and}\quad T_k(v_\eta)\to\delta_k \text{ in } L^{p(z)}(\Omega,\omega).\tag{3.7}$$

With the help of Equation (3.4), we can conclude that there exists a constant C_7 that is independent of both k and η, implying that

$$\|\nabla T_k(v_\eta)\|_{L^{p(z)}(\Omega,\omega)} \le C_7 k^{\Lambda^+/p^+} \quad \text{for } k\ge 1.\tag{3.8}$$

Given a ball B_R in Ω, if k is taken to be sufficiently large, by utilizing Equation (3.8) and invoking the Poincaré type inequality and Proposition 2, we arrive at the conclusion that

$$\begin{aligned}k\ \text{meas}\big(\{|v_\eta|>k\}\cap B_R\big) &= \int_{\{|v_\eta|>k\}\cap B_R}|T_k(v_\eta)|dz\\ &\le C_9\|\nabla T_k(v_\eta)\|_{L^{p(z)}(\Omega,\omega)} \le C_{10}k^{\frac{\Lambda^+}{p^+}}.\end{aligned}$$

Taking $\varLambda(z)$ such that $(1 < \varLambda(z) < p(z))$, we infer

$$\operatorname{meas}\big(\{|v_\eta| > k\} \cap B_R\big) \leq C_{10} \frac{1}{k^{1-\varLambda^+/p^+}} \to 0 \quad \text{as} \;\; k \to +\infty. \tag{3.9}$$

For each $\zeta > 0$, we obtain

$$\begin{aligned} \operatorname{meas}\big(\{|v_\eta - v_\theta| > \zeta\} \cap B_R\big) &\leq \operatorname{meas}\big(\{|v_\eta| > k\} \cap B_R\big) \\ &\quad + \operatorname{meas}\big(\{|v_\theta| > k\} \cap B_R\big) + \operatorname{meas}\big(\{|T_k(v_\eta) - T_k(v_\theta)| > \zeta\big). \end{aligned}$$

By the Equation (3.9), we can take a sufficiently large value of $k = k(m)$, where $m > 0$.

$$\operatorname{meas}\big(\{|v_\eta| > k\} \cap B_R\big) \leq m/3 \;\; \text{and} \;\; \operatorname{meas}\big(\{|v_\theta| > k\} \cap B_R\big) \leq m/3. \tag{3.10}$$

In other words, from the Equation (3.7), let $(T_k(v_\eta))_{\eta\in\mathbb{N}}$ is a Cauchy sequence in measure. Consequently, for every positive value of k and ζ, and for every positive value of m, there exists a specific value $\eta_0 = \eta_0(k, \zeta, m)$ such that

$$\operatorname{meas}\big\{|T_k(v_\eta) - T_k(v_\theta)| > \zeta\big\} \leq m/3 \quad \text{ for all } \eta \geq \eta_0(k, \zeta, m). \tag{3.11}$$

From the Equations (3.10) and (3.11), we conclude that for all positive values of ζ and m there exists a value $\eta_0 = \eta_0(k(m), \zeta, R)$ such that

$$\operatorname{meas}\big(\{|v_\eta - v_\theta| > \zeta\} \cap B_R\big) \leq m \qquad \forall \eta \geq \eta_0(k(m), \zeta, R).$$

This demonstrates that the sequence $(v_\eta)_\eta$ converges in measure and therefore converges a.e. to a measurable function v. As a result, we can state that

$$T_k(v_\eta) \rightharpoonup T_k(v) \quad \text{ in } \;\; W_0^{1,p(z)}(\varOmega, \omega),$$

and by means of the dominated convergence Theorem of Lebesgue, we arrive at

$$T_k(v_\eta) \to T_k(v) \;\; \text{in} \;\; L^{p(z)}(\varOmega, \omega) \text{ and a.e in } \varOmega. \tag{3.12}$$

Step 4: Strong convergence of truncations

In the following, we use the symbol $m(\eta)$ to represent various functions with real values that goes to zero as η approaches infinity.
Take $s > r > 0$ and define $\mathcal{A}_\eta := v_\eta - T_s(v_\eta) + T_r(v_\eta) - T_r(v)$. Next, let $\mathcal{B}_\eta$ be defined as $T_{2r}(\mathcal{A}_\eta)$. Using $\mathcal{B}_\eta$ as a test function in (3.1), the following result can be obtained

$$\begin{aligned} &\int_\varOmega \varPhi(z, T_\eta(v_\eta), \nabla v_\eta) \nabla \mathcal{B}_\eta dz + \int_\varOmega g_\eta(z, v_\eta, \nabla v_\eta) \mathcal{B}_\eta dz \\ &\quad + \frac{1}{\eta} \int_\varOmega |v_\eta|^{p(z)-2} v_\eta \mathcal{B}_\eta dz = \rho \int_\varOmega \frac{|T_\eta(v_\eta)|^{p(z)-2} T_\eta(v_\eta)}{|z|^{p(z)} + 1/\eta} \mathcal{B}_\eta dz + \int_\varOmega f_\eta \mathcal{B}_\eta dz. \end{aligned}$$

If $\mathcal{G} = 4r + s$, it is easy to see that $\nabla \mathcal{B}_\eta = 0$ on $\{|v_\eta| \geq \mathcal{G}\}$, and the sign of $\mathcal{B}_\eta$ is the same as that of v_η on $\{|v_\eta| > r\}$, (More precisely, if v_η is greater than

r, then $v_\eta - T_s(v_\eta) \geq 0$ and $T_r(v_\eta) - T_r(v) \geq 0$, it follows that $\mathcal{B}_\eta \geq 0$. In the same way, we prove that $\mathcal{B}_\eta \leq 0$ on the set $\{v_\eta < -r\}$). We find

$$\begin{aligned}
&\int_{\{|v_\eta| \leq \mathcal{G}\}} \Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta)) \nabla \mathcal{B}_\eta dz + \int_\Omega g_\eta(z, v_\eta, \nabla v_\eta) \mathcal{B}_\eta dz \\
&+\frac{1}{\eta} \int_{\{|v_\eta| \leq r\}} |v_\eta|^{p(z)-2} v_\eta \mathcal{B}_\eta dz = \rho \int_\Omega \frac{|T_\eta(v_\eta)|^{p(z)-2} T_\eta(v_\eta)}{|z|^{p(z)} + 1/\eta} \mathcal{B}_\eta dz + \int_\Omega f_\eta \mathcal{B}_\eta dz.
\end{aligned}$$

Due to Young's inequality, we get

$$\begin{aligned}
\rho \int_{\{|v_\eta| > r\}} \frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)} + 1/\eta} |\mathcal{B}_\eta| dz \leq & \int_{\{|v_\eta| > r\}} |T_\eta(v_\eta)|^{p(z)} |\mathcal{B}_\eta| dz \\
& + C_{11} \int_{\{|v_\eta| > r\}} \frac{|\mathcal{B}_\eta|}{|z|^{p^2(z)}} dz,
\end{aligned}$$

and as $\mathcal{B}_\eta = T_r(v_\eta) - T_r(v)$ on $\{|v_\eta| \leq r\}$, therefore

$$\begin{aligned}
&\int_{\{|v_\eta| \leq \mathcal{G}\}} \Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta)) \nabla \mathcal{B}_\eta dz + \int_{\{|v_\eta| \leq r\}} |g_\eta(z, v_\eta, \nabla v_\eta)| (T_r(v_\eta) - T_r(v)) dz \\
&\quad + \frac{1}{\eta} \int_{\{|v_\eta| \leq r\}} |T_r(v_\eta)|^{p(z)-2} T_r(v_\eta) (T_r(v_\eta) - T_r(v)) dz \qquad (3.13) \\
&\leq \rho \int_{\{|v_\eta| \leq r\}} \frac{|T_r(v_\eta)|^{p(z)-2} T_r(v_\eta)}{|z|^{p(z)} + 1/\eta} (T_r(v_\eta) - T_r(v)) dz \\
&\quad + \int_\Omega f_\eta \mathcal{B}_\eta dz + C_{11} \int_{\{v_\eta > r\}} \frac{|\mathcal{B}_\eta|}{|z|^{p^2(z)}} dz.
\end{aligned}$$

Now, let's examine each term in the previous inequality. Considering the second and third terms on the left-hand side of Equation (3.13), in accordance with Lebesgue's dominated convergence theorem, we can deduce that

$$\begin{aligned}
|g_\eta(z, v_\eta, \nabla v_\eta)| T_r(v_\eta) &\to |g_\eta(z, v, \nabla v)| T_r(v) \quad \text{in} \quad L^1(\Omega), \\
|T_r(v_\eta)|^{p(z)-2} T_r(v_\eta) &\to |T_r(v)|^{p(z)-2} T_r(v) \quad \text{in} \quad L^1(\Omega),
\end{aligned}$$

and from the fact that $T_r(v_\eta) \rightharpoonup T_r(v)$ weak $- *$ in $L^\infty(\Omega)$, hence

$$\begin{aligned}
m_1(\eta) &= \int_{\{|v_\eta| \leq r\}} |g_\eta(z, v_\eta, \nabla v_\eta)| (T_r(v_\eta) - T_r(v)) dz \to 0 \quad \text{as} \quad \eta \to \infty, \\
m_2(\eta) &= \frac{1}{\eta} \int_{\{|v_\eta| \leq r\}} |T_r(v_\eta)|^{p(z)-2} T_r(v_\eta) (T_r(v_\eta) - T_r(v)) dz \to 0 \quad \text{as} \quad \eta \to \infty.
\end{aligned}$$

For the terms of the second member of (3.13) one has

$$\begin{aligned}
m_3(\eta) &= \Big| \int_{\{|v_\eta| \leq r\}} \frac{|T_r(v_\eta)|^{p(z)-2} T_r(v_\eta)}{|z|^{p(z)} + 1/\eta} (T_r(v_\eta) - T_r(v)) dz \Big| \\
&\leq \max\left(r^{p^- - 1}, r^{p^+ - 1}\right) \int_{\{|v_\eta| \leq r\}} \frac{|T_r(v_\eta) - T_r(v)|}{|z|^{p(z)}} dz \to 0 \quad \text{as} \quad \eta \to \infty,
\end{aligned}$$

thus, we get

$$\int_\Omega f_\eta \mathcal{B}_\eta dz = \int_\Omega f T_{2r}(v - T_s(v))dz + m_4(\eta), \tag{3.14}$$

$$\int_{\{|v_\eta|>r\}} \frac{|\mathcal{B}_\eta|}{|z|^{p^2(z)}} dz = \int_{\{|v|>r\}} \frac{|T_{2r}(v - T_s(v))|}{|z|^{p^2(z)}} dz + m_5(\eta). \tag{3.15}$$

According to (3.13)–(3.15), we infer

$$\begin{aligned} &\int_{\{|v_\eta|\le \mathcal{G}\}} \Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta))\nabla \mathcal{B}_\eta dz \\ &\qquad \le \int_\Omega f T_{2r}(v - T_s(v))dz + C_{12}\int_{\{|v|>r\}} \frac{|T_{2r}(v - T_s(v))|}{|z|^{p^2(z)}} dz + m_6(\eta). \end{aligned} \tag{3.16}$$

Conversely, we obtain

$$\begin{aligned} \int_{\{|v_\eta|\le \mathcal{G}\}} &\Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta))\nabla \mathcal{B}_\eta dz = \int_\Omega \big(\Phi(z, T_r(v_\eta), \nabla T_r(v_\eta)) - \Phi(z, T_r(v_\eta), \\ &\nabla T_r(v))\big)(\nabla T_r(v_\eta) - \nabla T_r(v))dz + \int_\Omega \Phi\big(z, T_r(v_\eta), \nabla T_r(v)\big) \\ &\times (\nabla T_r(v_\eta) - \nabla T_r(v))dz + \int_{\{|v_\eta|>r\}} \Phi(z, T_r(v_\eta), \nabla T_r(v_\eta))\nabla T_r(v)dz \\ &+ \int_{\{r<|v_\eta|\le \mathcal{G}\}} \Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta))\nabla \mathcal{B}_\eta dz. \end{aligned} \tag{3.17}$$

Regarding the second and third terms on the right-hand side of Equation (3.17), thanks to Lebesgue's dominated convergence theorem, we have the convergence $T_r(v_\eta) \to T_r(v)$ in $L^{p(z)}(\Omega, \omega)$, which leads to $\Phi(z, T_r(v_\eta), \nabla T_r(v)) \to \Phi(z, T_r(v), \nabla T_r(v))$ in $L^{p'(z)}(\Omega, \omega^*)$. Additionally, since $\nabla T_r(v_\eta) \rightharpoonup \nabla T_r(v)$ in $L^{p(z)}(\Omega, \omega)$, it results that

$$m_7(\eta) = \int_\Omega \Phi(z, T_r(v_\eta), \nabla T_r(v))(\nabla T_r(v_\eta) - \nabla T_r(v))dz \to 0 \ \text{ as } \ \eta \to \infty,$$

and as $\Phi(z, s, 0) = 0$, we have

$$\int_{\{|v_\eta|>r\}} \Phi(z, T_r(v_\eta), \nabla T_r(v_\eta))\nabla T_r(v)dz = \int_{\{|v_\eta|>r\}} \Phi(z, T_r(v_\eta), 0)\nabla T_r(v)dz = 0.$$

About the last term on the right side of (3.17), thanks to (2.3) we have $(\Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta)))_\eta$ is bounded in $L^{p'(z)}(\Omega, \omega^*)$. This means that there exists a function $\Theta_\eta \in L^{p'(z)}(\Omega, \omega^*)$ such that the absolute value of $\Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta))$ converges to Θ_η in the $L^{p'(z)}(\Omega, \omega^*)$, it follows that

$$\begin{aligned} \lim_{\eta\to\infty} \int_{\{r<|v_\eta|\le \mathcal{G}\}} &\Phi(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta))\nabla \mathcal{B}_\eta dz = \lim_{\eta\to\infty} \int_{\{r<|v_\eta|\le \mathcal{G}\}\cap\{|\mathcal{A}_\eta|\le 2r\}} \Phi\big(z, T_\mathcal{G}(v_\eta), \nabla T_\mathcal{G}(v_\eta)\big) \\ &\times (\nabla v_\eta - \nabla T_s(v_\eta) - \nabla T_r(v))dz \ge -\int_{\{r<|v|\le \mathcal{G}\}} \Theta_\eta |\nabla T_r(v)|dz = 0. \end{aligned} \tag{3.18}$$

By (3.16)–(3.18), we deduced that

$$\int_{\Omega}(\Phi(z,T_r(v_\eta),\nabla T_r(v_\eta))-\Phi(z,T_r(v_\eta),\nabla T_r(v)))(\nabla T_r(v_\eta)-\nabla T_r(v))dz$$
$$\leq 2r\int_{\{|v|>s\}}|f|dz+2rC_{13}\int_{\{|v|>s\}}\frac{1}{|z|^{p^2(z)}}dz+m_8(\eta).$$

Letting η and then s tend to infinity in the above inequality, and using (3.12), we derive

$$\lim_{\eta\to\infty}\Big(\int_{\Omega}\big(\Phi(z,T_r(v_\eta),\nabla T_r(v_\eta))-\Phi(z,T_r(v_\eta),\nabla T_r(v))\big)(\nabla T_r(v_\eta)-\nabla T_r(v))dz$$
$$+\int_{\Omega}(|T_r(v_\eta)|^{p(z)-2}T_r(v_\eta)-|T_r(v_\eta)|^{p(z)-2}T_r(v_\eta))(T_r(v_\eta)-T_r(v))dz\Big)=0.$$

Using Lemma 3, we get

$$T_r(v_\eta)\to T_r(v) \text{ strongly in } W_0^{1,p(z)}(\Omega,\omega) \text{ and } \nabla v_\eta\to\nabla v \text{ a.e. in } \Omega. \tag{3.19}$$

Step 5: Equi-integrability of the nonlinearitie functions

Here, we will demonstrate that

$$g_\eta(z,v_\eta,\nabla v_\eta)\to g(z,v,\nabla v) \text{ strongly in } L^1(\Omega), \tag{3.20}$$
$$\frac{|T_\eta(v_\eta)|^{p(z)-2}T_\eta(v_\eta)}{|z|^{p(z)}+1/\eta}\to\frac{|v|^{p(z)-2}v}{|z|^{p(z)}} \text{ strongly in } L^1(\Omega),$$
$$\frac{1}{\eta}|v_\eta|^{p(z)-2}v_\eta\to 0 \quad \text{strongly in } L^1(\Omega). \tag{3.21}$$

First of all, we have

$$g_\eta(z,v_\eta,\nabla v_\eta)\to g(z,v,\nabla v) \text{ a.e. in } \Omega, \tag{3.22}$$
$$\frac{1}{\eta}|v_\eta|^{p(z)-2}v_\eta\to 0 \text{ a.e. in } \Omega, \tag{3.23}$$
$$\frac{|T_\eta(v_\eta)|^{p(z)-2}T_\eta(v_\eta)}{|z|^{p(z)}+1/\eta}\to\frac{|v|^{p(z)-2}v}{|z|^{p(z)}} \text{ a.e. in } \Omega.$$

Hence, to prove the uniform equi-integrability of these functions, it is sufficient to apply Vitali's theorem. Using $T_1(G_\ell(v_\eta))$ as the test function in Equation (3.1), where G_t ($t>0$) is the truncation function defined by $G_t(s)=s-T_k(s)$, we get

$$\alpha\int_{\{\ell<|v_\eta|\leq\ell+1\}}|\nabla v_\eta|^{p(z)}\omega dz$$
$$+\int_{\{|v_\eta|\geq\ell\}}|g_\eta(z,v_\eta,\nabla v_\eta)||T_1(G_\ell(v_\eta))|dz+\frac{1}{\eta}\int_{\{|v_\eta|\geq\ell+1\}}|v_\eta|^{p(z)-1}dz$$
$$\leq\rho\int_{\{|v_\eta|\geq\ell\}}\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}|T_1(G_\ell(v_\eta))|dz+\int_{\{|v_\eta|\geq\ell\}}|f_\eta|dz.$$

Using Young inequality, we get

$$\begin{aligned}&\rho\int_{\{|v_\eta|\ge\ell\}}\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+\frac{1}{\eta}}|T_1(G_\ell(v_\eta))|dz\\&\quad\le\frac{1}{3}\int_{\{|v_\eta|\ge\ell\}}|T_\eta(v_\eta)|^{p(z)}|T_1(G_\ell(v_\eta))|dz+C_{12}\int_{\{|v_\eta|\ge\ell\}}\frac{|T_1(G_\ell(v_\eta))|}{|z|^{p^2(z)}}dz.\end{aligned}$$

As a result,

$$\begin{aligned}&\frac{1}{3}\int_{\{|v_\eta|\ge\ell+1\}}|g_\eta(z,v_\eta,\nabla v_\eta)|dz+\rho\int_{\{|v_\eta|\ge\ell+1\}}\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+\frac{1}{\eta}}dz\\&+\frac{1}{\eta}\int_{\{|v_\eta|\ge\ell+1\}}|v_\eta|^{p(z)-1}dz\le 2C_{12}\int_{\{|v_\eta|\ge\ell\}}\frac{|T_1(G_\ell(v_\eta))|}{|z|^{p^2(z)}}dz+\int_{\{|v_\eta|\ge\ell\}}|f_\eta|dz.\end{aligned}$$

Then, for all $\tau>0$, there exists $\ell(\tau)>0$, where

$$\int_{\{|v_\eta|\ge\ell(\tau)\}}|g_\eta(z,v_\eta,\nabla v_\eta)|dz+\int_{\{|v_\eta|\ge\ell(\tau)\}}\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}dz+\frac{1}{\eta}\int_{\{|v_\eta|\ge\ell(\tau)\}}|v_\eta|^{p(z)-1}dz\le\frac{\tau}{2}. \quad (3.24)$$

Hence, for all measurable subset $E\subseteq\Omega$, we obtain

$$\begin{aligned}&\int_E|g_\eta(z,v_\eta,\nabla v_\eta)|dz+\int_E\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}dz+\frac{1}{\eta}\int_E|v_\eta|^{p(z)-1}dz\\&\le\int_E|g_\eta(z,v_\eta,\nabla v_\eta)|dz+\int_E\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}dz+\frac{1}{\eta}\int_E|T_\eta(v_\eta)|^{p(z)-1}dz\\&+\int_{\{|v_\eta|\ge\ell(\tau)\}}|g_\eta(z,v_\eta,\nabla v_\eta)|dz+\int_{\{|v_\eta|\ge\ell(\tau)\}}\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}dz+\frac{1}{\eta}\int_{\{|v_\eta|\ge\ell(\tau)\}}|v_\eta|^{p(z)-1}dz.\end{aligned}$$

In view of (3.19), there exists $\gamma(\tau)>0$, where: for each $E\subseteq\Omega$ such that meas$(E)\le\gamma(\tau)$

$$\int_E|g_\eta(z,v_\eta,\nabla v_\eta)|dz+\int_E\frac{|T_{\ell(\tau)}(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}dz+\frac{1}{\eta}\int_E|T_{\ell(\tau)}(v_\eta)|^{p(z)-1}dz\le\frac{\tau}{2}. \quad (3.25)$$

Finally, by combining (3.24)–(3.25), one easily has

$$\int_E|g_\eta(z,v_\eta,\nabla v_\eta)|dz+\int_E\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+\frac{1}{\eta}}dz+\frac{1}{\eta}\int_E|v_\eta|^{p(z)-1}dz\le\tau$$

with meas$(E)\le\gamma(\tau)$. This means that $(g_\eta(z,v_\eta,\nabla v_\eta))_\eta$, $(|v_\eta|^{p(z)-2}v_\eta)_\eta$ and $\left(\frac{|T_\eta(v_\eta)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}\right)_\eta$ are equi-integrability. In virtue (3.22)–(3.23) and Vitali's theorem, one has the convergence given in (3.20)–(3.21).

Step 6: Passage to the limit

Let's assume $\mathcal{G}=r+\|\varphi\|_\infty$ and $\varphi\in W_0^{1,p(z)}(\Omega,\omega)\cap L^\infty(\Omega)$. Using $T_r(v_\eta-\varphi)$ as a test function in (3.1), we have

$$\begin{aligned}&\int_\Omega \Phi(z,T_\eta(v_\eta),\nabla v_\eta)\nabla T_r(v_\eta-\varphi)dz+\int_\Omega g_\eta(z,T_\eta(v_\eta),\nabla v_\eta)T_r(v_\eta-\varphi)dz\\&\qquad+\frac{1}{\eta}\int_\Omega |v_\eta|^{p(z)-2}v_\eta T_r(v_\eta-\varphi)dz\\&\quad=\rho\int_\Omega \frac{|T_\eta(v_\eta)|^{p(z)-2}T_\eta(v_\eta)}{|z|^{p(z)}+1/\eta}T_r(v_\eta-\varphi)dz+\int_\Omega f_\eta T_r(v_\eta-\varphi)dz.\end{aligned}$$

Firstly, we have $\{|v_\eta-\varphi|\le k\}\subseteq\{|v_\eta|\le\mathcal{G}\}$. Thus,

$$\begin{aligned}\int_\Omega \Phi(z,T_\eta(v_\eta),\nabla v_\eta)\nabla T_r(v_\eta-\varphi)dz&=\int_\Omega(\Phi(z,T_{\mathcal{G}}(v_\eta),\nabla T_{\mathcal{G}}(v_\eta))\\&\quad-\Phi(z,T_{\mathcal{G}}(v_\eta),\nabla\varphi))(\nabla T_{\mathcal{G}}(v_\eta)-\nabla\varphi)\chi_{\{|v_\eta-\varphi|\le r\}}dz\\&\quad+\int_\Omega \Phi(z,T_{\mathcal{G}}(v_\eta),\nabla\varphi)(\nabla T_{\mathcal{G}}(v_\eta)-\nabla\varphi)\chi_{\{|v_\eta-\varphi|\le r\}}dz.\end{aligned}$$

It is therefore obvious that

$$\begin{aligned}&\lim_{\eta\to\infty}\int_\Omega \Phi(z,T_{\mathcal{G}}(v_\eta),\nabla\varphi)(\nabla T_{\mathcal{G}}(v_\eta)-\nabla\varphi)\chi_{\{|v_\eta-\varphi|\le r\}}dz\\&\qquad\qquad=\int_\Omega \Phi(z,T_{\mathcal{G}}(v),\nabla\varphi)(\nabla T_{\mathcal{G}}(v)-\nabla\varphi)\chi_{\{|v-\varphi|\le r\}}dz.\end{aligned}$$

By applying the Fatou Lemma, we infer

$$\begin{aligned}&\liminf_{\eta\to\infty}\int_\Omega \Phi(z,T_\eta(v_\eta),\nabla v_\eta)\nabla T_r(v_\eta-\varphi)dz\\&\ge\int_\Omega(\Phi(z,T_{\mathcal{G}}(v),\nabla T_{\mathcal{G}}(v))-\Phi(z,T_{\mathcal{G}}(v),\nabla\varphi))(\nabla T_{\mathcal{G}}(v)-\nabla\varphi)\chi_{\{|v-\varphi|\le r\}}dz\\&\quad+\int_\Omega \Phi(z,T_{\mathcal{G}}(v),\nabla\varphi)(\nabla T_{\mathcal{G}}(v)-\nabla\varphi)\chi_{\{|v-\varphi|\le r\}}dz\\&=\int_\Omega \Phi(z,T_{\mathcal{G}}(v),\nabla T_{\mathcal{G}}(v))(\nabla T_{\mathcal{G}}(v)-\nabla\varphi)\chi_{\{|v-\varphi|\le r\}}dz\\&=\int_\Omega \Phi(z,v,\nabla v)\nabla T_r(v-\varphi)dz.\end{aligned}$$

Conversely, we can conclude that $T_r(v_\eta-\varphi)\rightharpoonup T_r(v-\varphi)$ weak-* in $L^\infty(\Omega)$ and due to (3.20)–(3.23), we infer

$$\begin{aligned}&\int_\Omega |g(z,v_\eta,\nabla v_\eta)|T_r(v_\eta-\varphi)dz\to\int_\Omega |g(z,v,\nabla v)|T_r(v-\varphi)dz,\\&\frac{1}{\eta}\int_\Omega |v_\eta|^{p(z)-1}v_\eta T_r(v_\eta-\varphi)dz\to 0,\end{aligned}$$

$$\begin{aligned}&\int_\Omega \frac{|T_\eta(v_\eta)|^{p(z)-2}T_\eta(v_\eta)}{|z|^{p(z)}+1/\eta}T_r(v_\eta-\varphi)dz\to\int_\Omega \frac{|v|^{p(z)-2}v}{|z|^{p(z)}}T_r(v-\varphi)dz,\\&\int_\Omega f_\eta T_r(v_\eta-\varphi)dz\to\int_\Omega fT_r(v-\varphi)dz.\end{aligned}$$

So, having put all the terms together, we finish the proof of Theorem 1.

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Appendix

The purpose of this section is to prove that the operator $B_\eta = A_\eta + G_\eta$, satisfies both the coercive and pseudo-monotonicity properties.

Proof of Lemma 5. In the light of the Hölder's inequality and (2.3), we obtain

$$
\begin{aligned}
|\langle A_\eta v,\varphi\rangle| &= \Big| \int_\Omega \Phi(z,T_\eta(v),\nabla v)\nabla\varphi dz + \frac{1}{\eta}\int_\Omega |v|^{p(z)-2} v\varphi dz\Big| \\
&\le \Big(\int_\Omega |\Phi(z,T_\eta(v),\nabla v)|^{p'(z)}\omega^{1-p'(z)}dz\Big)^{\frac{1}{p'_-}} \|\omega^{\frac{1}{p(z)}}\nabla\varphi\|_{L^{p(z)}(\Omega)} \\
&\quad + \frac{1}{\eta}\Big(\int_\Omega |v|^{(p(z)-1)p(z)}dz\Big)^{\frac{1}{p'_-}} \|\varphi\|_{p(z)} \\
&\le \beta\Big(\int_\Omega (R^{p(z)} + |T_\eta(v)|^{p(z)} + \omega|\nabla v|^{p(z)})\Big)^{\frac{1}{p'_-}} \|\nabla\varphi\|_{L^{p(z)}(\Omega,\omega)} \\
&\quad + \frac{1}{\eta}\Big(\int_\Omega |v|^{p(z)}dz\Big)^{\frac{1}{p'_-}} \|\varphi\|_{p(z)} \le C_4\|\varphi\|_{W_0^{1,p(z)}(\Omega,\omega)}.
\end{aligned}
$$

Thus, by Proposition 5, it can be deduced that the operator B_η is bounded.

Regarding the coercivity, thanks to (2.1) and Proposition 5 and for any $v \in W_0^{1,p(z)}(\Omega,\omega)$ we get

$$
\begin{aligned}
\langle B_\eta v, v\rangle = &\int_\Omega \Phi(z,T_\eta(v),\nabla v)\nabla v dz + \int_\Omega g_\eta(z,v,\nabla v)v dz + \frac{1}{\eta}\int_\Omega |v|^{p(z)}dz \\
&- \rho\int_\Omega \frac{|T_\eta(v)|^{p(z)-1}}{|z|^{p(z)}+1/\eta}|v|dz \ge \beta\|v\|^\lambda_{W_0^{1,p(z)}(\Omega,\omega)} - C_0\|v\|_{W_0^{1,p(z)}(\Omega,\omega)},
\end{aligned}
$$

with $\beta = \min\big(\alpha, 1/\eta\big)$ and

$$
\lambda = \begin{cases} p^+ & \text{if} \quad \|v\|_{W_0^{1,p(z)}(\Omega,\omega)} < 1, \\ p_- & \text{if} \quad \|v\|_{W_0^{1,p(z)}(\Omega,\omega)} \ge 1. \end{cases}
$$

Hence, $\langle B_\eta v, v\rangle / \|v\|_{W_0^{1,p(z)}(\Omega,\omega)} \to +\infty$ as $\|v\|_{W_0^{1,p(z)}(\Omega,\omega)} \to \infty$.

We still need to demonstrate that B_η is pseudo-monotone. To do so, consider a sequence $(v_n)_{n\in\mathbb{N}}$ in $W_0^{1,p(z)}(\Omega,\omega)$ such that

$$
\begin{cases} v_n \rightharpoonup v \text{ in } W_0^{1,p(z)}(\Omega,\omega), \quad B_\eta v_n \rightharpoonup \chi \text{ in } W_0^{-1,p'(z)}(\Omega,\omega^*), \\ \limsup\limits_{n\to\infty}\langle B_\eta v_n, v_n\rangle \le \langle \chi, v\rangle. \end{cases} \tag{3.26}
$$

We will prove that $\chi = B_\eta v$ and $\langle B_\eta v_n, v_n\rangle \to \langle\chi, v\rangle$ as $n \to +\infty$. Given the compact embedding $W_0^{1,p(z)}(\Omega,\omega) \hookrightarrow L^{p(z)}(\Omega)$, we can infer that a subsequence, which we will still denote as $(v_n)_{n\in\mathbb{N}}$, of v_n converges to v in $L^{q(z)}(\Omega)$. Since the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(z)}(\Omega,\omega)$ and the growth condition (2.3) holds, we can conclude that the sequence $(\Phi(z,T_\eta(v_n),\nabla v_n))_n$ is also bounded in $L^{p'(z)}(\Omega,\omega^*)$. This implies that there exists a function $\phi_\eta \in L^{p'(z)}(\Omega,\omega^*)$ such that

$$
\Phi(z,T_\eta(v_n),\nabla v_n) \rightharpoonup \phi_\eta \ \text{ in } \ L^{p'(z)}(\Omega,\omega^*) \ \text{ as } \ n\to\infty. \tag{3.27}
$$

Similarly, since $\big(g_n(z, v_\eta, \nabla v_\eta)\big)_{\eta\in\mathbb{N}^*}$ is bounded in $L^{p'(z)}(\Omega, \omega^*)$, then there exists a measurable function ψ_η in $L^{p'(z)}(\Omega, \omega^*)$, such that

$$g_n(z, T_\eta(v_n), \nabla v_n) \rightharpoonup \psi_\eta \quad \text{in} \quad L^{p'(z)}(\Omega, \omega^*) \quad \text{as} \quad n \to \infty. \tag{3.28}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$\frac{|T_\eta(v_n)|^{p(z)-2}T_\eta(v_n)}{|z|^{p(z)} + 1/\eta} \to \frac{|T_\eta(v)|^{p(z)-2}T_\eta(v)}{|z|^{p(z)} + 1/\eta} \quad \text{in} \quad L^{p'(z)}(\Omega, \omega^*). \tag{3.29}$$

Also, we have

$$\frac{1}{\eta}|v_n|^{p(z)-2}v_n \rightharpoonup \frac{1}{\eta}|v|^{p(z)-2}v \quad \text{in} \quad L^{p'(z)}(\Omega, \omega^*). \tag{3.30}$$

Thus, for any $\varphi \in W_0^{1,p(z)}(\Omega, \omega)$, we have

$$\begin{aligned}\lim_{n\to\infty}\langle B_\eta v_n, \varphi\rangle &= \lim_{n\to\infty}\int_\Omega \Phi(z, T_\eta(v_n), \nabla v_n)\nabla\varphi dz + \lim_{n\to\infty}\int_\Omega g_\eta(z, v_n, \nabla v_n)\varphi dz\\ &\quad + \lim_{n\to\infty}\frac{1}{\eta}\int_\Omega |v|^{p(z)-2}v\varphi dz - \lim_{n\to\infty}\rho\int_\Omega \frac{|T_\eta(v_n)|^{p(z)-2}T_n(v_n)}{|z|^{p(z)} + 1/\eta}\varphi dz \\ &= \int_\Omega \phi_\eta\nabla\varphi dz + \int_\Omega \psi_\eta\varphi dz + \frac{1}{\eta}\int_\Omega |v|^{p(z)-2}v\varphi dz - \rho\int_\Omega \frac{|T_\eta(v)|^{p(z)-2}T_\eta(v)}{|z|^{p(z)} + \frac{1}{\eta}}\varphi dz.\end{aligned} \tag{3.31}$$

Having in mind (3.26)–(3.31), we obtain

$$\begin{aligned}\limsup_{\eta\to\infty}\langle B_\eta v_n, v_n\rangle &= \limsup_{n\to\infty}\Big\{\int_\Omega \Phi(z, T_\eta(v_n), \nabla v_n)\nabla v_n dz\\ &\quad + \int_\Omega |g_\eta(z, v_n, \nabla v_n)|v_n dz + \frac{1}{\eta}\int_\Omega |v_n|^{p(z)}dz - \rho\int_\Omega \frac{|T_\eta(v_n)|^{p(z)-2}T_\eta(v_n)}{|z|^{p(z)} + 1/\eta}v_n dz\Big\}\\ &\le \int_\Omega \phi_\eta\nabla v dz + \int_\Omega \psi_\eta v dz + \frac{1}{\eta}\int_\Omega |v|^{p(z)}dz - \rho\int_\Omega \frac{|T_\eta(v)|^{p(z)-2}T_\eta(v)}{|z|^{p(z)} + 1/\eta}v dz.\end{aligned}$$

Thanks to (3.28) and (3.29), we have

$$\begin{aligned}&\int_\Omega g_\eta(z, v_n, \nabla v_n)v_n dz \to \int_\Omega \psi_\eta v dz \quad \text{as } n \to \infty,\\ &\int_\Omega \frac{|T_\eta(v_n)|^{p(z)-2}T_\eta(v_n)}{|z|^{p(z)} + 1/\eta}v_n dz \to \int_\Omega \frac{|T_\eta(v)|^{p(z)-2}T_\eta(v)}{|z|^{p(z)} + 1/\eta}v dz \text{ as } n \to \infty.\end{aligned} \tag{3.32}$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\Big(\int_\Omega \Phi(z, T_\eta(v_n), \nabla v_n)\nabla v_n dz + \frac{1}{\eta}\int_\Omega |v_n|^{p(z)}dz\Big)&\\ \le \int_\Omega \phi_\eta\nabla v dz + \frac{1}{\eta}\int_\Omega |v|^{p(z)}dz.&\end{aligned} \tag{3.33}$$

On the other hand, in view of (2.4) we have

$$\int_\Omega (\Phi(z,T_\eta(v_n),\nabla v_n)-\Phi(z,T_\eta(v_n),\nabla v)(\nabla v_n-\nabla v)dz \\ +\frac{1}{\eta}\int_\Omega (|v_n|^{p(z)-2}v_n-|v|^{p(z)-2}v)(v_n-v)dz\geq 0,$$

then

$$\int_\Omega \Phi(z,T_\eta(v_n),\nabla v_n)\nabla v_n dz+\frac{1}{\eta}\int_\Omega |v_n|^{p(z)}dz\geq \int_\Omega \Phi(z,T_\eta(v_n),\nabla v_n)\nabla v dz \\ +\frac{1}{\eta}\int_\Omega |v_n|^{p(z)-2}v_n v dz+\int_\Omega \Phi(z,T_\eta(v_n),\nabla v)(\nabla v_n-\nabla v)dz \\ +\frac{1}{\eta}\int_\Omega |v|^{p(z)-2}v(v_n-v)dz.$$

By the Lebesgue's dominated convergence theorem, we can conclude that $T_\eta(v_n)\to T_\eta(v)$ in $L^{p(z)}(\Omega,\omega)$.

As a result, $\Phi(z,T_\eta(v_n),\nabla v)\to \Phi(z,T_\eta(v),\nabla v)$ in $L^{p'(z)}(\Omega,\omega^*)$. By utilizing Equation (3.27), we can derive that

$$\liminf_{n\to\infty}\Big(\int_\Omega \Phi(z,T_\eta(v_n),\nabla v_n)\nabla v_n dz+\frac{1}{\eta}\int_\Omega |v_n|^{p(z)}dz\Big) \\ \geq \int_\Omega \phi_\eta \nabla v dz+\frac{1}{\eta}\int_\Omega |v|^{p(z)}dz.$$

We conclude, taking into account (3.33), that

$$\lim_{k\to\infty}\Big(\int_\Omega \Phi(z,T_\eta(v_n),\nabla v_n)\nabla v_n dz+\frac{1}{\eta}\int_\Omega |v_n|^{p(z)}dz\Big) \\ =\int_\Omega \phi_\eta \nabla v dz+\frac{1}{\eta}\int_\Omega |v|^{p(z)}dz. \quad (3.34)$$

Therefore, by combining (3.31)–(3.32), we infer $\langle B_\eta v_n, v_n\rangle\to\langle \chi, v\rangle$ as $n\to\infty$. With the help of (3.34), we deduce

$$\lim_{n\to+\infty}\Big(\int_\Omega \big(\Phi(z,T_\eta(v_n),\nabla v_n)-\Phi(z,T_\eta(v_n),\nabla v)\big)\big(\nabla v_n-\nabla v\big)dz \\ +\frac{1}{\eta}\int_\Omega (|v_n|^{p(z)-2}v_n-|v|^{p(z)-2}v)(v_\eta-v)dz\Big)=0.$$

Thus, from Lemma 3, we have $v_n\to v$ in $W_0^{1,p(z)}(\Omega,\omega)$, $\nabla v_n\to\nabla v$ a.e. in Ω, then,

$$\Phi(z,T_\eta(v_\eta),\nabla v_\eta)\rightharpoonup \Phi(z,T_\eta(v),\nabla v) \ \text{ in } \ L^{p'(z)}(\Omega,\omega^*).$$

By means Equations (3.28) and (3.30), we are able to derive $\chi=B_\eta v$, leading us to conclude the proof of Lemma 5.