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An Accurate Numerical Scheme for Three-Dimensional Variable-Order Time-Fractional Partial Differential Equations in Two Types of Space Domains

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Abstract. We consider the discretization method for solving three-dimensional variable-order (3D-VO) time-fractional partial differential equations. The proposed method is developed based on discrete shifted Hahn polynomials (DSHPs) and their operational matrices. In the process of method implementation, the modified operational matrix (MOM) and complement vector (CV) of integration and pseudo-operational matrix (POM) of VO fractional derivative plays an important role in the accuracy of the method. Further, we discuss the error of the approximate solution. At last, the methodology is validated by well test examples in two types of space domains. In order to evaluate the accuracy and applicability of the approach, the results are compared with other methods.

Keywords: discrete shifted Hahn polynomials, variable-order Caputo fractional derivative, operational matrix, three-dimensional partial differential equations.

AMS Subject Classification: 65M70; 65M15.

1 Introduction

Nowadays, the VO-fractional differential equations have appeared in the modelling of plenty of physical phenomena [16, 17]. The researchers found that the VO-fractional derivative describes complex physical models more accurately than the constant-order derivative [21, 23]. In the meantime, numerical methods for solving VO-fractional differential equations worked very powerfully. Therefore, several numerical methods have been introduced, such as fractional-order Taylor wavelets [24], spline finite difference scheme [13], modified wavelet method [6], Genocchi collocation method [5], etc.

This paper presents the numerical framework based on discrete shifted Hahn polynomials for solving 3D-VO time-fractional partial differential equations. One of the important features of discrete polynomials is the approximation of continuous functions with high accuracy. So that the coefficients in the approximation process are accurately evaluated. The discrete shifted Hahn polynomials have been applied in a few numerical approaches among discrete polynomials. For instance, Mohammadi et al. [14] applied these polynomials for solving optimal control of fractional Volterra integro-differential equations. Authors in [4] provided a method based on Legendre-Gauss-Lobatto quadrature and discrete shifted Hahn polynomials for solving Caputo-Fabrizio fractional partial integro-differential equations.

Due to the complexity of 3D-time-fractional partial differential equations and the high volume of calculations, a small number of papers have been published in this field, which can be found in [1, 19, 22, 25]. In this study, we focus on the numerical solution of the following 3D-VO time-fractional partial differential equations:

$$\begin{aligned} D_t^{\gamma(\mathbf{x},t)}u(\mathbf{x},t) =& \mu_1 \Delta u(\mathbf{x},t) + \mu_2 \nabla u(\mathbf{x},t) + \boldsymbol{\Upsilon}(u(\mathbf{x},t)) + f(\mathbf{x},t), \\ \mathbf{x} =& (x,y) \in \Omega \subset R^2, \quad t \in [0,1], \quad 0 < \gamma(\mathbf{x},t) \le 1, \end{aligned} \tag{1.1}$$

with initial condition

$$u(x,y,0) = \varphi(x,y), \quad x,y \in \Omega,$$

and Dirichlet boundary conditions

$$\begin{aligned} u(x,0,t) &= \psi_1(x,t), \quad u(x,1,t) = \psi_2(x,t), \\ u(0,y,t) &= \psi_3(y,t), \quad u(1,y,t) = \psi_4(y,t), \quad x,y \in \partial\Omega, \quad t \in [0,1], \end{aligned}$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and $\nabla u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ represent Laplace and Gradient operators, respectively. And also, $D_t^{\gamma(\mathbf{x},t)}$ denotes the VO-Caputo fractional derivative which is defined as follows [18]:

$$D_t^{\gamma(\mathbf{x},t)}u(\mathbf{x},t) = \begin{cases} \frac{1}{\Gamma(q-\gamma(\mathbf{x},t))} \int_0^t (t-s)^{q-\gamma(\mathbf{x},t)-1} \dfrac{\partial^q u(\mathbf{x},s)}{\partial s^q}, & q-1 < \gamma(\mathbf{x},t) < q, \\ \dfrac{\partial^q u(\mathbf{x},t)}{\partial t^q}, & \gamma(\mathbf{x},t) = q \in \mathbb{N}. \end{cases}$$

Given the existence of a VO-fractional derivative in the proposed equation, it is impossible to provide an analytical method to obtain an exact solution.

Due to this subject, numerical methods to solve these equations received much attention. Some of these methods are as follows:

a) Gu and Sun [8], presented a meshless method for solving the three-dimensional VO time-fractional diffusion equation.

b) In [11], the authors applied the RBF-based differential quadrature approach for solving the 2D-VO time-fractional advection-diffusion equation.

c) Shekari et al. [20], provided the moving least squares meshless approach for the numerical solution of the 2D-VO time-fractional nonlinear diffusion-wave equation.

A small number of numerical methods are proposed to solve these equations. Therefore, this issue motivated us to provide a high-precision numerical method to solve the presented problems. The method provided here is the composition of DSHPs properties with the operational matrices for finding the approximate solution of the problem in two types of space domains. From outcomes illustrated in different space domains, it can be understood that the method is powerful in solving problems and can be utilized to solve VO-fractional multi-dimensional problems.

The rest of this paper is organized as follows. In Section 2, we provide a brief summary of DSHPs and their properties. The method of finding the required matrices is discussed in Section 3. The discretization approach for solving the proposed problem is described in Section 4. Section 5 discusses the error of approximate solution. The numerical experiments to confirm the accuracy and efficiency of the method are presented in Section 6. At last, Section 7 gives brief conclusions about the presented technique.

2 Discrete shifted Hahn polynomials

The analytical formula of Hahn polynomials on the interval $[0, M]$ is defined with the help of hypergeometric series as follows [4, 14]:

$$\mathcal{H}^{\alpha,\beta}_{m,M}(\xi) = {}_3F_2(-m, m+\alpha+\beta+1, -\xi; \alpha+1, -M; 1)$$
$$= \sum_{k=0}^{m} \frac{(-m)_k(m+\alpha+\beta+1)_k(-\xi)_k}{k!(\alpha+1)_k(-M)_k}, \ m = 0, 1, \ldots, M \in \mathbb{Z}^+, \ \alpha, \beta > -1.$$

where $(.)_k$ denotes the Pochhammer symbol. To simplify the above formula, we rewrite the explicit analytical formula as below:

$$\mathcal{H}^{\alpha,\beta}_{m,M}(\xi) = \sum_{k=0}^{m} h_{k,m}\xi^k, \quad m = 0, 1, \ldots, M, \tag{2.1}$$

in which

$$h_{k,m} = \sum_{i=k}^{m} \frac{(-1)^i(-m)_i(m+\alpha+\beta+1)_i\mathcal{S}(k,i)}{i!(\alpha+1)_i(-M)_i},$$

and $\mathcal{S}(k,i) = \frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}j^k$.

Remark 1. By considering the change of variable $\xi = Mx$, the shifted Hahn polynomials $\mathbf{H}^{\alpha,\beta}_m(x)$ are obtained on the interval $[0, 1]$.

Any continuous function $f(x,t)$ in the interval Ω can be approximated by truncated shifted Hahn series as follows:

$$\begin{aligned} f(x,y,t) \simeq & \sum_{m=0}^{M}\sum_{n=0}^{M}\sum_{k=0}^{M} f_{mnk}\mathbf{H}_m^{\alpha,\beta}(x)\mathbf{H}_n^{\alpha,\beta}(y)\mathbf{H}_k^{\alpha,\beta}(t) \\ =&(\mathbf{H}^{\alpha,\beta}(x))^T\mathbf{R}^{\alpha,\beta}(y)\mathbf{Q}^{\alpha,\beta}(t)\mathbf{F}, \end{aligned} \tag{2.2}$$

where $F = [f_{000}, f_{001}, \ldots, f_{00M}, f_{100}, f_{101}, \ldots, f_{10M}, \ldots, f_{MM0}, f_{MM1}, \ldots, f_{MMM}]^T_{(M+1)^3\times 1}$,

$$\begin{aligned} \mathbf{H}^{\alpha,\beta}(x) &= \left[\mathbf{H}_0^{\alpha,\beta}(x)\ \mathbf{H}_1^{\alpha,\beta}(x)\ \ldots\ \mathbf{H}_M^{\alpha,\beta}(x)\right]^T_{1\times(M+1)}, \\ \mathbf{R}^{\alpha,\beta}(y) &= diag\left[(\mathbf{H}^{\alpha,\beta}(y))^T\ (\mathbf{H}^{\alpha,\beta}(y))^T\ \ldots\ (\mathbf{H}^{\alpha,\beta}(y))^T\right]_{(M+1)\times(M+1)^2}, \\ \mathbf{Q}^{\alpha,\beta}(t) &= diag\left[(\mathbf{H}^{\alpha,\beta}(t))^T\ (\mathbf{H}^{\alpha,\beta}(t))^T\ \ldots\ (\mathbf{H}^{\alpha,\beta}(t))\right]_{(M+1)^2\times(M+1)^3}. \end{aligned}$$

According to discrete orthogonal polynomials, the components of the coefficient vector are obtained as follows:

$$f_{mnk} := \frac{\sum_{x=0}^{M}\sum_{y=0}^{M}\sum_{t=0}^{M} f(x,y,t)\mathbf{H}_m^{\alpha,\beta}(x)\mathbf{H}_m^{\alpha,\beta}(y)\mathbf{H}_m^{\alpha,\beta}(t)w(x,y,t)}{\tau_m\tau_n\tau_k},$$

where $w(x,y,t)$ is a real non-negative weight function:

$$\begin{aligned} w(x,y,t) &= \binom{\alpha+Mx}{Mx}\binom{\alpha+My}{My}\binom{\alpha+Mt}{Mt} \\ &\quad\times\binom{\beta+M-Mx}{M-Mx}\binom{\beta+M-My}{M-My}\binom{\beta+M-Mt}{M-Mt}, \\ \tau_m &= \frac{(-1)^m(m+\alpha+\beta+1)_{M+1}(\beta+1)_m m!}{(2m+\alpha+\beta+1)(\alpha+1)_m(-M)_m M!}, \\ \tau_n &= \frac{(-1)^n(n+\alpha+\beta+1)_{M+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-M)_n M!}, \\ \tau_k &= \frac{(-1)^k(k+\alpha+\beta+1)_{M+1}(\beta+1)_k k!}{(2k+\alpha+\beta+1)(\alpha+1)_k(-M)_k M!}. \end{aligned}$$

In addition, the MATLAB code of the discrete Hahn polynomials vector is as follows:

```
for m=1:M+1
H(m,1)=symsum((pochhammer(-m+1,k)*pochhammer(m-1+alpha+beta+1,k)
*pochhammer(-x,k))/(pochhammer(alpha+1,k)*pochhammer(-M,k)
*factorial(k)),k,0,m-1);
end
H=expand(H);
```

3 Required matrices

The main aim of this section is to present the obtaining algorithm of operational matrices used in the process of the numerical method.

3.1 MOM and CV of integration

The modified operational matrix and complement vector of integration for DSHPs are obtained as follows [4]:

$$\int_0^x \mathbf{H}^{\alpha,\beta}(\eta)d\eta = \Theta\mathbf{H}^{\alpha,\beta}(x) + \mathbf{W}(x). \tag{3.1}$$

Here, Θ and $\mathbf{W}(x)$ denote the modified operational matrix and complement vector of integration, respectively. Due to Equation (2.1), the modified operational matrix and complement vector of $\mathbf{R}^{\alpha,\beta}(y)$ and $\mathbf{Q}^{\alpha,\beta}(t)$ are obtained as follows:

$$\int_0^y \mathbf{R}^{\alpha,\beta}(\eta)d\eta = \mathbf{R}^{\alpha,\beta}(y)P + \mathbf{V}(y),$$

where $P = I_{(M+1)\times(M+1)} \otimes \Theta,\ \mathbf{V}(y) = I_{(M+1)\times(M+1)} \otimes \mathbf{W}(y)$, and O denotes the zero matrix with $(M+1)\times(M+1)$ dimension. And also, we get

$$\int_0^t \mathbf{Q}^{\alpha,\beta}(\eta)d\eta = \mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t),$$

where $S = I_{(M+1)^2\times(M+1)^2} \otimes \Theta,\ \mathbf{Z}(t) = I_{(M+1)\times(M+1)} \otimes \mathbf{W}(t)$.

3.2 POM of VO-fractional derivative

This section provides the pseudo-operational matrix of VO fractional derivative in the VO-Caputo sense. Therefore, we get

$$D^{\gamma(\mathbf{x},t)}\mathbf{H}^{\alpha,\beta}(t) = t^{-\gamma(\mathbf{x},t)}\Lambda(\mathbf{x},t)\mathbf{H}^{\alpha,\beta}(t). \tag{3.2}$$

In this formula $\Lambda(\mathbf{x},t)$ denotes the VO-fractional derivative pseudo-operational matrix. To reach our aim, we apply the analytic formula of DSHPs and VO-Caputo definition:

$$\begin{aligned} D^{\gamma(\mathbf{x},t)}\mathbf{H}_m^{\alpha,\beta}(t) &= D^{\gamma(\mathbf{x},t)}\left(\sum_{k=0}^{m} M^k h_{k,m} t^k\right) = \sum_{k=0}^{m} M^k h_{k,m} \\ &\times \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma(\mathbf{x},t))} t^{k-\gamma(\mathbf{x},t)} = t^{-\gamma(\mathbf{x},t)} \sum_{k=0}^{m} y_{k,m,M}^{\gamma(\mathbf{x},t)} \left(\sum_{i=0}^{M} a_i \mathbf{H}_i^{\alpha,\beta}(t)\right), \end{aligned}$$

where $y_{k,m,M}^{\gamma(\mathbf{x},t)} = \frac{\Gamma(k+1)M^k h_{k,m}}{\Gamma(k+1-\gamma(\mathbf{x},t))}$ and

$$a_i = \frac{\langle t^k, \mathbf{H}_k^{\alpha,\beta}(t)\rangle}{\tau_m} := \frac{\sum_{t=0}^{M} t^k w(t)\mathbf{H}_m^{\alpha,\beta}(t)}{\tau_m},\ w(t) = \binom{\alpha+Mt}{Mt}\binom{\beta+M-Mt}{M-Mt}.$$

Then, we obtain

$$D^{\gamma(\mathbf{x},t)}\mathbf{H}_m^{\alpha,\beta}(t) = t^{-\gamma(\mathbf{x},t)} \sum_{i=0}^{M} \lambda_{i,k,m,M}^{\gamma(\mathbf{x},t)} \mathbf{H}_m^{\alpha,\beta}(t),\ \lambda_{i,k,m,M}^{\gamma(\mathbf{x},t)} = \sum_{k=0}^{m} a_i y_{k,m,M}^{\gamma(\mathbf{x},t)}.$$

Next, the corresponding pseudo-operational matrix of VO-fractional derivative of $\mathbf{Q}^{\alpha,\beta}(t)$ is obtained as follows:

$$D^{\gamma(\mathbf{x},t)}\mathbf{Q}^{\alpha,\beta}(t) = t^{-\gamma(\mathbf{x},t)}\mathbf{Q}^{\alpha,\beta}(t)\Psi(\mathbf{x},t),$$

where $\Psi(\mathbf{x},t) = I_{(M+1)^2\times(M+1)^2} \otimes \Lambda^T(\mathbf{x},t)$.

4 Discretization approach

In this section, we offer the numerical approach based on MOM and CV of integration and POM of VO-fractional derivative. To achieve this purpose, we consider the following relation:

$$\frac{\partial^5 u(\mathbf{x},t)}{\partial x^2 \partial y^2 \partial t} \simeq (\mathbf{H}^{\alpha,\beta}(x))^T \mathbf{R}^{\alpha,\beta}(y)\mathbf{Q}^{\alpha,\beta}(t)\mathbf{U}, \tag{4.1}$$

where vector U is defined as similar to F in Equation (2.2). In the next step, we need to provide the approximation of other unknown functions in Equation (1.1). Therefore, by integrating Equation (4.1) with respect to t and using MOM and CV of integration, we get

$$\frac{\partial^4 u(\mathbf{x},t)}{\partial x^2 \partial y^2} \simeq (\mathbf{H}^{\alpha,\beta}(x))^T \mathbf{R}^{\alpha,\beta}(y)\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} + \varphi(x,y). \tag{4.2}$$

By repeating the above procedure with respect to variable y, we obtain:

$$\begin{aligned}\frac{\partial^3 u(\mathbf{x},t)}{\partial x^2 \partial y} \simeq& (\mathbf{H}^{\alpha,\beta}(x))^T \left(\mathbf{R}^{\alpha,\beta}(y)P + \mathbf{V}(y)\right)\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} \\ &+ \int_0^y \varphi(x,\eta)d\eta + \frac{\partial^3 u(x,0,t)}{\partial x^2 \partial y}.\end{aligned} \tag{4.3}$$

As you can see, function $\frac{\partial^3 u(x,0,t)}{\partial x^2 \partial y}$ is unknown. Therefore, to calculate this function, take integral from Equation (4.3) to variable y in the range 0 to 1:

$$\begin{aligned}\frac{\partial^3 u(x,0,t)}{\partial x^2 \partial y} \simeq& \frac{\partial^2 \psi_2(x,t)}{\partial x^2} - \frac{\partial^2 \psi_1(x,t)}{\partial x^2} - (\mathbf{H}^{\alpha,\beta}(x))^T \Big(\Big[\int_0^1 \mathbf{R}^{\alpha,\beta}(y)dy\Big]P \\ &+ \Big[\int_0^1 \mathbf{V}(y)dy\Big]\Big)\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} + \int_0^1\int_0^y \varphi(x,\eta)d\eta dy.\end{aligned}$$

Now, by placing the above expression in Equation (4.3), we will have:

$$\begin{aligned}\frac{\partial^3 u(\mathbf{x},t)}{\partial x^2 \partial y} \simeq& (\mathbf{H}^{\alpha,\beta}(x))^T \left(\mathbf{R}^{\alpha,\beta}(y)P + \mathbf{V}(y)\right)\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} + \int_0^y \varphi(x,\eta)d\eta \\ &+ \Bigg\{\frac{\partial^2 \psi_2(x,t)}{\partial x^2} - \frac{\partial^2 \psi_1(x,t)}{\partial x^2} - (\mathbf{H}^{\alpha,\beta}(x))^T \left(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y)dy\right]P + \left[\int_0^1 \mathbf{V}(y)dy\right]\right) \\ &\times \left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} + \int_0^1\int_0^y \varphi(x,\eta)d\eta dy\Bigg\}.\end{aligned} \tag{4.4}$$

By integrating Equation (4.4) with respect to y, we have

$$\begin{aligned}\frac{\partial^2 u(\mathbf{x},t)}{\partial x^2} &\simeq (\mathbf{H}^{\alpha,\beta}(x))^T\left(\mathbf{R}^{\alpha,\beta}(y)P^2+\mathbf{V}(y)P+\left[\int_0^y \mathbf{V}(\eta)d\eta\right]\right)\\ &\times\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^y\int_0^z \varphi(x,\eta)d\eta dz+y\Big\{\frac{\partial^2\psi_2(x,t)}{\partial x^2}\\ &-\frac{\partial^2\psi_1(x,t)}{\partial x^2}-(\mathbf{H}^{\alpha,\beta}(x))^T\left(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y)dy\right]P+\left[\int_0^1 \mathbf{V}(y)dy\right]\right)\\ &\times\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^1\int_0^y \varphi(x,\eta)d\eta dy\Big\}+\frac{\partial^2\psi_1(x,t)}{\partial x^2}.\end{aligned}$$

To continue the calculation process, it is necessary to integrate the above expression with respect to variable x:

$$\begin{aligned}\frac{\partial u(\mathbf{x},t)}{\partial x} &\simeq \left((\mathbf{H}^{\alpha,\beta}(x))^T\Theta^T+\mathbf{W}^T(x)\right)\left(\mathbf{R}^{\alpha,\beta}(y)P^2+\mathbf{V}(y)P+\left[\int_0^y\mathbf{V}(\eta)d\eta\right]\right)\\ &\times\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^x\int_0^y\int_0^z\varphi(\xi,\eta)d\eta dz d\xi\\ &+y\left\{\frac{\partial\psi_2(x,t)}{\partial x}-\left.\frac{\partial\psi_2(x,t)}{\partial x}\right|_{x=0}-\frac{\partial\psi_1(x,t)}{\partial x}+\left.\frac{\partial\psi_1(x,t)}{\partial x}\right|_{x=0}\right.\\ &-\left((\mathbf{H}^{\alpha,\beta}(x))^T\Theta^T+\mathbf{W}^T(x)\right)\left(\left[\int_0^1\mathbf{R}^{\alpha,\beta}(y)dy\right]P+\left[\int_0^1\mathbf{V}(y)dy\right]\right)\\ &\left.\times\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^x\int_0^1\int_0^y\varphi(\xi,\eta)d\eta dy d\xi\right\}\\ &+\frac{\partial\psi_1(x,t)}{\partial x}-\left.\frac{\partial\psi_1(x,t)}{\partial x}\right|_{x=0}+\frac{\partial u(0,y,t)}{\partial x}. \qquad (4.5)\end{aligned}$$

In Equation (4.5), function $\frac{\partial u(0,y,t)}{\partial x}$ is indeterminate. Hence, to calculate this function, we integrate Equation (4.5) from variable x in the range 0 to 1 as follows:

$$\begin{aligned}\frac{\partial u(0,y,t)}{\partial x} &\simeq \psi_4(y,t)-\psi_3(y,t)-\left(\left[\int_0^1(\mathbf{H}^{\alpha,\beta}(x))^Tdx\right]\Theta^T+\left[\int_0^1\mathbf{W}^T(x)dx\right]\right)\\ &\times\left(\mathbf{R}^{\alpha,\beta}(y)P^2+\mathbf{V}(y)P+\left[\int_0^y\mathbf{V}(\eta)d\eta\right]\right)\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}\\ &-\int_0^1\int_0^x\int_0^y\int_0^z\varphi(\xi,\eta)d\eta dz d\xi dx-y\left\{\psi_2(1,t)-\psi_2(0,t)-\left.\frac{\partial\psi_2(x,t)}{\partial x}\right|_{x=0}\right.\\ &-\psi_1(1,t)+\psi_1(0,t)+\left.\frac{\partial\psi_1(x,t)}{\partial x}\right|_{x=0}-\left(\left[\int_0^1(\mathbf{H}^{\alpha,\beta}(x))^Tdx\right]\Theta^T\right.\\ &\quad\left.+\left[\int_0^1\mathbf{W}^T(x)dx\right]\right)\left(\left[\int_0^1\mathbf{R}^{\alpha,\beta}(y)dy\right]P+\left[\int_0^1\mathbf{V}(y)dy\right]\right)\\ &\left.\times\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^1\int_0^x\int_0^1\int_0^y\varphi(\xi,\eta)d\eta d\xi dy dx\right\}\end{aligned}$$

$$- \psi_1(1,t) + \psi_1(0,t) + \left.\frac{\partial \psi_1(x,t)}{\partial x}\right|_{x=0}. \tag{4.6}$$

So, by replacing Equation (4.6) in Equation (4.5), we get the approximation of $\frac{\partial u(\mathbf{x},t)}{\partial x}$ as:

$$\begin{aligned}
\frac{\partial u(\mathbf{x},t)}{\partial x} &\simeq \left((\mathbf{H}^{\alpha,\beta}(x))^T \Theta^T + \mathbf{W}^T(x)\right)\left(\mathbf{R}^{\alpha,\beta}(y)P^2 + \mathbf{V}(y)P + \left[\int_0^y \mathbf{V}(\eta)d\eta\right]\right) \\
&\times \left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} + \int_0^x\int_0^y\int_0^z \varphi(\xi,\eta)d\eta dz d\xi \\
&+ y\left\{\frac{\partial \psi_2(x,t)}{\partial x} - \left.\frac{\partial \psi_2(x,t)}{\partial x}\right|_{x=0} - \frac{\partial \psi_1(x,t)}{\partial x} + \left.\frac{\partial \psi_1(x,t)}{\partial x}\right|_{x=0}\right. \\
&- \left((\mathbf{H}^{\alpha,\beta}(x))^T \Theta^T + \mathbf{W}^T(x)\right)\left(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y)dy\right]P + \left[\int_0^1 \mathbf{V}(y)dy\right]\right) \\
&\times \left.\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} + \int_0^x\int_0^1\int_0^y \varphi(\xi,\eta)d\eta d\xi dy\right\} \\
&+ \frac{\partial \psi_1(x,t)}{\partial x} - \left.\frac{\partial \psi_1(x,t)}{\partial x}\right|_{x=0} \\
&+ \left\{\psi_4(y,t) - \psi_3(y,t) - \left(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx\right]\Theta^T + \left[\int_0^1 \mathbf{W}^T(x)dx\right]\right)\right. \\
&\times \left(\mathbf{R}^{\alpha,\beta}(y)P^2 + \mathbf{V}(y)P + \left[\int_0^y \mathbf{V}(\eta)d\eta\right]\right)\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} \\
&- \int_0^1\int_0^x\int_0^y\int_0^z \varphi(\xi,\eta)d\eta dz d\xi dx - y\left\{\psi_2(1,t) - \psi_2(0,t) - \left.\frac{\partial \psi_2(x,t)}{\partial x}\right|_{x=0}\right. \\
&- \psi_1(1,t) + \psi_1(0,t) + \left.\frac{\partial \psi_1(x,t)}{\partial x}\right|_{x=0} \\
&- \left(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx\right]\Theta^T + \left[\int_0^1 \mathbf{W}^T(x)dx\right]\right) \\
&\times \left(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y)dy\right]P + \left[\int_0^1 \mathbf{V}(y)dy\right]\right)\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} \\
&+ \left.\left.\int_0^1\int_0^x\int_0^1\int_0^y \varphi(\xi,\eta)d\eta d\xi dy dx\right\} - \psi_1(1,t) + \psi_1(0,t) + \left.\frac{\partial \psi_1(x,t)}{\partial x}\right|_{x=0}\right\}.
\end{aligned} \tag{4.7}$$

Finally, by integrating Equation (4.7) with respect to x, the approximate solution of the problem is obtained

$$\begin{aligned}
u(\mathbf{x},t) &\simeq \left((\mathbf{H}^{\alpha,\beta}(x))^T (\Theta^T)^2 + \mathbf{W}^T(x)\Theta^T + \left[\int_0^x \mathbf{W}^T(\xi)d\xi\right]\right) \\
&\times \left(\mathbf{R}^{\alpha,\beta}(y)P^2 + \mathbf{V}(y)P + \left[\int_0^y \mathbf{V}(\eta)d\eta\right]\right)\left(\mathbf{Q}^{\alpha,\beta}(t)S + \mathbf{Z}(t)\right)\mathbf{U} \\
&+ \int_0^x\int_0^\varepsilon\int_0^y\int_0^z \varphi(\xi,\eta)d\eta dz d\xi d\varepsilon + y\left\{\psi_2(x,t) - \psi_2(0,t)\right. \\
&- x\left.\frac{\partial \psi_2(x,t)}{\partial x}\right|_{x=0} - \psi_1(x,t) - \psi_1(0,t) + x\left.\frac{\partial \psi_1(x,t)}{\partial x}\right|_{x=0}
\end{aligned}$$

$$
\begin{aligned}
&- \Bigg((\mathbf{H}^{\alpha,\beta}(x))^T (\Theta^T)^2 + \mathbf{W}^T(x)\Theta^T + \left[\int_0^x \mathbf{W}^T(\xi) d\xi \right] \Bigg) \\
&\times \left(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y) dy \right] P + \left[\int_0^1 \mathbf{V}(y) dy \right] \right) \left(\mathbf{Q}^{\alpha,\beta}(t) S + \mathbf{Z}(t) \right) \mathbf{U} \\
&+ \int_0^x \int_0^\varepsilon \int_0^1 \int_0^y \varphi(\xi,\eta) d\eta dy d\xi d\varepsilon \Bigg\} + \psi_1(x,t) - \psi_1(0,t) - x \left. \frac{\partial \psi_1(x,t)}{\partial x} \right|_{x=0} \\
&+ x \Bigg\{ \psi_4(x,t) - \psi_3(x,t) - \left(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx \right] \Theta^T + \left[\int_0^1 \mathbf{W}^T(x) dx \right] \right) \\
&\times \left(\mathbf{R}^{\alpha,\beta}(y) P^2 + \mathbf{V}(y) P + \left[\int_0^y \mathbf{V}(\eta) d\eta \right] \right) \left(\mathbf{Q}^{\alpha,\beta}(t) S + \mathbf{Z}(t) \right) \mathbf{U} \\
&- \int_0^1 \int_0^x \int_0^y \int_0^z \varphi(\xi,\eta) d\eta dz d\xi dx - y \Bigg\{ \psi_2(1,t) - \psi_2(0,t) - \left. \frac{\partial \psi_2(x,t)}{\partial x} \right|_{x=0} \\
&- \psi_1(1,t) + \psi_1(0,t) + \left. \frac{\partial \psi_1(x,t)}{\partial x} \right|_{x=0} \\
&- \left(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx \right] \Theta^T + \left[\int_0^1 \mathbf{W}^T(x) dx \right] \right) \\
&\times \left(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y) dy \right] P + \left[\int_0^1 \mathbf{V}(y) dy \right] \right) \left(\mathbf{Q}^{\alpha,\beta}(t) S + \mathbf{Z}(t) \right) \mathbf{U} \\
&+ \int_0^1 \int_0^x \int_0^1 \int_0^y \varphi(\xi,\eta) d\eta d\xi dy dx \Bigg\} - \psi_1(1,t) + \psi_1(0,t) + \left. \frac{\partial \psi_1(x,t)}{\partial x} \right|_{x=0} \Bigg\} \\
&+ \psi_3(y,t). \qquad (4.8)
\end{aligned}
$$

Now, to approximate the other functions in Equation (1.1), it is necessary to integrate from Equation (4.2) concerning variable x:

$$
\begin{aligned}
\frac{\partial^3 u(\mathbf{x},t)}{\partial x \partial y^2} \simeq{}& \left((\mathbf{H}^{\alpha,\beta}(x))^T \Theta^T + \mathbf{W}^T(x) \right) \mathbf{R}^{\alpha,\beta}(y) \left(\mathbf{Q}^{\alpha,\beta}(t) S + \mathbf{Z}(t) \right) \mathbf{U} \\
&+ \int_0^x \varphi(\xi,y) d\xi + \frac{\partial^3 u(0,y,t)}{\partial x \partial y^2}. \qquad (4.9)
\end{aligned}
$$

To calculate the unknown function $\frac{\partial^3 u(0,y,t)}{\partial x \partial y^2}$, we take an integral from Equation (4.9) concerning with x in the range 0 to 1:

$$
\begin{aligned}
\frac{\partial^3 u(0,y,t)}{\partial x \partial y^2} \simeq{}& \frac{\partial^2 \psi_4(y,t)}{\partial y^2} - \frac{\partial^2 \psi_3(y,t)}{\partial y^2} - \Bigg(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx \right] \Theta^T \\
&+ \left[\int_0^1 \mathbf{W}^T(x) dx \right] \Bigg) \mathbf{R}^{\alpha,\beta}(y) \left(\mathbf{Q}^{\alpha,\beta}(t) S + \mathbf{Z}(t) \right) \mathbf{U} - \int_0^1 \int_0^x \varphi(\xi,y) d\xi dx. \qquad (4.10)
\end{aligned}
$$

Then, by substituting Equation (4.10) in Equation (4.9), we get

$$
\begin{aligned}
\frac{\partial^3 u(\mathbf{x},t)}{\partial x \partial y^2} \simeq{}& \left((\mathbf{H}^{\alpha,\beta}(x))^T \Theta^T + \mathbf{W}^T(x) \right) \mathbf{R}^{\alpha,\beta}(y) \left(\mathbf{Q}^{\alpha,\beta}(t) S + \mathbf{Z}(t) \right) \mathbf{U} \\
&+ \int_0^x \varphi(\xi,y) d\xi + \Bigg\{ \frac{\partial^2 \psi_4(y,t)}{\partial y^2} - \frac{\partial^2 \psi_3(y,t)}{\partial y^2} - \Bigg(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx \right] \Theta^T
\end{aligned}
$$

$$+\left[\int_0^1 \mathbf{W}^T(x)dx\right]\Big)\mathbf{R}^{\alpha,\beta}(y)\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}-\int_0^1\int_0^x \varphi(\xi,y)d\xi dx\Big\}. \quad (4.11)$$

Next, we take the integral from Equation (4.11) with respect to x:

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial y^2} \simeq \Big((\mathbf{H}^{\alpha,\beta}(x))^T(\Theta^T)^2+\mathbf{W}^T(x)\Theta^T+\left[\int_0^x \mathbf{W}^T(\xi)d\xi\right]\Big)$$
$$\times \mathbf{R}^{\alpha,\beta}(y)\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^x\int_0^\varepsilon \varphi(\xi,y)d\xi d\varepsilon + x\Big\{\frac{\partial^2\psi_4(y,t)}{\partial y^2}$$
$$-\frac{\partial^2\psi_3(y,t)}{\partial y^2}-\Big(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx\right]\Theta^T+\left[\int_0^1 \mathbf{W}^T(x)dx\right]\Big)\mathbf{R}^{\alpha,\beta}(y)$$
$$\times\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^1\int_0^x \varphi(\xi,y)d\xi dx\Big\}+\frac{\partial^2\psi_3(y,t)}{\partial y^2}. \quad (4.12)$$

At the end of this process, we take the integral from Equation (4.12) with respect to y:

$$\frac{\partial u(\mathbf{x},t)}{\partial y} \simeq \Big((\mathbf{H}^{\alpha,\beta}(x))^T(\Theta^T)^2+\mathbf{W}^T(x)\Theta^T+\left[\int_0^x \mathbf{W}^T(\xi)d\xi\right]\Big)$$
$$\times\left(\mathbf{R}^{\alpha,\beta}(y)P+\mathbf{V}(y)\right)\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^y\int_0^x\int_0^\varepsilon \varphi(\xi,\eta)d\xi d\varepsilon d\eta$$
$$+x\Big\{\frac{\partial\psi_4(y,t)}{\partial y}-\frac{\partial\psi_4(y,t)}{\partial y}\Big|_{y=0}-\frac{\partial\psi_3(y,t)}{\partial y}+\frac{\partial\psi_3(y,t)}{\partial y}\Big|_{y=0}$$
$$-\Big(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx\right]\Theta^T+\left[\int_0^1 \mathbf{W}^T(x)dx\right]\Big)\left(\mathbf{R}^{\alpha,\beta}(y)P+\mathbf{V}(y)\right)$$
$$\times\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}+\int_0^y\int_0^1\int_0^x \varphi(\xi,\eta)d\xi dx d\eta\Big\}$$
$$+\frac{\partial\psi_3(y,t)}{\partial y}-\frac{\partial\psi_3(y,t)}{\partial y}\Big|_{y=0}+\frac{\partial u(x,0,t)}{\partial y}. \quad (4.13)$$

It is noted that, the function $\frac{\partial u(x,0,t)}{\partial y}$ is unknown. So, we take integral from Equation (4.13) concerning with y in the range 0 to 1:

$$\frac{\partial u(x,0,t)}{\partial y} \simeq \psi_2(x,t)-\psi_1(x,t)-\Big((\mathbf{H}^{\alpha,\beta}(x))^T(\Theta^T)^2+\mathbf{W}^T(x)\Theta^T$$
$$+\left[\int_0^x \mathbf{W}^T(\xi)d\xi\right]\Big)\Big(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y)dy\right]P+\left[\int_0^1 \mathbf{V}(y)dy\right]\Big)\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}$$
$$-\int_0^1\int_0^y\int_0^x\int_0^\varepsilon \varphi(\xi,\eta)d\xi d\varepsilon d\eta dy - x\Big\{\psi_4(1,t)-\psi_4(0,t)-\frac{\partial\psi_4(y,t)}{\partial y}\Big|_{y=0}$$
$$-\psi_3(1,t)+\psi_3(0,t)+\frac{\partial\psi_3(y,t)}{\partial y}\Big|_{y=0}-\Big(\left[\int_0^1 (\mathbf{H}^{\alpha,\beta}(x))^T dx\right]\Theta^T$$
$$+\left[\int_0^1 \mathbf{W}^T(x)dx\right]\Big)\Big(\left[\int_0^1 \mathbf{R}^{\alpha,\beta}(y)dy\right]P+\left[\int_0^1 \mathbf{V}(y)dy\right]\Big)\left(\mathbf{Q}^{\alpha,\beta}(t)S+\mathbf{Z}(t)\right)\mathbf{U}$$

$$+\int_0^1\int_0^y\int_0^1\int_0^x \varphi(\xi,\eta)d\xi dx d\eta dy\Big\} - \psi_3(1,t)+\psi_3(0,t)+\left.\frac{\partial\psi_3(y,t)}{\partial y}\right|_{y=0}.$$

At this point, it is necessary to approximate the part of the Caputo derivative in the problem. For this purpose, we use Equation (4.8) and the properties of the Caputo derivative. Consequently, by placing the approximations obtained above in Equation (1.1) and using nodal points, we reach the system of algebraic equations. By solving the system of equations, the vector of unknown coefficients is obtained. By placing this vector in Equation (4.8), the approximate solution of the problem is achieved.

Algorithm	
Input:	Space domain $\Omega\subset R^2$, time interval $[0,1]$, variable order $0<\gamma(\mathbf{x},t)\le 1$, the number of basis function M, known functions $f(\mathbf{x},t)$, $\boldsymbol{\Upsilon}(u(\mathbf{x},t))$, $\varphi(x,y)$, $\psi_i(x,t), i=1,2$ and $\psi_j(y,t), i=2,4$.
Output:	The approximate solution $u(\mathbf{x},t)$.
Step 1:	Choose the value of M.
Step 2:	Compute the modified operational matrix Θ and complement vector $\mathbf{W}(x)$ of integration corresponding to M by using Eq. (3.1).
Step 3:	Compute the pseudo-operational matrix of VO fractional derivative $\Lambda(\mathbf{x},t)$ corresponding to M by using Eq. (3.2).
Step 4:	Compute the approximation of the appeared functions in Eq. (1.1) by using Steps 2 and 3.
Step 5:	Substitute the approximation functions in Eq. (1.1) and use nodal points to obtain a system of algebraic equations.
Step 6:	Calculate the unknown matrix U and replace it in Eq. (4.8).

5 Error estimation

This section investigates the error of the approximate solution.

Lemma 1. *Assume that $P_m g(x)=\sum_{m=0}^{M}\varepsilon_m\mathbf{H}_m^{\alpha,\beta}(x)$ is the approximation of $g(x)\in H^{\mu}(0,1)$. Then, the following relation is obtained for the truncated error [9]:*

$$\|g-P_m g\|_{L^\infty(0,1)}\le\sqrt{2}CM^{\frac{1}{2}-\mu}\,|||g|||_{H^{\mu;M}(0,1)}\,.$$

Here, C denotes the positive constant dependent on $\mu\ge 1$ and independent of M.

Suppose that $G_{mnk}(x,y,t)=\sum_{m=0}^{M}\sum_{n=0}^{M}\sum_{k=0}^{M}d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x)\mathbf{H}_n^{\alpha,\beta}(y)\mathbf{H}_k^{\alpha,\beta}(t)$ is an approximation of $G(x,y,t)\in H^{\mu}(\Pi)$, $\Pi=[0,1]\times[0,1]\times[0,1]$. Then, it can be written:

$$\begin{aligned}G(x,y,t)-G_{mnk}(x,y,t)&=[G(x,y,t)-PG_{Mnk}(x,y,t)]\\&+[PG_{Mnk}(x,y,t)-PG_{MMk}(x,y,t)]+[PG_{MMk}(x,y,t)-G_{mnk}(x,y,t)]\end{aligned}\quad(5.1)$$

where

$$G(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x)\mathbf{H}_n^{\alpha,\beta}(y)\mathbf{H}_k^{\alpha,\beta}(t),$$

$$PG_{Mnk}(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x)\mathbf{H}_n^{\alpha,\beta}(y)\mathbf{H}_k^{\alpha,\beta}(t),$$

$$PG_{MMk}(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{M}\sum_{k=0}^{\infty} d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x)\mathbf{H}_n^{\alpha,\beta}(y)\mathbf{H}_k^{\alpha,\beta}(t).$$

Therefore, we utilize Equation (5.1) to obtain the upper bound of truncated error

$$\begin{aligned}
\|G - G_{mnk}\|_{L^\infty(\Pi)} &\le \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left\| \sum_{m=0}^{\infty} d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x) - \sum_{m=0}^{M} d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x)\right\|_{L^\infty(0,1)} \\
&\times \left\|\mathbf{H}_n^{\alpha,\beta}(y)\right\|_{L^\infty(0,1)} \left\|\mathbf{H}_k^{\alpha,\beta}(t)\right\|_{L^\infty(0,1)} + \sum_{m=0}^{M}\sum_{k=0}^{\infty}\left\| \sum_{n=0}^{\infty} d_{mnk}\mathbf{H}_n^{\alpha,\beta}(y)\right. \\
&\left. - \sum_{n=0}^{M} d_{mnk}\mathbf{H}_n^{\alpha,\beta}(y)\right\|_{L^\infty(0,1)} \left\|\mathbf{H}_m^{\alpha,\beta}(x)\right\|_{L^\infty(0,1)} \left\|\mathbf{H}_k^{\alpha,\beta}(t)\right\|_{L^\infty(0,1)} \\
&+ \sum_{m=0}^{M}\sum_{n=0}^{M} \left\|\mathbf{H}_m^{\alpha,\beta}(x)\right\|_{L^\infty(0,1)} \left\|\mathbf{H}_n^{\alpha,\beta}(y)\right\|_{L^\infty(0,1)} \\
&\times \left\| \sum_{k=0}^{\infty} d_{mnk}\mathbf{H}_k^{\alpha,\beta}(t) - \sum_{k=0}^{M} d_{mnk}\mathbf{H}_k^{\alpha,\beta}(t)\right\|_{L^\infty(0,1)}.
\end{aligned}$$

By considering the following formulas

$$R_{nk}(x) = d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x), \quad R_{mk}(y) = d_{mnk}\mathbf{H}_n^{\alpha,\beta}(y), \quad R_{mn}(t) = d_{mnk}\mathbf{H}_k^{\alpha,\beta}(t),$$

$$ER_{nk}(x) = \sum_{m=0}^{\infty} d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x) - \sum_{m=0}^{M} d_{mnk}\mathbf{H}_m^{\alpha,\beta}(x),$$

$$ER_{mk}(y) = \sum_{n=0}^{\infty} d_{mnk}\mathbf{H}_n^{\alpha,\beta}(y) - \sum_{n=0}^{M} d_{mnk}\mathbf{H}_n^{\alpha,\beta}(y),$$

$$ER_{mn}(t) = \sum_{k=0}^{\infty} d_{mnk}\mathbf{H}_k^{\alpha,\beta}(t) - \sum_{k=0}^{M} d_{mnk}\mathbf{H}_k^{\alpha,\beta}(t)$$

and also Lemma 1, we conclude:

$$\begin{aligned}
&\|G - G_{mnk}\|_{L^\infty(\Pi)} \\
&\le \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \sqrt{2}CM^{\frac{1}{2}-\mu} \, |||R_{nk}|||_{H^{\mu;M}(0,1)} \left\|\mathbf{H}_n^{\alpha,\beta}(y)\right\|_{L^\infty(0,1)} \left\|\mathbf{H}_k^{\alpha,\beta}(t)\right\|_{L^\infty(0,1)} \\
&+ \sum_{m=0}^{M}\sum_{k=0}^{\infty} \sqrt{2}CM^{\frac{1}{2}-\mu} \, |||R_{mk}|||_{H^{\mu;M}(0,1)} \left\|\mathbf{H}_m^{\alpha,\beta}(x)\right\|_{L^\infty(0,1)} \left\|\mathbf{H}_k^{\alpha,\beta}(t)\right\|_{L^\infty(0,1)} \\
&+ \sum_{m=0}^{M}\sum_{n=0}^{M} \sqrt{2}CM^{\frac{1}{2}-\mu} \left\|\mathbf{H}_m^{\alpha,\beta}(x)\right\|_{L^\infty(0,1)} \left\|\mathbf{H}_n^{\alpha,\beta}(y)\right\|_{L^\infty(0,1)} |||R_{mn}|||_{H^{\mu;M}(0,1)}.
\end{aligned}$$

Due to the above outcome and the assumptions of Lemma 1, it can be observed that by increasing the M, the truncated error tends to zero.

6 Numerical results

In this section, we examine some numerical examples to confirm the effectiveness of the methodology. It should be noted that to demonstrate the accuracy of the proposed method, we use

$$L_2\text{−}error = \sqrt{\sum_{i=1}^{N} |u(\mathbf{x}_i) - u_M(\mathbf{x}_i)|^2},\ L_\infty\text{−}error = \max_{1\le i\le N} |u(\mathbf{x}_i) - u_M(\mathbf{x}_i)|,$$

$$RMS = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |u(\mathbf{x}_i) - u_M(\mathbf{x}_i)|^2}.$$

So that, in the above formulas $u(\mathbf{x}_i)$ and $u_M(\mathbf{x}_i)$ are the exact and approximate solutions at a specific time, respectively. Also, we solve the problems in different domains, which are defined as follows:

- Rectangular domain (rectangular): $\Omega_1 = \left\{(x, y) \in R^2 : 0 \le x, y \le 1\right\}$.
- Non-regular domain (circular):

$$\Omega_2 = \left\{(x, y) \in R^2 : (x - 0.5)^2 + (y - 0.5)^2 \le 0.25\right\}.$$

Example 1. Consider the following three-dimensional variable-order time-fractional partial differential equations [11]:

$$D_t^{\gamma(\mathbf{x},t)} u(\mathbf{x}, t) = \frac{\partial^2 u(\mathbf{x}, t)}{\partial x^2} + \frac{\partial^2 u(\mathbf{x}, t)}{\partial y^2} - \frac{\partial u(\mathbf{x}, t)}{\partial x} - \frac{\partial u(\mathbf{x}, t)}{\partial y} + \frac{2t^{2-\gamma(\mathbf{x},t)}}{\Gamma(3 - \gamma(\mathbf{x}, t))} + 2x + 2y - 4,$$

The corresponding initial and boundary conditions of this problem are computed due to the exact solution $u(\mathbf{x}, t) = x^2 + y^2 + t^2$. By implementing the present method for this example, we get the exact solution. From Table 1, we realize that our discretization method for different choices of $\alpha(\mathbf{x}, t)$ is more accurate in comparison with the method in [11].

Example 2. Consider the following three-dimensional non-linear variable-order time-fractional advection-reaction-diffusion equation [15]:

$$D_t^{\gamma(\mathbf{x},t)} u(\mathbf{x}, t) = \frac{\partial^2 u(\mathbf{x}, t)}{\partial x^2} + \frac{\partial^2 u(\mathbf{x}, t)}{\partial y^2} + \frac{\partial u(\mathbf{x}, t)}{\partial x} + \frac{\partial u(\mathbf{x}, t)}{\partial y} + u(\mathbf{x}, t)(1 - u(\mathbf{x}, t)) + f(\mathbf{x}, t),$$

where $f(\mathbf{x}, t) = \frac{2t^{2-\gamma(\mathbf{x},t)}}{\Gamma(3-\gamma(\mathbf{x},t))} x^2y^2 + x^4y^4t^4 + t^2(-2xy^2 - 2y^2 + x^2(-2 - y^2)) - 2x^2yt^2$. The corresponding initial and boundary conditions of this problem are

Table 1. The errors of the approximate solution for diverse choices of $\alpha(\mathbf{x},t)$ with $M=2$ in the rectangular domain Ω_1 of Example 1.

Present method $\gamma(\mathbf{x},t)=0.5, 0.8, 0.8-0.1\cos(xt)\sin(x)-0.1\cos(yt)\sin(y)$			
t	L_2-error	L_∞-error	RMS
0.2	9.2550×10^{-41}	9.1835×10^{-41}	2.7904×10^{-41}
0.4	6.3791×10^{-40}	3.6734×10^{-40}	1.9233×10^{-40}
0.6	8.4668×10^{-40}	7.3468×10^{-40}	2.5528×10^{-40}
0.8	4.1070×10^{-40}	3.6734×10^{-40}	1.2383×10^{-40}
1.0	1.1020×10^{-39}	7.3468×10^{-40}	3.3227×10^{-40}
Ref. [11] $\gamma(\mathbf{x},t)=0.5$			
t	L_2-error	L_∞-error	RMS
0.2	1.6997×10^{-4}	2.9999×10^{-5}	1.5452×10^{-5}
0.4	1.7686×10^{-4}	2.1276×10^{-4}	1.6078×10^{-5}
0.6	1.7975×10^{-4}	3.1766×10^{-5}	1.6341×10^{-5}
0.8	1.8144×10^{-4}	3.2083×10^{-5}	1.6494×10^{-5}
1.0	1.8256×10^{-4}	3.2296×10^{-5}	1.6596×10^{-5}
Ref. [11] $\gamma(\mathbf{x},t)=0.8$			
t	L_2-error	L_∞-error	RMS
0.2	1.1577×10^{-3}	2.0416×10^{-4}	1.0536×10^{-4}
0.4	1.2020×10^{-3}	2.1276×10^{-4}	1.0938×10^{-4}
0.6	1.2147×10^{-3}	2.1522×10^{-4}	1.1054×10^{-4}
0.8	1.2210×10^{-3}	2.1641×10^{-4}	1.1109×10^{-4}
1.0	1.2248×10^{-3}	2.1712×10^{-4}	1.1143×10^{-4}
Ref. [11] $\gamma(\mathbf{x},t)=0.8-0.1\cos(xt)\sin(x)-0.1\cos(yt)\sin(y)$			
t	L_2-error	L_∞-error	RMS
0.2	6.6238×10^{-4}	1.1668×10^{-4}	6.0217×10^{-5}
0.4	6.9398×10^{-4}	1.2258×10^{-4}	6.3089×10^{-5}
0.6	7.1434×10^{-4}	1.2612×10^{-4}	6.4940×10^{-5}
0.8	7.3611×10^{-4}	1.2974×10^{-4}	6.6919×10^{-5}
1.0	7.6200×10^{-4}	1.3394×10^{-4}	6.9272×10^{-5}

Table 2. The Maximum absolute error of the approximate solution for $\gamma(\mathbf{x},t)=0.9$ in the rectangular domain Ω_1 of Example 2.

	Present method		
	M	L_∞-error	CPU
$y=t=0.5,\ 0\le x\le 1$	2	6.14273×10^{-14}	1.848613
$x=t=0.5,\ 0\le y\le 1$	2	6.05582×10^{-14}	1.848613
	Ref. [15]		
	N	L_∞-error	CPU
$y=t=0.5,\ 0\le x\le 1$	5	0.00033	7.47
	7	0.000028	10.54
	9	0.0000022	12.35
$x=t=0.5,\ 0\le y\le 1$	5	0.00038	7.47
	7	0.000032	10.53
	9	0.0000026	12.35

computed due to the exact solution $u(\mathbf{x},t)=x^2y^2t^2$. Due to the method, for $M=2$ and $\gamma(\mathbf{x},t)=0.9$ in the rectangular domain Ω_1, we get

$$u(\mathbf{x},t)=1.0000000001095t^2x^2y^2+\cdots\times 10^{-13}.$$

In order to compare the results with method [15], we listed the maximum absolute error and CPU time for $\gamma(\mathbf{x},t)=0.9$ and $M=2$ in Table 2. The results illustrate that the proposed method is more efficient than method [15]. Furthermore, the plots of the absolute error in the different domains are shown in Figures 1 and 2. From these figures, we can understand that the proposed method is powerful and efficient for solving nonlinear problems.

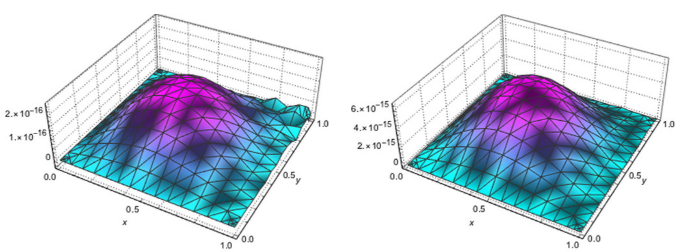

Figure 1. The graphs of the absolute error at different time $t=0.5$ (left) and $t=1$ (right) with $\gamma(x,t)=0.55+0.45\sin(xyt)$ and $M=2$ on regular domain Ω_1 of Example 2.

Example 3. We consider three-dimensional non-linear variable-order time-fractional nonlinear Fisher's equation [15]:

$$D_t^{\gamma(\mathbf{x},t)}u(\mathbf{x},t)=\frac{\partial^2 u(\mathbf{x},t)}{\partial x^2}+\frac{\partial^2 u(\mathbf{x},t)}{\partial y^2}+u^2(\mathbf{x},t)(1-u(\mathbf{x},t))+f(\mathbf{x},t),$$

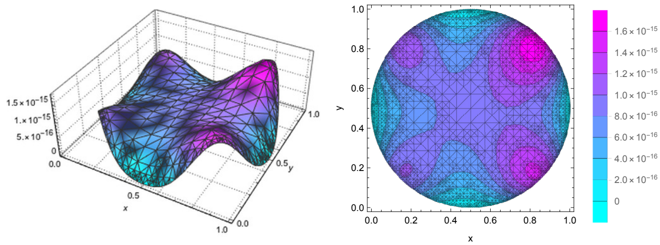

Figure 2. The graphs of the absolute error (left) and contour plot (right) for $\gamma(x,t) = 0.55 + 0.45\sin(xyt)$ and $M = 2$ at $t = 0.5$ on non-regular domain Ω_2 of Example 2.

Table 3. The Maximum absolute error of the approximate solution for $\gamma(\mathbf{x},t) = 0.9$ in the rectangular domain Ω_1 of Example 3.

	Present method		
	M	L_∞-error	CPU
$y = t = 0.5,\ 0 \le x \le 1$	2	5.909235×10^{-6}	1.294003
	3	2.176107×10^{-7}	6.679956
$x = t = 0.5,\ 0 \le y \le 1$	2	5.909235×10^{-6}	1.284393
	3	1.257769×10^{-7}	6.622301
	Ref. [15]		
	N	L_∞-error	CPU
$y = t = 0.5,\ 0 \le x \le 1$	5	0.00026	7.12
	7	0.000041	9.26
	9	0.0000067	10.55
$x = t = 0.5,\ 0 \le y \le 1$	5	0.0003	7.10
	7	0.000039	9.41
	9	0.0000064	10.48

where

$$f(\mathbf{x},t)=\exp(xy)\left(\frac{2t^{2-\gamma(\mathbf{x},t)}}{\Gamma(3-\gamma(\mathbf{x},t))}+\exp(xy)t^4(-1+\exp(xy)t^2)-\frac{t^2}{2}(x^2+y^2)\right).$$

The corresponding initial and boundary conditions of this problem are computed due to the exact solution $u(\mathbf{x},t) = t^2\exp(xy)$. Fisher's equation has appeared in different mathematical models, such as flame propagation [7], nuclear reactor theory [3], branching Brownian motion process [2] and chemical kinetics [12]. We have solved this problem with regard to the presented method. The outcomes are demonstrated in Figure 3 and Table 3. The maximum absolute error and CPU time (in seconds) in the rectangular domain Ω_1 with various choices of M are listed in Table 3. This table shows that the proposed method with a smaller number of basis functions has higher accuracy than method [15]. Moreover, the graphs of the absolute error on non-regular domain Ω_2 are plotted in Figure 3.

Example 4. Consider the following three-dimensional non-linear variable-order

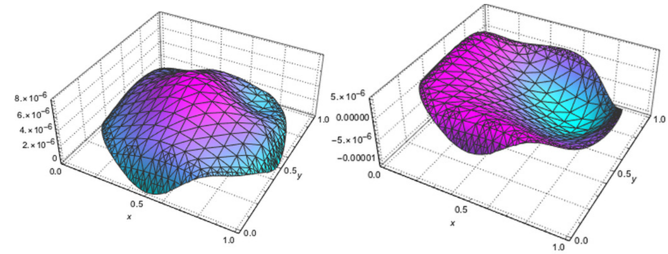

Figure 3. The graphs of the absolute error with $\gamma(x,t)=0.2$ (left) and $\gamma(x,t)=1-0.75\exp(-(xyt)^2)$ (right) at $t=1$ and $M=3$ on non-regular domain Ω_2 of Example 3.

time-fractional advection-diffusion equation [10]:

$$D_t^{\gamma(\mathbf{x},t)}u(\mathbf{x},t) = 5\frac{\partial^2 u(\mathbf{x},t)}{\partial x^2} + 5\frac{\partial^2 u(\mathbf{x},t)}{\partial y^2} - \frac{\partial u(\mathbf{x},t)}{\partial x} - \frac{\partial u(\mathbf{x},t)}{\partial y} + \sin\left(u(\mathbf{x},t)\right)) + f(\mathbf{x},t),$$

where

$$f(\mathbf{x},t) = \left(2t^{1-\gamma(\mathbf{x},t)}E_{1,2-\gamma(\mathbf{x},t)}(2t)\right)(\sin(x)+\sin(y)) + \exp(2t)\left(5(\sin(x)+\sin(y))+\cos(x)+\cos(y)\right).$$

The corresponding initial and boundary conditions of this problem are computed due to the exact solution $u(\mathbf{x},t)=(\sin(x)+\sin(y))\exp(2t)$. The obtained approximate solution of the provided method for $\gamma(\mathbf{x},t)=1$ with $M=2$ is as follows:

$$u(\mathbf{x},t) = \exp(2t)\sin(x) + \exp(2t)\sin(y) + \cdots \times 10^{-13}.$$

Also, the different errors are provided in the rectangular domain Ω_1 for this example in Table 4. The result demonstrates that the present method at various times is more accurate than the method in [10]. Moreover, Figures 4 and 5 can be seen that the method on non-regular domain Ω_2 also has high accuracy.

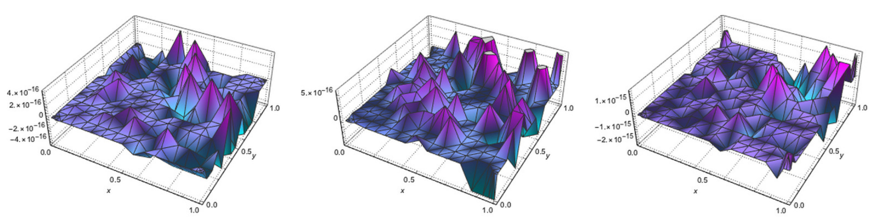

Figure 4. The graphs of the absolute error with $t=0.3$ (left), $t=0.5$ (centre) and $t=1$ (right) with $\gamma(x,t)=1-0.75\exp(-(xyt)^2)$ and $M=2$ on regular domain Ω_1 of Example 4.

Table 4. The error of the approximate solution for $\gamma(\mathbf{x},t) = 0.25 + 0.25\sin^2(xyt)$ in the rectangular domain Ω_1 of Example 4.

	Present method		
t	L_2-error	L_∞-error	RMS
0.2	1.5324×10^{-18}	7.0858×10^{-19}	4.6204×10^{-18}
0.4	1.5324×10^{-18}	7.0858×10^{-19}	4.6204×10^{-18}
0.6	1.5324×10^{-18}	7.0858×10^{-19}	4.6204×10^{-18}
0.8	1.5324×10^{-18}	7.0858×10^{-19}	4.6204×10^{-18}
1.0	1.5324×10^{-18}	7.0858×10^{-19}	4.6204×10^{-18}
	Ref. [10]		
t	L_2-error	L_∞-error	RMS
0.2	1.4857×10^{-4}	3.4763×10^{-5}	1.4857×10^{-5}
0.4	7.4543×10^{-4}	2.0172×10^{-4}	7.4543×10^{-5}
0.6	1.2883×10^{-3}	3.3247×10^{-4}	1.2883×10^{-4}
0.8	9.6180×10^{-4}	3.1914×10^{-4}	9.6198×10^{-5}
1.0	9.3911×10^{-3}	3.0913×10^{-4}	9.3911×10^{-5}

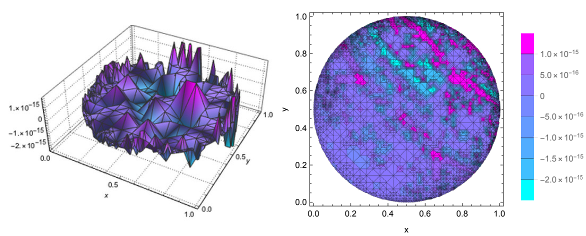

Figure 5. The graphs of the absolute error (left) and contour plot (right) for $\gamma(x,t) = 0.8 - 0.1\cos(xt)\sin(x) - 0.1\cos(yt)\sin(y)$ and $M = 2$ at $t = 1$ on non-regular domain Ω_2 of Example 4.

7 Conclusions

In this paper, by combining the modified operational matrix and complement vector of integration and pseudo-operational matrix of VO-fractional derivative with the discretization method, we introduced a new technique for solving three-dimensional variable-order time-fractional partial differential equations. In the proposed method algorithm, MOM and CV of integration and problem conditions are used, and this process is very effective in the accuracy of the approximate solution. The method is implemented in several numerical examples where the results are presented in the form of tables and figures. It should be noted that the figures are drawn in the form of two types of space domains. The results illustrate that the method introduced in two types of space domains has high accuracy and efficiency and is in high agreement with the analytical solution.

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