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Regularizing Effect in Singular Semilinear Problems

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Abstract. We analyze how different relations in the lower order terms lead to the same regularizing effect on singular problems whose model is $-\Delta u+g(x,u) = f(x)/u^{\gamma}$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded open set of $\mathbb{R}^N$, $\gamma > 0$, $f(x)$ is a nonnegative function in $L^1(\Omega)$ and $g(x,s)$ is a Carathéodory function. In a framework where no $H_0^1(\Omega)$ solution is expected, we prove its existence (regularizing effect) whenever the datum f interacts conveniently either with the boundary of the domain or with the lower order term.

Keywords: nonlinear elliptic equations, singular problem, regularizing effect.

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1 Introduction

In this paper, we study the following boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + g(x,u) = f(x)/u^{\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and $M(x)$ is a bounded elliptic matrix, i.e., there exist $\alpha, \beta > 0$ such that

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta \tag{1.2}$$

for every $\xi \in \mathbb{R}^N$ and for almost every x in Ω. We also assume that $f \in L^1(\Omega)$ is a nonnegative function and that $g(x,s)$ is a Carathéodory function (that is, measurable with respect to x for every $s \in \mathbb{R}$ and continuous with respect to s for almost every $x \in \Omega$).

The scope of this paper is to analyze the existence of solutions to (1.1) in $H_0^1(\Omega)$ in a wider range of values of the parameter γ or functions g than currently known (regularizing effect). Therefore, we put in evidence that in spite of the fact that the datum f only belongs to $L^1(\Omega)$, the interplay given by f and the boundary of Ω or with the lower order term provides a regularizing effect on the problem (1.1). We review now the literature of problems related to (1.1) in order to present our main results. Then we will carry out an exhaustive analysis of our hypotheses which will show that they are natural with respect to such literature.

The boundary value problem

$$\begin{cases} -\text{div}(M(x)\nabla u) = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $h(x,s)$ singular at $s = 0$ has been extensively studied. In [8] the authors dealt with some singular problems including the cases $h(x,s) = f(x)e^{1/s}$ or $h(x,s) = f(x)/s^\gamma$ for a regular function $f(x)$ and they proved the existence of classical solution to problem (1.3) with $M(x)$ being the identity matrix. Similar results were proved in [7, 17] for a regular matrix $M(x)$ and a regular function $h(x,s)$ uniformly bounded for $s > 1$ with $\lim_{s\to 0} h(x,s) = +\infty$ uniformly for $x \in \overline{\Omega}$. Furthermore, in [7] it is proved some continuity properties of the solution if $h(x,s)$ does not depend on x.

In [13], the authors studied the problem (1.3) with $h(x,s) = f(x)/s^\gamma$ and $f(x)$ a positive Hölder continuous function in $\overline{\Omega}$ and it is showed that this problem has a classical solution which may not be in $H_0^1(\Omega)$. Concretely, it is proved that the solution belongs to $H_0^1(\Omega)$ if, and only if, $\gamma < 3$. Moreover, they established that for $\gamma > 1$ the solution is not in $\mathcal{C}^1(\overline{\Omega})$ (confront these results with Theorem 1 below). Some extensions may be found, in the case $h(x,s) = f(x)\tilde{h}(s)$, among others in [11,12] for $\Omega = \mathbb{R}^N$ and in [18] for bounded domains. In this last case $f(x)$ may be singular at the boundary.

We highlight the paper [4], in which the authors extensively studied problem (1.3) in the case $h(x,s) = f(x)/s^\gamma$ with $f \in L^m(\Omega)$ for $m \geq 1$ and existence results depending on γ and on the summability of f are obtained. For $\gamma = 1$ and $f \in L^1(\Omega)$, they proved the existence of a solution belonging to $H_0^1(\Omega)$. A similar result for the case $\gamma < 1$ is proved but they imposed more summability on f, namely $f \in L^m(\Omega)$ with $m \geq C(N,\gamma) > 1$. Finally, for the case $\gamma > 1$ and $f \in L^1(\Omega)$ it was proved the existence of a solution u belonging to $H^1_{\text{loc}}(\Omega)$ satisfying that $u^{(\gamma+1)/2}$ belongs to $H_0^1(\Omega)$.

In [2], the authors partially improved the results in [4] for the case $\gamma > 1$ by adding more restrictive hypotheses. Specifically, in a regular domain and for $f \in L^m(\Omega)$ greater than a positive constant the existence of a finite energy solution to (1.1) is proved, with $g \equiv 0$, for every $1 < \gamma < (3m-1)/(m+1)$. These results seem to be optimal since, for $f \in L^\infty(\Omega)$, it is proved in [13] that such solution belongs to $H_0^1(\Omega)$ for all $\gamma < 3$. The existence of energy solutions is also discussed in [3] for elliptic systems involving a singular equation related to (1.1), for which, with frozen unknown, they prove existence and uniqueness of bounded solutions.

We also have to mention that existence of solution for problem (1.3) in the case $h(x,s) = f(x)/s^\gamma$ with $f \in L^m(\Omega)$ for $m \geq 1$ is obtained in [9,10] where the notion of solution is understood in a different sense from the one studied in the paper [4]. We point out that the cases $h(x,s) = f(x)/s^\gamma + \mu$ and $h(x,s) = \mu\tilde{h}(s)$ with μ a nonnegative Radon measure have been studied in [14,15]. Moreover the case of a variable exponent $\gamma = \gamma(x)$, i.e., $h(x,s) = cf(x)/s^{\gamma(x)}$ is considered in [6].

Now, we present our principal results. Our approach is twofold, on one hand we extend to the problem (1.1) some known results for (1.3) and on the other hand we analyze the regularizing effect produced by different interplays of $f(x)$, illustrated here according to whether γ is greater or less than one. Firstly, we prove some regularity and non-regularity results for the problem (1.1) when $\gamma > 1$ depending on the interplay of the behavior of the datum $f(x)$ near the boundary of Ω and the behavior of $g(x,s)$ when s is near zero. Secondly, we study the problem (1.1) with $\gamma \leq 1$ and $g(x,s) = a(x)\tilde{g}(s)$ according to the interplay between $f(x)$ and $a(x)$.

In the first case ($\gamma > 1$), we will assume that there exists $r > -1$ such that, the function $f(x)$ satisfies, for some $m_1 > 0$, that

$$f(x) \geq m_1 \varphi_1^r \text{ a.e. in } \Omega, \tag{1.4}$$

where φ_1 denotes a positive eigenfunction associated to the first eigenvalue of the operator $-\mathrm{div}(M(x)\nabla\cdot)$ with zero Dirichlet boundary condition. The relation between φ_1 and a solution u to problem (1.3) when $h(x,s) = f(x)/s^\gamma$ and $\gamma > 1$ was highlighted in [13] where the authors proved that $u^{\frac{\gamma+1}{2}}/\varphi_1$ is bounded by two positive constants. They also observed that this result can be slightly improved when (1.4) is imposed (with $0 < r < \gamma+1$). Thus, we remark that hypothesis (1.4) (and also (1.8) below) is quite natural and more general with respect to the previous results in [13].

Regarding the function $g\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ we assume that is a Carathéodory function verifying that

$$g(x,s) \text{ is nonnegative and increasing in } s \text{ for a.e. } x \in \Omega, \tag{1.5}$$

$$g(\cdot,s) \in L^1_{\mathrm{loc}}(\Omega) \text{ for each } s \gg 0 \text{ fixed.} \tag{1.6}$$

Moreover, we suppose that there exists some $0 < s_0 < 1$ and some $c_1, c_2 > 0$ such that

$$\begin{cases} g(x,s) \leq c_1 s^{\frac{r-2\gamma}{2+r}}, & \text{if } r \geq 2\gamma, \\ g(x,s) \leq c_1 (s+c_2)^{\frac{r-2\gamma}{2+r}}, & \text{if } r < 2\gamma, \end{cases} \tag{1.7}$$

for every $0 \leq s \leq s_0$ and almost every $x \in \Omega$. A simple model of function g is $g(x,s) = a(x)s^t$ with $a \in L^\infty(\Omega)$ and $t \geq \max\{0, (r-2)\gamma/(2+r)\}$.

Finally, the regularity result is obtained when there exists $m_2 > 0$ and an open neighborhood of $\partial\Omega$ in Ω, denoted by Γ, such that

$$f(x) \le m_2 \varphi_1^r \text{ a.e. in } \Gamma. \tag{1.8}$$

The main result of the paper in the case $\gamma > 1$ is the following one.

Theorem 1. *Assume that Ω satisfies the interior sphere condition, $\gamma > 1$, $M(x)$ verifies (1.2) and that $g(x,s)$ satisfies (1.5) and (1.6). Assume also that there exists $r > -1$ such that $0 \le f \in L^1(\Omega)$ satisfies (1.4) and $g(x,s)$ verifies (1.7). Then, there exists $u \in H^1_{\mathrm{loc}}(\Omega)$ solution to (1.1) such that the function $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$ and:*

i) If $\gamma > \max\{1, r+1\}$, then $u \notin \mathcal{C}^1(\overline{\Omega})$.

ii) If $1 < \gamma < 2r+3$ and $f(x)$ satisfies (1.8), then $u \in H_0^1(\Omega)$.

iii) If $\gamma \ge 2r+3$, $f(x)$ satisfies (1.8) and u is bounded in Γ, then $u \notin H_0^1(\Omega)$.

We remark that, under more restrictive hypotheses, Theorem 1 improves the results in [2] when $-1 < r < 0$. Indeed, (1.8) implies that $f \in L^m(\Gamma)$ for every $m < \frac{1}{-r}$ and we establish the existence of a solution in $H_0^1(\Omega)$ for $1 < \gamma < 2r+3$. In [2], for $f \in L^{1/-r}(\Omega)$ the authors obtain this existence result only for $1 < \gamma < (3+r)/(1-r)$ (note that $(3+r)/(1-r) < 2r+3$ if $-1 < r < 0$).

Third item of Theorem 1 gives, in some sense, the sharpness of the exponent $2r+3$ in order to obtain energy solutions. We remark, that condition u bounded in Γ can be removed under additional conditions on f and g. Indeed, whenever f satisfies (1.8) in Ω then $u \in L^\infty(\Omega)$. Also arguing as in [3] it is possible to prove that solutions are bounded if $g(x,s)s^\gamma \ge f(x)$ for s large.

In the second case ($\gamma \le 1$), we are inspired by [1]. We assume the particular case $g(x,s) = a(x)\tilde{g}(s)$, where $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ is a function satisfying that

$$\tilde{g} \text{ is continuous, increasing and odd and we denote } \tilde{g}_\infty = \lim_{s\to+\infty} \tilde{g}(s). \tag{1.9}$$

We also assume that

$$0 \le a(x), f(x) \in L^1(\Omega) \tag{1.10}$$

and the "Q-condition":

$$\text{there exists } Q \in (0, \tilde{g}_\infty) \text{ such that } f(x) \le Q a(x) \text{ a.e. in } \Omega. \tag{1.11}$$

Notice that (1.11) is now quite natural in order to obtain more regularity since this regularizing phenomenon was first pointed out in the literature by D. Arcoya and L. Boccardo in [1].

Our main result of the paper in the case $\gamma \le 1$ is the following one.

Theorem 2. *Assume that $\gamma \le 1$, $M(x)$ satisfies (1.2) and $g(x,u) = a(x)\tilde{g}(u)$ where $\tilde{g}$ verifies (1.9). Assume also that $a(x)$ and $f(x)$ both satisfy (1.10) and (1.11). Then the problem (1.1) has a unique solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.*

The plan of the paper is the following. In Section 2, we establish the definition of solution that we adopt in this paper and we prove some preliminary results mainly related with Theorem 1. Then, in Section 3, we prove Theorem 1, which is based on a comparison principle between the approximated solutions and a suitable power of φ_1 proved in the preceding section. Finally, Section 4 is devoted to the proof of Theorem 2.

2 Preliminaries

The concept of solution we adopt is gathered in the following definition.

Definition 1. A function $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $u \geq 0$ a.e. in Ω, satisfying also that $g(\cdot, u) \in L^1_{\mathrm{loc}}(\Omega)$, $f/u^\gamma \in L^1_{\mathrm{loc}}(\Omega)$ is a *supersolution* to problem (1.1) if

$$\int_\Omega M(x)\nabla u \nabla \phi + \int_\Omega g(x,u)\phi \geq \int_\Omega \frac{f}{u^\gamma}\phi, \ \forall 0 \leq \phi \in \mathcal{C}^1_c(\Omega).$$

When the reverse inequality is satisfied and $u^\tau \in H^1_0(\Omega)$ for some $\tau > 0$, we understand that u is a *subsolution* for problem (1.1).

A function $u \in H^1_{\mathrm{loc}}(\Omega)$ is a *solution* for (1.1) if it is both a subsolution and a supersolution for such a problem. If, in addition, $u \in H^1_0(\Omega)$, we say that u is a *finite energy solution* for problem (1.1).

Let us clarify that the function $\dfrac{f}{u^\gamma}\phi$ takes the value $+\infty$ in the case $u = 0$ and $f\phi \neq 0$ while takes the value zero whenever $f\phi = 0$.

Remark 1. A sufficient condition to obtain $f/u^\gamma \in L^1_{\mathrm{loc}}(\Omega)$ is that u be uniformly bounded from below by a positive constant in every subset compactly contained in Ω. Namely, for all $\omega \subset\subset \Omega$ there exists some $c_\omega > 0$ such that $u \geq c_\omega > 0$ in ω.

Remark 2. Arguing as in [5, Appendix], if $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1) with $g(\cdot, u) \in L^1(\Omega)$, then $\dfrac{f}{u^\gamma}\phi \in L^1(\Omega)$ for all $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega M(x)\nabla u \nabla \phi + \int_\Omega g(x,u)\phi = \int_\Omega \frac{f}{u^\gamma}\phi, \ \forall \phi \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

In order to prove our main results we proceed as usual by approximation. For any $k > 0$ we set $T_k(s) = \min\{k, \max\{s, -k\}\}$ and $G_k(s) = s - T_k(s)$.

The proofs of Theorem 1 and Theorem 2 rely on approximating the problem (1.1) by a certain sequence of approximated problems

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_n) + g_n(x, u_n) = f_n(x)/\Big(|u_n| + \dfrac{1}{n}\Big)^\gamma & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

and on the fact that the sequence of solutions to (2.1) converges, as $n \to \infty$, to a solution to (1.1).

In the next result, we summarize the main existence results for the approximated problems (2.1).

Lemma 1. *Assume that* $0 \le f_n \in L^\infty(\Omega)$ *and* $g_n(x,s)$ *is a Carathéodory function with* $g_n(x,s)s \ge 0$ *for every* $s \in \mathbb{R}$ *and a.e.* $x \in \Omega$ *and* g_n *is bounded for* s *in bounded sets. There exists* $0 \le u_n \in H_0^1(\Omega)$ *solution of* (2.1) *for every fixed* $n \in \mathbb{N}$. *In addition, for* $\gamma \le 1$ *we have that* $u_n \in L^\infty(\Omega)$. *Moreover, in the case* $\gamma > 1$, *the existence of solution* $u_n \in H_0^1(\Omega)$ *is deduced even for* $f_n \in L^1(\Omega)$.

Proof. The existence of $u_n \in H_0^1(\Omega)$ for $f_n \in L^\infty(\Omega)$ is consequence of the Schauder Theorem. In the case $\gamma \le 1$ using Stampacchia Theorem (see [16]), we can assure that $u_n \in L^\infty(\Omega)$. Moreover, for $f_n \in L^1(\Omega)$ we can use the previous existence result approaching $f_n(x)$ by $f_{n,m}(x) = T_m(f_n(x))$ and passing to the limit as $m \to \infty$. Here, to obtain the a priori estimate in $H_0^1(\Omega)$ it is key to use the fact that $\gamma > 1$.

Finally, let us remark that $u_n \ge 0$ since f_n is nonnegative and $g_n(x,s)s \ge 0$. □

The rest of the section is devoted to the case $\gamma > 1$ where we approximate the nonlinearity $g(x,s)$ by a suitable sequence of Carathéodory functions g_n defined in $\Omega \times \mathbb{R}$. Specifically, we define

$$g_n(x,s) = \begin{cases} T_n(g(x,s)), & s \ge 1/n, \\ n\, s\, T_n(g(x,s)), & 0 < s < 1/n, \\ 0, & s \le 0. \end{cases} \tag{2.2}$$

Observe that $g_n(x,s)$ is increasing in s for a.e. $x \in \Omega$ when (1.5) is satisfied and that $g_n(x,s) \le g(x,s)$ for $s \ge 0$.

According to whether r, given by (1.4), is positive or negative, we also approximate or not the datum $f(x)$. In order to deal with both cases simultaneously we define $\chi(r) = 0$ for $r \le 0$ and $\chi(r) = c_1 + 1$ for $r > 0$, where c_1 is given by (1.7). Thus, we approximate $f(x)$ by $f_n(x)$ as follows

$$f_n(x) = f(x) + \chi(r)/n^{r(\gamma+1)/(2+r)}. \tag{2.3}$$

Following the ideas in [2], we prove that a certain power of an approximation of φ_1 is a subsolution of (2.1) in the following result.

Lemma 2. *Assume that* $\gamma > 1$, $M(x)$ *verifies* (1.2), $g(x,s)$ *satisfies* (1.5) *and there exists* $r > -1$ *such that* $0 \le f(x) \in L^1(\Omega)$ *verifies* (1.4) *and* $g(x,s)$ *verifies* (1.7). *Then, there exist* $C > 0$ *(independent of* n*) and* $n_0 \in \mathbb{N}$ *such that, for every* $n \ge n_0$, *the function*

$$z_n(x) = \left(C\varphi_1(x) + 1/n^{(\gamma+1)/(2+r)}\right)^{\frac{2+r}{\gamma+1}} - 1/n$$

is a subsolution of (2.1) *with* g_n *and* f_n *given by* (2.2) *and* (2.3) *respectively.*

Moreover, denoting u_n *to the solution of* (2.1) *given by Lemma 1, we have*

$$z_n \le u_n \text{ a.e. in } \Omega.$$

Proof. First, let us note that $z_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ since $\varphi_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Let us denote

$$w_n(x) = C\varphi_1(x) + 1/n^{(\gamma+1)/(2+r)}.$$

On the one hand, we have

$$\nabla z_n = \frac{C(2+r)}{\gamma+1} w_n^{(1+r-\gamma)/(\gamma+1)} \nabla\varphi_1$$

and thus

$$\begin{aligned} -\mathrm{div}(M(x)\nabla z_n) =& -\nabla\left(\frac{C(2+r)}{\gamma+1} w_n^{(1+r-\gamma)/(\gamma+1)}\right) M(x)\nabla\varphi_1 \qquad (2.4)\\ &+ \frac{C(2+r)}{\gamma+1} w_n^{(1+r-\gamma)/(\gamma+1)}\big(-\mathrm{div}(M(x)\nabla\varphi_1)\big)\\ =&\frac{C^2(2+r)(\gamma-r-1)}{(\gamma+1)^2} w_n^{(r-2\gamma)/(\gamma+1)} M(x)\nabla\varphi_1\nabla\varphi_1\\ &+ \frac{C\lambda_1(2+r)}{\gamma+1} w_n^{(1+r-\gamma)/(\gamma+1)}\varphi_1\\ =&\frac{C}{w_n^{(2\gamma-r)/(\gamma+1)}}\left[\frac{C(2+r)(\gamma-r-1)}{(\gamma+1)^2} M(x)\nabla\varphi_1\nabla\varphi_1 + \frac{\lambda_1(2+r)}{\gamma+1} w_n\varphi_1\right]\\ \le&\frac{C^2}{w_n^{(2\gamma-r)/(\gamma+1)}}\left[\frac{\beta(2+r)|\gamma-r-1|}{(\gamma+1)^2}\|\nabla\varphi_1\|^2_{L^\infty(\Omega)}\right.\\ &\left.+ \frac{\lambda_1(2+r)}{\gamma+1}\left(\|\varphi_1\|^2_{L^\infty(\Omega)} + \|\varphi_1\|_{L^\infty(\Omega)}\right)\right]\\ \equiv&\frac{C^2 b}{w_n^{(2\gamma-r)/(\gamma+1)}} = \frac{C^2 b w_n^r}{w_n^{\gamma(2+r)/(\gamma+1)}} = \frac{C^2 b w_n^r}{(z_n+1/n)^\gamma}. \end{aligned}$$

Now, since $g(x,s)$ satisfies (1.7), we deduce in the set $\{z_n(x) < s_0\}$

$$g_n(x,z_n) \le g(x,z_n) \le \tilde c\left(z_n + \frac{1}{n}\right)^{\frac{r-2\gamma}{2+r}} = \frac{\tilde c\left(z_n+\frac{1}{n}\right)^{\frac{\gamma+1}{2+r}r}}{\left(z_n+\frac{1}{n}\right)^\gamma} = \frac{\tilde c w_n^r}{\left(z_n+\frac{1}{n}\right)^\gamma}, \quad (2.5)$$

where $\tilde c = c_1$ in the case $r > 0$ and $\tilde c = C$ in the case $-1 < r \le 0$. Actually we claim that, using (1.7), given $-1 < r \le 0$, for every fixed small C, and for n large enough there exists $s_0(C) \in (0,1)$ with $g(x,s) \le C(s+\frac{1}{n})^{\frac{r-2\gamma}{2+r}}$ for every $0 < s < s_0(C)$ and $(C\varphi_1(x))^{\frac{2+r}{\gamma+1}} < s_0(C)$. Indeed, observe that since $1+r-\gamma < 0$, it is enough to take

$$\|\varphi_1\|^{\frac{2+r}{\gamma+1}}_{L^\infty(\Omega)} < \frac{c_2}{c_1^{\frac{2+r}{2\gamma-r}} - C^{\frac{2+r}{2\gamma-r}}}, \left(C\|\varphi_1\|_{L^\infty(\Omega)}\right)^{\frac{2+r}{\gamma+1}} < s_0(C) < \frac{c_2\,(C/c_1)^{\frac{2+r}{2\gamma-r}}}{1-(C/c_1)^{\frac{2+r}{2\gamma-r}}}.$$

Thus, given $C > 0$ small and $s_0(C)$ as above, we can choose $n_0 \in \mathbb{N}$ large enough such that

$$z_n(x) \le s_0(C),\ \forall x \in \Omega,\ \forall n \ge n_0.$$

Combining the inequalities (2.4) and (2.5), and taking into account (1.4) we have, for C small enough, that

$$\begin{aligned} -\operatorname{div}(M(x)\nabla z_n) + g_n(x, z_n) &\leq \frac{(C^2b+\tilde{c})w_n^r}{(z_n+1/n)^\gamma} \\ &\leq \frac{(C^2b+\tilde{c})\left[\frac{C^r}{m_1}f(x) + \frac{\chi(r)/(c_1+1)}{n^{r(\gamma+1)/(r+2)}}\right]}{(z_n+1/n)^\gamma} \leq \frac{f_n(x)}{(z_n+1/n)^\gamma}, \end{aligned} \tag{2.6}$$

i.e., that z_n is a subsolution of (2.1). Here we have used that $(C^2b+\tilde{c})C^r \to 0$ as $C \to 0$ (in the case $-1 < r \leq 0$ we have that $\tilde{c} = C$).

Now, we take $(z_n - u_n)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the inequality (2.6) satisfied by z_n and in the problem (2.1) that satisfies u_n. Then, we subtract them, and applying (1.2) and that $g_n(x, s)$ is increasing in s by (1.5) it follows that

$$\begin{aligned} \alpha \int_\Omega |\nabla(z_n - u_n)^+|^2 &\leq \alpha \int_\Omega |\nabla(z_n - u_n)^+|^2 + \int_\Omega [g_n(x, z_n) - g_n(x, u_n)]\,(z_n - u_n)^+ \\ &\leq \int_\Omega f_n(x)\left[\frac{1}{(z_n+1/n)^\gamma} - \frac{1}{(u_n+1/n)^\gamma}\right](z_n - u_n)^+ \leq 0. \end{aligned}$$

Therefore, we deduce that $(z_n - u_n)^+ = 0$ a.e. in Ω and we can conclude that $z_n \leq u_n$ a.e. in Ω. □

In [4] it is proved existence of a solution to (1.1) with $g(x, s) = 0$. In order to do that the authors approached the problem (1.1) with $g(x, s) = 0$ by a suitable sequence of approximated problems such that, its corresponding sequence of solutions $\{v_n\}$ is an increasing sequence. Then, they apply the strong maximum principle to v_1 and, in this way, they obtain the uniform lower boundedness of v_n in every subset compactly contained in Ω.

Here, we cannot obtain that $\{u_n\}$, the sequence of solutions to (2.1) with g_n and f_n given by (2.2) and (2.3) respectively, is an increasing sequence. In fact, we could not even apply the strong maximum principle to any u_n. However, Lemma 2 allows us to obtain an uniform lower bound for u_n, for $n \geq n_0$, and this suffices to prove the existence of a solution of (1.1). The first part of this proof is similar to [4, Lemma 4.1], but we include it here for the convenience of the reader.

Theorem 3. *Assume that* $\gamma > 1$, $M(x)$ *verifies* (1.2) *and* $g(x, s)$ *satisfies* (1.5) *and* (1.6). *Assume also that there exists* $r > -1$ *such that* $0 \leq f(x) \in L^1(\Omega)$ *satisfies* (1.4) *and* $g(x, s)$ *verifies* (1.7).

Then, there exists $u \in H^1_{\mathrm{loc}}(\Omega)$ *solution of* (1.1) *satisfying that the function* $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$. *Moreover, if* u_n *satisfies* (2.1) *with* g_n *and* f_n *given by* (2.2) *and* (2.3) *respectively, then* $u_n \to u$ a.e. *in* Ω.

Remark 3. If $\gamma \leq 1$ the existence and regularity results cotained in [4] are still true for (1.1) with the hypotheses of this theorem.

Proof. First of all, let us note that since $\varphi_1 \in \mathcal{C}(\Omega)$ is positive and the function $s \mapsto (c + s^{\frac{\gamma+1}{2+r}})^{\frac{2+r}{\gamma+1}} - s$ with $c > 0$ is greater than a positive constant in $[0,1]$, we deduce thanks to the Lemma 2 that

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega : u_n \geq c_\omega > 0 \text{ in } \omega \text{ for every } n \geq n_0, \tag{2.7}$$

where $n_0 \in \mathbb{N}$ is given by the Lemma 2. So, in what follows, let us fix $n \geq n_0$.

Now, we claim that the sequence $\{u_n^{(\gamma+1)/2}\}_{n\geq n_0}$ is bounded in $H_0^1(\Omega)$. For $\gamma > 1$ we take $T_k(u_n)^\gamma \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $k > 0$ as test function in (2.1) and, using (1.2) and (1.5), we obtain

$$\begin{aligned}\alpha\gamma \int_\Omega |\nabla T_k(u_n)| T_k(u_n)^{\gamma-1} &\leq \int_\Omega \frac{f_n T_k(u_n)^\gamma}{(u_n + 1/n)^\gamma} \\ &\leq \int_\Omega \frac{f_n u_n^\gamma}{(u_n+1/n)^\gamma} \leq \int_\Omega f_n \leq \int_\Omega [f + 1 + c_1].\end{aligned}$$

Since

$$\int_\Omega |\nabla T_k(u_n)| T_k(u_n)^{\gamma-1} = \frac{4}{(\gamma+1)^2} \int_\Omega \left|\nabla T_k(u_n)^{\frac{\gamma+1}{2}}\right|^2,$$

we deduce that $\left\{T_k(u_n)^{\frac{\gamma+1}{2}}\right\}_{n\geq n_0}$ is bounded in $H_0^1(\Omega)$ by a constant independent of k, so we can apply Fatou Lemma to conclude that $\{u_n^{(\gamma+1)/2}\}_{n\geq n_0}$ is bounded in $H_0^1(\Omega)$. Moreover, by the Sobolev embedding we have that the sequence $\{u_n\}_{n\geq n_0}$ is bounded in $L^\tau(\Omega)$ with $\tau = 2^\star(\gamma+1)/2$.

After this, we will prove that $\{u_n\}_{n\geq n_0}$ is bounded in $H^1_{\mathrm{loc}}(\Omega)$.

Let $\phi \in \mathcal{C}_0^1(\Omega)$ and let $\omega = \{\phi \neq 0\}$. Choosing $T_k(u_n)\phi^2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $k > 0$ as test function in (2.1), we have, recalling (1.2), (1.5) and (2.7),

$$\begin{aligned}\alpha \int_\Omega |\nabla T_k(u_n)|^2 \phi^2 + 2\int_\Omega M(x)\nabla u_n \nabla\phi\ T_k(u_n)\phi &\leq \int_\Omega \frac{f_n T_k(u_n)\phi^2}{(u_n+1/n)^\gamma} \\ &\leq \int_\Omega \frac{f_n \phi^2}{(u_n+1/n)^{\gamma-1}} \leq \frac{1}{c_\omega^{\gamma-1}} \int_\Omega f_n \phi^2.\end{aligned}$$

By (1.2) and by Young inequality, we deduce

$$\begin{aligned}-2\int_\Omega M(x)\nabla u_n \nabla\phi\ T_k(u_n)\phi &\leq 2\beta \int_\Omega |\nabla u_n|\, |\nabla\phi|\, u_n\, |\phi| \\ &\leq \frac{\alpha}{2}\int_\Omega |\nabla u_n|^2\phi^2 + \frac{2\beta^2}{\alpha}\int_\Omega |\nabla\phi|^2 u_n^2\end{aligned}$$

and thus

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2\phi^2 \leq \frac{1}{c_\omega^{\gamma-1}}\int_\Omega f_n\phi^2 + \frac{\alpha}{2}\int_\Omega |\nabla u_n|^2\phi^2 + \frac{2\beta^2}{\alpha}\int_\Omega |\nabla\phi|^2 u_n^2.$$

Taking limits when $k \to +\infty$ and applying Fatou Lemma in the left hand side integral we obtain

$$\alpha \int_\Omega |\nabla u_n|^2\phi^2 \leq \frac{1}{c_\omega^{\gamma-1}}\int_\Omega f_n\phi^2 + \frac{\alpha}{2}\int_\Omega |\nabla u_n|^2\phi^2 + \frac{2\beta^2}{\alpha}\int_\Omega |\nabla\phi|^2 u_n^2.$$

Finally, due to the boundedness of $\{u_n\}_{n\geq n_0}$ in $L^\tau(\Omega)$ with $\tau \geq 2$ we have

$$\begin{aligned}\frac{\alpha}{2}\int_\Omega |\nabla u_n|^2\phi^2 &\leq \frac{1}{c_\omega^{\gamma-1}}\int_\Omega f_n\phi^2 + \frac{2\beta^2}{\alpha}\int_\Omega |\nabla\phi|^2 u_n^2\\ &\leq \frac{\|\phi\|^2_{L^\infty(\Omega)}}{c_\omega^{\gamma-1}}\int_\Omega [f+1+c_1] + \frac{2\beta^2}{\alpha}\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega u_n^2 \leq C(f,\phi,\omega),\end{aligned}$$

so that the sequence $\{u_n\}_{n\geq n_0}$ is bounded in $H^1_{\mathrm{loc}}(\Omega)$.

Thanks to this boundedness, there exists a function $u \in H^1_{\mathrm{loc}}(\Omega)$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\{u_n\}$ converges to u weakly in $H^1_{\mathrm{loc}}(\Omega)$ and a.e. in Ω. As a consequence of (2.7), u also satisfies that

$$\forall\omega \subset\subset \Omega,\ \exists c_\omega : u \geq c_\omega > 0 \text{ in } \omega. \tag{2.8}$$

Now, we prove that $g(x,u) \in L^1(\Omega)$. Since $g_n(x,u_n)$ is bounded by a constant in the set $\{u_n \leq s_0\}$ due to (1.7) it follows that

$$\int_\Omega g_n(x,u_n) = \int_{\{u_n\leq s_0\}} g_n(x,u_n) + \int_{\{u_n>s_0\}} g_n(x,u_n) \leq C_1 + \int_{\{u_n>s_0\}} g_n(x,u_n).$$

Taking $\psi(u_n) = T_1\left(\max\{0, \frac{2}{s_0}(u_n - \frac{s_0}{2})\}\right) \in H^1_0(\Omega)\cap L^\infty(\Omega)$ as test function in (2.1) and dropping positive terms we get

$$\begin{aligned}\int_{\{u_n>s_0\}} g_n(x,u_n) &\leq \int_{\{u_n>\frac{s_0}{2}\}} \frac{f_n}{(u_n+1/n)^\gamma}\\ &\leq \left(\frac{2}{s_0}\right)^\gamma \int_{\{u_n>\frac{s_0}{2}\}} f_n \leq \left(\frac{2}{s_0}\right)^\gamma \int_\Omega [f+1+c_1],\end{aligned}$$

so the sequence $\{g_n(x,u_n)\}$ is bounded in $L^1(\Omega)$ and thus $g(x,u) \in L^1(\Omega)$ as a consequence of Fatou Lemma.

To conclude the proof it only remains to pass to the limit on n in the equation satisfied by u_n

$$\int_\Omega M(x)\nabla u_n\nabla\phi + \int_\Omega g_n(x,u_n)\phi = \int_\Omega \frac{f_n\phi}{(u_n+1/n)^\gamma},\ \forall\phi\in\mathcal{C}^1_c(\Omega).$$

Let us fix $\phi \in \mathcal{C}^1_0(\Omega)$. First, since $u_n \rightharpoonup u$ in $H^1_{\mathrm{loc}}(\Omega)$, it is satisfied

$$\lim_{n\to\infty}\int_\Omega M(x)\nabla u_n\nabla\phi = \int_\Omega M(x)\nabla u\nabla\phi.$$

Furthermore, as u_n satisfies (2.7), we deduce

$$\left|\frac{f_n\phi}{(u_n+1/n)^\gamma}\right| \leq \frac{\|\phi\|_{L^\infty(\Omega)}}{c_\omega^\gamma}(f+1+c_1) \in L^1(\Omega),$$

where ω is the set $\{\phi \neq 0\}$. Thus, since also $u_n \to u$ a.e in Ω, we can apply Lebesgue Theorem and it follows that

$$\lim_{n\to\infty}\int_\Omega \frac{f_n\phi}{(u_n+1/n)^\gamma} = \int_\Omega \frac{f\phi}{u^\gamma}.$$

To obtain the limit of $\int_\Omega g_n(x,u_n)\phi$ we use Vitali Theorem. In order to do that we fix $\omega \subset\subset \Omega$ and $\varepsilon > 0$. For $E \subset \omega$, we have by (1.5)

$$\begin{aligned}\int_E g_n(x,u_n) &= \int_{E\cap\{u_n\le k\}} g_n(x,u_n) + \int_{E\cap\{u_n>k\}} g_n(x,u_n)\\ &\le \int_E g(x,k) + \int_{\{u_n>k\}} g_n(x,u_n).\end{aligned} \tag{2.9}$$

On the one hand, if we use $T_1(G_{k-1}(u_n)) \in H_0^1(\Omega)\cap L^\infty(\Omega)$ for $k\ge 2$ as test function in (2.1) and we drop positive terms, we obtain

$$\int_{\{u_n>k\}} g_n(x,u_n) \le \int_{\{u_n>k-1\}} \frac{f_n}{\left(u_n+\frac{1}{n}\right)^\gamma} \le \int_{\{u_n>k-1\}} f_n \le \int_{\{u_n>k-1\}} (f+1+c_1),$$

because $\left(u_n+\frac{1}{n}\right)^\gamma \ge 1$ on the set $\{u_n > k-1\}$. Since $f\in L^1(\Omega)$ and $\{u_n\}_{n\ge n_0}$ is bounded in $L^1(\Omega)$, there exists $k_1 \ge 2$ such that

$$\int_{\{u_n>k\}} g_n(x,u_n) \le \frac{\varepsilon}{2},\ \forall k\ge k_1,\ \forall n\ge n_0. \tag{2.10}$$

On the other hand, by (1.6) there exists $k_0 > k_1$ such that $g(x,k_0)\in L^1_{\text{loc}}(\Omega)$. Then, by the absolute continuity of the integral, there exists $\delta > 0$ such that

$$\int_E g(x,k_0) < \frac{\varepsilon}{2},\ \forall E\subset\omega \text{ with } \operatorname{meas}(E) < \delta. \tag{2.11}$$

Thus, joining (2.9), (2.10) and (2.11), for every $E\subset\omega$ such that $\operatorname{meas}(E)<\delta$ we have

$$\int_E g_n(x,u_n) \le \int_E g(x,k_0) + \int_{\{u_n>k_0\}} g_n(x,u_n) < \varepsilon,\ \forall n\ge n_0,$$

i.e., the sequence $\{g_n(x,u_n)\}_{n\ge n_0}$ is equiintegrable in each $\omega\subset\subset\Omega$. As we also have that $g_n(x,u_n)\to g(x,u)$ a.e. in Ω since $\operatorname{meas}\{u=0\}=0$ by (2.8), we can apply Vitali Theorem to obtain

$$\lim_{n\to\infty}\int_\Omega g_n(x,u_n)\phi = \int_\Omega g(x,u)\phi$$

and thus the proof is concluded. □

3 Regularizing effect due to the behavior of the data at the boundary of Ω

In this section we prove Theorem 1.

Proof of Theorem 1. In the first place, since $\gamma > 1$ we can apply Theorem 3 to assure the existence of a solution $u\in H^1_{\text{loc}}(\Omega)$ of (1.1) such that $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$ which is also the a.e. limit in Ω of the sequence $\{u_n\}$ of solutions of (2.1).

In order to prove item *i)*, i.e., that $u \notin \mathcal{C}^1(\overline{\Omega})$ if $\gamma > r+1$ we follow the ideas in [13]. Arguing by contradiction, suppose that $u \in \mathcal{C}^1(\overline{\Omega})$ for $\gamma > r+1$. First, observe that if $x_0 \in \partial\Omega$ and we denote by $\vec{n}$ the inner normal to $\partial\Omega$ at x_0, then

$$\lim_{s\to 0^+} \frac{\varphi_1(x_0+s\vec{n})}{s} = \lim_{s\to 0^+} \frac{\varphi_1(x_0+s\vec{n})-\varphi_1(x_0)}{s} = \nabla\varphi_1(x_0)\cdot\vec{n} > 0.$$

Now, due to Lemma 2 we have

$$M\varphi_1^{\frac{2+r}{\gamma+1}} \le u \text{ a.e. in } \Omega$$

for some $M > 0$ since $u_n \to u$ a.e. in Ω. Let us remark that $\gamma > r+1$ implies that $t := (2+r)(\gamma+1) < 1$. Since $u \in \mathcal{C}(\overline{\Omega})$, then $u(x_0) = 0$ and for $s > 0$ it follows that

$$\frac{u(x_0+s\vec{n})-u(x_0)}{s} \ge M\varphi_1(x_0+s\vec{n})^{t-1}\frac{\varphi_1(x_0+s\vec{n})}{s}.$$

Therefore, we have

$$\lim_{s\to 0^+} \frac{u(x_0+s\vec{n})-u(x_0)}{s} = +\infty,$$

which contradicts that $u \in \mathcal{C}^1(\overline{\Omega})$.

Now, we deal with item *ii)* and we prove that $u \in H_0^1(\Omega)$ if $1 < \gamma < 2r+3$. Taking $\left(T_k(u_n)+\frac{1}{n}\right)^\theta - \frac{1}{n^\theta} \in H_0^1(\Omega)\cap L^\infty(\Omega)$ with $\theta > \max\left\{0, \gamma - \frac{(r+1)(\gamma+1)}{2+r}\right\}$ as test function in (2.1), we obtain after applying (1.2) and dropping a positive term

$$\begin{aligned} \alpha\theta\int_\Omega (T_k(u_n)+1/n)^{\theta-1}|\nabla T_k(u_n)|^2 &\le \int_\Omega f_n \frac{(T_k(u_n)+1/n)^\theta - 1/n^\theta}{(u_n+1/n)^\gamma} \\ &\le \int_\Omega f_n (u_n+1/n)^{\theta-\gamma}. \end{aligned} \tag{3.1}$$

First of all, let us note that

$$\begin{aligned} &\alpha\theta\int_\Omega (T_k(u_n)+1/n)^{\theta-1}|\nabla T_k(u_n)|^2 \\ &\quad= \frac{4\alpha\theta}{(\theta+1)^2}\int_\Omega \left|\nabla\left((T_k(u_n)+1/n)^{(\theta+1)/2} - 1/n^{\frac{\theta+1}{2}}\right)\right|^2. \end{aligned} \tag{3.2}$$

If we take $\theta < \gamma$, we can apply the Lemma 2 to deduce

$$\int_\Omega f_n (u_n+1/n)^{\theta-\gamma} \le \int_\Omega f_n \left(C\varphi_1(x)+1/n^{(\gamma+1)/(2+r)}\right)^{(2+r)(\theta-\gamma)/\gamma+1}. \tag{3.3}$$

On one hand, there is $C_1 > 0$ such that $\varphi_1 > C_1$ in $\Omega\backslash\Gamma$ (Γ given by (1.8)) since $\varphi_1 > 0$ in Ω, $\varphi_1 \in \mathcal{C}(\Omega)$ and $\Omega\backslash\Gamma$ is closed. Therefore, we have

$$\int_{\Omega\backslash\Gamma} f_n \left(C\varphi_1(x)+\frac{1}{n^{(\gamma+1)/(2+r)}}\right)^{\frac{(2+r)(\theta-\gamma)}{\gamma+1}} \le C_2\int_\Omega (f+1+c_1). \tag{3.4}$$

On the other hand, applying hypothesis (1.8), and using the definition of f_n given by (2.3), it follows that

$$\begin{aligned}
&\int_\Gamma f_n\left(C\varphi_1(x)+1/n^{(\gamma+1)/(2+r)}\right)^{(2+r)(\theta-\gamma)/(\gamma+1)}\\
&\le \int_\Gamma \left(m_2\varphi_1(x)^r+\frac{\chi(r)}{n^{r(\gamma+1)/(2+r)}}\right)\left(C\varphi_1(x)+\frac{1}{n^{(\gamma+1)/(2+r)}}\right)^{\frac{(2+r)(\theta-\gamma)}{\gamma+1}}\\
&\le C_3\int_\Gamma \left(\varphi_1(x)^r+\frac{\chi(r)}{n^{r(\gamma+1)/(2+r)}}\right)\left(\varphi_1(x)+\frac{\chi(r)}{n^{(\gamma+1)/(2+r)}}\right)^{\frac{(2+r)(\theta-\gamma)}{\gamma+1}}\\
&\le C_4\int_\Gamma \left(\varphi_1(x)+\frac{\chi(r)}{n^{(\gamma+1)/(2+r)}}\right)^r\left(\varphi_1(x)+\frac{\chi(r)}{n^{(\gamma+1)/(2+r)}}\right)^{\frac{(2+r)(\theta-\gamma)}{\gamma+1}}\\
&= C_4\int_\Gamma \left(\varphi_1(x)+\chi(r)/n^{(\gamma+1)/(2+r)}\right)^{r+(2+r)(\theta-\gamma)/(\gamma+1)}.
\end{aligned} \tag{3.5}$$

In addition, let us note that

$$\int_\Gamma \varphi_1(x)^{r+(2+r)(\theta-\gamma)/(\gamma+1)}<+\infty$$

since $r+(2+r)(\theta-\gamma)/(\gamma+1)>-1$ because $\theta>\gamma-(r+1)(\gamma+1)/(2+r)$ and $\partial\Omega$ satisfies the interior sphere condition.

In this way, we can deduce from (3.1)–(3.5) that the sequence

$$\left\{(T_k(u_n)+1/n)^{(\theta+1)/2}-1/n^{(\theta+1)/2}\right\}$$

is bounded in $H_0^1(\Omega)$ by a constant independent of k. For this reason, we can use Fatou Lemma to assure

$$\left\{(u_n+1/n)^{(\theta+1)/2}-1/n^{(\theta+1)/2}\right\}$$

is bounded in $H_0^1(\Omega)$ and thus, up to a subsequence, we can assume that it converges weakly in $H_0^1(\Omega)$. Since $u_n\to u$ a.e. in Ω, this weak limit has to be equal to $u^{\frac{\theta+1}{2}}$ and, consequently $u^{\frac{\theta+1}{2}}\in H_0^1(\Omega)$.

Finally, let us note that

$$\theta\in\,]\max\{0,\gamma-(r+1)(\gamma+1)/(2+r)\},\gamma[$$

if, and only if,

$$(\theta+1)/2\in\,]\max\{0.5,(\gamma+1)/(2(2+r))\},(\gamma+1)/2[$$

and that

$$1\in\,]\max\{0.5,(\gamma+1)/(2(2+r))\},(\gamma+1)/2[$$

if, and only if, $1<\gamma<2r+3$.

Finally, we prove item *iii)* i.e., $u\notin H_0^1(\Omega)$ if $\gamma\ge 2r+3$ and u is bounded in Γ. We argue by contradiction, so we assume that $u\in H_0^1(\Omega)$.

In that case, $z(x) = (K\varphi_1(x))^{(2+r)/(\gamma+1)}$ is a supersolution of

$$\begin{cases} -\mathrm{div}(M(x)\nabla z) + g(x,z) = f(x)/z^\gamma & \text{in } \Gamma, \\ z = 0 & \text{on } \partial\Gamma \cap \partial\Omega, \\ z = u & \text{on } \partial\Omega \cap \Omega, \end{cases} \tag{3.6}$$

for large K. Indeed, we get

$$\begin{aligned} -\mathrm{div}(M(x)\nabla z) =& \frac{K\left[\frac{K(2+r)(\gamma-r-1)}{(\gamma+1)^2} M(x)\nabla\varphi_1\nabla\varphi_1 + \frac{\lambda_1(2+r)}{\gamma+1}(K\varphi_1)\varphi_1\right]}{(K\varphi_1)^{\frac{2\gamma-r}{\gamma+1}}} \\ \geq& \frac{K^2 b}{(K\varphi_1)^{\frac{2\gamma-r}{\gamma+1}}} = \frac{K^2 b (K\varphi_1)^r}{(K\varphi_1)^{\frac{\gamma(2+r)}{\gamma+1}}} = \frac{K^2 b(K\varphi_1)^r}{z^\gamma}, \end{aligned} \tag{3.7}$$

where b is a positive constant. This inequality is possible since $\gamma > r+1$, $\varphi_1 \in \mathcal{C}^\infty(\overline{\Omega})$, $\varphi_1 > 0$ in Ω and $M(x)\nabla\varphi_1\nabla\varphi_1 > 0$ in $\partial\Omega$ by (1.2) and by Hopf Lemma.

Now, we use that $g(x,z) \geq 0$ by (1.5), the inequality (3.7) and the hypotheses (1.8) to deduce for K large enough that

$$-\mathrm{div}(M(x)\nabla z) + g(x,z) \geq \frac{K^2 b(K\varphi_1)^r}{z^\gamma} \geq \frac{\frac{K^{2+r}b}{m_2} f(x)}{z^\gamma} \geq \frac{f(x)}{z^\gamma},\ x \in \Gamma, \tag{3.8}$$

i.e., z is a supersolution of (3.6). Now, we take $(u-z)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the problem satisfied by u in Γ and in the inequality (3.8) satisfied by z and subtracting we yield to

$$\begin{aligned} \alpha \int_\Gamma |\nabla(u-z)^+|^2 \leq \alpha \int_\Gamma |\nabla(u-z)^+|^2 + \int_\Gamma (g(x,u) - g(x,z))(u-z)^+ \\ \leq \int_\Gamma \left(f(x)/u^\gamma - f(x)/z^\gamma\right)(u-z)^+ \leq 0. \end{aligned}$$

Therefore, we deduce that $u \leq z$ a.e. in Γ.

We recall that $g(x,u) \in L^1(\Omega)$. Thus we can take as test function in (1.1) $T_k(u)$, for some $k \geq K\varphi_1^{\frac{2+r}{\gamma+1}}$, (see Remark 2) and, due to (1.2) and (1.4), we obtain for some $K_1 > 0$ that

$$\begin{aligned} \beta\|u\|^2_{H_0^1(\Omega)} + k\|g(x,u)\|_{L^1(\Omega)} \geq \beta \int_{\{u\leq k\}} |\nabla u|^2 + \int_\Omega g(x,u)T_k(u) \\ \geq \int_\Omega \frac{f}{u^\gamma} T_k(u) \geq \int_{\{u \leq k\}} f u^{1-\gamma} \geq \int_\Gamma f u^{1-\gamma} \geq K_1 \int_\Gamma \varphi_1^{r + \frac{(1-\gamma)(2+r)}{\gamma+1}} = +\infty. \end{aligned}$$

The last equality is due to $\gamma \geq 2r+3$, since in this case $r + \frac{(1-\gamma)(2+r)}{\gamma+1} \leq -1$. This is a contradiction which assures that $u \notin H_0^1(\Omega)$ and we conclude the proof. □

4 Regularizing effect thanks to the Q-condition

In this section, we prove Theorem 2.

Proof of Theorem 2. Inspired by [1], we define the approximated problems

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) + a_n(x)\tilde{g}(u_n) = \dfrac{f_n(x)}{(|u_n| + 1/n)^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

$$f_n(x) = \frac{f(x)}{1 + (1/n)f(x)}, \qquad a_n(x) = \frac{a(x)}{1 + (Q/n)a(x)}.$$

Note that as the function $s \mapsto \dfrac{s}{1 + s/n}$ is increasing, we deduce by (1.11)

$$f_n(x) \le Q a_n(x) \text{ a.e. in } \Omega. \tag{4.2}$$

Since $a_n(x)$ and $f_n(x)$ are nonnegative functions by (1.10) and $\tilde{g}(s)s \ge 0$ for all $s \in \mathbb{R}$ by (1.9), we can apply Lemma 1 to assure the existence of $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution of (4.1), i.e., satisfying

$$\int_\Omega M(x)\nabla u_n \nabla\phi + \int_\Omega a_n(x)\tilde{g}(u_n)\phi = \int_\Omega \frac{f_n(x)\phi}{(|u_n|+1/n)^\gamma}, \ \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{4.3}$$

Moreover, these hypotheses allow us to prove that $u_n \ge 0$ for all $n \in \mathbb{N}$ by taking $u_n^- \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (4.3).

The scheme of the rest of the proof is as follows:

Step 1. $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$.

Step 2. Control of the right hand side integral of (4.3).

Step 3. Passing to the limit in (4.3).

Step 1. In this step we apply the ideas in [1]. To obtain the boundedness of $\{u_n\}$ in $L^\infty(\Omega)$ we use $G_k(u_n) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (4.1), with $k = \max\{1, \tilde{g}^{-1}(Q)\}$. Let us remark that we are allowed to write $\tilde{g}^{-1}(Q)$ since by (1.9) $\tilde{g}$ has an inverse $\tilde{g}^{-1}$ in $(-\tilde{g}_\infty, \tilde{g}_\infty)$ and $0 < Q < \tilde{g}_\infty$. Therefore, taking $G_k(u_n)$ as test function we get thanks to (1.2) and to (4.2)

$$\begin{aligned} \alpha \int_\Omega |\nabla G_k(u_n)|^2 + \int_\Omega a_n(x)\tilde{g}(u_n)G_k(u_n) &\le \int_\Omega \frac{f_n(x)G_k(u_n)}{(u_n + 1/n)^\gamma} \\ &\le \int_\Omega \frac{Q a_n(x)G_k(u_n)}{(u_n + 1/n)^\gamma} \le \int_\Omega Q a_n(x) G_k(u_n), \end{aligned}$$

where in the last inequality we have used that $\left(u_n + \frac{1}{n}\right)^\gamma \ge u_n^\gamma \ge k^\gamma \ge 1$ on the set $\{u_n \ge k\}$. Thus, we obtain

$$\alpha \int_\Omega |\nabla G_k(u_n)|^2 + \int_\Omega a_n(x)[\tilde{g}(u_n) - Q]G_k(u_n) \le 0.$$

Since the second integral of the previous inequality is nonnegative because $\tilde{g}(u_n) \ge Q$ on the set $\{u_n \ge k\}$ we conclude that $\|G_k(u_n)\|_{H_0^1(\Omega)} = 0$. Then, $\{u_n\}$ is bounded in $L^\infty(\Omega)$ with $\|u_n\|_{L^\infty(\Omega)} \le k$.

Now, using $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (4.1) and using this boundedness of $\{u_n\}$ in $L^\infty(\Omega)$, we can deduce by (1.2), (1.10) and (1.9)

$$\alpha \int_\Omega |\nabla u_n|^2 \le \int_\Omega M(x)\nabla u_n \nabla u_n + \int_\Omega a_n(x)\tilde{g}(u_n)u_n$$
$$= \int_\Omega \frac{f_n(x)u_n}{(u_n+1/n)^\gamma} \le \int_\Omega f_n(x)u_n^{1-\gamma} \le \int_\Omega f(x)k^{1-\gamma}.$$

It should be noted that we have been able to obtain this a priori bound since $\gamma \le 1$.

Thus, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Therefore, there exists a subsequence, still denoted by $\{u_n\}$, which converges weakly in $H_0^1(\Omega)$ and a.e. to some $0 \le u \in H_0^1(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le k$.

Step 2. In this part, we follow the ideas in [9]. We introduce for $\delta > 0$ the function

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \le s \le \delta, \\ -s/\delta + 2, & \text{if } \delta \le s \le 2\delta, \\ 0, & \text{if } 2\delta \le s. \end{cases}$$

Taking $Z_\delta(u_n)\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (4.1), where ϕ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\phi \ge 0$, one has

$$\int_\Omega M(x)\nabla u_n \nabla \phi \, Z_\delta(u_n) + \int_\Omega a_n(x)\tilde{g}(u_n)Z_\delta(u_n)\phi$$
$$= \frac{1}{\delta}\int_{\{\delta \le u_n \le 2\delta\}} M(x)\nabla u_n \nabla u_n \phi + \int_\Omega \frac{f_n(x)}{(u_n+1/n)^\gamma} Z_\delta(u_n)\phi.$$

Since $Z_\delta(u_n) = 1$ in $\{u_n \le \delta\}$ and the first integral of the right hand side is positive, we deduce the inequality

$$0 \le \int_{\{u_n \le \delta\}} \frac{f_n(x)}{(u_n+1/n)^\gamma}\phi \le \int_\Omega M(x)\nabla u_n \nabla \phi \, Z_\delta(u_n) + \int_\Omega a_n(x)\tilde{g}(u_n)Z_\delta(u_n)\phi.$$

Using that $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and converges weakly in $H_0^1(\Omega)$ and a.e. in Ω to u, we can easily pass to the limit in n to obtain

$$0 \le \limsup_{n\to+\infty} \int_{\{u_n \le \delta\}} \frac{f_n(x)}{(u_n+1/n)^\gamma}\phi \le \int_\Omega M(x)\nabla u \nabla \phi Z_\delta(u) + \int_\Omega a(x)\tilde{g}(u)Z_\delta(u)\phi.$$

Now, we pass to the limit as δ tends to 0. Let us note that $Z_\delta(u) \to \chi_{\{u=0\}}$. We use the fact that $\tilde{g}(0) = 0$, since $\tilde{g}$ is odd by (1.9), and we also use that $\nabla u = 0$ a.e. in $\{u = 0\}$, since $u \in H_0^1(\Omega)$. This allow us to conclude

$$\limsup_{n\to+\infty} \int_{\{u_n \le \delta\}} \frac{f_n(x)}{(u_n+1/n)^\gamma}\phi \to 0 \text{ as } \delta \to 0. \tag{4.4}$$

Step 3. Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\phi \ge 0$. Since $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ a.e. in Ω and $\{u_n\}$ is bounded in $L^\infty(\Omega)$, we can pass to the limit in the left hand side of (4.3) to assure

$$\int_\Omega M(x)\nabla u_n \nabla \phi + \int_\Omega a_n(x)\tilde{g}(u_n)\phi \to \int_\Omega M(x)\nabla u \nabla \phi + \int_\Omega a(x)\tilde{g}(u)\phi. \tag{4.5}$$

Now, we choose $\delta_m \to 0$ such that $\text{meas}\{u = \delta_m\} = 0$ (observe that this is posible since the set $\{\delta > 0 : \text{meas}\{u = \delta\} > 0\}$ is at most countable) and we split the right hand side integral of (4.3) into two parts, namely

$$\int_\Omega \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi = \int_{\{u_n \le \delta_m\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi + \int_{\{u_n > \delta_m\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi. \quad (4.6)$$

With respect to the second integral of the right hand side of (4.6), we express it as

$$\int_{\{u_n > \delta_m\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi = \int_\Omega \frac{f_n(x)}{(u_n + 1/n)^\gamma}\chi_{\{u_n > \delta_m\}}\phi.$$

Now, for every fixed m we have

$$0 \le \frac{f_n(x)}{(u_n + 1/n)^\gamma}\chi_{\{u_n > \delta_m\}}\phi \le \frac{f(x)}{\delta_m^\gamma}\phi \in L^1(\Omega)$$

and

$$\frac{f_n(x)}{(u_n + 1/n)^\gamma}\chi_{\{u_n > \delta_m\}} \to \frac{f(x)}{u^\gamma}\chi_{\{u > \delta_m\}} \text{ a.e. } x \in \Omega \text{ when } n \to \infty.$$

Thus, one has by Lebesgue Theorem

$$\int_{\{u_n > \delta_m\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi \to \int_{\{u > \delta_m\}} \frac{f(x)}{u^\gamma}\phi \text{ as } n \to +\infty.$$

Observe that thanks to (4.3), (4.5) and (4.6) we get

$$\lim_{n\to\infty} \int_{\{u_n \le \delta_m\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi = \int_\Omega M(x)\nabla u \nabla \phi + \int_\Omega a(x)\tilde{g}(u)\phi - \int_{\{u > \delta_m\}} \frac{f(x)}{u^\gamma}\phi$$

and, using (4.4), we obtain

$$\begin{aligned} 0 &= \lim_{m\to\infty}\lim_{n\to\infty} \int_{\{u_n \le \delta_m\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi \qquad (4.7) \\ &= \int_\Omega M(x)\nabla u \nabla \phi + \int_\Omega a(x)\tilde{g}(u)\phi - \lim_{m\to\infty} \int_{\{u > \delta_m\}} \frac{f(x)}{u^\gamma}\phi. \end{aligned}$$

In particular, using Fatou Lemma we deduce that $\frac{f(x)}{u^\gamma}\phi \in L^1(\{u > 0\})$ and then, using Lebesgue Theorem

$$\lim_{m\to\infty} \int_{\{u > \delta_m\}} \frac{f(x)}{u^\gamma}\phi = \int_{\{u > 0\}} \frac{f(x)}{u^\gamma}\phi. \quad (4.8)$$

Now we observe that (4.4) also implies that $\int_{\{u=0\}} \frac{f(x)}{u^\gamma}\phi = 0$ or equivalently that $\text{meas}\{u = 0, f\phi \ne 0\} = 0$. Indeed, note that for every $\delta > 0$ it follows that

$$\int_{\{u=0\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\chi_{\{u_n \le \delta\}}\phi \le \int_{\{u_n \le \delta\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi.$$

Moreover, for every $\delta > 0$

$$\frac{f_n(x)}{(u_n + 1/n)^\gamma}\chi_{\{u_n \le \delta\}} \to \frac{f(x)}{u^\gamma} \text{ a.e. in } \{u = 0\} \text{ when } n \to \infty.$$

Then, we can apply Fatou Lemma to obtain

$$\int_{\{u=0\}} \frac{f(x)}{u^\gamma}\phi \le \limsup_{n\to+\infty} \int_{\{u_n \le \delta\}} \frac{f_n(x)}{(u_n + 1/n)^\gamma}\phi, \ \forall \delta > 0.$$

In view of (4.4), this implies

$$\int_{\{u=0\}} \frac{f(x)}{u^\gamma}\phi = 0$$

and, as a consequence

$$\int_{\{u>0\}} \frac{f(x)}{u^\gamma}\phi = \int_\Omega \frac{f(x)}{u^\gamma}\phi.$$

This, combined with (4.7) and(4.8) give us

$$\int_\Omega M(x)\nabla u \nabla \phi + \int_\Omega a(x)\tilde{g}(u)\phi = \int_\Omega \frac{f(x)}{u^\gamma}\phi, \ \forall 0 \le \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

and, in this way,

$$\int_\Omega M(x)\nabla u \nabla \phi + \int_\Omega a(x)\tilde{g}(u)\phi = \int_\Omega \frac{f(x)}{u^\gamma}\phi, \ \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Moreover, $g(x, u) = a(x)\tilde{g}(u) \in L^1(\Omega)$ since $u \in L^\infty(\Omega)$ and $\frac{f}{u^\gamma} \in L^1_{\mathrm{loc}}(\Omega)$ since $\int_\Omega \frac{f(x)}{u^\gamma}|\phi| < +\infty$ for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Thus, it is proved that the function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1), as desired. Uniqueness is deduced, as usual, from (1.9). Indeed, given $u_1, u_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ any two solutions to (1.1) with $g(x, u) = a(x)\tilde{g}(u)$ where $\tilde{g}$ verifies (1.9), then taking $(u_1 - u_2)^+$ as test function in the problems satisfied by u_1 and u_2, subtracting and taking into account (1.2), (1.9) and (1.10), we yield to

$$\begin{aligned} &\alpha \int_\Omega |\nabla(u_1 - u_2)^+|^2 \le \int_\Omega M(x)\nabla(u_1 - u_2)\nabla(u_1 - u_2)^+ \\ &+ \int_\Omega a(x)(\tilde{g}(u_1) - \tilde{g}(u_2))(u_1 - u_2)^+ = \int_\Omega f(x)\left(1/u_1^\gamma - 1/u_2^\gamma\right)(u_1 - u_2)^+ \le 0. \end{aligned}$$

this implies that $(u_1 - u_2)^+ = 0$, i.e., $u_1 \le u_2$, and since both are arbitrary solution we also have the reverse inequality and the proof is finished. □

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